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The Weyl Tensor of Riemannian Manifolds and some Topological Invariants in Dimension Four

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Introduction

This dissertation, articulated in four chapters, provides an introduction to some topics related to the study of the *Weyl tensor*[1] of Riemannian manifolds[2], with special emphasis on Einstein manifolds[3]. Moreover, particular attention is given to the four–dimensional case.

The Weyl tensor arises from the following well–known orthogonal decomposition of the Riemann curvature tensor (in dimension $n \geq 4$) of a Riemannian manifold (M, g),

$$\mathrm{Riem} = \frac{\mathrm{R}}{2n(n-1)} g \owedge g + \frac{1}{n-2} \overset{\circ}{\mathrm{Ric}} \owedge g + \mathrm{Weyl}\,,$$

where the Ricci tensor[4] Ric is its $(1,3)$–trace, the scalar curvature R its complete trace and $\overset{\circ}{\mathrm{Ric}} = \mathrm{Ric} - \mathrm{R} g/n$ denotes the trace–free component of Ric.

Einstein manifolds are manifolds (M, g) whose metrics satisfy $\mathrm{Ric} = \lambda g$, for some constant $\lambda \in \mathbb{R}$ (i.e. $\overset{\circ}{\mathrm{Ric}} = 0$) and as such, due to the above decomposition of the Riemann tensor, they have their curvature completely determined by the Weyl tensor (and the constant λ). These manifolds play a fundamental role in general relativity as the *Einstein's field equation* (in a 4–dimensional Lorentzian manifold[5], see for instance [8]) reads

$$\mathrm{Ric} - \frac{\mathrm{R}}{2} g + \Lambda g = kT\,,$$

where Λ is the so–called *cosmological constant*, k is the *Einstein's gravitational constant* and T is the *stress–momentum tensor*. In the vacuum there holds $T = 0$, thus $\mathrm{Ric} = (\mathrm{R}/2 - \Lambda)g = \lambda g$ and the solutions are 4–dimensional Einstein (Lorentzian) metrics. It is an extremely interesting fact that the Einstein's field equation arises as the Euler–Lagrange[6] equation of the *Einstein–Hilbert action*[7] which, in the vacuum and ignoring the physical constants, is given (in any dimension n) by

$$\mathfrak{S}(g) = \mathrm{Vol}_g(M)^{-\frac{n-2}{n}} \int_M \mathrm{R}_g \, \mathrm{d}V_g\,, \tag{1}$$

see for instance [9]. The metrics on an n–dimensional differentiable manifold M such that the first variation of this functional is zero are exactly the Einstein metrics. This holds also in the Riemannian setting, that is, considering the functional of Riemannian metrics.

[1] After the German mathematician Hermann Klaus Hugo Weyl (1885–1955) [36].
[2] After the German mathematician Georg Friedrich Bernhard Riemann (1826–1866) [37].
[3] After the German physicist Albert Einstein (1879–1955) [38].
[4] After the Italian mathematician Gregorio Ricci–Curbastro (1853–1925) [39].
[5] After the Dutch physicist Hendrik Antoon Lorentz (1853–1928) [40].
[6] After the Swiss mathematician Leonhard Euler (1707–1783) [41] and the French naturalised Italian mathematician Joseph–Louis Lagrange (1736–1813) [42].
[7] After Albert Einstein and the German mathematician David Hilbert (1862–1943) [43].

The four–dimensional case has some very peculiar features, that in higher dimensions either stop being valid or are too complicated to deal with explicitly in general. Namely, the very special further orthogonal decomposition of the Weyl tensor into its *self–dual* and *anti–self–dual* components $W = W^+ + W^-$, the *Chern–Gauß–Bonnet formula*[8] [10]

$$\begin{aligned}\chi(M) &= \frac{1}{32\pi^2}\int_M \left(|\mathrm{Riem}|^2 - 4|\mathrm{Ric}|^2 + \mathrm{R}^2\right) \mathrm{d}V_M \\ &= \frac{1}{192\pi^2}\int_M \left(6|\mathrm{Weyl}|^2 - 12|\mathring{\mathrm{Ric}}|^2 + \mathrm{R}^2\right) \mathrm{d}V_M\,, \end{aligned} \tag{2}$$

which, by means of such decomposition of the Weyl tensor, can also be written as

$$\chi(M) = \frac{1}{192\pi^2}\int_M \left(6|W^+|^2 + 6|W^-|^2 - 12|\mathring{\mathrm{Ric}}|^2 + \mathrm{R}^2\right) \mathrm{d}V_M \tag{3}$$

and the *Hirzebruch formula*[9] [17]

$$\tau(M) = \frac{1}{48\pi^2}\int_M \left(|W^+|^2 - |W^-|^2\right) \mathrm{d}V_M\,, \tag{4}$$

where $\chi(M)$ and $\tau(M)$ are topological invariants: respectively, the *Euler–Poincaré characteristic*[10] and the *signature* of the manifold M (see [6, 28]).
By means of these two results, one can obtain a (necessary only) condition for a four–dimensional manifold to be Einstein, namely the so–called *Hitchin–Thorpe inequality*[11] [18, 34]

$$\chi(M) \geq \frac{3}{2}|\tau(M)|\,, \tag{5}$$

that, in the case of nonnegative or nonpositive sectional curvature, can be improved to

$$\chi(M) \geq \left(\frac{3}{2}\right)^{\frac{3}{2}}|\tau(M)|\,.$$

The main references throughout all our work are the celebrated book by A. L. Besse "*Einstein manifolds*" [6] and the award–winning monograph by G. Catino and P. Mastrolia "*A perspective on canonical Riemannian metrics*" [9].

The first chapter is devoted to a quick summary of prerequisites of differential and Riemannian geometry. The number of independent components of the Weyl tensor and its connection with the *purity* of the Riemann tensor are discussed. Moreover, at the end of the chapter, we give a brief introduction to Cartan formalism[12], which will be necessary for the third chapter.

The second chapter discusses conformal transformations and locally conformally flat (LCF) manifolds. The conformal invariance of the Weyl tensor in dimension $n \geq 4$ and

[8] After the Chinese–American mathematician Shiing–Shen Chern (陳省身, 1911–2004) [44], the German mathematician Johann Carl Friedrich Gauß (1777–1855) [45] and the French mathematician Pierre Ossian Bonnet (1819–1892) [46].

[9] After the German mathematician Friedrich Ernst Peter Hirzebruch (1927–2012) [47].

[10] After Leonhard Euler and the French mathematician and physicist Jules Henri Poincaré (1854–1912) [48]

[11] After the English mathematician Nigel James Hitchin (1946) [49] and the American mathematician John Alden Thorpe (1936) [50].

[12] After the French mathematician Élie Cartan (1869–1951) [51].

of the Cotton tensor[13] in dimension $n = 3$ is shown, implying that the LCF manifolds of the corresponding dimension need to have these tensors trivial. Then, we show the Weyl–Schouten theorem[14] saying that such triviality is actually also a sufficient condition for a manifold to be LCF. Finally, the uniqueness of "global" conformal changes (if they exist) is briefly discussed.

The third chapter describes Chern's original proof of the Chern–Gauß–Bonnet theorem for even–dimensional Riemannian manifolds, following [10, 24] where more details have been added. Using a particular orthonormal frame, as presented in [7], the Chern–Gauß–Bonnet formula (2) in dimension four is then derived and some consequences are discussed. Here we also highlight the main idea of the previous proofs (historically relevant, being studied earlier) of generalisations of the "classical" Gauß–Bonnet theorem for a compact oriented Riemannian manifold embedded in a Euclidean space of higher dimension.

The fourth and last chapter deals with Einstein manifolds, with special attention to the dimension four. Satisfying $\mathrm{Ric} = \lambda g$ for some constant $\lambda \in \mathbb{R}$, Einstein manifolds "stay in the middle" between *constant curvature manifolds* (with $\mathrm{Riem} = \frac{\lambda}{2(n-1)} g \owedge g$), which are completely classified and *constant scalar curvature manifolds* (with $\mathrm{R} = \lambda n$), hence they are neither "too" nor "too little" rigid.

We start by showing the computation of the first variation of the Einstein–Hilbert action (1), whose nullity characterises Einstein manifolds.
Then, we "improve" the orthogonal decomposition of the Weyl tensor in dimension $n = 4$. The standard decomposition of the space of algebraic curvature tensors consists of the irreducible components under the action of the group $O(n)$, but in dimension $n = 4$ the space of the Weyl tensors can be further refined by considering its irreducible orthogonal components under the action of $SO(4)$ (if $n \neq 4$, the action of $SO(n)$ does not provide any new decomposition). Once applied to Riemannian manifolds, this refined decomposition yields the so–called self–dual and anti–self–dual components $W^{\pm}$ of its Weyl tensor, as well as a convenient matrix representation of the curvature 2–form. Moreover, the Chern–Gauß–Bonnet formula (2) in dimension 4 can be rewritten in the form (3).
In Section 4.3, we introduce the signature $\tau(M)$, which is another topological invariant of a four–manifold M, and present (without proof, for which we refer the reader to [28]) the Hirzebruch theorem, showing the equality

$$\tau(M) = \frac{1}{48\pi^2} \int_M \Big(|W^+|^2 - |W^-|^2 \Big) \, \mathrm{d}V_M \, .$$

We then proceed by studying the *Weyl functional*,

$$\mathfrak{W}(g) = \int_M |\mathrm{Weyl}_g|^{n/2} \, \mathrm{d}V_g \, ,$$

which is quadratic in dimension four. We compute its first variation and we show that *conformal Einstein metrics* (i.e. metrics having an Einstein metric in their conformal class) and *half–conformally flat metrics* , (i.e. for which either W^+ or W^- vanishes) are critical metrics, in dimension four.
In the last section, we discuss four–dimensional Einstein manifolds. By combining the Chern–Gauß–Bonnet and Hirzebruch formulae (3), (4), we then obtain the Hitchin–Thorpe inequality (5)

$$\chi(M) \geq \frac{3}{2} |\tau(M)|$$

[13]After the French mathematician Émile Clément Cotton (1872–1950) [52].

[14]After Hermann Klaus Hugo Weyl and the Dutch mathematician Jan Arnoldus Schouten (1883–1971) [53].

which is a necessary condition for a compact oriented 4–dimensional manifold to be Einstein. We underline that this inequality is a necessary (only) condition for a compact and oriented 4–dimensional manifold to be Einstein and no sufficient conditions are known up to now, unless we restrict ourselves to more rigid classes of Riemannian manifolds (e.g. Kähler manifolds[15], see [7]). In light of that, the chapter and the thesis end by providing some examples of manifolds which *do not* admit any Einstein metric, being the converse problem very difficult, the readers of the book of A. L. Besse [6] "*[...] are offered a meal in a starred restaurant in exchange for a new example [of an Einstein manifold]*".

[15] After the German mathematician Erich Kähler (1906–2000) [54].

Chapter 1

Some prerequisites of Riemannian geometry

1.1 Differential manifolds

Definition 1.1.1. An n–*dimensional topological manifold with boundary* is a topological space that is Hausdorff, admits a countable base and such that for every point $p \in M$ there exists an open neighbourhood $U \subseteq M$ of p and a homeomorphism $\varphi\colon U \to \Omega$, with Ω open set of $\mathbb{H}^n = \{ (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0 \}$. The pair (U, φ) is called a *coordinate chart* (or a *local chart*). If $U \cap \mathbb{H}^n = \emptyset$ we will call U an *interior chart*, otherwise we will call it a *boundary chart*.
A point $p \in M$ is said to be an *interior point* if it admits an interior chart as neighbourhood, otherwise it is called a *boundary point*. We will denote by $\mathring{M}$ and ∂M the set of interior and boundary points respectively.

Definition 1.1.2. Let $k \in \mathbb{N} \cup \{\infty\}$. Two charts (U, φ) and (V, ψ) are C^k–*compatible* if either $U \cap V = \emptyset$ or the transition map (which is a homeomorphism)

$$\varphi_\alpha \circ \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a C^k–diffeomorphism (which accounts for the boundary, in case there is one).
A C^k–*atlas* on M is a cover of M with C^k–compatible charts.
A chart is *compatible* with a C^k–atlas if it is C^k–compatible with each of its charts.
Two C^k–atlases are *equivalent* if their union is another C^k–atlas.
A C^k–atlas is *maximal* if it contains all charts compatible with it.
A C^k–*differential structure* on M is a maximal C^k–atlas on M.

Definition 1.1.3. A C^k–*differential manifold with boundary* is a topological manifold with boundary where a C^k–differential structure has been chosen.

Definition 1.1.4. A *topological* (resp. *differential*) *manifold (without boundary)* is a topological (resp. *differential*) manifold M with $\partial M = 0$.

Remark 1.1.5. Every manifold with boundary M is the disjoint union of $\mathring{M}$ and ∂M which are, respectively, an n–dimensional and an $(n-1)$–dimensional manifold without boundary.

We refer the reader to [1, 12, 23] for the basic notions recalled in this section; in particular, for the standard definitions and properties of C^∞ functions, tangent, cotangent and tensor *bundles, vector fields, tensor fields, differential forms, differential operators, etc. We will also assume familiarity with the canonical topological, differential and Riemannian structures of standard spaces like $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$.*

We will write x^i to denote the i–th coordinate induced by a chart (U, φ), with associated local vector fields $\frac{\partial}{\partial x^i}$ (which we will mostly denote by ∂_i) and 1–forms $\mathrm{d}x^i$. We will denote by $C^\infty(M, N)$ the space of C^∞ functions between the two differential manifolds M and N and $C^\infty(M) := C^\infty(M, \mathbb{R})$.

In the entire text, unless stated otherwise, all manifolds will be without boundaries, connected, C^∞ and n–dimensional. We shall also use the Einstein convention for summation over repeated indices, for example, a vector field X over M will be written in local coordinates as $X = X^i\partial_i = X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$.

Let $\pi\colon E \to B$ be a vector bundle of rank k. We will denote by $\Gamma(E)$ the space of its (global) *sections*, that is, functions $s\colon B \to E$ such that $\pi \circ s = \mathrm{id}$. A *(local) frame* for the vector bundle on a subset $U \subseteq B$ is a choice of k linearly independent sections for every point of U. If $U = B$ the frame is said to be a *global frame* and the bundle a *trivial bundle*. The most common vector bundles we will deal with are:

- the line bundle $M \times \mathbb{R}$, with sections $\Gamma(M \times \mathbb{R}) = C^\infty(M)$;
- the tangent bundle TM, whose sections are the vector fields;
- the sphere bundle $\mathbb{S}M$, whose sections are the unitary vector fields;
- the cotangent bundle TM^*, whose sections, $\Gamma(TM^*) = \Omega^1(M)$, are the differential 1–forms;
- the vector bundle $T^r_s M = TM^{*\otimes s} \otimes TM^{\otimes r}$, whose sections are the tensor (fields) of type (r, s) (r is the number of contravariant components and s the number of covariant components, $T^1_0 M = TM$, $T^0_1 M = TM^*$ and $T^0_0 M = M \times \mathbb{R}$);
- the bundle of the alternating k–forms $\Lambda^k M$, who sections, $\Gamma(\Lambda^k M) = \Omega^k(M)$, are the differential k–forms;
- the bundle of the symmetric k–forms $S^k M$, with sections $\Gamma(S^k M) = \Sigma^k(M)$.

If $T \in \Gamma(T^r_s M)$, we will denote by T_p the tensor at the point $p \in M$, which is an element of the vector space $T^r_s M_p = T_pM^{*\otimes s} \otimes T_pM^{\otimes r}$. There is a natural linear isomorphism between $T^r_s M_p$ and the space of multilinear functions from $T_pM^{\oplus s} \oplus T_pM^{*\oplus r}$ to $\mathbb{R}$. In local coordinates an (r, s)–tensor T is therefore given by

$$T = T^{i_1\dots i_r}_{j_1\dots j_s}\, \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}}\,,$$

where

$$T^{i_1\dots i_r}_{j_1\dots j_s} := T\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}}, \mathrm{d}x^{i_1}, \dots, \mathrm{d}x^{i_r}\right).$$

These considerations let us see a tensor $T \in \Gamma(T^r_s M)$ as a $C^\infty(M)$–linear map from the $C^\infty(M)$–module $\Gamma(TM^{\oplus s} \oplus TM^{*\oplus r})$ to $C^\infty(M)$.

1.1.1 Symmetric and skew–symmetric forms on vector spaces

Let V be a real vector space of dimension n. We denote by $\Lambda^k(V^*)$ and $S^k(V^*)$ respectively the sets of the *alternating* and *symmetric* multilinear forms on V^k. For any such form η, we call *degree* of η the value $|\eta| = k$. Recall that

$$\begin{cases} \dim \Lambda^k(V^*) = \dbinom{n}{k} & \text{for } 0 \leq k \leq n\,, \\ \dim S^k(V^*) = \dbinom{n+k-1}{k} & \text{for } k \geq 0\,. \end{cases} \tag{1.1}$$

We define the *exterior* (or *wedge*) product of a k–form η and an s–form ζ as the $(k+s)$–form

$$\begin{aligned} (\eta \wedge \zeta)(\alpha_1, \dots, \alpha_{k+s}) &= \frac{1}{k!s!} \sum_{\sigma \in \Sigma_{k+s}} \mathrm{sgn}(\sigma)(\eta \otimes \zeta)(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k+s)}) \\ &= \frac{1}{k!s!} \sum_{\sigma \in \Sigma_{k+s}} \mathrm{sgn}(\sigma)\, \eta(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)})\zeta(\alpha_{\sigma(k+1)}, \dots, \alpha_{\sigma(k+s)})\,, \end{aligned} \tag{1.2}$$

with Σ_{k+s} the set of permutation of $k+s$ elements and $\alpha_i \in V$ for $i \in \{1, \dots, k+s\}$.
This product endows the space

$$\Lambda(V^*) \coloneqq \bigoplus_{k=0}^{n} \Lambda^k(V^*)$$

with a supercommutative algebra structure, as the product satisfies

$$\eta \wedge \zeta = (-1)^{|\eta||\zeta|}\, \zeta \wedge \eta\,.$$

In the particular case of two 1–forms, formula (1.2) becomes

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha\,,$$

for $\alpha, \beta \in \Lambda^1(V^*)$.

Definition 1.1.6. Two nonzero n–forms $\eta, \zeta \in \Lambda^n(V^*)$ are *equioriented* if $\eta = \lambda\zeta$ for a positive number λ; if instead λ is negative the forms are said to have the *reverse orientation.* An *orientation* on V is the choice of a nonzero $\eta \in \Lambda^n(V^*)$. The pair (V, η) is said to be an *oriented vector space.* We will say that a basis $\{u^i\}_{i=1}^n$ of V^* is *oriented* if the form $u^1 \wedge \cdots \wedge u^n$ is equioriented with η.

On a vector space V with a scalar product g we will limit ourselves to considering only orientations Θ that can be expressed as

$$\Theta = \vartheta^1 \wedge \cdots \wedge \vartheta^n\,,$$

for an orthonormal basis $\{\vartheta^i\}_{i=1}^n$ of V^*.
Then, in any other oriented basis $\{u^i\}_{i=1}^n$ such an orientation can be expressed as

$$\Theta = \sqrt{\det g_{ij}}\, u^1 \wedge \cdots \wedge u^n\,.$$

Definition 1.1.7. Let (V, g, Θ) be an oriented n–dimensional vector space, as above and $k \in \{0, \dots, n\}$. Define the linear *Hodge operator*[1] $\star\colon \Lambda^k(V^*) \to \Lambda^{n-k}(V^*)$ by

$$\zeta \wedge \star\eta = g(\zeta, \eta)\, \Theta\,,$$

for $\zeta, \eta \in \Lambda^k(V^*)$.

Reversing the orientation on V changes the sign for the Hodge operator. In particular, if $\eta, \zeta \in \Lambda^k(V^*)$, then $\eta \wedge \star\zeta = \zeta \wedge \star\eta$ and $\eta \wedge \star\eta = |\eta|^2\, \Theta$. Note also that $\star 1 = \Theta$. The explicit formula for $\star$ on an oriented basis can be given as

$$\star(u^{i_1} \wedge \cdots \wedge u^{i_k}) = \frac{\sqrt{\det g_{ij}}}{(n-k)!} \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma)\, g^{i_1\sigma(1)} \cdots g^{i_k\sigma(k)}\, u^{\sigma(k+1)} \wedge \cdots \wedge u^{\sigma(n)}\,. \tag{1.3}$$

Where Σ_n is the set of permutation of $\{1, \dots, n\}$. and $\{u^i\}_{i=1}^n$ is a basis of V^*
For an orthonormal basis $\{\vartheta^i\}_{i=1}^n$ of V^* formula (1.3) just reads

$$\star(\vartheta^{i_1} \wedge \cdots \wedge \vartheta^{i_k}) = \operatorname{sgn}(\widetilde{\sigma})\, \vartheta^{i_{k+1}} \wedge \cdots \wedge \vartheta^{i_n}\,,$$

where $(i_{k+1}, \dots, i_n)$ are the ordered remaining indices and $\widetilde{\sigma}$ is the permutation $(i_1 \dots i_n)$.

Remark 1.1.8. Applying $\star$ twice yields the formula

$$\star\star\,\eta = (-1)^{k(n-k)}\eta$$

for every $\eta \in \Lambda^k(V^*)$, which implies that $\star$ is an isomorphism with inverse $\star^{-1} = (-1)^{k(n-k)}\star$. We also remark that if n is even then $\star\colon \Lambda^{n/2}(V^*) \to \Lambda^{n/2}(V^*)$ is an automorphism with inverse $\star^{-1} = (-1)^{n/2}\star$. In particular if n is a multiple of 4, then $\star^{-1} = \star$.

1.1.2 Differential forms on manifolds

As already mentioned, let $\Lambda^k M := \Lambda^k(TM^*)$, $S^k M := S^k(TM^*)$ and call $\Omega^k(M) = \Gamma(\Lambda^k M)$ and $\Sigma^k(M) = \Gamma(S^k M)$, the spaces of differential k–forms on M and symmetric k–forms on M respectively. We define the *differential* operator d by

$$\begin{aligned} \mathrm{d}\eta(X_0, X_1, \dots, X_k) &= \sum_{i=0}^k (-1)^i X_i(\eta(X_0, \dots, \widehat{X}_i, \dots, X_k)) \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \eta([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k)\,, \end{aligned}$$

where $\widehat{X}_i$ indicates that the i–th field is missing, $\eta \in \Omega^k(M)$, $X_i \in \Gamma(TM)$ for $i \in \{0, 1, \dots, n\}$ and $[\cdot, \cdot]$ is the Lie brackets[2] on TM defined by

$$[X, Y] = X \circ Y - Y \circ X\,.$$

In local coordinates

$$\begin{aligned} \mathrm{d}\eta &= \mathrm{d} \sum_{1 \le i_1 < \cdots < i_k \le n} \eta_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} \mathrm{d}\eta_{i_1 \dots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\,. \end{aligned}$$

[1] After the British mathematician Sir William Vallance Douglas Hodge (1903–1975) [55].
[2] After the Norwegian mathematician Marius Sophus Lie (1842–1899) [56].

