The Chebyshev Problem

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1 The Chebyshev problem

A closed subset C of an Hilbert space H with norm $\|\cdot\|$ is called *Chebyshev set* if for every point $x \in H$ there exists a *unique* point $\pi(x)$ with minimum distance from x, that is,

$$\|x - \pi(x)\| = d(x, C) \text{ and } \|x - y\| > \|x - \pi(x)\| \text{ if } y \in C \text{ and } y \neq \pi(x).$$

In such case, the function $x \mapsto \pi(x)$ is called *metric projection* on C.

One of the fundamental and well known properties of an Hilbert space is that every closed and convex subset is a Chebyshev set.
It is a natural question asking if the converse is true:

Every Chebyshev set is convex?

Several authors proved that under various extra hypotheses, the answer is positive. Anyway, Klee in [6] showed some evidence for the existence of a non–convex Chebyshev set.
At the moment the problem is open.

The term "Chebyshev set" was introduced by Efimov and Stechkin [4] in 1961 trying to understand if the set of rational functions

$$R_{n,m} = \left\{ \frac{a_0 + a_1 x ... + a_n x^n}{b_0 + b_1 x ... + b_m x^m} \;\middle|\; a_i, b_i \in \mathbb{R} \right\}$$

had unique projection in $L^p(0,1)$ (Chebyshev [3] proved such result in $C([0,1])$).
Even if the Chebyshev problem was explicitly formulated in the actual form in [4] and independently in the works of Klee, several authors considered similar questions since the beginning of the century. It is hence difficult to decide if there was a real "author" of the problem.

We show briefly a short historical list of the main contributions:

- In 1934 Bunt [2] proves that a Chebyshev set in the Euclidean plane is convex.
- In 1938 Kritikos [7], moving by Bunt's results, extends the theorem to $\mathbb{R}^n$.
- In 1961 Efimov and Stechkin [4] show that in a general Hilbert space a Chebyshev set which is *approximately compact* is convex.

  A subset C of a normed space $(X, \|\cdot\|)$ is called approximately compact if given any $x \in X$ and a sequence of points $y_n \in C$ with $\|x - y_n\| \to d(x, C)$, there exists a subsequence y_{n_k} and a point $y \in C$ such that $y_{n_k} \to y$.
- in Klee [5] proves that Chebyshev set which is weakly closed is convex.

- In 1969 Asplund [1] shows that a Chebyshev set with a continuous metric projection is convex.

By the result of Asplund, the Chebyshev problem can then be reformulated as follows:

Every Chebyshev set has a continuous metric projections?

There are several hypotheses which imply the continuity of the metric projection, notably the weakly closedness of the set C in the Hilbert space H.
The fact that in $\mathbb{R}^n$ the weak and the strong topology coincide, makes such hypothesis superfluous and gives an heuristic motivation for the strong difference between the finite and the infinite dimensional case.

At the moment, to our knowledge, the problem to prove the same conclusion for closed subsets of a Hilbert space (even separable), not weakly closed and sharing the unique projection property, is completely open. Also a counterexample was not found, though actually most of the work concentrated on proving the assertion not on disproving it.
For further details and references, see the rather comprehensive article of Vlasov [2].

2 A proof for weakly closed subsets C

We are going to show a proof (up to our knowledge original), in collaboration with G. Alberti, that a Chebyshev subset C of the Hilbert space H which is weakly closed, it is convex. In particular, this proof works for $\mathbb{R}^n$.

Considering the distance function from C, $d_C(x) : H \to \mathbb{R}$, we have that $d_C(x) = \|x - \pi(x)\|$, by the uniqueness of the point of minimum distance, at every point $x \in H \setminus C$ the distance function d_C is differentiable and the modulus of the gradient is 1. Defining the function

$$W(x) = \frac{\|x\|^2 - d_C^2(x)}{2},$$

we then get,

$$\nabla W(x) = x - d_C(x)\nabla d_C(x) = \pi(x),$$

at every point of $H \setminus C$, by the properties of the distance function.
Moreover, the function $W : H \to \mathbb{R}$ is convex, being $\sup$ of linear continuous functionals on H,

$$W(x) = \frac{\|x\|^2 - \inf_{y\in C} \|x - y\|^2}{2} = \sup_{y\in C} \frac{2\langle x, y\rangle - \|y\|^2}{2}.$$

We study the following ODE *system*,

$$\begin{cases} \dot{x}(t) = \nabla d_C(x(t)), \\ x(0) = y. \end{cases} \tag{2.1}$$

Suppose that the system (2.1) has a unique maximal solution for every initial point $y \in H \setminus C$, then for $t \leq 0$, the solution $x(t)$ follows the segment connecting y to its projection $\pi(y)$ on C. This implies, considering the solution curve for $t \geq 0$, that this latter is also a straight halfline, moreover all its points $x(t)$ have a common projection $\pi(x(t)) \in C$.
From this fact we can conclude that for every point y not belonging to C there exists a separating hyperplane between y and C, which implies that C is convex. Indeed, by choosing a hyperplane crossing orthogonally the "interior" of the segment connecting y and $\pi(y)$, there cannot be points of C in the halfspace containing y, otherwise such points would be definitely closer to $x(t)$ than $\pi(x(t))$, as $t \to +\infty$.

We notice now that the existence and uniqueness of solutions for every initial point $y \in H \setminus C$ would follow from the same result for the solutions of the following ODE system, being connected by a continuous perturbation and reparametrization of the solutions,

$$\begin{cases} \dot{x}(t) = -\nabla W(x(t)) \\ x(0) = y. \end{cases} \tag{2.2}$$

Hence, we can concentrate on the problem of the existence and uniqueness of the solutions for this system.

We first see that if the set C is weakly closed, then the map $\pi : H \to C$ is continuous and so the function $\nabla W = \pi$. If $x_n \to x$, it is easy to see that the sequence $\pi(x_n)$ is bounded, then by the compactness of the balls in the weak topology of H, there exists a subsequence x_{n_k} such that $\pi(x_{n_k})$ weakly–converges to some point z which belongs to C, since it is weakly–closed. As the norm of H is lower semicontinuous in the weak topology, we conclude that

$$\|x - z\| \leq \liminf_{k\to\infty} \|x_{n_k} - \pi(x_{n_k})\| = \liminf_{k\to\infty} d_C(x_{n_k}) = d_C(x),$$

hence, by the uniqueness of the projection on C, we have $z = \pi(x)$.
Setting $y_{n_k} = x_{n_k} - \pi(x_{n_k})$, we see that y_{n_k} weakly–converges to $x - \pi(x)$ and $\|y_{n_k}\|$ converges to $\|x - \pi(x)\|$, which, in a Hilbert space, implies that y_{n_k} converges *in the norm* of H to $x - \pi(x)$, hence $\pi(x_{n_k})$ converges to $\pi(x)$ in the norm of H.
The continuity of the projection map $\pi : H \to C$ follows by this argument.

Getting back to system (2.2), uniqueness is guaranteed by the convexity of the function W (implying the monotonicity of $\nabla W = \pi$), while the continuity of ∇W gives the existence of solutions by the usual argument for ODE's (see [1]).

The convexity of C then follows, as we said above.

References

[1] H. H. Brézis, *Opérateurs maximaux monotones et semi–groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, North–Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[2] L. P. Vlasov, *Approximative properties of sets in normed linear spaces*, Russ. Math. Surveys **28** (1973), no. 6, 1–66.