Definition 1.1.9. Let M be a differential manifold.
Two charts (U, φ), (V, φ) are *equioriented* if $U \cap V \neq \emptyset$ and the Jacobian determinant of the transition map (which is always nonzero) is a positive function.
An atlas is *oriented* if all of its charts with nonempty intersections are equioriented.
Two atlases are *equioriented* if their union is an oriented atlas.
An *orientation* on M is a maximal oriented atlas on M.
The manifold M is said to be *orientable* if it admits an orientation.

Definition 1.1.10. An *oriented manifold* is an orientable manifold where an orientation has been chosen. If M is an oriented manifold, we will denote by $-M$ its *reverse orientation*, that is, the oriented manifold obtained by reversing the orientation of every chart of M.

Remark 1.1.11. An orientation on M induces an orientation on ∂M by restriction of the atlas. If the dimension is even, we define that as the standard orientation on ∂M, otherwise, as the reverse orientation.

It can be shown that every connected orientable manifold has exactly two possible orientations and that a connected nonorientable manifold has a universal (differential) covering of degree 2 which is an orientable manifold. it follows that every simply connected differential manifold is orientable.

Definition 1.1.12. A diffeomorphism $f\colon M \to N$ between two oriented manifolds is said to *orientation–preserving* if the induced atlas from M to N is equioriented with the atlas on N. It is said to be *orientation–reversing* if the induced atlas has the reverse orientation.

Definition 1.1.13. A *volume form* on the n–dimensional manifold M is a global frame of $\Omega^n(M)$, that is, a never vanishing differential n–form.

If η and η' are two volume forms, then there exists a never vanishing C^∞ function f such that $\eta' = f\omega$ and if M is connected such function has constant sign.
It can be shown that choosing an orientation on M is equivalent to choosing a volume form and that a diffeomorphism preserves (resp. reverses) the orientation if and only if it "pulls–back" the volume form of the image manifold onto the volume form of the domain without (resp. with) applying a change of sign.

Let $\eta \in \Omega^n(M)$ be any n–form on M with compact support contained in a chart (U, φ). Then there exists a function $f \in C^\infty(\mathbb{R}^n)$ such that $(\varphi^{-1})^*\eta = f \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ and we define

$$\int_M \eta \coloneqq \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\,.$$

In general, if η is a compactly supported n–form and $\{\varrho_\alpha\}_{\alpha\in A}$ is a partition of unity subordinated to an oriented atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$, we define

$$\int_M \eta \coloneqq \sum_{\alpha\in A} \int_M \varrho_\alpha \eta\,.$$

It can then be shown that this integral is independent of the choice of the partition of unity and the oriented atlas. If $-M$ is the same manifold with the reverse orientation, then $\int_{-M} \eta = -\int_M \eta$. More generally, if $f\colon M \to N$ is any diffeomorphism between two oriented, connected manifolds, then

$$\int_M f^*\eta = \pm \int_N \eta\,, \tag{1.4}$$

where the sign depends on whether f preserves or reverses the orientation.

For a differential form $\eta \in \Omega^{n-1}(M)$ on a differential manifold with boundary M and inclusion map $\iota\colon \partial M \hookrightarrow M$, we define

$$\eta|_{\partial M} \coloneqq \iota^* \eta \,,$$

and

$$\int_{\partial M} \eta \coloneqq \int_{\partial M} \eta|_{\partial M} \ .$$

Theorem 1.1.14 (Stokes theorem[3])**.** *Let M be an oriented n–dimensional manifold with boundary and $\eta \in \Omega^{n-1}(M)$ a differential form with compact support. Then*

$$\int_M \mathrm{d}\eta = \int_{\partial M} \eta \,.$$

Definition 1.1.15. A differential form $\eta \in \Omega^k(M)$ is said to be *closed* if $\mathrm{d}\eta = 0$, *exact* if $\eta = \mathrm{d}\zeta$ for some differential form $\zeta \in \Omega^{k-1}(M)$ and *locally exact* if for every $p \in M$ there exists a neighbourhood U of p such that $\eta|_U$ is exact.

As $\mathrm{d}^2 = 0$, every exact differential form is also closed. We denote by $\mathrm{d}_k \coloneqq \mathrm{d}|_{\Omega^k(M)}$.

Definition 1.1.16. We define for $k \in \mathbb{N}$ the *k–th de Rham cohomology group*[4] as the real vector space

$$H^k(M) \coloneqq \frac{\ker \mathrm{d}_k}{\operatorname{im} \mathrm{d}_{k-1}}$$

and we call its dimension

$$\beta_k(M) \coloneqq \dim H^k(M)$$

the *k–th Betti number*[5].

The de Rham groups can be "pasted together" into a graded algebra

$$H(M) = \bigoplus_{k=0}^{n} H^k(M) \,,$$

with a so–called *cup product*

$$[\eta] \smile [\zeta] \coloneqq [\eta \wedge \zeta]$$

for $[\eta], [\zeta] \in H(M)$.

Theorem 1.1.17 (Poincaré lemma)**.** *Let $k \geq 1$ and U be a star–convex open set of $\mathbb{R}^n$. Then any closed k–form on U is exact.*

Corollary 1.1.18. *Let $k \geq 1$, then any closed k–form on M is locally exact.*

Definition 1.1.19. For an n–dimensional manifold M with finite Betti numbers, the *Euler–Poincaré characteristic* of M is defined as

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \beta_k(M) \,.$$

[3]After the Irish mathematician and physicist Sir George Gabriel Stokes (1819–1903) [57].
[4]After the Swiss mathematician Georges de Rham (1903–1990) [58].
[5]After the Italian mathematician Enrico Betti Glaoui (1823–1892) [59].

We recall that if M is a compact 2–dimensional manifold without boundary (that is, a closed surface) of *genus g* (the number of "holes" if the surface is oriented, roughly speaking), then $\chi(M) = 2 - 2g$.

Remark 1.1.20. It can be shown that the de Rham cohomology groups, the Betti numbers and the Euler characteristic are topological invariants. In particular, if M and N are two n–dimensional, complete differential manifold *without boundary*,

- if M is compact, every $H^k(M)$ is finite–dimensional and $H^k(M)$ and $H^{n-k}(M)$ are isomorphic (by *Poincaré duality*, see [16, Section 3.3]),
- $H^0(M) = \mathbb{R}$, if M is connected,
- $H^1(M) = 0$, if M is simply connected,
- $H^n(M) = \mathbb{R}$, if M is connected, compact and orientable,
- $H^n(M) = 0$, if M is noncompact or nonorientable,
- $\chi(M) = 0$, if M is an odd–dimensional compact manifold, by the first point,
- $\chi(M \times N) = \chi(M)\chi(N)$, if M and N are compact (by the *Künneth formula*[6], see [16, Section 3.B]),
- $\chi(M\#N) = \chi(M) + \chi(N) - \chi(\mathbb{S}^n)$, if M and N are compact and connected.

Moreover, we recall the Euler–Poincaré characteristic of some common manifolds, namely, for every positive integer k one has $\chi(\mathbb{S}^{2k}) = 2$, $\chi(\mathbb{RP}^{2k}) = 1$, $\chi(\mathbb{CP}^k) = k + 1$.

1.1.3 The Frobenius theorem

We state the *Frobenius theorem*[7] in its classical form, about the existence of solutions of *overdetermined* systems of first–order differential equations on $\mathbb{R}^n$. We refer the reader to [23, Proposition 19.29] for a proof and to [23, Theorem 19.21] for a geometric version of the theorem.

Theorem 1.1.21. *Let $X_i \colon V \times U \subseteq \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be C^∞ vector fields, for $i \in \{1, \dots, m\}$ and V, U open sets. Then, for every $(x_0, y_0) \in V \times U$ there exist a connected neighbourhood $W \subseteq V$ of x_0 and a unique C^∞ function $u \colon W \to U$ such that*

$$\begin{cases} \dfrac{\partial u}{\partial x^i}(x) = X_i(x, u(x)) & \text{for every } x \in W \text{ and } i \in \{1, \dots, m\}\,, \\ u(x_0) = y_0\,, \end{cases} \tag{1.5}$$

if and only if the following integrability condition is satisfied on $V \times U$:

$$\frac{\partial X_i}{\partial x^j} + X_j^k \frac{\partial X_i}{\partial y^k} = \frac{\partial X_j}{\partial x^i} + X_i^k \frac{\partial X_j}{\partial y^k}\,. \tag{1.6}$$

Remark 1.1.22. If u is a solution of equation (1.5) then the two sides of equation (1.6) are just the second derivatives of u with switched indices, so the necessity of its validity follows from the Schwarz theorem[8].

[6] After the German mathematician Hermann Lorenz Künneth (1892–1975) [60].
[7] After the German mathematician Ferdinand Georg Frobenius (1849–1917) [61].
[8] After the German mathematician Karl Hermann Amandus Schwarz (1843–1921) [62].

Remark 1.1.23. The integrability condition (1.6) may be also written as

$$\frac{\partial}{\partial x^j} X_i(x, u(x)) = \frac{\partial}{\partial x^i} X_j(x, u(x))$$

where every instance of $\partial u / \partial x^i$ has been substituted by X_i.

1.2 Riemannian manifolds

A good reference for this section is the book [22].

Definition 1.2.1. Let M be a differential manifold. A *(Riemannian) metric* on M is a positive definite bilinear symmetric form $g \in \Sigma^2(M)$. A *Riemannian manifold* is a pair (M, g) with g a Riemannian metric on M.

Theorem 1.2.2 (Theorem "zero" of Riemannian geometry)**.** *Every differential manifold admits a Riemannian metric.*

Definition 1.2.3. Since a metric is identifiable with a scalar product on every tangent plane, we can define for every $v, w \in T_p M$ the norm and angles between (nonzero) vectors as

$$|v| = \sqrt{g(v, v)}\,, \qquad \angle(v, w) = \arccos g\left(\frac{v}{|v|}, \frac{w}{|w|}\right).$$

The components of the metric in a chart $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$ define an invertible matrix $(g_{ij})_{i,j=1}^n$. We will denote by g^{ij} the components of the inverse matrix and with g^{-1} the 2–vector with such components.

Definition 1.2.4. Let (M, g) be an oriented Riemannian manifold. The *Riemannian volume form* $\mathrm{d}V_M$ is defined as

$$\mathrm{d}V_M = \sqrt{\det g_{ij}}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\,,$$

for every oriented chart $(U, (x^1, \ldots, x^n))$.

It can be shown that such form is globally defined and that in an oriented frame $\{e_i\}_{i=1}^n$ with dual frame $\{\vartheta^i\}_{i=1}^n$ it satisfies

$$\mathrm{d}V_M = \vartheta^1 \wedge \cdots \wedge \vartheta^n\,. \tag{1.7}$$

Having a canonical volume form, we can introduce as in Definition 1.1.7 the Hodge operator $\star$, which we extend pointwise to a C^∞–linear map on the whole space $\Omega^k(M)$ for $0 \leq k \leq n$ by

$$\zeta \wedge \star\eta = g(\zeta, \eta)\, \mathrm{d}V_M\,, \tag{1.8}$$

for $\zeta, \eta \in \Omega^k(M)$.

Definition 1.2.5. Let (M, g) be an oriented Riemannian manifold. We define a scalar product on $\Omega^k(M)$ by

$$\langle \zeta, \eta \rangle := \int_M \zeta \wedge \star\eta = \int_M g(\zeta, \eta)\, \mathrm{d}V_M\,,$$

for every $\zeta, \eta \in \Omega^k(M)$ with at least one of them having compact support.

Definition 1.2.6. Let M be a differential manifold and $\pi\colon E \to M$ a vector bundle. A *connection* on the bundle E is a map

$$\begin{aligned}\nabla\colon \Gamma(TM)\times\Gamma(E) &\to \Gamma(E)\\ (X,\eta) &\mapsto \nabla_X\eta\,,\end{aligned}$$

such that

- ∇ is $C^\infty(M)$–linear in X,
- ∇ is linear in η,
- for every $X \in \Gamma(TM)$, $\eta \in \Gamma(E)$ and $f \in C^\infty(M)$, there holds

$$\nabla_X(f\eta) = X(f) + f\nabla_X\eta\,.$$

We will call $\nabla_X\eta$ the *covariant derivative of η with respect to X*.
If $E = TM$, we will call ∇ an *affine connection.*[9]

For every $\eta \in \Gamma(E)$, we will denote by $\nabla\eta$ the $C^\infty(M)$–linear map

$$\nabla\eta\colon X \in \Gamma(TM) \to \nabla_X\eta \in \Gamma(E)\,.$$

Remark 1.2.7. It can be shown that the value $\nabla_X\eta\big|_p$ only depends on the value of η along a curve passing through p with velocity X at p.

We set $\nabla_i \coloneqq \nabla_{\frac{\partial}{\partial x^i}}$ and define the *Christoffel symbols*[10] for a connection ∇ as

$$\nabla_i\xi_j = \Gamma^k_{ij}\xi_k\,,$$

with $\{\xi_j\}_{j=1}^m$ a local frame for E.
In this way, for every section $\eta = \eta^j\xi_j$, we have

$$\nabla_X\eta = X^i(\partial_i\eta^k + \Gamma^k_{ij}\eta^j)\xi_k\,.$$

Any affine connection can be uniquely extended to the whole tensor bundle $\Gamma(T^r_sM)$ by imposing that

- $\nabla\colon f \mapsto \mathrm{d}f$ for all $f \in C^\infty(M)$ (we, however, will reserve the symbol ∇f for the vector satisfying $g(\nabla f, X) = \mathrm{d}f(X) = X(f)$ for all $X \in \Gamma(TM)$),
- $\nabla(T\otimes S) = \nabla T \otimes S + T \otimes \nabla S$ for all $T, S \in \Gamma(T^r_sM)$,
- ∇ commutes with the contractions.

For every $T \in \Gamma(T^r_sM)$ we will denote by $\nabla^\ell T$ the ℓ–times application of ∇ to T and by

$$\nabla_{k_1\dots k_\ell}T^{i_1\dots i_r}_{j_1\dots j_s} = (\nabla^\ell T)^{i_1\dots i_r}_{k_1\dots k_\ell j_1\dots j_s}$$

its components.

[9] The reason for this name will become clear in light of Lemma 2.1.1.
[10] After the German mathematician Elwin Bruno Christoffel (1829–1900) [63].

Definition 1.2.8. We define the divergence $\operatorname{div} T$ of a tensor $T \in \Gamma(T^r_s M)$, with $r \geq 1$, as the contraction of ∇T between its first covariant entry and its first contravariant entry; in coordinates

$$\operatorname{div} T = \nabla_k T^{k i_2 \dots i_r}_{j_1 \dots j_s} \,.$$

If $\omega \in \Gamma(T^0_s M)$ is an s–form, with $s \geq 1$, we define

$$\operatorname{div} \omega = g^{hk} \nabla_h \omega_{k j_2 \dots k_s} \,.$$

For every tensor $T \in \Gamma(T^r_s M)$ we define the *Laplacian*[11] ΔT as the trace of $\nabla^2 T$ on its first two covariant entries; in coordinates

$$\Delta T = g^{ij} \nabla_{ij} T^{i_1 \dots i_r}_{j_1 \dots j_s} \,.$$

Definition 1.2.9. For an affine connection ∇, we define the *torsion tensor*

$$T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \,, \tag{1.9}$$

and, given a metric g, the *non–metricity* tensor

$$M^{\nabla, g}(X, Y, Z) = (\nabla_X g)(Y, Z) \coloneqq X g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \,. \tag{1.10}$$

We will say that ∇ is *symmetric* if $T^{\nabla} = 0$ and *metric compatible* if $M^{\nabla, g} = 0$.

Remark 1.2.10. Symmetry and metric compatibility can be explicitly expressed using the Christoffel symbols, since for $T^{\nabla} = 0$, formula (1.9) is equivalent to

$$\Gamma^k_{ij} = \Gamma^k_{ji} \,,$$

and for $M^{\nabla, g} = 0$ equation (1.10) becomes

$$\Gamma_{ijk} + \Gamma_{ikj} = \partial_i g_{jk} \,.$$

Theorem 1.2.11 (Fundamental theorem of Riemannian geometry)**.** *On every Riemannian manifold (M, g) there exists a unique connection ∇, which we will call the* Levi–Civita connection[12] *on (M, g), that is symmetric and metric compatible.*

Remark 1.2.12. The explicit expression for the Levi–Civita connection through its Christoffel symbols and the metric g is given by

$$\Gamma_{ijk} = \tfrac{1}{2} \big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \big) \,. \tag{1.11}$$

An important equation linking the Levi–Civita connection (and in general, any symmetric connection) with the differential operator d is the following,

$$\mathrm{d}\eta(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i} \eta)(X_0, \dots, \widehat{X}_i, \dots, X_k) \,, \tag{1.12}$$

for any $\eta \in \Omega^k(M)$.

One of the most fundamental results for the Levi–Civita connection (and in general, for any metric connection) is the *divergence theorem.*

[11] After the French mathematician Pierre–Simon, marquis de Laplace (1749–1827) [64].

[12] After the Italian mathematician Tullio Levi–Civita (1873–1941) [65].

Theorem 1.2.13 (divergence theorem). *Let (M, g) be an oriented n–dimensional Riemannian manifold and $X \in \Gamma(TM)$ a vector field with compact support. Then,*

$$\int_M \mathrm{div} X \, \mathrm{d}V_M = \int_{\partial M} g(X, \nu) \, \mathrm{d}V_{\partial M} \, ,$$

where $\nu\colon \partial M \to TM$ the outward–pointing unit normal vector

In the whole thesis we will use exclusively the Levi–Civita connection.

Definition 1.2.14. The *Riemann operator* is the tensor defined by

$$R(X, Y)Z = \nabla^2_{Y,X} Z - \nabla^2_{X,Y} Z = [\nabla_Y, \nabla_X] Z - \nabla_{[Y,X]} Z \, , \qquad \text{for } X, Y, Z \in \Gamma(TM)$$

and the *Riemann tensor* Riem is the $(0, 4)$–version of the Riemann operator, denoted by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W) \, , \qquad \text{for } X, Y, Z, W \in \Gamma(TM) \, .$$

Remark 1.2.15. In local coordinates, by means of the Christoffel symbols, the Riemann operator can be written as

$$R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^s_{ik} \Gamma^l_{sj} - \Gamma^s_{jk} \Gamma^l_{si} \, .$$

Proposition 1.2.16 (Symmetries of the Riemann tensor). *The following properties hold:*

- *skew–symmetry in the first two entries*

$$\begin{aligned} R(X, Y, Z, W) &= -R(Y, X, Z, W) \, , \\ R_{ijkl} &= -R_{jikl} \, ; \end{aligned}$$

- *skew–symmetry in the last two entries*

$$\begin{aligned} R(X, Y, Z, W) &= -R(X, Y, W, Z) \, , \\ R_{ijkl} &= -R_{ijlk} \, ; \end{aligned}$$

- *symmetry between the first and second pair*

$$\begin{aligned} R(X, Y, Z, W) &= R(Z, W, X, Y) \, , \\ R_{ijkl} &= R_{klij} \, , \end{aligned}$$

for all $X, Y, Z, W \in \Gamma(TM)$ and $i, j, k, l \in \{1, \dots, n\}$.

Proposition 1.2.17 (Bianchi identities[13]). *The following properties hold:*

- *first (or algebraic) Bianchi identity*

$$\begin{aligned} R(X, Y)Z + R(Z, X)Y + R(Y, Z)X &= 0 \, , \\ R_{ijkl} + R_{kijl} + R_{jkil} &= 0 \, ; \end{aligned}$$

- *second (or differential) Bianchi identity*

$$\begin{aligned} \nabla_X R(Y, Z) + \nabla_Z R(X, Y) + \nabla_Y R(Z, X) &= 0 \, , \\ \nabla_i R_{jklm} + \nabla_k R_{ijlm} + \nabla_j R_{kilm} &= 0 \, , \end{aligned} \tag{1.13}$$

[13] After the Italian mathematician Luigi Bianchi (1856–1928) [66].

for all $X, Y, Z \in \Gamma(TM)$ *and* $i, j, k, l, m \in \{1, \ldots, n\}$.

Due to the symmetries of the $(0,4)$–Riemann tensor, it is possible to define an associated bilinear symmetric form $\mathcal{R}$ and a linear self–adjoint map $\mathscr{R}$ on $\Gamma(\Lambda^2(TM))$, which we will call the *curvature form* and *curvature operator* respectively, by linear extension from

$$R(X, Y, Z, W) = \mathcal{R}(X \wedge Y, Z \wedge W) = \tfrac{1}{2} g(\mathscr{R}(X \wedge Y), Z \wedge W)$$

with $X, Y, Z, W \in \Gamma(TM)$, where we are considering the extension of the scalar product g to $\Gamma(\Lambda^2(TM))$.

From the Riemann curvature tensor one can define the so–called *sectional curvature* of any 2–plane $\pi = \langle v, w \rangle \subseteq T_pM$ at $p \in M$,

$$\mathrm{Sec}(v, w) := \frac{R_p(v, w, v, w)}{|v|^2|w|^2 - \big(g_p(v, w)\big)^2} = \frac{g_p(\mathscr{R}_p(v \wedge w), v \wedge w)}{g_p(v \wedge w, v \wedge w)}\,,$$

which has a more direct geometrical interpretation, as it is equal to the standard Gaußian curvature at p of the 2–dimensional submanifold locally swept out by the geodesics tangent to the 2–plane π around p, once embedding such surface in $\mathbb{R}^3$, if possible (see [22, Proposition 8.29]). We define $\mathrm{Sec}(v, w) := 0$ if v and w are linearly dependent.

Other forms of curvature are obtained through one or two applications of the trace operator to the Riemann tensor, leading to the definition of the *Ricci curvature tensor* R (or Ric), the *scalar curvature* R and the trace–free Ricci tensor $\overset{\circ}{\mathrm{Ric}}$, as follows:

$$\begin{aligned}
\mathrm{Ric}(X, Y) &:= (\mathrm{tr}^{1,3} R)(X, Y) = g^{ik} R\Big(\frac{\partial}{\partial x^i}, X, \frac{\partial}{\partial x^k}, Y\Big)\,;\\
\mathrm{R} &:= \mathrm{tr}\,\mathrm{Ric} = g^{ik} g^{jl} R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)\,;\\
\overset{\circ}{\mathrm{Ric}}(X, Y) &:= \mathrm{Ric}(X, Y) - \frac{\mathrm{R}}{n} g(X, Y)\,,
\end{aligned}$$

for $X, Y \in \Gamma(TM)$.

We conclude this section by recalling some well–known facts about *flat* and, more in general, manifolds with *constant (sectional) curvature* (see [22], for instance).

Theorem 1.2.18. *Let (M, g) be an n–dimensional Riemannian manifold. The following properties are equivalent:*

(i) *M is flat, i.e.,* $\mathrm{Riem} = 0$*;*

(ii) *every point $p \in M$ admits a neighbourhood isometric to an open set of $\mathbb{R}^n$;*

(iii) *every point $p \in M$ admits a local coordinate chart in which $g_{ij} = \delta_{ij}$;*

(iv) *every point $p \in M$ admits a local coordinate chart such that $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ is an orthonormal frame;*

(v) *every point $p \in M$ admits a local coordinate chart in which the Christoffel symbols Γ^k_{ij} vanish everywhere.*

Theorem 1.2.19. *Let (M, g) be a complete n–dimensional Riemannian manifold with constant curvature k, then the universal covering of M is*

- *$\mathbb{R}^n$, with its canonical metric, if $k = 0$;*
- *$\mathbb{S}^n$, with its canonical metric, if $k > 0$;*
- *$\mathbb{H}^n$, with its canonical metric, if $k < 0$.*

1.3 Algebraic curvature tensors

The properties of the Riemann tensor may be categorised into *algebraic* and *differential*. The ones which are algebraic in nature do not really depend upon the manifold structure but are rather pointwise properties valid in the more general setting of the *algebraic curvature tensors*. These are tensors on a real vector space endowed with a scalar product satisfying all the algebraic symmetries of the Riemann tensor. We will compute the dimension of the vector space of these tensors, which corresponds to finding the number of independent components of the Riemann tensor; we then present its orthogonal decomposition to introduce the "Weyl–like" tensors, by means of the *Kulkarni–Nomizu product*;[14] and express a few initial properties of those tensors which will motivate the next chapter.

Let (V, g) denote an n–dimensional vector space with a scalar product g.

Definition 1.3.1. Given $h, k \in S^2(V^*)$ define their *Kulkarni–Nomizu* product as

$$(h \owedge k)(x, y, z, w) = h(x, z)k(y, w) + h(y, w)k(x, z) - h(x, w)k(y, z) - h(y, z)k(x, w)\,,$$

for $x, y, z, w \in V$.

Remark 1.3.2. The natural extension of g onto $\Lambda^2(V)$ is given by

$$\begin{aligned} g(x \wedge y, z \wedge w) &= g(x \otimes y - y \otimes x, z \otimes w - w \otimes z) \\ &= 2g(x, z)g(y, w) - 2g(x, w)g(y, z) \\ &= (g \owedge g)(x, y, z, w)\,. \end{aligned}$$

Definition 1.3.3. Let $\mathcal{F}^4(V)$ denote the set of $(0, 4)$–tensors on V satisfying the Riemann symmetries in Proposition 1.2.16, that is, $P \in \mathcal{F}^4(V)$ if and only if

(i) $P(x, y, z, w) = -P(y, x, z, w)\,,$

(ii) $P(x, y, z, w) = -P(x, y, w, z)\,,$

(iii) $P(x, y, z, w) = P(z, w, x, y)\,,$

for $x, y, z, w \in V$.
For each $P \in \mathcal{F}^4(V)$ we define the correspondent operator $\mathcal{P} \in S^2(\Lambda^2(V)^*)$ and the map $\mathscr{P} \colon \Lambda^2(V) \to \Lambda^2(V)$ by linear extension from

$$P(x, y, z, w) = \mathcal{P}(x \wedge y, z \wedge w) = \tfrac{1}{2} g(\mathscr{P}(x \wedge y), z \wedge w)\,, \qquad \text{for } x, y, z, w \in V\,.$$

Then, we denote by Pic the $(1, 3)$–*trace of* P, with P its *complete trace*, with $\mathring{\mathrm{Pic}}$ the *trace–free component* of Pic and with Pec its *sectional curvature*, that is, given a basis $\{e_i\}_{i=1}^n$ and four vectors x, y, v, w such that v and w are independent,

- $\mathrm{Pic}(x, y) = (\mathrm{tr}^{1,3} P)(x, y) = g^{ik} P(e_i, x, e_k, y)\,,$
- $\mathrm{P} = \mathrm{tr}\,\mathrm{Pic} = \mathrm{tr}(\mathrm{tr}^{1,3} P) = g^{ik} g^{jl} P(e_i, e_j, e_k, e_j)\,,$
- $\mathring{\mathrm{Pic}} = \mathrm{Pic} - \dfrac{\mathrm{P}}{n} g\,,$

[14] After the Indian mathematician Ravindra Shripad Kulkarni (1942) [67] and the Japanese–American mathematician Katsumi Nomizu (野水 克己, 1924–2008) [68].

- $\mathrm{Pec}(v,w) = P(v,w,v,w)/\frac{1}{2}(g \owedge g)(v,w,v,w) = g(\mathscr{P}(v\wedge w), v\wedge w)/g(v\wedge w, v\wedge w)$.

Moreover, we also define $\mathrm{Pec}(v,w) := 0$ if v and w are linearly dependent.
We call P an *algebraic curvature tensor* if it satisfies the relations (i)–(ii)–(iii) above and the first Bianchi identity:

(iv) $P(x,y,z,w) + P(z,x,y,w) + P(y,z,x,w) = 0$,

for all $x,y,z,w \in V$.
We denote by $C^4(V)$ the set of all algebraic curvature tensors on V.

Remark 1.3.4. The denominator in the definition of Pec is given by

$$\tfrac{1}{2}(g \owedge g)(v,w,v,w) = \tfrac{1}{2}\Big(2g(v,v)g(w,w) - 2\big(g(v,w)\big)^2\Big) = |v|^2|w|^2 - \big(g(v,w)\big)^2,$$

which is the area of the parallelogram with sides the vectors v and w in π; in particular, if $\{e_1, e_2\}$ is an orthonormal basis of π, then $\frac{1}{2}(g \owedge g)(e_1,e_2,e_1,e_2) = 1$ and

$$\mathrm{Pec}(e_1,e_2) = P(e_1,e_2,e_1,e_2).$$

It is easy to verify that we can always limit ourselves to orthonormal bases as the value of $\mathrm{Pec}(v,w)$ is only dependent on the 2–plane spanned by v,w; indeed, given another basis $\{u_1,u_2\}$ of $\pi = \langle v,w\rangle$ and writing $v = v^iu_i$ and $w = w^iu_i$, we get

$$\begin{aligned}
\mathrm{Pec}(v,w) &= \frac{P(v^iu_i, w^ju_j, v^ku_k, w^lu_l)}{\frac{1}{2}(g \owedge g)(v^iu_i, w^ju_j, v^ku_k, w^lu_l)}\\
&= \frac{P(u_1,u_2,u_1,u_2)\big((v^1)^2(w^2)^2 + (v^2)^2(w^1)^2 - 2v^1v^2w^1w^2\big)}{\frac{1}{2}(g \owedge g)(u_1,u_2,u_1,u_2)\big((v^1)^2(w^2)^2 + (v^2)^2(w^1)^2 - 2v^1v^2w^1w^2\big)}\\
&= \frac{P(u_1,u_2,u_1,u_2)}{\frac{1}{2}(g \owedge g)(u_1,u_2,u_1,u_2)}\\
&= \mathrm{Pec}(u_1,u_2).
\end{aligned}$$

Remark 1.3.5. Since $\mathcal{P}$ is symmetric, the operator $\mathscr{P}$ is self–adjoint and can be diagonalised.

Definition 1.3.6. We say that $P \in \mathcal{F}^4(V)$ is *simple* if so is a basis of eigenvectors of $\mathscr{P}$, i.e., there exist vectors $\{x_i, y_i\}_{i=1}^n$ such that $\{x_i \wedge y_j\}_{i,j}$ is an eigenbasis for $\mathscr{P}$.
We say that P is *pure* if it is simple and $\{x_i, y_j\}_{i,j}$ is an orthonormal basis of V.

Remark 1.3.7. If $h,k \in S^2(V^*)$ then $h \owedge k \in C^4(V)$. In fact, for $h \owedge k$ properties (i)-(ii)-(iii) in Definition 1.3.3 are easily checked and

$$\begin{aligned}
&(h \owedge k)(x,y,z,w) + (h \owedge k)(z,x,y,w) + (h \owedge k)(y,z,x,w)\\
&= h(x,z)k(y,w) + h(y,w)k(x,z) - h(x,w)k(y,z) - h(y,z)k(x,w)\\
&\quad + h(z,y)k(x,w) + h(x,w)k(z,y) - h(z,w)k(x,y) - h(x,y)k(z,w)\\
&\quad + h(y,x)k(z,w) + h(z,w)k(y,x) - h(y,w)k(z,x) - h(z,x)k(y,w)\\
&= 0.
\end{aligned}$$

We would now like to compute the dimension of $C^4(V)$.

Definition 1.3.8. We define the *Bianchi map* $b\colon \mathcal{F}^4(V) \to \Lambda^4(V^*)$ as

$$b(A)(x,y,z,w) = \tfrac{1}{3}[A(x,y,z,w) + A(z,x,y,w) + A(y,z,x,w)]\,,$$

for every $x,y,z,w \in V$.

Remark 1.3.9. We recall that $\Lambda^4(V^*) \subseteq \mathcal{F}^4(V) \subseteq T^0_4(V)$.

Proposition 1.3.10. *The Bianchi map is well–defined, self–adjoint,*

$$\ker(b) = C^4(V)\,, \qquad \operatorname{im}(b) = \Lambda^4(V^*) \qquad \textit{and} \qquad \mathcal{F}^4(V) = C^4(V) \oplus^{\perp} \Lambda^4(V^*)\,.$$

Remark 1.3.11. If $n \in \{2,3\}$ then $\mathcal{F}^4(V) = C^4(V)$, indeed by the symmetry properties of the algebraic curvature tensors the Bianchi identity in these dimensions is identically zero.

Proof. Let $x,y,z,w \in V$ and $A, B \in \mathcal{F}^4(V)$, then in any basis of V

$$\begin{aligned}
g\big(A, b(B)\big) &= A^{ijkl} b(B)_{ijkl} \\
&= \tfrac{1}{3} A^{ijkl}(B_{ijkl} + B_{kijl} + B_{jkil}) \\
&= \tfrac{1}{3} (A^{ijkl}B_{ijkl} + A^{ijkl}B_{kijl} + A^{ijkl}B_{jkil}) \\
&= \tfrac{1}{3} (A^{ijkl}B_{ijkl} + A^{jkil}B_{ijkl} + A^{kijl}B_{ijkl}) \\
&= \tfrac{1}{3} (A^{ijkl} + A^{kijl} + A^{jkil})B_{ijkl} \\
&= g(A)^{ijkl} B_{ijkl} \\
&= g\big(b(A), B\big)\,,
\end{aligned}$$

hence, the Bianchi map is g–self–adjoint.
To check that $b(A)$ belongs to $\Lambda^4(V^*)$ we first notice that

$$b(A)(x,y,z,w) = b(A)(z,x,y,w) = b(A)(y,z,x,w)\,, \tag{1.14}$$

then

$$\begin{aligned}
b(A)(w,x,y,z) &= \tfrac{1}{3}[A(w,x,y,z) + A(y,w,x,z) + A(x,y,w,z)] \\
&= \tfrac{1}{3}[-A(y,z,x,w) - A(z,x,y,w) - A(x,y,z,w)] \\
&= -b(A)(x,y,z,w)\,.
\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}
b(A)(x,x,z,w) &= \tfrac{1}{3}[A(x,x,z,w) + A(z,x,x,w) + A(x,z,x,w)] \\
&= \tfrac{1}{3}[A(z,x,x,w) - A(z,x,x,w)] \\
&= 0\,.
\end{aligned} \tag{1.16}$$

Then, from equation (1.14) and (1.16), we have

$$b(A)(x,x,z,w) = b(A)(z,x,x,w) = b(A)(x,z,x,w) = 0$$

and finally, by using equation (1.15), we also obtain

$$b(A)(x,y,z,x) = b(A)(y,x,z,x) = b(A)(y,z,x,x) = 0\,,$$

hence, $b(A) \in \Lambda^4(V^*)$.
Clearly $\ker(b) = C^4(V)$. To see that the map b is a projection onto $\Lambda^4(V^*)$, we consider $A \in \Lambda^4(V^*)$ and evaluate

$$\begin{aligned} b(A)(x,y,z,w) &= \frac{1}{3}[A(x,y,z,w) + A(z,x,y,w) + A(y,z,x,w)] \\ &= \frac{1}{3}[A(x,y,z,w) + A(x,y,z,w) + A(x,y,z,w)] \\ &= A(x,y,z,w)\,, \end{aligned}$$

thus $\mathrm{im}(b) = \Lambda^4(V^*)$ and $b^2(A) = b(A)$ for every $A \in \mathcal{F}^4(V)$.
As a consequence $A \in \mathcal{F}^4(V)$ can be decomposed in $A = (A - b(A)) + b(A)$ with $A - b(A) \in \ker(b)$ since $b(A - b(A)) = b(A) - b(A) = 0$.
If $A \in \ker(b)$ and $B = b(C) \in \mathrm{im}(b)$ then

$$g(A,B) = g(A,b(C)) = g(b(A),C) = 0$$

and $\mathcal{F}^4(V) = \ker(b) \oplus^\perp \mathrm{im}(b) = C^4(V) \oplus^\perp \Lambda^4(V^*)$. ☐

Proposition 1.3.12. $\dim C^4(V) = \dfrac{n^2(n^2-1)}{12}$.

Proof. From $S^2(\Lambda^2(V)^*) \simeq \mathcal{F}^4(V) = C^4(V) \oplus^\perp \Lambda^4(V^*)$ and formulae (1.1) it follows

$$\dim \mathcal{F}^4(V) = \dim S^2(\Lambda^2(V)^*) = \frac{n(n-1)(n^2-n+2)}{8} \tag{1.17}$$

and

$$\begin{aligned} \dim C^4(V) &= \dim \mathcal{F}^4(V) - \dim \Lambda^4(V^*) \\ &= \frac{n(n-1)(n^2-n+2)}{8} - \frac{n(n-1)(n-2)(n-3)}{24} \\ &= \frac{n^2(n^2-1)}{12}\,. \end{aligned}$$

☐

We will now establish some relations between a tensor P and its traces Pic and P.
Let v, w be two independent vectors and choose an orthonormal basis $\{e_i\}_{i=1}^n$. We have

$$\begin{aligned} \mathrm{Pic}(v,w) &= \sum_{i=1}^n P(e_i,v,e_i,w)\,, \\ \mathrm{Pic}(v,v) &= \sum_{i=1}^n P(e_i,v,e_i,v) = \sum_{i=1}^n \mathrm{Pec}(v,e_i)\cdot \tfrac{1}{2}(g \owedge g)(v,e_i,v,e_i)\,, \end{aligned} \tag{1.18}$$

and observe that by choosing $\{e_i\}_{i=2}^n$ so that they complete $\{v/|v|\}$ to an orthonormal basis of V, expression (1.18) reduces to

$$\mathrm{Pic}(v,v) = |v|^2 \sum_{i=2}^n \mathrm{Pec}(v,e_i)\,.$$

Writing $\widetilde{\mathrm{Pec}}(v,w) = \mathrm{Pec}(v,w)\cdot\frac{1}{2}(g \owedge g)(v,w,v,w)$ and choosing a complete orthonormal basis $\{e_i\}_{i=i}^n$, from equality (1.18) we obtain

$$\mathrm{Pic}(v,w) = \frac{1}{2}\sum_{i=1}^n \Big[\widetilde{\mathrm{Pec}}(v+w,e_i) - \widetilde{\mathrm{Pec}}(v,e_i) - \widetilde{\mathrm{Pec}}(w,e_i)\Big]$$

and

$$\mathrm{P} = \sum_{i,j=1}^{n} P(e_i, e_j, e_i, e_j) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathrm{Pec}(e_i, e_j) = 2 \sum_{\substack{i,j=1 \\ i < j}}^{n} \mathrm{Pec}(e_i, e_j)\,, \tag{1.19}$$

which shows that we can express Pic and P through Pec.

Proposition 1.3.13. *If for $P, P' \in C^4(V)$ we have $\mathrm{Pec} = \mathrm{Pec}'$, then $P = P'$.*

Proof. First notice that

$$P(x+z, y, x+z, y) = P(x, y, x, y) + P(z, y, z, y) + 2P(x, y, z, y)\,,$$

thus any term as $P(x, y, z, y)$ with at least one repetition can be obtained by combining sectional curvatures. Then we evaluate

$$\begin{aligned} P(x+z, y+w, x+z, y+w) &\\ = 2[&P(x,y,z,w) + P(x,w,z,y)] \\ &+ 2[P(x,y,z,y) + P(x,y,x,w) + P(x,w,z,w) + P(z,y,z,w)] \\ &+ P(x,y,x,y) + P(x,w,x,w) + P(z,y,z,y) + P(z,w,z,w)\,, \end{aligned}$$

$$\begin{aligned} P(x+w, y+z, x+w, y+z) &\\ = 2[&P(x,y,w,z) + P(x,z,w,y)] \\ &+ 2[P(x,y,w,y) + P(x,y,x,z) + P(x,z,w,z) + P(w,y,w,z)] \\ &+ P(x,y,x,y) + P(x,z,x,z) + P(w,y,w,y) + P(w,z,w,z)\,. \end{aligned}$$

From the two equalities, we get

$$\begin{cases} P(x,y,z,w) + P(x,w,z,y) = A \\ P(x,y,w,z) + P(x,z,w,y) = B \end{cases}$$

where A and B can be expressed in terms of sectional curvatures. Then, by means of the Bianchi identity, one can solve the system, obtaining

$$P(x,y,z,w) = \frac{1}{3}(A - B)\,,$$

which proves the result. We underline that expanding also $P(x+y, z+w, x+y, z+w)$ does not provide a new independent equation; it is necessary to use the Bianchi identity in order to conclude the argument.

The explicit formula for $P(x,y,z,w)$ is then

$$\begin{aligned} P(x,y,z,w) = \frac{1}{6}\Big\{ &\widetilde{\mathrm{Pec}}(x+z, y+w) - \widetilde{\mathrm{Pec}}(x+w, y+z) \\ &+ \widetilde{\mathrm{Pec}}(x+w, y) + \widetilde{\mathrm{Pec}}(x, y+z) + \widetilde{\mathrm{Pec}}(x+w, z) + \widetilde{\mathrm{Pec}}(w, y+z) \\ &- \widetilde{\mathrm{Pec}}(x+z, y) - \widetilde{\mathrm{Pec}}(x, y+w) - \widetilde{\mathrm{Pec}}(x+z, w) - \widetilde{\mathrm{Pec}}(z, y+w) \\ &+ \widetilde{\mathrm{Pec}}(x, w) + \widetilde{\mathrm{Pec}}(y, z) - \widetilde{\mathrm{Pec}}(x, z) - \widetilde{\mathrm{Pec}}(y, w) \Big\}\,, \end{aligned}$$

with $\widetilde{\mathrm{Pec}}(x,y) = P(x,y,x,y) = \mathrm{Pec}(x,y) \cdot \frac{1}{2}(g \owedge g)(x,y,x,y)\,.$ ☐

Lemma 1.3.14. *If* Pec *is constant over all* 2*–planes, that is,* $\mathrm{Pec} = K$*, then* $P = \frac{K}{2} g \owedge g$*.*

Proof. The two algebraic curvature tensors P and $\frac{K}{2} g \owedge g$ have the same associated sectional curvatures, hence they coincide. ☐

Dimension 2. If $n = 2$ then $\dim C^4(V) = 2^2(2^2 - 1)/12 = 1$ so it should in principle be possible to express $P \in C^4(V)$ in terms of its complete trace P, indeed taking an orthonormal basis $\{e_1, e_2\}$ of the unique 2–plane in V (which is V itself) we get

$$P(e_1, e_2, e_1, e_2) = \mathrm{Pec}(e_1, e_2) = \frac{1}{2}\mathrm{P}$$

which is the only value of P required to describe it, as all other values are either 0, equals to it, or to its opposite. Then, as a consequence of the previous lemma, we get

$$P = \frac{\mathrm{P}}{4} g \owedge g\,,$$

hence,

$$C^4(V) = \langle g \owedge g \rangle\,.$$

Before discussing dimensions 3 and higher, let us take a closer look at the properties of the Kulkarni–Nomizu product.

Proposition 1.3.15. *If $\alpha \in \mathbb{R}$, $h, k \in S^2(V^*)$ and $P \in \mathcal{F}^4(V)$, then the following equalities hold:*

(i) $h \owedge k = k \owedge h\,;$

(ii) $\alpha h \owedge k = h \owedge \alpha k = \alpha(h \owedge k)\,;$

(iii) $\mathrm{tr}^{1,3}(h \owedge g) = (\mathrm{tr}\, h)g + (n-2)h\,;$

$(*)$ $\mathrm{tr}^{1,3}(g \owedge g) = 2(n-1)g\,;$

(iv) $\mathrm{tr}\,\mathrm{tr}^{1,3}(h \owedge g) = 2(n-1)\,\mathrm{tr}\, h\,;$

$(*)$ $\mathrm{tr}\,\mathrm{tr}^{1,3}(g \owedge g) = 2n(n-1)\,;$

(v) $g(h \owedge g, P) = 4g(h, \mathrm{Pic})\,;$

$(*)$ $g(h \owedge g, g \owedge g) = 8(n-1)\,\mathrm{tr}\, h\,;$

$(**)$ $|h \owedge g|^2 = 4\big((n-2)|h|^2 + (\mathrm{tr}\, h)^2\big)\,;$

$(***)$ $|g \owedge g|^2 = 8n(n-1)\,;$

(vi) $|h \owedge k|^2 = 4\Big(|h|^2|k|^2 + \big(g(h,k)\big)^2 - 2\big(g(h^2,k^2)\big)^2\Big)\,,$

where for a tensor $p \in S^2(V^*)$, we define $p^2 = \mathrm{tr}^{2,3}(p \otimes p)$, i.e., $(p^2)_{ij} = p_{ik}g^{kl}p_{lj}$.

Remark 1.3.16. Property (v) may be interpreted by saying that the operators $P \mapsto 4\,\mathrm{Pic}$ and $h \mapsto h \owedge g$ between $S^2(V^*)$ and $\mathcal{F}^4(V)$ are g–adjoint.

Remark 1.3.17. If $\mathrm{tr}\, h = 0$ then $\mathrm{tr}\,\mathrm{tr}^{1,3}(h \owedge g) = 0$ and $(h \owedge g) \perp (g \owedge g)$.

Proof of Proposition 1.3.15. Properties (i) and (ii) are straightforward. Computing in any basis

$$\begin{aligned}\mathrm{tr}^{1,3}(h \owedge g)_{jl} &= (h \owedge g)^i{}_{jil}\\ &= h^i_i g_{jl} + h_{jl} g^i_i - h^i_l g_{ji} - h_{ji} g^i_l\\ &= (\mathrm{tr}\, h) g_{jl} + h_{jl}(\mathrm{tr}\, g) - h_{lj} - h_{jl}\\ &= (\mathrm{tr}\, h) g_{jl} + (n-2) h_{jl}\,,\end{aligned}$$

gives relations (iii), $(iii)^*$, (iv) and $(iv)^*$.
Now we evaluate

$$
\begin{aligned}
g(h \owedge g, P) &= (h \owedge g)^{ijkl} P_{ijkl} \\
&= (h^{ik}g^{jl} + h^{jl}g^{ik} - h^{il}g^{jk} - h^{jk}g^{il}) P_{ijkl} \\
&= h^{ik}(\operatorname{tr}^{1,3} P)_{ik} + h^{jl}(\operatorname{tr}^{2,4} P)_{jl} - h^{il}(\operatorname{tr}^{2,3} P)_{il} - h^{jk}(\operatorname{tr}^{1,4} P)_{jk} \\
&= h^{ik}\operatorname{Pic}_{ik} + h^{jl}\operatorname{Pic}_{jl} + h^{il}\operatorname{Pic}_{il} + h^{jk}\operatorname{Pic}_{jk} \\
&= 4g(h, \operatorname{Pic})\,,
\end{aligned}
$$

which is (v) and also implies equalities $(v)^*$ and $(v)^{**}$, as

$$
\begin{aligned}
g(h \owedge g, g \owedge g) &= 4g(h, \operatorname{tr}^{1,3}(g \owedge g)) = 8(n-1)g(h,g) = 8(n-1)\operatorname{tr} h\,, \\
|h \owedge g|^2 &= g(h \owedge g, h \owedge g) = 4g(h, \operatorname{tr}^{1,3}(h \owedge g)) \\
&= 4\big((\operatorname{tr} h)g(h,g) + (n-2)g(h,h)\big) = 4\big((\operatorname{tr} h)^2 + (n-2)|h|^2\big)
\end{aligned}
$$

and similarly for $(v)^{***}$. Finally, about formula (vi), we compute in an orthonormal basis $\{e_i\}_{i=1}^n$,

$$
\begin{aligned}
|h \owedge k|^2 &= \sum_{i,j,k,l=1}^n \big((h \owedge k)_{ijkl}\big)^2 \\
&= \sum_{i,j,k,l=1}^n (h_{ik}k_{jl} + h_{jl}k_{ik} - h_{il}k_{jk} - h_{jk}k_{il})^2 \\
&= \sum_{i,j,k,l=1}^n (h_{ik}^2k_{jl}^2 + h_{jl}^2k_{ik}^2 + h_{il}^2k_{jk}^2 + h_{jk}^2k_{il}^2) \\
&\quad + 2\sum_{i,j,k,l=1}^n (h_{ik}k_{jl}h_{jl}k_{ik} + h_{il}k_{jk}h_{jk}k_{il}) \\
&\quad - 2\sum_{i,j,k,l=1}^n (h_{ik}k_{jl}h_{il}k_{jk} + h_{ik}k_{jl}h_{jk}k_{il} + h_{jl}k_{ik}h_{il}k_{jk} + h_{jl}k_{ik}h_{jk}k_{il}) \\
&= 4\sum_{i,j,k,l=1}^n h_{ij}^2k_{kl}^2 + 4\sum_{i,j,k,l=1}^n h_{ij}k_{ij}h_{kl}k_{kl} - 8\sum_{i,j,k,l=1}^n h_{ij}h_{jk}k_{il}k_{lk} \\
&= 4\Big(|h|^2|k|^2 + \big(g(h,k)\big)^2 - 2\big(g(h^2,k^2)\big)^2\Big)\,.
\end{aligned}
$$

□

Dimension 3. If $n = 3$ then $\dim C^4(V) = 3^2(3^2-1)/12 = 6 = 3(3+1)/2 = \dim S^2(V^*)$ so it should again be possible in principle to express $P \in C^4(V)$ in terms of its $(1,3)$–trace Pic. Take indeed any 2–plane π expressed using an orthonormal basis by $\pi = \langle e_1, e_2\rangle$ and add a vector $\{e_3\}$ so that $\{e_1, e_2, e_3\}$ is an orthonormal basis of V. Then,

$$
\begin{aligned}
\operatorname{Pic}(e_1,e_1) &= \operatorname{Pec}(e_1,e_2) + \operatorname{Pec}(e_1,e_3)\,, \\
\operatorname{Pic}(e_2,e_2) &= \operatorname{Pec}(e_1,e_2) + \operatorname{Pec}(e_2,e_3)\,, \\
\operatorname{Pic}(e_3,e_3) &= \operatorname{Pec}(e_1,e_3) + \operatorname{Pec}(e_2,e_3)\,,
\end{aligned}
$$

hence, solving in terms of $\operatorname{Pec}(e_1,e_2)$, which is the sectional curvature of π, we have

$$
\operatorname{Pec}(e_1,e_2) = \operatorname{Pic}(e_1,e_1) + \operatorname{Pic}(e_2,e_2) - \operatorname{Pic}(e_3,e_3)\,.
$$

Thus, any sectional curvature can be obtained through Pic, hence, it determines the tensor P.

To get an explicit formula of P in terms of Pic and g, the idea is to orthogonally decompose P applying "successive divisions" by g, that is, finding a and B such that

$$P = ag \owedge g + B \owedge g \tag{1.20}$$

with $\operatorname{tr} B = 0$, so that $ag \owedge g \perp B \owedge g$ (Remark 1.3.17).
Suppose that formula (1.20) is correct, then using formulae in Proposition 1.3.15 we compute

$$\begin{aligned} \mathrm{Pic} &= \operatorname{tr}^{1,3} P = a \operatorname{tr}^{1,3}(g \owedge g) + \operatorname{tr}^{1,3}(B \owedge g) = 2a(n-1)g + (n-2)B\,, \\ \mathrm{P} &= \operatorname{tr} \mathrm{Pic} = 2an(n-1)\,, \end{aligned}$$

hence,

$$\begin{cases} a = \dfrac{\mathrm{P}}{2n(n-1)} \\ B = \dfrac{1}{n-2}\Big(\mathrm{Pic} - \dfrac{\mathrm{P}}{n} g\Big) = \dfrac{1}{n-2} \mathring{\mathrm{Pic}} \end{cases}$$

giving the formula

$$P = \frac{\mathrm{P}}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{\mathrm{Pic}} \owedge g\,, \tag{1.21}$$

and since we are in dimension $n = 3$,

$$P = \frac{\mathrm{P}}{12} g \owedge g + \mathring{\mathrm{Pic}} \owedge g\,. \tag{1.22}$$

Formula (1.22) does constitute a valid decomposition formula for P in terms of g and Pic. Consider indeed the tensor

$$W = P - \frac{\mathrm{P}}{12} g \owedge g - \mathring{\mathrm{Pic}} \owedge g\,,$$

by construction we have $\operatorname{tr}^{1,3} W = 0$ and since $W \in C^4(V)$, every trace $\operatorname{tr}^{i,j} W = 0$, $i, j \in \{1,2,3,4\}$, $i \neq j$. In dimension $n = 3$ this must imply $W = 0$, in fact, in an orthonormal basis $\{e_1, e_2, e_3\}$,

$$\begin{gathered} W_{1212} \overset{(\operatorname{tr}^{1,3}=0)}{=} -W_{3232} \overset{(\operatorname{tr}^{2,4}=0)}{=} W_{3131} \overset{(\operatorname{tr}^{1,3}=0)}{=} -W_{2121} = -W_{1212} \implies W_{1212} = 0\,, \\ W_{1213} \overset{(\operatorname{tr}^{1,3}=0)}{=} -W_{2223} - W_{3233} = 0\,, \end{gathered}$$

thus, $W_{ijkl} = 0$ for every $i, j, k, l \in \{1,2,3\}$.

General Case. If $n \geq 4$ there actually exist nontrivial completely trace–free tensors $W \in C^4(V)$, hence equation (1.21) is not always valid. We define nevertheless W^P as the *completely trace–free component* of P as

$$W^P \coloneqq P - \frac{\mathrm{P}}{2n(n-1)} g \owedge g - \frac{1}{n-2} \mathring{\mathrm{Pic}} \owedge g\,.$$

This gives the decomposition formula

$$P = \frac{\mathrm{P}}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{\mathrm{Pic}} \owedge g + W^P\,,$$

which again is an orthogonal decomposition of P, indeed as before $g(g \owedge g, \mathring{\mathrm{Pic}} \owedge g) = 0$, since $\operatorname{tr} \mathring{\mathrm{Pic}} = 0$ and

$$g(h \owedge g, W^P) = g(h, \operatorname{tr}^{1,3} W^P) = 0 \qquad \text{for all } h \in S^2(V^*)\,.$$

Definition 1.3.18. Call W a *Weyl tensor* on V if $W \in C^4(V)$ and is completely trace–free, i.e., $\operatorname{tr}^{1,3} W = 0$. We denote by $\mathcal{W}^4(V)$ the set of all Weyl tensors on V.

Definition 1.3.19. We denote by $S^2_0(V^*)$ the set of trace–free symmetric bilinear forms on V.

Remark 1.3.20. If $n = 2$ then $S^2_0(V^*) \owedge g = 0$, despite $S^2_0(V^*)$ being nontrivial, due to the high amount of constrictions on $C^4(V)$. Let indeed $h \in S^2_0(V^*)$, then in an orthonormal basis $\{e_1, e_2\}$ we have

$$(h \owedge g)_{1212} = h_{11}g_{22} + h_{22}g_{11} - h_{12}g_{21} - h_{21}g_{12} = h_{11} + h_{22} = \operatorname{tr} h = 0\,,$$

so $h \owedge g = 0$. However, if $n \geq 3$ the map $S^2(V^*) \ni h \mapsto h \owedge g \in C^4(V)$ is injective; let indeed $h \owedge g = 0$, then

$$0 = |h \owedge g|^2 = 4\big((n-2)|h|^2 + (\operatorname{tr} h)^2\big)\,,$$

which implies $h = 0$, since $n > 2$.

Remark 1.3.21. There holds

$$|\mathring{\mathrm{Pic}}|^2 = \left|\mathrm{Pic} - \frac{\mathrm{P}}{n} g\right|^2 = |\mathrm{Pic}|^2 + \frac{\mathrm{P}^2}{n^2}|g|^2 - 2\frac{\mathrm{P}}{n} g(\mathrm{Pic}, g) = |\mathrm{Pic}|^2 - \frac{\mathrm{P}^2}{n}\,.$$

All the previous results can be summarised in the following decomposition theorem.

Theorem 1.3.22. *Let (V, g) be a real n-dimensional vector space with scalar product g. The following decomposition formula holds:*

$$C^4(V) = \langle g \owedge g \rangle \oplus^{\perp} S^2_0(V^*) \owedge g \oplus^{\perp} \mathcal{W}^4(V)\,,$$

where if $n = 1$ all spaces are trivial.
If $n = 2$, then $S^2_0(V^*) \owedge g$ and $\mathcal{W}^4(V)$ are trivial and every $P \in C^4(V)$ can be decomposed as

$$P = \frac{\mathrm{P}}{4} g \owedge g\,, \tag{1.23}$$

hence,

$$|P| = \frac{|\mathrm{P}|}{4}|g \owedge g| = |\mathrm{P}|\,.$$

If $n = 3$, then $\mathcal{W}^4(V)$ is trivial and every $P \in C^4(V)$ can be decomposed as

$$P = \frac{\mathrm{P}}{12} g \owedge g + \mathring{\mathrm{Pic}} \owedge g\,,$$

hence,

$$|P|^2 = \frac{\mathrm{P}^2}{144}|g \owedge g|^2 + |\mathring{\mathrm{Pic}} \owedge g|^2 = \frac{1}{3}\mathrm{P}^2 + 4|\mathring{\mathrm{Pic}}|^2 = 4|\mathrm{Pic}|^2 - \mathrm{P}^2\,.$$

If $n \geq 4$ all spaces are nontrivial and every $P \in C^4(V)$ can be decomposed as

$$P = \frac{\mathrm{P}}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{\mathrm{Pic}} \owedge g + W^P \,, \tag{1.24}$$

hence,

$$\begin{aligned}
|P|^2 &= \frac{\mathrm{P}^2}{4n^2(n-1)^2} |g \owedge g|^2 + \frac{1}{(n-2)^2} |\mathring{\mathrm{Pic}} \owedge g|^2 + |W^P|^2 \\
&= \frac{2}{n(n-1)} \mathrm{P}^2 + \frac{4}{n-2} |\mathring{\mathrm{Pic}}|^2 + |W^P|^2 \\
&= |W^P|^2 + \frac{4}{n-2} |\mathrm{Pic}|^2 - \frac{2}{(n-1)(n-2)} \mathrm{P}^2 \,.
\end{aligned}$$

From equation (1.24), using $\mathring{\mathrm{Pic}} = \mathrm{Pic} - \mathrm{P} g/n$, for $n \geq 3$ we also get the orthogonal decomposition $C^4(V) = S^2(V^*) \owedge g \oplus^{\perp} \mathcal{W}^4(V)$ with

$$P = \frac{1}{n-2} \Big(\mathrm{Pic} - \frac{\mathrm{P}}{2(n-1)} g \Big) \owedge g + W^P \,.$$

Definition 1.3.23. If $n \geq 3$, we denote by

$$S^P = \frac{1}{n-2} \Big(\mathrm{Pic} - \frac{\mathrm{P}}{2(n-1)} g \Big)$$

the *Schouten tensor* of P.
We set $S^P := 0$ for any P, if $n \in \{1, 2\}$.

Then, for $n \geq 3$ we also have the decomposition

$$P = S^P \owedge g + W^P \,.$$

Remark 1.3.24. The dimensions of the spaces involved in the decomposition formula are given by

$$\begin{aligned}
\dim \langle g \owedge g \rangle &= \dim \mathbb{R} = 1 \,, \\
\dim S^2_0(V^*) &= \dim S^2(V^*) - \dim \langle g \owedge g \rangle = \frac{(n+2)(n-1)}{2}
\end{aligned}$$

and, for $n \geq 3$,

$$\dim \mathcal{W}^4(V) = \dim C^4(V) - \dim S^2(V^*) = \frac{n(n+1)(n+2)(n-3)}{12} \,.$$

Let us now discuss the *purity* of the Riemann tensor.
We recall that if $\{e_i\}_{i=1}^n$ is an orthonormal basis of V, then $\{e_i \wedge e_j\}_{i<j=1}^n$ is an orthonormal basis of $\Lambda^2(V)$ with respect to $\frac{1}{2} g$; take indeed $i < j$ and $k < l$, then

$$\tfrac{1}{2} g(e_i \wedge e_j, e_k \wedge e_l) = g(e_i, e_k) g(e_j, e_l) - g(e_i, e_l) g(e_j, e_k) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

and it easily follows that the scalar product is 1 if and only if $(i, j) = (k, l)$ and 0 otherwise.

Remark 1.3.25. Any symmetric form $h \in S^2(V^*)$ defines an adjoint operator $\hbar$ by

$$h(v,w) = g(\hbar(v),w) \qquad \text{for } v,w \in V\,,$$

such that the eigenvalue relation $\hbar(e) = \lambda e$ is equivalent to

$$h(e,w) = \lambda g(e,w)\,, \qquad \text{for every } w \in V\,.$$

Lemma 1.3.26. *Let $h \in S^2(V^*)$ and call $\hbar$ and $\mathscr{H}$ the operators defined by linear extension from*

$$\begin{aligned} h(v,w) &= g(\hbar(v),w) && \textit{for } v,w \in V\,,\\ (h \owedge g)(x,y,z,w) &= \tfrac{1}{2} g(\mathscr{H}(x\wedge y), z\wedge w) && \textit{for } x,y,z,w \in V\,. \end{aligned}$$

If $\{e_i\}_{i=1}^n$ is an orthonormal eigenbasis of $\hbar$ then $\{e_i \wedge e_j\}_{i<j=1}^n$ is an eigenbasis of $\mathscr{H}$.

Proof. Let $\hbar(e_i) = \lambda_i e_i$, we consider $i<j$, $k<l$ and evaluate

$$\begin{aligned} (h \owedge g)_{ijkl} &= h_{ik}g_{jl} + h_{jl}g_{ik} - h_{il}g_{jk} - h_{jk}g_{il}\\ &= \lambda_i g_{ik}g_{jl} + \lambda_j g_{jl}g_{ik} - \lambda_i g_{il}g_{jk} - \lambda_j g_{jk}g_{il}\\ &= (\lambda_i + \lambda_j)\tfrac{1}{2}(g \owedge g)_{ijkl}\,. \end{aligned}$$

Thus, by Remark 1.3.25 (with the metric $\frac{1}{2}g$), we obtain

$$\mathscr{H}(e_i \wedge e_j) = (\lambda_i + \lambda_j)(e_i \wedge e_j)\,.$$

$\square$

Lemma 1.3.27. *Let $P \in C^4(V)$ and $\{e_i\}_{i=1}^n$ an orthonormal basis of V, we call c, $\mathscr{C}$ and $\mathscr{W}$ the operators defined by linear extension from*

$$\begin{aligned} \mathrm{Pic}(v,w) &= g(c(v),w) && \textit{for } v,w \in V\,,\\ (\mathrm{Pic} \owedge g)(x,y,z,w) &= \tfrac{1}{2} g(\mathscr{C}(x\wedge y), z\wedge w) && \textit{for } x,y,z,w \in V\,,\\ W^P(x,y,z,w) &= \tfrac{1}{2} g(\mathscr{W}(x\wedge y), z\wedge w) && \textit{for } x,y,z,w \in V\,. \end{aligned}$$

Then $\{e_i \wedge e_j\}_{i<j=1}^n$ is an eigenbasis of $\mathscr{P}$ if and only if it is an eigenbasis of $\mathscr{C}$ and $\mathscr{W}$.

Proof. Clearly, any such eigenbasis of $\mathscr{C}$ and $\mathscr{W}$ is an eigenbasis of $\mathscr{P}$, by the decomposition (1.24) (as it is also an eigenbasis of the form associated to $\frac{1}{2}(g \owedge g)$). By the same formula, any such eigenbasis of $\mathscr{P}$ and $\mathscr{C}$ is an eigenbasis of $\mathscr{W}$. Hence, by Lemma 1.3.26, it is only necessary to prove that $\{e_i\}_{i=1}^n$ is an eigenbasis of c.
We let $\mathscr{P}(e_i \wedge e_j) = \lambda_{ij}(e_i \wedge e_j)$ and evaluate

$$\mathrm{Pic}_{ij} = \sum_{\substack{k=1\\k\neq i}}^n P_{kikj} = \sum_{\substack{k=1\\k\neq i}}^n \lambda_{ki}\tfrac{1}{2}(g \owedge g)_{kikj} = \Big(\sum_{\substack{k=1\\k\neq i}}^n \lambda_{ki}\Big) g_{ij}\,,$$

Thus, by Remark 1.3.25, we obtain

$$c(e_i) = \Big(\sum_{\substack{k=1\\k\neq i}}^n \lambda_{ki}\Big) e_i\,.$$

$\square$

Theorem 1.3.28. *If $W^P = 0$ then P is pure; i.e., there exists an eigenbasis $\{e_i \wedge e_j\}_{i<j=1}^n$ of $\mathscr{P}$ with $\{e_i\}_{i=1}^n$ orthonormal basis of V.*

Proof. Since c is self–adjoint, it admits an orthonormal eigenbasis $\{e_i\}_{i=1}^n$ of V. Then, the result follows from Lemmas 1.3.26 and 1.3.27, as $\mathscr{W} = 0$. □

Corollary 1.3.29. *In dimension 3 all algebraic curvature tensors $P \in C^4(V)$ are pure.*

Proof. In dimension 3 there are no nontrivial Weyl tensors. □

Remark 1.3.30. The Weyl tensor of $P \in C^4(V)$ is only dependent on the conformal class of g. Let indeed $\widetilde{g} = \lambda g$, with $\lambda > 0$ and denote by

$$\widetilde{\mathrm{Pic}} = \widetilde{\mathrm{tr}}^{1,3} P\,, \qquad\qquad \widetilde{\mathrm{P}} = \widetilde{\mathrm{tr}}\,\widetilde{\mathrm{Pic}}\,,$$

and $\widetilde{W}^P$ the tensor obtained by the decomposition (1.24) such that

$$P = \frac{\widetilde{\mathrm{P}}}{2n(n-1)}\widetilde{g} \owedge \widetilde{g} + \frac{1}{n-2}\left(\widetilde{\mathrm{Pic}} - \frac{\widetilde{\mathrm{P}}}{n}\widetilde{g}\right) \owedge \widetilde{g} + \widetilde{W}^P\,.$$

Since in any basis $(\widetilde{g}^{ij})_{i,j=1}^n$ is the inverse matrix of $(\widetilde{g}_{ij})_{i,j=1}^n = (\lambda g_{ij})_{i,j=1}^n$, that is, $\widetilde{g}^{ij} = \frac{1}{\lambda} g^{ij}$, we have

$$\begin{aligned}
\widetilde{\mathrm{Pic}} &= \widetilde{\mathrm{tr}}^{1,3} P = \frac{1}{\lambda}\,\mathrm{tr}^{1,3} P = \frac{1}{\lambda}\,\mathrm{Pic}\,,\\
\widetilde{\mathrm{P}} &= \widetilde{\mathrm{tr}}\,\widetilde{\mathrm{Pic}} = \frac{1}{\lambda}\,\mathrm{tr}\,\widetilde{\mathrm{Pic}} = \frac{1}{\lambda^2}\mathrm{P}\,,
\end{aligned}$$

so

$$\begin{aligned}
P - \widetilde{W}^P &= \frac{\widetilde{\mathrm{P}}}{2n(n-1)}\widetilde{g} \owedge \widetilde{g} + \frac{1}{n-2}\left(\widetilde{\mathrm{Pic}} - \frac{\widetilde{\mathrm{P}}}{n}\widetilde{g}\right) \owedge \widetilde{g}\\
&= \frac{\mathrm{P}/\lambda^2}{2n(n-1)}\lambda g \owedge \lambda g + \frac{1}{n-2}\left(\frac{1}{\lambda}\,\mathrm{Pic} - \frac{\mathrm{P}/\lambda^2}{n}\lambda g\right) \owedge \lambda g\\
&= \frac{\mathrm{P}}{2n(n-1)} g \owedge g + \frac{1}{n-2}\left(\mathrm{Pic} - \frac{\mathrm{P}}{n} g\right) \owedge g\\
&= P - W^P
\end{aligned}$$

and $W^P = \widetilde{W}^P$.

We end this section by expressing the consequences of the decomposition Theorem 1.3.22 for an n–dimensional Riemannian manifold (M, g) and its Riemann tensor Riem.

The Riemann tensor admits the orthogonal decomposition

$$\begin{aligned}
\mathrm{Riem} &= \frac{\mathrm{R}}{4} g \owedge g && \text{if } n = 2\,,\\
\mathrm{Riem} &= \frac{\mathrm{R}}{12} g \owedge g + \mathring{\mathrm{Ric}} \owedge g && \text{if } n = 3\,,\\
\mathrm{Riem} &= \frac{\mathrm{R}}{2n(n-1)} g \owedge g + \frac{1}{n-2}\mathring{\mathrm{Ric}} \owedge g + \mathrm{Weyl} && \text{if } n \geq 4\,,
\end{aligned} \tag{1.25}$$

where the *Weyl tensor* Weyl is an algebraic curvature tensor, *completely trace–free* (i.e., each of its traces is zero).
It follows that $|\mathrm{Riem}| = |\mathrm{R}|$, when $n = 2$ and, for $n \geq 3$ (setting $\mathrm{Weyl} = 0$, if $n = 3$),

$$|\mathrm{Riem}|^2 = \frac{4|\mathrm{Ric}|^2}{n-2} - \frac{2\mathrm{R}^2}{(n-1)(n-2)} + |\mathrm{Weyl}|^2 = \frac{4|\mathring{\mathrm{Ric}}|^2}{n-2} + \frac{2\mathrm{R}^2}{n(n-1)} + |\mathrm{Weyl}|^2 \tag{1.26}$$

as

$$|\mathring{\mathrm{Ric}}|^2 = |\mathrm{Ric}|^2 - \mathrm{R}^2/n\,. \tag{1.27}$$

Moreover, by defining for $n \geq 3$ the *Schouten tensor* S, which is clearly symmetric, as

$$S = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{R}}{2(n-1)} g\right),$$

we can also write the orthogonal decomposition as follows,

$$\begin{aligned} \mathrm{Riem} &= S \owedge g && \text{if } n = 3\,, \\ \mathrm{Riem} &= S \owedge g + \mathrm{Weyl} && \text{if } n \geq 4\,. \end{aligned}$$

For completeness, we also define $\mathrm{Weyl} := 0$ if $n \in \{1, 2, 3\}$, $S := 0$ if $n = 1$ and $S := \mathrm{R}g/4$ if $n = 2$. Then, the decomposition

$$\mathrm{Riem} = S \owedge g + \mathrm{Weyl}$$

holds in every dimension. In local coordinates,

$$\begin{aligned} (n-1)(n-2)R_{ijkl} = {} & (n-1)(n-2)W_{ijkl} - \mathrm{R}g_{ik}g_{jl} + \mathrm{R}g_{il}g_{jk} \\ & + (n-1)(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il})\,. \end{aligned} \tag{1.28}$$

1.4 The Cartan formalism

The formalism we have been using up to now (and that we will continue to use in most of the dissertation) to deal with the curvature is the standard (*global*, coordinate–free) *Koszul formalism*.[15] A (local) alternative consists in describing everything by looking at the behaviour of a local *moving frame*, in particular an orthonormal frame. This is the so–called *Cartan formalism*.

We work with vectors and matrices having differential forms as entries; such elements belong to $\Omega^k(M, E)$, which we call the space of E–valued differential k–forms, with $E = \mathbb{R}^n$ and $E = \mathbb{R}^{n\times m}$, respectively.

In this section indices do not denote tensor components but rather the "position" in a matrix or vector. The convention of summation over repeated indices is anyway still always adopted.

We extend to these spaces the exterior product through the usual operations between matrices and vectors, as well as the differential d, as follows,

$$\begin{aligned} (\eta\wedge\zeta)^i_j &:= \eta^i_r \wedge \zeta^r_j \qquad \text{and} \qquad (\eta\wedge z)^i := \eta^i_r \wedge z^r\,, \\ (\mathrm{d}\eta)^i_j &:= \mathrm{d}\eta^i_j \qquad \text{and} \qquad (\mathrm{d}z)^j := \mathrm{d}z^j\,, \end{aligned}$$

for $\eta \in \Omega^k(M, \mathbb{R}^{n\times m})$, $\zeta \in \Omega^s(M, \mathbb{R}^{m\times l})$, $z \in \Omega^s(M, \mathbb{R}^m)$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$.
We still denote by $|\eta| = k$ and $|z| = s$, the *degree* of $\eta \in \Omega^k(M, \mathbb{R}^{n\times m})$ and $z \in \Omega^s(M, \mathbb{R}^m)$, respectively.

[15] After the French mathematician Jean–Louis Koszul (1921–2018) [69].

Remark 1.4.1. The usual relation $\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha$ for ordinary forms α and β, although still valid component–wise, has the following analogue in this setting

$$(\eta \wedge \zeta)^T = (-1)^{|\eta||\zeta|}\zeta^T \wedge \eta^T ,$$

for $\eta \in \Omega^k(M, \mathbb{R}^{n\times m})$, $\zeta \in \Omega^s(M, \mathbb{R}^{m\times l})$.
In particular, one should not be surprised to have $\eta \wedge \eta \neq 0$ for some 1–form η.

We now fix a metric g and a local frame $\{e_i\}_{i=1}^n$ on an open set $U \subseteq M$, with associated coframe $\{\vartheta^i\}_{i=1}^n$. Notice that $\vartheta = (\vartheta^i)_{i=1}^n$ may be regarded as an $\mathbb{R}^n$–valued differential 1–form.

Definition 1.4.2. We define the local 1–form $\omega = (\omega^i_j)_{i,j=1}^n$ and 2–form $\Omega = (\Omega^i_j)_{i,j=1}^n$ by

$$\nabla e_j = \omega^i_j \otimes e_i \,, \tag{1.29}$$

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega \,, \tag{1.30}$$

where ∇ is the Levi–Civita connection of the Riemannian manifold (M, g).
We call ω and Ω the *(Levi–Civita) connection* 1*–form* and the *(Levi–Civita) curvature* 2*–form*, associated to the frame $\{e_i\}_{i=1}^n$ in U, respectively.

Remark 1.4.3. Equation (1.29) explicitly defines the 1–form ω^i_j as

$$\omega^i_j(X) = \vartheta^i(\nabla_X e_j) = \Gamma^i_{kj}X^k \tag{1.31}$$

for any $X = X^k e_k \in \Gamma(TM)$, where Γ^i_{kj} are the Christoffel symbols of ∇ in the local basis $\{e_i\}_{i=1}^n$ (notice that they are not necessarily symmetric).

Equation (1.30) is called the *second structural equation*, the *first structural equation* being

$$\mathrm{d}\vartheta = -\omega \wedge \vartheta \,.$$

They together,

$$\begin{cases} \mathrm{d}\vartheta = -\omega \wedge \vartheta \,, \\ \mathrm{d}\omega = \Omega - \omega \wedge \omega \,, \end{cases} \tag{1.32}$$

are called *Cartan structural equations*. Once differentiated, they give

$$\begin{cases} \quad 0 = \Omega \wedge \vartheta \,, \\ \mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega \,, \end{cases} \tag{1.33}$$

which are called the *Bianchi identities in the Cartan formalism.*

Proof of the first structural equation. Take any two vectors $X = X^i e_i$ and $Y = Y^i e_i$ then

$$\begin{aligned} (\nabla_X \vartheta^i)(Y) &= X\big(\vartheta^i(Y)\big) - \vartheta^i(\nabla_X Y) \\ &= X(Y^i) - \vartheta^i\big(X(Y^j)e_j + Y^j\omega^s_j(X)e_s\big) \\ &= X(Y^i) - X(Y^i) - Y^j\omega^i_j(X) \\ &= -\omega^i_j(X)\vartheta^j(Y) \end{aligned}$$

hence, using equation (1.12), we obtain

$$\begin{aligned} \mathrm{d}\vartheta^i(X, Y) &= (\nabla_X \vartheta^i)(Y) - (\nabla_Y \vartheta^i)(X) \\ &= -\omega^i_j(X)\vartheta^j(Y) + \omega^i_j(Y)\vartheta^j(X) \\ &= -(\omega^i_j \wedge \vartheta^j)(X, Y) \,. \end{aligned}$$

□

Proof of the Bianchi identities. We simply take the differential of both sides of equations (1.32),

$$\begin{cases} 0 = \mathrm{d}(-\omega \wedge \vartheta)\,, \\ 0 = \mathrm{d}(\Omega - \omega \wedge \omega)\,, \end{cases}$$

then, we have

$$0 = -\mathrm{d}\omega \wedge \vartheta + \omega \wedge \mathrm{d}\vartheta = -\Omega \wedge \vartheta + \omega \wedge \omega \wedge \vartheta - \omega \wedge \omega \wedge \vartheta = -\Omega \wedge \vartheta$$

and

$$\begin{aligned} 0 &= \mathrm{d}\Omega - \mathrm{d}\omega \wedge \omega + \omega \wedge \mathrm{d}\omega \\ &= \mathrm{d}\Omega - \Omega \wedge \omega + \omega \wedge \Omega + \omega \wedge \omega \wedge \omega - \omega \wedge \omega \wedge \omega \\ &= \mathrm{d}\Omega - \Omega \wedge \omega + \omega \wedge \Omega\,. \end{aligned}$$

□

We now see how the connection and curvature forms transform under a change of the local frame.

Proposition 1.4.4 (Transformation laws)**.** *Changing to a local frame $\{\widetilde{e}_i\}_{i=1}^n$ via a transformation $\widetilde{e}_i = f_i^j e_j$, the forms ω and Ω transform according to the relations*

$$\begin{cases} \widetilde{\omega} = f^{-1}\omega f + f^{-1}\mathrm{d}f\,, \\ \widetilde{\Omega} = f^{-1}\Omega f\,. \end{cases} \tag{1.34}$$

Proof. We compute

$$\begin{aligned} \nabla\widetilde{e}_j &= \nabla f_j^k e_k = \mathrm{d}f_j^k \otimes e_k + f_j^k \omega_k^s \otimes e_s \\ &= \mathrm{d}f_j^k \otimes (f^{-1})_k^i \widetilde{e}_i + f_j^k \omega_k^s \otimes (f^{-1})_s^i \widetilde{e}_i \\ &= ((f^{-1})_k^i \mathrm{d}f_j^k + (f^{-1})_s^i \omega_k^s f_j^k) \otimes \widetilde{e}_i\,, \end{aligned}$$

showing the relation for $\widetilde{\omega}$. Then we have

$$\begin{aligned} \widetilde{\Omega} &= \mathrm{d}\widetilde{\omega} + \widetilde{\omega} \wedge \widetilde{\omega} \\ &= \mathrm{d}f^{-1} \wedge \omega f + f^{-1}\mathrm{d}\omega f - f^{-1}\omega \wedge \mathrm{d}f + \mathrm{d}f^{-1} \wedge \mathrm{d}f \\ &\quad + (f^{-1}\omega f + f^{-1}\mathrm{d}f) \wedge (f^{-1}\omega f + f^{-1}\mathrm{d}f) \\ &= f^{-1}(\mathrm{d}\omega + \omega \wedge \omega) f + \mathrm{d}f^{-1} \wedge (\omega f + \mathrm{d}f) + f^{-1}\big(-\omega + \omega + \mathrm{d}f f^{-1}\big) \wedge \mathrm{d}f \\ &\quad + f^{-1}\mathrm{d}f f^{-1} \wedge \omega f \\ &= f^{-1}\Omega f + \mathrm{d}f^{-1} \wedge (\omega f + \mathrm{d}f) - f^{-1} f \mathrm{d}f^{-1} \wedge \mathrm{d}f - f^{-1} f \mathrm{d}f^{-1} \wedge \omega f \\ &= f^{-1}\Omega f\,, \end{aligned}$$

where we used the identity

$$\mathrm{d}f f^{-1} = \mathrm{d}(f f^{-1}) - f\mathrm{d}f^{-1} = -f\mathrm{d}f^{-1}\,.$$

This proves also the relation for $\widetilde{\Omega}$. □

Remark 1.4.5. Restricting ourselves to using only local orthonormal frames implies that the transformation matrix f is orthogonal. Furthermore, in such frames the forms ω and Ω are skew–symmetric, that is,

$$\begin{cases} \omega^i_j = -\omega^j_i \,, \\ \Omega^i_j = -\Omega^j_i \,. \end{cases}$$

Taking, indeed, any vector field X one obtains

$$\begin{aligned} 0 = Xg(e_i, e_j) &= g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) \\ &= g(\omega^s_i(X)e_s, e_j) + g(\omega^s_j(X)e_s, e_i) \\ &= \omega^j_i(X) + \omega^i_j(X) \end{aligned}$$

then, the skew–symmetry of Ω follows from equation (1.30), as

$$\Omega^j_i = \mathrm{d}\omega^j_i + \omega^j_s \wedge \omega^s_i = -\mathrm{d}\omega^i_j - \omega^i_s \wedge \omega^s_j = -\Omega^i_j \,.$$

Remark 1.4.6. We remark that equations (1.32) and (1.33) are a consequence of the symmetry and metric compatibility of the Levi–Civita connection, respectively and that they in general contain extra "error terms" given by the torsion and the covariant derivative of the metric, for a general connection (see [11]).

Remark 1.4.7. Returning back to the Koszul formalism, the 1–forms ω^i_j and 2–forms Ω^i_j are given by

$$\begin{cases} \omega^i_j = \Gamma^i_{kj}\vartheta^k \,, \\ \Omega^i_j = \frac{1}{2} R^i{}_{jkl}\vartheta^k \wedge \vartheta^l \,. \end{cases} \tag{1.35}$$

and a version of Ω can be globally defined as $\Omega(v,w) = \mathscr{R}(v \wedge w)$.
Furthermore, the Bianchi identities in equation (1.33) correspond to the standard Bianchi identities.

Proof of the Koszul relations. We express everything in Koszul formalism, starting with equation (1.31)

$$\omega^i_j(X) = \vartheta^i(\nabla_X e_j) = \vartheta^i(X^k \Gamma^t_{kj} e_t) = \Gamma^i_{kj} X^k = \Gamma^i_{kj}\vartheta^k(X) \,,$$

then with equation (1.30)

$$\nabla_X \nabla_Y e_j = \nabla_X \omega^i_j(Y) e_i = X\omega^i_j(Y) e_i + \omega^i_j(Y)\omega^k_i(X) e_k$$

which gives

$$\begin{aligned} R(X,Y)e_j &= \nabla_Y \nabla_X e_j - \nabla_X \nabla_Y e_j - \nabla_{[Y,X]} e_j \\ &= Y\omega^i_j(X)e_i + \omega^i_j(X)\omega^k_i(Y)e_k - X\omega^i_j(Y)e_i - \omega^i_j(Y)\omega^k_i(X)e_k - \omega^i_j([Y,X])e_i \\ &= -\mathrm{d}\omega^i_j(X,Y)e_i - (\omega \wedge \omega)^k_j(X,Y)e_k \\ &= -\Omega^i_j(X,Y)e_i \,, \end{aligned}$$

hence,

$$\begin{aligned} \Omega^i_j(X,Y) &= -\vartheta^i(R(X,Y)e_j) = -X^k Y^l \vartheta^i(R_{klj}{}^s e_s) = R^i{}_{jkl} X^k Y^l \\ &= \tfrac{1}{2} R^i{}_{jkl}(\vartheta^k(X)\vartheta^l(Y) - \vartheta^k(Y)\vartheta^l(X)) \\ &= \tfrac{1}{2} R^i{}_{jkl}(\vartheta^k \wedge \vartheta^l)(X,Y) \,. \end{aligned}$$

Moreover, we recall that

$$\mathscr{R}(e_i \wedge e_j) = \sum_{1\leq k<l\leq n} R_{ij}{}^{kl} e_k \wedge e_l = \tfrac{1}{2} R_{ij}{}^{kl} e_k \wedge e_l \,,$$

by definition. □

Proof of the Bianchi relations. By using the symmetries of the Riemann tensor, we have

$$\begin{aligned}
0 = (\Omega \wedge \vartheta)^i &= \tfrac{1}{2} R^i{}_{jkl} \vartheta^k \wedge \vartheta^l \wedge \vartheta^j \\
&= \sum_{1\leq j<k<l\leq n} (R^i{}_{jkl} - R^i{}_{kjl} - R^i{}_{lkj})\, \vartheta^j \wedge \vartheta^k \wedge \vartheta^l \\
&= \sum_{1\leq j<k<l\leq n} (R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj})\, \vartheta^j \wedge \vartheta^k \wedge \vartheta^l \,,
\end{aligned}$$

that is, the first (algebraic) Bianchi identity $R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} = 0$.
About the second (differential) Bianchi identity, first we compute

$$\begin{aligned}
0 &= \mathrm{d}\Omega^i_j - \Omega^i_k \wedge \omega^k_j + \omega^i_k \wedge \Omega^k_j \\
&= \tfrac{1}{2} \mathrm{d}R^i{}_{jst} \wedge \vartheta^s \wedge \vartheta^t + \tfrac{1}{2} R^i{}_{jst} \mathrm{d}\vartheta^s \wedge \vartheta^t - \tfrac{1}{2} R^i{}_{jst} \vartheta^s \wedge \mathrm{d}\vartheta^t \\
&\quad - \tfrac{1}{2} R^i{}_{kst} \vartheta^s \wedge \vartheta^t \wedge \omega^k_j + \tfrac{1}{2} R^k{}_{jst} \vartheta^s \wedge \vartheta^t \wedge \omega^i_k \\
&= \tfrac{1}{2} \mathrm{d}R^i{}_{jst} \wedge \vartheta^s \wedge \vartheta^t - \tfrac{1}{2} R^i{}_{jst} \omega^s_r \wedge \vartheta^r \wedge \vartheta^t + \tfrac{1}{2} R^i{}_{jst} \vartheta^s \wedge \omega^t_r \wedge \vartheta^r \\
&\quad - \tfrac{1}{2} R^i{}_{kst} \vartheta^s \wedge \vartheta^t \wedge \omega^k_j + \tfrac{1}{2} R^k{}_{jst} \vartheta^s \wedge \vartheta^t \wedge \omega^i_k \\
&= \tfrac{1}{2} (\mathrm{d}R^i{}_{jst} + R^r{}_{jst} \omega^i_r - R^i{}_{rst} \omega^r_j - R^i{}_{jrt} \omega^r_s - R^i{}_{jsr} \omega^r_t) \wedge \vartheta^s \wedge \vartheta^t \,,
\end{aligned}$$

then, we just observe that the term in parentheses is the expression of $\nabla_r R^i{}_{jst} \vartheta^r$, hence,

$$\begin{aligned}
0 &= \tfrac{1}{2} \nabla_r R^i{}_{jst} \vartheta^r \wedge \vartheta^s \wedge \vartheta^t \\
&= \sum_{1\leq r<s<t\leq n} (\nabla_r R^i{}_{jst} + \nabla_t R^i{}_{jrs} + \nabla_s R^i{}_{jtr})\, \vartheta^r \wedge \vartheta^s \wedge \vartheta^t \,,
\end{aligned}$$

that is, $\nabla_r R^i{}_{jst} + \nabla_t R^i{}_{jrs} + \nabla_s R^i{}_{jtr} = 0$ and we are done. □

Chapter 2

The Weyl tensor and LCF manifolds

In this chapter we introduce the Weyl tensor and show its invariance under conformal changes. Then, on the converse, we will prove the Weyl–Schouten theorem stating that a Weyl–flat metric is locally conformal to a flat metric, in dimension at least four.
In dimensions $n = 2$ and $n = 3$ the Weyl tensor is trivial, hence, the respective conditions differ. In particular, every 2–manifold is locally conformally flat, while in dimension $n = 3$ the role of the Weyl tensor is taken by the Cotton tensor (which in higher dimensions is proportional to its divergence, hence it is zero if $\mathrm{Weyl} = 0$).

2.1 Transformation rules under a conformal change

Recalling the final part of Section 1.3, we define the *Weyl tensor* Weyl (or W) and the Schouten tensor S of an n–dimensional Riemannian manifold (M, g) as the tensors giving the decomposition, for $n \geq 4$:

$$\mathrm{Riem} = S \owedge g + \mathrm{Weyl}\,, \tag{2.1}$$

where, for $n \geq 3$

$$S = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{R}}{2(n-1)}g\right).$$

We also define, for convenience, $W := 0$ if $n \in \{1, 2, 3\}$, $S := 0$ if $n = 1$ and $S := \mathrm{R}g/4$ if $n = 2$, so that formula (2.1) holds in every dimension.

First, we would like to prove, arguing similarly to Remark 1.3.30, that W is conformally invariant; differently from such computation, $\widetilde{\mathrm{Riem}} \neq \mathrm{Riem}$, as in this case Riem also is dependent on the metric g.

Lemma 2.1.1. *If ∇ and $\widetilde{\nabla}$ are two connections on (M, g), then*

$$T\colon (X, Y) \mapsto T(X, Y) = \widetilde{\nabla}_X Y - \nabla_X Y \qquad \textit{for } X, Y \in \Gamma(TM)$$

defines a $(2, 1)$–tensor. If, in addition, ∇ and $\widetilde{\nabla}$ are torsion–free, then T is symmetric and

$$\widetilde{R}(X,Y)Z - R(X,Y)Z = \nabla_Y T(X,Z) - \nabla_X T(Y,Z) + T(Y, T(X,Z)) - T(X, T(Y,Z))\,.$$

Proof. It is sufficient to check the $C^\infty(M)$–linearity on the second entry of T,

$$T(X, fY) = \widetilde{\nabla}_X(fY) - \nabla_X(fY) = (Xf)Y + f\widetilde{\nabla}_X Y - (Xf)Y - f\nabla_X Y = fT(X,Y)\,.$$

If ∇ and $\widetilde{\nabla}$ are torsion–free, then

$$T(Y,X) = \widetilde{\nabla}_Y X - \nabla_Y X = \widetilde{\nabla}_X Y - [X,Y] - \nabla_X Y + [X,Y] = T(X,Y)\,.$$

We now evaluate

$$\begin{aligned}\widetilde{\nabla}_Y \widetilde{\nabla}_X Z &= \widetilde{\nabla}_Y (\nabla_X Z + T(X,Z))\\ &= \nabla_Y \nabla_X Z + \nabla_Y (T(X,Z)) + T(Y, \nabla_X Z) + T\big(Y, T(X,Z)\big)\\ &= \nabla_Y \nabla_X Z + \nabla_Y T(X,Z) + T(\nabla_Y X, Z) + T(X, \nabla_Y Z)\\ &\quad + T(Y, \nabla_X Z) + T\big(Y, T(X,Z)\big)\,,\end{aligned}$$

hence,

$$\begin{aligned}\widetilde{R}(X,Y)Z = {}& \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z - T([X,Y],Z)\\ &+ \nabla_Y T(X,Z) - \nabla_X T(Y,Z)\\ &+ T(\nabla_Y X, Z) + T(X, \nabla_Y Z) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z)\\ &+ T(Y, \nabla_X Z) - T(X, \nabla_Y Z)\\ &+ T\big(Y, T(X,Z)\big) - T\big(X, T(Y,Z)\big)\\ = {}& R(X,Y)Z + \nabla_Y T(X,Z) - \nabla_X T(Y,Z) + T\big(Y, T(X,Z)\big) - T\big(X, T(Y,Z)\big)\end{aligned}$$

and we are done. □

Definition 2.1.2. A *conformal change* is a transformation of the metric $g \mapsto ug$ on a manifold M, with u a positive C^∞ function.

Remark 2.1.3. We will always express such a function u in the form $u = e^{2\varphi}$, with $\varphi \in C^\infty(M)$, to simplify the computations.

Theorem 2.1.4. *Let $\widetilde{g} = e^{2\varphi} g$, with $\varphi \in C^\infty(M)$ and denote by $\widetilde{\nabla}$, $\widetilde{\Gamma}^k_{ij}$, $\widetilde{\mathrm{Riem}}$, $\widetilde{\mathrm{Ric}}$, $\widetilde{\mathrm{R}}$, $\widetilde{S}$, $\widetilde{W}$ the associated items. Then, the following relations hold:*

(i) $\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + T^k_{ij}\,,$

(ii) $\widetilde{\nabla}_X Y = \nabla_X Y + T(X,Y)\,,$

(iii) $\widetilde{\mathrm{Riem}} = e^{2\varphi}(\mathrm{Riem} - A \mathbin{\bigcirc\mkern-15mu\wedge} g)\,,$

(iv) $\widetilde{\mathrm{Ric}} = \mathrm{Ric} - \Delta\varphi g - (n-2)\Big(\frac{1}{2}|\nabla\varphi|^2 g + A\Big)\,,$

(v) $\widetilde{\mathrm{R}} = e^{-2\varphi}\Big\{\mathrm{R} - 2(n-1)\Big(\Delta\varphi + (n-2)\frac{1}{2}|\nabla\varphi|^2\Big)\Big\}\,,$

(vi) $\widetilde{S} = S - A\,,$ *if* $n \geq 3$ *and* $\widetilde{S} = S - \frac{1}{2}\Delta\varphi g\,,$ *if* $n = 2$,

(vii) $\widetilde{W} = e^{2\varphi} W\,,$

with

$$T(X,Y) = \mathrm{d}\varphi(X)Y + \mathrm{d}\varphi(Y)X - g(X,Y)\nabla\varphi$$

and

$$A = \nabla^2\varphi - \mathrm{d}\varphi \otimes \mathrm{d}\varphi + \tfrac{1}{2}|\nabla\varphi|^2 g\,.$$

Proof. By the Christoffel symbols formula (1.11),

$$
\begin{aligned}
\widetilde{\Gamma}_{ijk} &= \tfrac{1}{2}(\partial_i \widetilde{g}_{jk} + \partial_j \widetilde{g}_{ki} - \partial_k \widetilde{g}_{ij}) \\
&= \tfrac{1}{2}\big(\partial_i(e^{2\varphi} g_{jk}) + \partial_j(e^{2\varphi} g_{ki}) - \partial_k(e^{2\varphi} g_{ij})\big) \\
&= e^{2\varphi}\big((\partial_i\varphi g_{jk} + \partial_i g_{jk}) + (\partial_j\varphi g_{ki} + \partial_j g_{ki}) - (\partial_k\varphi g_{ij} + \partial_k g_{ij})\big) \\
&= e^{2\varphi}(\Gamma_{ijk} + \partial_i\varphi g_{jk} + \partial_j\varphi g_{ki} - \partial_k\varphi g_{ij})\,,
\end{aligned}
$$

thus,

$$
\begin{aligned}
\widetilde{\Gamma}_{ij}^k &= \widetilde{g}^{ks}\widetilde{\Gamma}_{ijs} = e^{-2\varphi} g^{ks}\widetilde{\Gamma}_{ijs} \\
&= g^{ks}\Gamma_{ijs} + g^{ks}(\partial_i\varphi g_{js} + \partial_j\varphi g_{si} - \partial_s\varphi g_{ij}) \\
&= \Gamma_{ij}^k + \partial_i\varphi\delta_j^k + \partial_j\varphi\delta_i^k - g_{ij}\partial^k\varphi \\
&= \Gamma_{ij}^k + T_{ij}^k\,,
\end{aligned}
$$

giving relations (i) and (ii).
Now we compute, by means of Lemma 2.1.1 in a normal chart centred at a point p,

$$
\begin{aligned}
&e^{-2\varphi}\widetilde{\mathrm{Riem}}_{ijkl} - \mathrm{Riem}_{ijkl} \\
&= \nabla_j T_{ikl} - \nabla_i T_{jkl} + T_{ik}^s T_{sjl} - T_{jk}^s T_{sil} \\
&= \partial_j T_{ikl} - \partial_i T_{jkl} \\
&\quad + (\partial_i\varphi\delta_k^s + \partial_k\varphi\delta_i^s - \delta_{ik}\partial^s\varphi)(\partial_s\varphi\delta_{jl} + \partial_j\varphi\delta_{sl} - \delta_{sj}\partial_l\varphi) \\
&\quad - (\partial_j\varphi\delta_k^s + \partial_k\varphi\delta_j^s - \delta_{jk}\partial^s\varphi)(\partial_s\varphi\delta_{il} + \partial_i\varphi\delta_{sl} - \delta_{si}\partial_l\varphi) \\
&= \partial_{ji}^2\varphi\delta_{kl} + \partial_{jk}^2\varphi\delta_{il} - \partial_{jl}^2\varphi\delta_{ik} - \partial_{ij}^2\varphi\delta_{kl} - \partial_{ik}^2\varphi\delta_{jl} + \partial_{il}^2\varphi\delta_{jk} \\
&\quad + \partial_i\varphi\partial_j\varphi\delta_{kl}(1-1) + \partial_i\varphi\partial_k\varphi\delta_{jl}(2-1) + \partial_i\varphi\partial_l\varphi\delta_{jk}(-1+0) \\
&\quad + \partial_j\varphi\partial_k\varphi\delta_{il}(1-2) + \partial_j\varphi\partial_l\varphi\delta_{ik}(0+1) + \partial_k\varphi\partial_l\varphi\delta_{ij}(-1+1) \\
&\quad - \partial^s\varphi\partial_s\varphi\delta_{ik}\delta_{jl} + \partial^s\varphi\partial_s\varphi\delta_{jk}\delta_{il} \\
&= -(\partial_{ik}^2\varphi\delta_{jl} + \partial_{jl}^2\varphi\delta_{ik} - \partial_{jk}^2\varphi\delta_{il} - \partial_{il}^2\varphi\delta_{jk}) \\
&\quad + \partial_i\varphi\partial_k\varphi\delta_{jl} + \partial_j\varphi\partial_l\varphi\delta_{ik} - \partial_i\varphi\partial_l\varphi\delta_{jk} - \partial_j\varphi\partial_k\varphi\delta_{il} \\
&\quad - (\partial^s\varphi\partial_s\varphi\delta_{ik}\delta_{jl} + \partial^s\varphi\partial_s\varphi\delta_{jk}\delta_{il}) \\
&= -(\nabla^2\varphi \owedge g)_{ijkl} + \big((\mathrm{d}\varphi\otimes\mathrm{d}\varphi)\owedge g\big)_{ijkl} - \tfrac{1}{2}|\nabla\varphi|^2(g\owedge g)_{ijkl} \\
&= -(A\owedge g)_{ijkl}\,,
\end{aligned}
$$

proving (iii).

Equality (iv) simply follows by relation (iii) and computing

$$
\widetilde{\mathrm{Ric}} = e^{-2\varphi}\,\mathrm{tr}^{1,3}\,\widetilde{\mathrm{Riem}} = \mathrm{Ric} - \mathrm{tr}^{1,3}(A\owedge g) = \mathrm{Ric} - (\mathrm{tr}\,A)g - (n-2)A\,,
$$

with

$$
\begin{aligned}
\mathrm{tr}\,A &= \mathrm{tr}\,\nabla^2\varphi - \mathrm{tr}(\mathrm{d}\varphi\otimes\mathrm{d}\varphi) + \tfrac{1}{2}|\nabla\varphi|^2\,\mathrm{tr}\,g \\
&= \Delta\varphi - |\nabla\varphi|^2 + \tfrac{n}{2}|\nabla\varphi|^2 \\
&= \Delta\varphi + (n-2)\tfrac{1}{2}|\nabla\varphi|^2\,.
\end{aligned}
$$

Similarly for relation (v),

$$
e^{2\varphi}\widetilde{\mathrm{R}} = \mathrm{tr}\,\widetilde{\mathrm{Ric}} = \mathrm{R} - 2(n-1)\,\mathrm{tr}\,A = \mathrm{R} - 2(n-1)\Big(\Delta\varphi + (n-2)\tfrac{1}{2}|\nabla\varphi|^2\Big)\,.
$$

For the Schouten tensor in (vi), we let $n \geq 3$ and evaluate

$$\begin{aligned}
\widetilde{S} &= \frac{1}{n-2}\left(\widetilde{\mathrm{Ric}} - \frac{\widetilde{\mathrm{R}}}{2(n-1)}\widetilde{g}\right) \\
&= \frac{1}{n-2}\left(\mathrm{Ric} - (\operatorname{tr} A)g - (n-2)A - \frac{e^{-2\varphi}(\mathrm{R} - 2(n-1)(\operatorname{tr} A))}{2(n-1)}e^{2\varphi}g\right) \\
&= S - A\,,
\end{aligned}$$

then, we can compute the Weyl tensor,

$$e^{-2\varphi}\widetilde{W} = e^{-2\varphi}(\widetilde{R} - \widetilde{S} \owedge \widetilde{g}) = R - A \owedge g - e^{-2\varphi}(S - A) \owedge e^{2\varphi}g = R - S \owedge g = W$$

hence getting (vii). $\square$

Remark 2.1.5. As a consequence, the $(1,3)$–form of the Weyl tensor is conformally invariant, indeed

$$\widetilde{W}^{l}_{ijk} = \widetilde{W}_{ijks}\widetilde{g}^{sl} = e^{2\varphi}W_{ijks}e^{-2\varphi}g^{sl} = W^{l}_{ijk}\,.$$

Before introducing the *Cotton tensor*, we recall the contracted forms of the differential Bianchi identity.

Proposition 2.1.6. *By contracting the differential Bianchi identity* (1.13) *we obtain*

$$\operatorname{div}\mathrm{Riem}_{jkl} = \nabla_k \mathrm{Ric}_{lj} - \nabla_l \mathrm{Ric}_{kj}\,, \tag{2.2}$$

and contracting again we obtain the so–called Schur's lemma[1]*,*

$$2\operatorname{div}\mathrm{Ric} = \mathrm{dR}\,. \tag{2.3}$$

Definition 2.1.7. A symmetric $(0,2)$–tensor B is called *Codazzi*[2] if the tensor ∇B is completely symmetric, that is,

$$\nabla_X B(Y,Z) - \nabla_Y B(X,Z) = 0\,.$$

The *Cotton tensor* is defined as

$$C(X,Y,Z) = \nabla_X S(Y,Z) - \nabla_Y S(X,Z)\,.$$

The Cotton tensor thus "measures" the defect of the Schouten tensor from being a Codazzi tensor. Explicitly, in local coordinates, we have, if $n \geq 3$,

$$C_{ijk} = \frac{1}{n-2}\Big(\nabla_i \mathrm{Ric}_{jk} - \nabla_j \mathrm{Ric}_{ik} - \frac{1}{2(n-1)}(\partial_i \mathrm{R} g_{jk} - \partial_j \mathrm{R} g_{ik})\Big)\,,$$

and we set $C := 0$, if $n \in \{1,2\}$.
We observe that, immediately from its definition and the symmetry of the Schouten tensor, the Cotton tensor satisfies

- $C_{ijk} + C_{jik} = 0\,,$
- $C_{ijk} + C_{jki} + C_{kij} = 0\,.$

[1] After the Russian mathematician Issai Schur (Иса́й Шур, 1875–1941) [70].
[2] After the Italian mathematician Delfino Codazzi (1824–1873) [71].

Hence, like the Weyl tensor, the Cotton tensor is completely trace–free. Indeed, if $n \geq 3$, we have

$$
\begin{aligned}
C^i{}_{ji} &= \frac{1}{n-2}\Big(\nabla^i \operatorname{Ric}_{ji} - \nabla_j \operatorname{Ric}^i_i - \frac{1}{2(n-1)}(\partial^i \mathrm{R} g_{ji} - \partial_j \mathrm{R} g^i_i)\Big) \\
&= \frac{1}{n-2}\Big(\operatorname{div}\operatorname{Ric}_j - \partial_j \mathrm{R} - \frac{1}{2(n-1)}(\partial_j \mathrm{R} - n\partial_j \mathrm{R})\Big) \\
&= \frac{1}{n-2}\Big(\operatorname{div}\operatorname{Ric}_j - \frac{1}{2}\partial_j \mathrm{R}\Big) = 0\,,
\end{aligned}
$$

by Schur's lemma, formula (2.3) and $C_{ji}{}^i = -C^i{}_{ij} - C^i{}_{ji} = 0$ as $C^i{}_{ij} = 0$.

Remark 2.1.8. If B is a Codazzi tensor, then $\operatorname{div} B = \mathrm{d}\operatorname{tr} B$ (that is, "$\operatorname{tr} \nabla B = \nabla \operatorname{tr} B$"), as

$$\nabla^j B_{ji} = \nabla_i B^j_j = \partial_i \operatorname{tr} B\,.$$

In general, the Cotton tensor is nontrivial, hence the Schouten tensor is not necessarily Codazzi. Despite that, it still satisfies

$$\operatorname{div} S = \mathrm{d}\operatorname{tr} S\,,$$

indeed,

$$\operatorname{tr} S = \frac{1}{n-2}\Big(\operatorname{tr}\operatorname{Ric} - \frac{1}{2(n-1)}\mathrm{R}\operatorname{tr} g\Big) = \frac{1}{n-2}\Big(\mathrm{R} - \frac{n}{2(n-1)}\mathrm{R}\Big) = \frac{1}{2(n-1)}\mathrm{R}$$

and using Schur's lemma, formula (2.3),

$$
\begin{aligned}
\operatorname{div} S &= \frac{1}{n-2}\Big(\operatorname{div}\operatorname{Ric} - \frac{1}{2(n-1)}\operatorname{div}(\mathrm{R} g)\Big) \\
&= \frac{1}{n-2}\Big(\frac{1}{2}\mathrm{dR} - \frac{1}{2(n-1)}\mathrm{dR}\Big) \\
&= \frac{1}{2(n-1)}\mathrm{dR}\,. \qquad (2.4)
\end{aligned}
$$

If $n \geq 4$ the Cotton tensor is directly related to the Weyl tensor as follows.

Theorem 2.1.9. *There holds the relation*

$$\operatorname{div} W_{jkl} = (n-3) C_{klj}\,. \qquad (2.5)$$

Proof. The equality is trivial if $n \in \{1,2,3\}$. We let $n \geq 4$ and we first compute in general

$$
\begin{aligned}
\operatorname{div}(h \owedge g)_{jkl} &= \nabla^i(h_{ik}g_{jl} + h_{jl}g_{ik} - h_{il}g_{jk} - h_{jk}g_{il}) \\
&= \nabla^i h_{ik}g_{jl} + \nabla^i h_{jl}g_{ik} - \nabla^i h_{il}g_{jk} - \nabla^i h_{jk}g_{il} \\
&= \nabla^i h_{ik}g_{jl} - \nabla^i h_{il}g_{jk} + \nabla_k h_{jl} - \nabla_l h_{jk} \\
&= \operatorname{div} h_k g_{jl} - \operatorname{div} h_l g_{jk} + \nabla_k h_{jl} - \nabla_l h_{jk}\,,
\end{aligned}
$$

hence, from equation (2.4) and the definition of the Cotton tensor,

$$
\begin{aligned}
\operatorname{div}(S \owedge g)_{jkl} &= \operatorname{div} S_k g_{jl} - \operatorname{div} S_l g_{jk} + \nabla_k S_{jl} - \nabla_l S_{jk} \\
&= \frac{1}{2(n-1)}(\partial_k \mathrm{R} g_{jl} - \partial_l \mathrm{R} g_{jk}) + C_{kjl}\,. \qquad (2.6)
\end{aligned}
$$

Then, by equations (2.2) and (2.6), we conclude

$$\begin{aligned}
\operatorname{div} W_{jkl} &= \operatorname{div} \mathrm{Riem}_{jkl} - \operatorname{div}(S \owedge g)_{jkl} \\
&= \nabla_k \mathrm{Ric}_{lj} - \nabla_l \mathrm{Ric}_{kj} - C_{kjl} - \frac{1}{2(n-1)}(\partial_k \mathrm{R} g_{jl} - \partial_l \mathrm{R} g_{jk}) \\
&= (n-2) C_{klj} - C_{klj} + \frac{1}{2(n-1)}(\partial_k \mathrm{R} g_{lj} - \partial_l \mathrm{R} g_{kj} - \partial_k \mathrm{R} g_{lj} + \partial_l \mathrm{R} g_{kj}) \\
&= (n-3) C_{klj} \,.
\end{aligned}$$

□

Remark 2.1.10. It is worth computing here the divergence of the Cotton tensor, which is the "double divergence" of the Weyl tensor, for $n \geq 3$,

$$\operatorname{div} C_{ij} = \frac{1}{2(n-1)(n-2)}\left(2(n-1)(R_{ikjl}R^{kl} - R_{ik}R^k_j + \Delta R_{ij}) - \Delta \mathrm{R} g_{ij} - (n-2)\nabla_{ij}\mathrm{R}\right). \tag{2.7}$$

Indeed, using equation (2.4) and the definition of the Schouten tensor, we have

$$\begin{aligned}
\nabla^k C_{kij} &= \nabla^k_k S_{ij} - \nabla_{ki} S^k_j \\
&= \Delta S_{ij} - \nabla_{ik} S^k_j + R_{ik}{}^{kl} S_{lj} + R_{ikjl} S^{kl} \\
&= \Delta S_{ij} - \frac{1}{2(n-1)} \nabla_{ij} \mathrm{R} - R_{ik} S^k_j + R_{ikjl} S^{kl} \\
&= \frac{1}{n-2} \Delta R_{ij} - \frac{1}{2(n-1)(n-2)} \Delta \mathrm{R} g_{ij} - \frac{1}{2(n-1)} \nabla_{ij} \mathrm{R} \\
&\quad - \frac{1}{n-2} R_{ik} R^k_j + \frac{1}{2(n-1)(n-2)} \mathrm{R} R_{ij} + \frac{1}{n-2} R_{ikjl} R^{kl} - \frac{1}{2(n-1)(n-2)} \mathrm{R} R_{ij} \\
&= \frac{1}{n-2} \Delta R_{ij} - \frac{1}{2(n-1)(n-2)} \Delta \mathrm{R} g_{ij} - \frac{1}{2(n-1)} \nabla_{ij} \mathrm{R} \\
&\quad - \frac{1}{n-2} R_{ik} R^k_j + \frac{1}{n-2} R_{ikjl} R^{kl} \,.
\end{aligned}$$

Remark 2.1.11. As a consequence of Theorem 2.1.9, if $n \geq 4$ and $W = 0$, then $C = 0$. This is, however, not necessarily true if $n = 3$.

Theorem 2.1.12. *Let $\widetilde{g} = e^{2\varphi} g$, as in Theorem 2.1.4, then*

$$\begin{aligned}
\widetilde{C}(X,Y,Z) &= C(X,Y,Z) - W(X,Y,Z,\nabla\varphi) \,, \\
\widetilde{C}_{ijk} &= C_{ijk} - W^l_{ijk} \partial_l \varphi \,,
\end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$.

Remark 2.1.13. If $n = 3$ this clearly shows that the Cotton tensor is conformally invariant.

Proof of Theorem 2.1.12. If $n \in \{1, 2\}$ all tensors are trivial, thus we let $n \geq 3$. We recall that the transformation formula for the Schouten tensor and the Levi–Civita connection under the conformal change $g \mapsto \widetilde{g} = e^{2\varphi} g$, are given by

$$\begin{aligned}
\widetilde{S} &= S - A \,, \\
\widetilde{\nabla}_X Y &= \nabla_X Y + T(X,Y) \,,
\end{aligned}$$

where

$$
\begin{aligned}
A_{ij} &= \nabla_i \partial_j \varphi - \partial_i \varphi \partial_j \varphi + \tfrac{1}{2} |\nabla \varphi|^2 g_{ij} \, , \\
T_{ij}^k &= \partial_i \varphi \delta_j^k + \partial_j \varphi \delta_i^k - g_{ij} \partial^k \varphi \, .
\end{aligned}
$$

Considering local normal coordinates centred at a point p, we compute

$$
\begin{aligned}
\nabla_i A_{jk} &= \nabla_{ijk} \varphi - \nabla_i (\partial_j \varphi \partial_k \varphi) + \tfrac{1}{2} \nabla_i |\nabla \varphi|^2 \delta_{jk} \\
&= \nabla^2_{ij} \partial_k \varphi - \partial^2_{ij} \varphi \partial_k \varphi - \partial_j \varphi \partial^2_{ik} \varphi + \tfrac{1}{2} \partial_i |\nabla \varphi|^2 \delta_{jk} \, ,
\end{aligned}
$$

hence,

$$
\begin{aligned}
\nabla_i A_{jk} - \nabla_j A_{ik} &= \nabla^2_{ij} \partial_k \varphi - \nabla^2_{ji} \partial_k \varphi - \partial_j \varphi \partial^2_{ik} \varphi + \partial_i \varphi \partial^2_{jk} \varphi + \tfrac{1}{2} \partial_i |\nabla \varphi|^2 \delta_{jk} - \tfrac{1}{2} \partial_j |\nabla \varphi|^2 \delta_{ik} \\
&= R_{ijkl} \partial^l \varphi - \partial_j \varphi \partial^2_{ik} \varphi + \partial_i \varphi \partial^2_{jk} \varphi + \tfrac{1}{2} \partial_i |\nabla \varphi|^2 \delta_{jk} - \tfrac{1}{2} \partial_j |\nabla \varphi|^2 \delta_{ik} \, ,
\end{aligned}
$$

then,

$$
\begin{aligned}
T^l_{ij} A_{lk} &= (\partial_i \varphi \delta^l_j + \partial_j \varphi \delta^l_i - \delta_{ij} \partial^l \varphi)(\partial^2_{lk} \varphi - \partial_l \varphi \partial_k \varphi + \tfrac{1}{2} |\nabla \varphi|^2 \delta_{lk}) \\
&= \partial_i \varphi \partial^2_{jk} \varphi + \partial_j \varphi \partial^2_{ik} \varphi - \delta_{ij} \partial^l \varphi \partial^2_{lk} \varphi \\
&\quad - (\partial_i \varphi \partial_j \varphi \partial_k \varphi + \partial_j \varphi \partial_i \varphi \partial_k \varphi - \delta_{ij} \partial_k \varphi |\nabla \varphi|^2) \\
&\quad + \tfrac{1}{2} |\nabla \varphi|^2 (\partial_i \varphi \delta_{jk} + \partial_j \varphi \delta_{ik} - \partial_k \varphi \delta_{ij}) \\
&= \partial_i \varphi \partial^2_{jk} \varphi + \partial_j \varphi \partial^2_{ik} \varphi - \delta_{ij} \partial^l \varphi \partial^2_{lk} \varphi - 2 \partial_i \varphi \partial_j \varphi \partial_k \varphi \\
&\quad + \tfrac{1}{2} |\nabla \varphi|^2 (\partial_i \varphi \delta_{jk} + \partial_j \varphi \delta_{ik} + \partial_k \varphi \delta_{ij}) \, .
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
\widetilde{C}_{ijk} - C_{ijk} &= \widetilde{\nabla}_i \widetilde{S}_{jk} - \widetilde{\nabla}_j \widetilde{S}_{ik} - \nabla_i S_{jk} + \nabla_j S_{ik} \\
&= \widetilde{\nabla}_i (S_{jk} - A_{jk}) - \widetilde{\nabla}_j (S_{ik} - A_{ik}) - \nabla_i S_{jk} + \nabla_j S_{ik} \\
&= \nabla_j A_{ik} - \nabla_i A_{jk} + T^t_{ij} A_{tk} + T^t_{ik} A_{jt} - T^t_{ji} A_{tk} - T^t_{jk} A_{it} \\
&\quad - T^t_{ij} S_{tk} - T^t_{ik} S_{jt} + T^t_{ji} S_{tk} + T^t_{jk} S_{it} \\
&= \nabla_j A_{ik} - \nabla_i A_{jk} + T^t_{ik} A_{jt} - T^t_{jk} A_{it} - T^t_{ik} S_{jt} + T^t_{jk} S_{it} \\
&= -(R_{ijkl} \partial^l \varphi - \partial_j \varphi \partial^2_{ik} \varphi + \partial_i \varphi \partial^2_{jk} \varphi + \tfrac{1}{2} \partial_i |\nabla \varphi|^2 \delta_{jk} - \tfrac{1}{2} \partial_j |\nabla \varphi|^2 \delta_{ik}) \\
&\quad + \partial_i \varphi \partial^2_{kj} \varphi + \partial_k \varphi \partial^2_{ij} \varphi - \delta_{ik} \partial^t \varphi \partial^2_{tj} \varphi - 2 \partial_i \varphi \partial_k \varphi \partial_j \varphi \\
&\quad + \tfrac{1}{2} |\nabla \varphi|^2 (\partial_i \varphi \delta_{kj} + \partial_k \varphi \delta_{ij} + \partial_j \varphi \delta_{ik}) \\
&\quad - (\partial_j \varphi \partial^2_{ki} \varphi + \partial_k \varphi \partial^2_{ji} \varphi - \delta_{jk} \partial^t \varphi \partial^2_{ti} \varphi - 2 \partial_j \varphi \partial_k \varphi \partial_i \varphi) \\
&\quad - \tfrac{1}{2} |\nabla \varphi|^2 (\partial_j \varphi \delta_{ki} + \partial_k \varphi \delta_{ji} + \partial_i \varphi \delta_{jk}) \\
&\quad - (\partial_i \varphi \delta^t_k + \partial_k \varphi \delta^t_i - \delta_{ik} \partial^t \varphi) S_{jt} + (\partial_j \varphi \delta^t_k + \partial_k \varphi \delta^t_j - \delta_{jk} \partial^t \varphi) S_{it} \\
&= -W_{ijkl} \partial^l \varphi - (S_{ik} \delta_{jl} + S_{jl} \delta_{ik} - S_{il} \delta_{jk} - S_{jk} \delta_{il}) \partial^l \varphi \\
&\quad - \tfrac{1}{2} \partial_i |\nabla \varphi|^2 \delta_{jk} + \tfrac{1}{2} \partial_j |\nabla \varphi|^2 \delta_{ik} - \delta_{ik} \partial^t \varphi \partial^2_{tj} \varphi + \delta_{jk} \partial^t \varphi \partial^2_{ti} \varphi \\
&\quad - \partial_i \varphi S_{jk} - \partial_k \varphi S_{ji} + \delta_{ik} \partial^t \varphi S_{jt} + \partial_j \varphi S_{ik} + \partial_k \varphi S_{ij} - \delta_{jk} \partial^t \varphi S_{it} \\
&= -W_{ijkl} \partial^l \varphi - (\partial_j \varphi S_{ik} + \partial^l \varphi S_{jl} \delta_{ik} - \partial^l \varphi S_{il} \delta_{jk} - \partial_i \varphi S_{jk}) \\
&\quad - \partial_i \varphi S_{jk} + \delta_{ik} \partial^l \varphi S_{jl} + \partial_j \varphi S_{ik} - \delta_{jk} \partial^l \varphi S_{il} \\
&= -W_{ijkl} \partial^l \varphi \, .
\end{aligned}
$$

□

Remark 2.1.14. If $n \geq 4$, the result could have been obtained also by using equation (2.5), indeed

$$
\begin{aligned}
&(n-3)(\widetilde{C}_{ijk} - C_{ijk}) \\
&= \widetilde{\operatorname{div} W}_{kij} - \operatorname{div} W_{kij} = e^{-2\varphi} g^{sl} \widetilde{\nabla}_s e^{2\varphi} W_{lkij} - g^{sl} \nabla_s W_{lkij} \\
&= g^{sl}(\nabla_s W_{lkij} - T^t_{sl} W_{tkij} - T^t_{sk} W_{ltij} - T^t_{si} W_{lktj} - T^t_{sj} W_{lkit}) - g^{sl} \nabla_s W_{lkij} \\
&\quad + 2g^{sl} \partial_s \varphi W_{lkij} \\
&= -g^{sl} W_{tkij}(\partial_s \varphi \delta^t_l + \partial_l \varphi \delta^t_s - g_{ls} \partial^t \varphi) - g^{sl} W_{ltij}(\partial_s \varphi \delta^t_k + \partial_k \varphi \delta^t_s - g_{ks} \partial^t \varphi) \\
&\quad - g^{sl} W_{lktj}(\partial_s \varphi \delta^t_i + \partial_i \varphi \delta^t_s - g_{is} \partial^t \varphi) - g^{sl} W_{lkit}(\partial_s \varphi \delta^t_j + \partial_j \varphi \delta^t_s - g_{js} \partial^t \varphi) \\
&\quad + 2\partial^l \varphi W_{lkij} \\
&= -W_{tkij}(\partial^t \varphi + \partial^t \varphi - n \partial^t \varphi) - W_{ltij}(\partial^l \varphi \delta^t_k + \partial_k \varphi g^{tl} - \delta^l_k \partial^t \varphi) \\
&\quad - W_{lktj}(\partial^l \varphi \delta^t_i + \partial_i \varphi g^{tl} - \delta^l_i \partial^t \varphi) - W_{lkit}(\partial^l \varphi \delta^t_j + \partial_j \varphi g^{tl} - \delta^l_j \partial^t \varphi) \\
&\quad + 2\partial^l \varphi W_{lkij} \\
&= (n-2) W_{tkij} \partial^t \varphi + 2\partial^l \varphi W_{lkij} \\
&\quad - (W_{ltij}(\partial^l \varphi \delta^t_k - \delta^l_k \partial^t \varphi) + W_{lktj}(\partial^l \varphi \delta^t_i - \delta^l_i \partial^t \varphi) + W_{lkit}(\partial^l \varphi \delta^t_j - \delta^l_j \partial^t \varphi) \\
&= n W_{lkij} \partial^l \varphi \\
&\quad - (W_{lkij} \partial^l \varphi - W_{ktij} \partial^t \varphi + W_{lkij} \partial^l \varphi - W_{iktj} \partial^t \varphi + W_{lkij} \partial^l \varphi - W_{jkit} \partial^t \varphi) \\
&= n W_{lkij} \partial^l \varphi - \partial^l \varphi (W_{lkij} - W_{klij} + W_{lkij} - W_{iklj} + W_{lkij} - W_{jkil}) \\
&= n W_{lkij} \partial^l \varphi - 3\partial^l \varphi W_{lkij} \\
&= (n-3) W_{lkij} \partial^l \varphi \,.
\end{aligned}
$$

2.2 LCF manifolds

Definition 2.2.1. A Riemannian manifold (M, g) is said to be *conformally flat* if there exists a positive function $u \in C^\infty(M)$ such that (M, ug) is flat.
A Riemannian manifold is said to be *locally conformally flat* (or *LCF*) if every point has a conformally flat neighbourhood.

Let us start by observing that speaking of 2–dimensional LCF manifolds is redundant.

Theorem 2.2.2. *Any* 2*–manifold is LCF.*

Proof. Let $\widetilde{g} = e^{2\varphi} g$ for $\varphi \in C^\infty(M)$. Since for $n = 2$ the curvature is completely determined by the scalar curvature as $\mathrm{Riem} = \mathrm{R} g \owedge g/4$, by relation (1.23), fixed some point $p \in M$, it is enough to show that $\widetilde{\mathrm{R}} = 0$ for some function φ defined in a neighbourhood of p.

By Theorem 2.1.4–(v), there holds $\widetilde{\mathrm{R}} = e^{-2\varphi}(\mathrm{R} - 2\Delta\varphi)$, hence we need to solve locally the PDE $\Delta\varphi = \mathrm{R}/2.$, which in coordinates reads

$$
g^{ij} \partial^2_{ij} \varphi - g^{ij} \Gamma^k_{ij} \partial_k \varphi = \frac{1}{2} \mathrm{R} \,. \tag{2.8}
$$

This is a uniform elliptic equation with smooth coefficients, which is well known to have a unique C^∞ solution in any smooth domain U compactly contained in the local chart, once we set a boundary data, let us say $\varphi = 0$ on ∂U (see [14], for instance). ☐

Remark 2.1.5 and Theorem 2.1.12 imply that an LCF manifold must have $W = 0$, if $n \geq 3$ and $C = 0$, if $n = 3$. The following theorem proves that these conditions are also sufficient to be locally conformally flat.

Theorem 2.2.3 (Weyl–Schouten theorem)**.** *Let* $\dim M \geq 3$*, then* (M, g) *is LCF if and only if*

- $C = 0$*, if* $\dim M = 3$,
- $W = 0$*, if* $\dim M \geq 4$.

Proof. If $n = 3$ the Cotton tensor is conformally invariant and if $n \geq 4$ a (version) of the Weyl tensor is conformally invariant, hence the conditions are necessary since for the Euclidean spaces these tensors are both zero.
Now, assuming that the conditions hold, both in the case $n = 3$ and $n \geq 4$, we have $W = C = 0$, as if $n = 3$ this is clearly true simply since $W = 0$, while if $n \geq 4$ we have that $W = 0$ implies $C = 0$, by relation (2.5). Moreover, being $W = 0$, the Riemann tensor simply reads as

$$\mathrm{Riem} = S \owedge g\,,$$

by the decomposition formula (2.1).

We consider a function $\varphi \in C^\infty(M)$ and set $\widetilde{g} = e^{2\varphi} g$, as in Theorem 2.1.4. Imposing $\widetilde{\mathrm{Riem}} = 0$ and using relation (v) of the same theorem, we obtain the following equation that we want to solve locally for φ,

$$0 = \widetilde{\mathrm{Riem}} = e^{2\varphi}(\mathrm{Riem} - A \owedge g) = e^{2\varphi}(S \owedge g - A \owedge g) = e^{2\varphi}(S - A) \owedge g\,,$$

where $A = \nabla \mathrm{d}\varphi - \mathrm{d}\varphi \otimes \mathrm{d}\varphi + \frac{1}{2}|\mathrm{d}\varphi|^2 g$. This then equivalent to $S = A$, that is,

$$\nabla \mathrm{d}\varphi = S + \mathrm{d}\varphi \otimes \mathrm{d}\varphi - \tfrac{1}{2}|\mathrm{d}\varphi|^2 g\,, \tag{2.9}$$

by Remark 1.3.20, since $n \geq 3$.
To solve equation (2.9) we now show that there exists a smooth 1–form ω solving

$$\nabla \omega = S + \omega \otimes \omega - \tfrac{1}{2}|\omega|^2 g \tag{2.10}$$

and that such ω is (locally) exact. To see this latter, we set

$$F(\omega) = S + \omega \otimes \omega - \tfrac{1}{2}|\omega|^2 g\,,$$

hence equation (2.10) becomes $\nabla \omega = F(\omega)$ and being $F(\omega)$ symmetric, we recall equation (1.12) to notice that a solution of equation (2.10) satisfies

$$\mathrm{d}\omega_{ij} = \nabla_i \omega_j - \nabla_j \omega_i = F(\omega)_{ij} - F(\omega)_{ji} = 0\,.$$

Hence, the solution ω is a closed 1–form, thus locally exact due to the Poincaré Lemma 1.1.18.

In order to deal with the existence point, we suppose again that ω solves equation (2.10) and compute

$$\begin{aligned}
\nabla_i F(\omega)_{jk} &= \nabla_i S_{jk} + \nabla_i(\omega_j \omega_k) - \tfrac{1}{2}\nabla_i(\omega^l \omega_l) g_{jk} \\
&= \nabla_i S_{jk} + \omega_j \nabla_i \omega_k + \omega_k \nabla_i \omega_j - g_{jk}\omega^l \nabla_i \omega_l \\
&= \nabla_i S_{jk} + \omega_j F(\omega)_{ik} + \omega_k F(\omega)_{ij} - g_{jk}\omega^l F(\omega)_{il} \\
&= \nabla_i S_{jk} + \omega_k F(\omega)_{ij} + \omega_j(S_{ik} + \omega_i\omega_k - \tfrac{1}{2}|\omega|^2 g_{ik}) - g_{jk}\omega^l(S_{il} + \omega_i\omega_l - \tfrac{1}{2}|\omega|^2 g_{il}) \\
&= \nabla_i S_{jk} + S_{ik} g_{jl}\omega^l - S_{il} g_{jk}\omega^l \\
&\quad + \omega_k F(\omega)_{ij} + \omega_i\omega_j\omega_k - \tfrac{1}{2}|\omega|^2 \omega_j g_{ik} - |\omega|^2 \omega_i g_{jk} + \tfrac{1}{2}|\omega|^2 \omega_i g_{jk} \\
&= \nabla_i S_{jk} + S_{ik} g_{jl}\omega^l - S_{il} g_{jk}\omega^l \\
&\quad + \omega_k F(\omega)_{ij} + \omega_i\omega_j\omega_k - \tfrac{1}{2}|\omega|^2 \omega_j g_{ik} - \tfrac{1}{2}|\omega|^2 \omega_i g_{jk}\,,
\end{aligned}$$

hence,

$$
\begin{aligned}
R_{ijkl}\omega^l &= \nabla^2_{ij}\omega_k - \nabla^2_{ji}\omega_k = \nabla_i F(\omega)_{jk} - \nabla_j F(\omega)_{ik} = C_{ijk} + (S \owedge g)_{ijkl}\omega^l \\
&= R_{ijkl}\omega^l + C_{ijk} - W_{ijkl}\omega^l \,.
\end{aligned}
$$

This indicates that $W = C = 0$ is necessary in order to satisfy equation (2.10). It also shows how

$$
\nabla_X F(\omega)(Y, Z) - \nabla_Y F(\omega)(X, Z) = R(X, Y, Z, \omega^\sharp) \tag{2.11}
$$

can be obtained by substituting $F(\omega)$ to $\nabla\omega$ whenever it occurs in the computations. This particular fact makes the condition $W = C = 0$ also sufficient, by the following Lemma 2.2.5, which concludes the proof. □

Remark 2.2.4. We observe that condition (2.11) resembles, in a Riemannian setting, the formulation of Frobenius theorem as in Remark 1.1.23. Indeed, considering the overdetermined system of first–order differential equations

$$
\frac{\partial \omega_k}{\partial x^j}(x) = G_{jk}(x, \omega(x))
$$

where

$$
G_{jk}(x, \omega(x)) = F(\omega)(x)_{jk} + \Gamma^l_{jk}(x)\omega_l(x)\,,
$$

one sees that writing $F(\omega)$ in place of $\nabla\omega$ is equivalent to writing $G_{jk}(x, \omega(x))$ in place of $\partial_j\omega_k(x)$. This is expressed more formally in the following lemma.

Lemma 2.2.5. *Let (M, g) be a Riemannian manifold and consider the equation*

$$
\nabla\omega = F(\omega)\,, \tag{2.12}
$$

where $F\colon \Gamma(T^0_1 M) \to \Gamma(T^0_2 M)$ is a C^∞ bundle morphism such that the equality

$$
\widehat{\nabla}_i F_{jk} - \widehat{\nabla}_j F_{ik} = R_{ijkl}\omega^l \tag{2.13}
$$

is satisfied by the quantity

$$
\widehat{\nabla}_i F_{jk} := (\nabla_i F_{jk})(\omega) + \frac{\partial F_{jk}}{\partial \omega_l}(\omega)\big(F(\omega)_{il} + \Gamma^t_{il}\omega_t\big)\,.
$$

Then, for every $p \in M$ and $\omega_0 \in T_pM^*$ there is a unique solution ω of equation (2.12) *in a connected open set U of p such that $\omega_p = \omega_0$.*

We notice that $\widehat{\nabla} F_{jk}$ is "almost equal" to the expression of $\nabla(F \circ (\cdot, \omega))$, except for the fact that $F(\omega)$ is in place of $\nabla\omega$.

Proof of Lemma 2.2.5. Equation (2.12) in a coordinate chart reads

$$
\partial_j\omega_k(x) = F(\omega)(x)_{jk} + \Gamma^l_{jk}(x)\omega_l(x) =: G_{jk}(x, \omega(x))\,,
$$

and Frobenius Theorem 1.1.21 guarantees that a unique solution exists, given that the condition

$$
\frac{\partial G_{jk}}{\partial x^i} + G_{il}\frac{\partial G_{jk}}{\partial \omega_l} = \frac{\partial G_{ik}}{\partial x^j} + G_{jl}\frac{\partial G_{ik}}{\partial \omega_l}
$$

is satisfied. Let us denote by

$$
\partial_i := \frac{\partial}{\partial x^i} \qquad \text{and} \qquad \widehat{\partial}^i := \frac{\partial}{\partial \omega_i}
$$

to compute

$$
\begin{aligned}
&\frac{\partial G_{jk}}{\partial x^i} + G_{il}\frac{\partial G_{jk}}{\partial \omega_l} - \frac{\partial G_{ik}}{\partial x^j} - G_{jl}\frac{\partial G_{ik}}{\partial \omega_l} \\
&= (\partial_i F_{jk} + \partial_i \Gamma^l_{jk}\omega_l) + (F_{il} + \Gamma^t_{il}\omega_t)(\widehat{\partial}^l F_{jk} + \Gamma^l_{jk}) \\
&\quad - (\partial_j F_{ik} + \partial_j \Gamma^l_{ik}\omega_l) - (F_{jl} + \Gamma^t_{jl}\omega_t)(\widehat{\partial}^l F_{ik} + \Gamma^l_{ik}) \\
&= \partial_i F_{jk} - \Gamma^l_{ij}F_{lj} - \Gamma^l_{ik}F_{lk} - \partial_j F_{ik} + \Gamma^l_{ij}F_{lj} + \Gamma^l_{jk}F_{il} \\
&\quad + \partial_i \Gamma^l_{jk}\omega_l - \partial_j \Gamma^l_{ik}\omega_l + \Gamma^t_{il}\Gamma^l_{jk}\omega_t - \Gamma^t_{jl}\Gamma^l_{ik}\omega_t \\
&\quad + \widehat{\partial}^l F_{jk}(F_{il} + \Gamma^t_{il}\omega_t) - \widehat{\partial}^l F_{ik}(F_{jl} + \Gamma^t_{jl}\omega_t) \\
&= \nabla_i F_{jk} - \nabla_j F_{ik} + \omega_l(\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{it}\Gamma^t_{jk} - \Gamma^l_{jt}\Gamma^t_{ik}) \\
&\quad + \widehat{\partial}^l F_{jk}(F_{il} + \Gamma^t_{il}\omega_t) - \widehat{\partial}^l F_{ik}(F_{jl} + \Gamma^t_{jl}\omega_t) \\
&= \nabla_i F_{jk} - \nabla_j F_{ik} - R_{ijkl}\omega^l + \widehat{\partial}^l F_{jk}(F_{il} + \Gamma^t_{il}\omega_t) - \widehat{\partial}^l F_{ik}(F_{jl} + \Gamma^t_{jl}\omega_t) \\
&= \widehat{\nabla}_i F_{jk} - \widehat{\nabla}_j F_{ik} - R_{ijkl}\omega^l \\
&= 0\,,
\end{aligned}
$$

which proves the lemma. □

Remark 2.2.6. The explicit computation that condition (2.13) is satisfied for the function F in the proof of Theorem 2.2.3 goes as follows,

$$
\begin{aligned}
\widehat{\nabla}_i F_{jk} &= \nabla_i F_{jk} + \widehat{\partial}^l F_{jk}(F_{il} + \Gamma^t_{il}\omega_t) \\
&= \nabla_i(S_{jk} + \omega_j\omega_k - \tfrac{1}{2}|\omega|^2 g_{jk}) \\
&\quad + \big[\widehat{\partial}^l(S_{jk} + \omega_j\omega_k - \tfrac{1}{2}|\omega|^2 g_{jk})\big](S_{il} + \omega_i\omega_l - \tfrac{1}{2}|\omega|^2 g_{il} + \Gamma^t_{il}\omega_t) \\
&= \nabla_i S_{jk} + \omega_j\nabla_i\omega_k + \omega_k\nabla_i\omega_j - g_{jk}\omega^l\nabla_i\omega_l \\
&\quad + (\omega_j\delta^l_k + \omega_k\delta^l_j - \omega^l g_{jk})(S_{il} + \omega_i\omega_l - \tfrac{1}{2}|\omega|^2 g_{il} + \Gamma^t_{il}\omega_t) \\
&= \nabla_i S_{jk} - \omega_j\Gamma^t_{ik}\omega_t - \omega_k\Gamma^t_{ij}\omega_t + g_{jk}\omega^l\Gamma^t_{il}\omega_t \\
&\quad + S_{ik}\omega_j + S_{ij}\omega_k - S_{il}g_{jk}\omega^l \\
&\quad + \omega_i\omega_j\omega_k + \omega_i\omega_j\omega_k - |\omega|^2\omega_i g_{jk} \\
&\quad - (\tfrac{1}{2}|\omega|^2\omega_j g_{ik} + \tfrac{1}{2}|\omega|^2\omega_k g_{ij} - \tfrac{1}{2}|\omega|^2\omega_i g_{jk}) \\
&\quad + \Gamma^t_{ik}\omega_t\omega_j + \Gamma^t_{ij}\omega_t\omega_k - \Gamma^t_{il}\omega_t\omega^l g_{jk} \\
&= \nabla_i S_{jk} + S_{ik}g_{jl}\omega^l - S_{il}g_{jk}\omega^l \\
&\quad + S_{ij}\omega_k + 2\omega_i\omega_j\omega_k - \tfrac{1}{2}|\omega|^2(\omega_j g_{ik} + \omega_k g_{ij} + \omega_i g_{jk})\,,
\end{aligned}
$$

hence,

$$
\begin{aligned}
\widehat{\nabla}_i F_{jk} - \widehat{\nabla}_j F_{ik} &= C_{ijk} + \omega^l(S_{ik}g_{jl} - S_{il}g_{jk} - S_{jk}g_{il} + S_{jl}g_{ik}) \\
&= C_{ijk} - W_{ijkl}\omega^l + R_{ijkl}\omega^l \\
&= R_{ijkl}\omega^l\,.
\end{aligned}
$$

On an LCF manifold (M, g) of dimension $n \geq 3$, the method used in Theorem 2.2.3 implies that one has to fix, for a point p, both $\varphi(p)$ and $\mathrm{d}\varphi_p$ in order to guarantee the uniqueness of the local conformal change $u = e^{2\varphi}$ (because of Poincaré lemma and Frobenius theorem) turning g into the flat metric $\widetilde{g}$, in a neighbourhood of p.

Then, one may wonder about uniqueness conditions for a *global conformal change* making the metric flat. If $\dim M = 2$, clearly uniqueness cannot be granted, as equation (2.8) on $\mathbb{R}^2$ reads

$$\Delta\varphi = 0\,,$$

which is solved by any harmonic function.
If $n \geq 3$ however, the next theorem gives that such uniqueness holds, up to a constant.

Theorem 2.2.7. *Let (M, g) be a conformally flat n–dimensional Riemannian manifold with $n \geq 3$. Then any two positive functions $u, u' \in C^\infty(M)$ such that (M, ug) and $(M, u'g)$ are flat, are proportional by a positive factor.*

Proof. We only need to prove that for a flat manifold such a conformal change can only be conveyed through multiplication by a positive factor, hence, the conformal change u'/u turning ug into $u'g$ will have to be a constant. Furthermore, we only need to prove this for the special case of $\mathbb{R}^n$, as a flat manifold would transfer its nonconstant conformal change to its universal Riemannian covering, which is $\mathbb{R}^n$ (see Theorem 1.2.19), by means of $\widetilde{u} = u \circ \pi\colon \mathbb{R}^n \to \mathbb{R}$, with $\pi\colon \mathbb{R}^n \to M$ the projection map of the covering.
Thus, let us assume g to be the canonical metric on $\mathbb{R}^n$ and $u = e^{2\varphi}$ to be a conformal change that makes $e^{2\varphi}g$ flat, then, equation (2.9) on $\mathbb{R}^n$ reads

$$\partial^2_{ij}\varphi = \partial_i\varphi\partial_j\varphi - \tfrac{1}{2}|\nabla\varphi|^2\delta_{ij}\,.$$

We set $f = 1/\sqrt{u} = e^{-\varphi}$ and compute

$$\frac{\partial}{\partial x^j}\partial_i f = \partial^2_{ji} f = (-\partial^2_{ji}\varphi + \partial_j\varphi\partial_i\varphi)e^{-\varphi} = \tfrac{1}{2}|\nabla\varphi|^2 e^{-\varphi}\delta_{ji} = \frac{|\nabla f|^2}{2f}\delta_{ji}\,,$$

so that the function $\partial_i f$ is only dependent on x^i. Setting $\mathbb{R} \ni x^i \mapsto A_i(x^i) = \partial_i f(x) \in \mathbb{R}$, for $i \in \{1, \dots, n\}$, we observe that the functions A_i have to be linear, as

$$\partial_1 A_1(x^1) = \ldots = \partial_n A_n(x^n) = \frac{|\nabla f|^2}{2f} = \text{constant} = k\,,$$

so $A_i(x^i) = kx^i + a^i$ for $(a^1, \dots, a^n) = a \in \mathbb{R}^n$.
Assuming $k \neq 0$, we have $A_i(-a^i/k) = 0$ and the contradiction

$$k = \frac{|\nabla f(-a/k)|^2}{2f(-a/k)} = \frac{\sum_{i=1}^n \big[A_i(-a^i/k)\big]^2}{2f(-a/k)} = 0\,.$$

Therefore, $k = 0$, hence $|\nabla f|^2 = 0$, f is constant and φ and u are constant. ☐

Chapter 3

The Chern–Gauß–Bonnet theorem

The famous "classical" Gauß–Bonnet theorem for a connected, compact and oriented surface S, without boundary, immersed in the 3–dimensional Euclidean space $\mathbb{R}^3$, says that

$$\int_S K \, \mathrm{d}V_S = 2\pi\chi(S) \,,$$

where K is the Gaußian curvature of S, that is, the determinant of the second fundamental form, or equivalently, the product of its two eigenvalues (the principal curvatures of the surface).
Actually, the result holds also for abstract Riemannian surfaces, substituting the Gaußian curvature K with $\mathrm{R}/2$ (half the scalar curvature, which coincides with K for an immersed surface), that is,

$$\int_S \mathrm{R} \, \mathrm{d}V_S = 4\pi\chi(S) \,. \tag{3.1}$$

This formula shows a wonderful relation between the curvature and the topology of the underlying surface, as the Euler–Poincaré characteristic is an "intrinsic" topological invariant. No matter how we deform the surface, we "produce the same amount" of new negative and positive curvature, as the *total curvature* (the Gauß–Bonnet integral) has to stay unaltered.

A generalisation of the theorem to hypersurfaces of even dimensions was given in 1926 by H. Hopf [19], stating that for a connected, compact and oriented n–dimensional manifold M without boundary, embedded in the $(n+1)$–dimensional Euclidean space $\mathbb{R}^{n+1}$, there holds

$$\int_M K \, \mathrm{d}V_M = \frac{\mathrm{Vol}(\mathbb{S}^n)}{2}\chi(M) = \frac{(2\pi)^{n/2}}{(n-1)!!}\chi(M) \,,$$

where K is the Gaußian curvature of the hypersurface (still, the determinant of the second fundamental form of the hypersurface). However, even if K can be expressed in terms of the curvature tensor of an abstract Riemannian manifold, hence suggesting the statement of the general theorem, Hopf's proof was "extrinsic", relying on the existence of a codimension–one embedding in $\mathbb{R}^{n+1}$ (we will be more precise and we will discuss such proof at the end of Section 3.2). It was later shown by C. B. Allendoerfer [2] and W. Fenchel [13] that any embedding in a Euclidean space of arbitrary codimension was sufficient, but *Nash embedding theorem*[1] [27] was not yet known at the time (they were anyway able to get the conclusion also for abstract Riemannian manifolds, by means of a different technique) and even more importantly, with the words of Michael Spivak [33],

[1]After the American mathematician John Forbes Nash, Jr. (1928–2015) [72].

"*[...] an intrinsic theorem ought to have an intrinsic proof*". S.–S. Chern provided such a proof of the right abstract generalisation of the theorem in 1944 [10], showing that

$$\int_M \mathrm{Pf}(\Omega) = (2\pi)^{n/2}\chi(M)\,,$$

where Ω is the *curvature* 2*–form* of the manifold (in the setting of *Cartan formalism*) and $\mathrm{Pf}(\Omega)$ is its *Pfaffian*[2], which we will define and discuss in the next sections.

Such Pfaffian is equal to a polynomial of order $n/2$ in the components of the Riemann curvature tensor, but in practice (because of the complexity in its computation) such an explicit expression is present in literature only for dimensions two, four and six (up to our knowledge). In dimension two, one recovers the Gauß–Bonnet formula (3.1) for surfaces, while in dimension four (which is relevant for us in view of the discussion about the Einstein manifolds in the next chapter), there holds

$$\int_M \Big(|\mathrm{Riem}|^2 - 4|\mathrm{Ric}|^2 + \mathrm{R}^2\Big)\,\mathrm{d}V_M = 32\pi^2\chi(M)$$

for every compact oriented 4–dimensional Riemannian manifold (M, g).

Finally, we mention that Chern's proof inspired a whole new theory of *characteristic classes*, which evolved into what is now referred to as *Chern–Weil theory*[3], in light of which the Chern–Gauß–Bonnet theorem may be written as

$$\int_M e(E) = \chi(E)\,,$$

where $e(E) \coloneqq \mathrm{Pf}(\Omega^E/2\pi) = \mathrm{Pf}(\Omega^E)/(2\pi)^{n/2}$ is the so–called *Euler class* of the vector bundle E over M and Ω^E is the curvature 2–form of any metric connection on E. We refer the interested reader to [24, 28, 33].

3.1 Preliminaries

We introduce some results and technical tools that we need for the proof of the theorem.

Lemma 3.1.1. *On every compact Riemannian manifold (M, g) there exists a vector field $X \in \Gamma(TM)$ with finitely many isolated zeroes.*

Proof. It is well known that there exists several *Morse functions* on any manifold M, that is, functions with only nondegenerate critical points p, i.e., such that $\nabla f_p = 0$ and $\det \nabla^2 f_p \neq 0$ (see [4, 25], for instance). It is then clear that they have finitely many isolated critical points, hence, the field $X = \nabla f$ has the required property. $\square$

Remark 3.1.2. If X is like above, then $X/|X|$ is a unit vector field on M with finitely many isolated singularities. We will denote by $\mathbb{S}M \subseteq \Gamma(TM)$ the set of all unit vector fields on M.

The following two definitions are well–posed (see [29], for instance).

[2] After the German mathematician Johann Friedrich Pfaff (1765–1825) [73], doctoral advisor of Johann Carl Friedrich Gauß.

[3] After Shiing–Shen Chern and the French mathematician André Weil (1906–1998) [74].

Definition 3.1.3. We define the *degree* of a smooth map $f\colon M \to N$ between two oriented n–dimensional differential manifolds as the number $\deg(f)$ satisfying

$$\int_M f^*\omega = \deg(f) \int_N \omega \tag{3.2}$$

for every compactly supported form $\omega \in \Omega^n(N)$.

Remark 3.1.4. Comparing equation (3.2) with equation (1.4), it is easy to see that the degree of a diffeomorphism between two connected manifolds is ± 1, with the sign depending on whether it preserves or reverses the orientation.

Definition 3.1.5. Let X be a vector field on an n–dimensional Riemannian manifold M having $z \in M$ as isolated zero. Choosing a (small) closed ball $\overline{B}$ around z such that it contains no other zeroes of X and $\mathbb{S}\overline{B} \cong \overline{B} \times \mathbb{S}^{n-1}$, we denote by $\pi^{\mathbb{S}^{n-1}} : \mathbb{S}\overline{B} \to \mathbb{S}^{n-1}$ the induced projection. We define the *index at z of the vector field X* as

$$\mathrm{ind}_z(X) = \deg(S_z)\,,$$

where the map $S_z : \partial\overline{B} \to \mathbb{S}^{n-1}$ is given by $S_z(p) = \pi^{\mathbb{S}^{n-1}}(X_p/|X_p|)$.

We are then ready to state the *Poincaré–Hopf index theorem*, whose proof can be found in [29, 35], for instance.

Theorem 3.1.6 (Poincaré–Hopf index theorem[4]). *Let (M, g) be a compact oriented Riemannian manifold and X a vector field on M with finitely many isolated zeroes $z_1, \dots, z_m$, such that X is "pointing outward" at every point of the boundary of M, if present. Then,*

$$\sum_{i=1}^{m} \mathrm{ind}_{z_i}(X) = \chi(M)\,,$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M.

We denote by Σ_n the symmetric group over n elements and with $n!!$ the product of all the integers from 1 up to n that have the same parity (odd or even) as n, that is, for even n, the double factorial is $n(n-2)(n-4)\cdots 4 \cdot 2$ and for odd n, it is $n(n-2)(n-4)\cdots 3 \cdot 1$. In particular, for $n = 2p$ we have $(2p)!! = 2^p p!$.

Definition 3.1.7. We define the *Pfaffian* of a skew–symmetric matrix $A \in \mathbb{R}^{n\times n}$ as

$$\mathrm{Pf}(A) = \begin{cases} \dfrac{1}{n!!} \displaystyle\sum_{\sigma\in\Sigma_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n/2} A^{\sigma(2i-1)}_{\sigma(2i)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Theorem 3.1.8. *If $A, B \in \mathbb{R}^{n\times n}$ with A skew–symmetric and B nonsingular then*

$$\mathrm{Pf}(B^T AB) = \det(B)\,\mathrm{Pf}(A)\,. \tag{3.3}$$

[4]After Jules Henri Poincaré and the German mathematician Heinz Hopf (1894–1971) [75].

Proof. The equality is trivial if n is odd. We let n be even and compute

$$\begin{aligned}
\mathrm{Pf}(B^T\!AB) &= \frac{1}{n!!}\sum_{\sigma\in\Sigma_n}\mathrm{sgn}(\sigma)\prod_{i=1}^{n/2}(B^T\!AB)^{\sigma(2i-1)}_{\sigma(2i)}\\
&= \frac{1}{n!!}\sum_{\sigma\in\Sigma_n}\mathrm{sgn}(\sigma)\prod_{i=1}^{n/2}(B^T)^{\sigma(2i-1)}_{s}A^s_tB^t_{\sigma(2i)}\\
&= \frac{1}{n!!}\sum_{\sigma,\tau\in\Sigma_n}\mathrm{sgn}(\sigma)\prod_{i=1}^{n/2}B^{\tau(2i-1)}_{\sigma(2i-1)}B^{\tau(2i)}_{\sigma(2i)}A^{\tau(2i-1)}_{\tau(2i)}\\
&= \frac{1}{n!!}\sum_{\sigma,\tau\in\Sigma_n}\mathrm{sgn}(\sigma)\prod_{i=1}^{n/2}B^{\tau(2i-1)}_{\sigma(2i-1)}B^{\tau(2i)}_{\sigma(2i)}\prod_{i=1}^{n/2}A^{\tau(2i-1)}_{\tau(2i)}\\
&= \frac{1}{n!!}\sum_{\sigma,\tau\in\Sigma_n}\mathrm{sgn}(\tau)\,\mathrm{sgn}(\sigma)\prod_{i=1}^{n}B^{\tau(i)}_{\sigma\tau(i)}\prod_{i=1}^{n/2}A^{\tau(2i-1)}_{\tau(2i)}\\
&= \frac{1}{n!!}\sum_{\tau\in\Sigma_n}\mathrm{sgn}(\tau)\det(B)\prod_{i=1}^{n/2}A^{\tau(2i-1)}_{\tau(2i)}\\
&= \det(B)\,\mathrm{Pf}(A)\,.
\end{aligned}$$

We explain the passage from the second to the third line above, where we transformed the sum over $s,t\in\{1,\dots,n\}$ (for each permutation $\sigma\in\Sigma_n$ and $i\in\{1,\dots,n/2\}$) to the sum over all permutations $\tau\in\Sigma_n$. The product is between terms each one consisting of the n^2 summands $\sum_{s,t=1}^n B^s_{\sigma(2i-1)}A^s_tB^t_{\sigma(2i)}$, hence giving a sum of terms like (with all the t_k distinct each other and the same for the s_k)

$$B^{s_1}_{\sigma(1)}A^{s_1}_{t_1}B^{t_1}_{\sigma(2)}\cdots B^{s_{n/2}}_{\sigma(n-1)}A^{s_{n/2}}_{t_{n/2}}B^{t_{n/2}}_{\sigma(n)}\,.$$

If some t_k coincides with s_k, then we have no contribution, as $A^{s_k}_{t_k}$ is zero, by the skew–symmetry of A, while if $t_k=s_m=\ell$, with $k\neq m$, supposing $k<m$, the term

$$\mathrm{sgn}(\sigma)B^{s_1}_{\sigma(1)}A^{s_1}_{t_1}B^{t_1}_{\sigma(2)}\cdots B^{s_k}_{\sigma(2k-1)}A^{s_k}_{t_k}B^{t_k}_{\sigma(2k)}\cdots B^{s_m}_{\sigma(2m-1)}A^{s_m}_{t_m}B^{t_m}_{\sigma(2m)}\cdots B^{s_{n/2}}_{\sigma(n-1)}A^{s_{n/2}}_{t_{n/2}}B^{t_{n/2}}_{\sigma(n)}$$

which is equal to

$$\mathrm{sgn}(\sigma)B^{s_1}_{\sigma(1)}A^{s_1}_{t_1}B^{t_1}_{\sigma(2)}\cdots B^{s_k}_{\sigma(2k-1)}A^{s_k}_{\ell}B^{\ell}_{\sigma(2k)}\cdots B^{\ell}_{\sigma(2m-1)}A^{\ell}_{t_m}B^{t_m}_{\sigma(2m)}\cdots B^{s_{n/2}}_{\sigma(n-1)}A^{s_{n/2}}_{t_{n/2}}B^{t_{n/2}}_{\sigma(n)}\,,$$

is cancelled by

$$\mathrm{sgn}(\overline{\sigma})B^{s_1}_{\overline{\sigma}(1)}A^{s_1}_{t_1}B^{t_1}_{\overline{\sigma}(2)}\cdots B^{s_k}_{\overline{\sigma}(2k-1)}A^{s_k}_{t_k}B^{t_k}_{\overline{\sigma}(2k)}\cdots B^{s_m}_{\overline{\sigma}(2m-1)}A^{s_m}_{t_m}B^{t_m}_{\overline{\sigma}(2m)}\cdots B^{s_{n/2}}_{\overline{\sigma}(n-1)}A^{s_{n/2}}_{t_{n/2}}B^{t_{n/2}}_{\overline{\sigma}(n)}$$

which is equal to

$$\mathrm{sgn}(\overline{\sigma})B^{s_1}_{\overline{\sigma}(1)}A^{s_1}_{t_1}B^{t_1}_{\overline{\sigma}(2)}\cdots B^{s_k}_{\overline{\sigma}(2k-1)}A^{s_k}_{\ell}B^{\ell}_{\overline{\sigma}(2m-1)}\cdots B^{\ell}_{\overline{\sigma}(2k)}A^{\ell}_{t_m}B^{t_m}_{\overline{\sigma}(2m)}\cdots B^{s_{n/2}}_{\overline{\sigma}(n-1)}A^{s_{n/2}}_{t_{n/2}}B^{t_{n/2}}_{\overline{\sigma}(n)}\,,$$

if $\overline{\sigma}$ differs from σ only for $\overline{\sigma}(2m-1)=\sigma(2k)$ and $\overline{\sigma}(2k)=\sigma(2m-1)$. Indeed, differing by a single transposition, the two permutations have opposite signs, implying that these two coincident terms cancel each other in the final sum. It follows that in every product above, we can assume that $t_1,\dots,t_{n/2},s_1,\dots,s_{n/2}$ are all distinct, hence they must be a permutation τ of the set $\{1,\dots,n\}$ and we get the expression at the third line. $\square$

Lemma 3.1.9. *Let n be even, then any skew–symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be written as*

$$B^T A B = \begin{pmatrix} S & & & & & \\ & \ddots & & & & \\ & & S & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

where B is nonsingular and

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The proof is a linear algebra exercise (see [33, Chapter 3, Section 3, Corollary 11]).

We observe that a matrix as above has determinant equal to 1 if there are no zeroes on the diagonal and it is zero otherwise. Similarly, its Pfaffian is equal to 1 if there are no zeroes on the diagonal and it is zero otherwise (it can be proved, for example, by induction).

Proposition 3.1.10. *For every skew–symmetric matrix $A \in \mathbb{R}^{n\times n}$ there holds*

$$\mathrm{Pf}(A)^2 = \det(A) .$$

Proof. If n is odd the equality is trivial, as any skew–symmetric matrix has determinant equal to zero. If n is even, recalling the above observation, the result follows if A is singular, hence we assume that A is a nonsingular skew–symmetric matrix. By Lemma 3.1.9 one has that such a matrix has no zeroes on the diagonal, thus

$$1 = \det \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix} = \det(B^T A B) = \det(B)^2 \det(A)$$

and, by equation (3.3),

$$1 = \mathrm{Pf} \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix} = \mathrm{Pf}(B^T A B) = \det(B)\,\mathrm{Pf}(A)$$

and we are done. □

3.2 Proof of the Chern–Gauß–Bonnet theorem

From now on, we assume that

- (M, g) is a compact oriented Riemannian manifold,
- $\dim M = 2p$, even,
- all the local frames $\{e_i\}_{i=1}^{2p}$ that we will consider are orthonormal and their coframes $\{\vartheta^i\}_{i=1}^{2p}$ are oriented with respect to the volume form $\mathrm{d}V_M$ (see Definition 1.1.6 and equation (1.7)).

We then recall here some notions of Cartan formalism from Section 1.4. In particular, the defining equation (1.29) of the connection 1–form ω

$$\nabla e_j = \omega^i_j \otimes e_i\,, \tag{3.4}$$

the Cartan structural equations (1.32) and the Bianchi identities (1.33)

$$\begin{cases} \mathrm{d}\vartheta = -\omega \wedge \vartheta \\ \mathrm{d}\omega = \Omega - \omega \wedge \omega \end{cases} \qquad \begin{cases} 0 = \Omega \wedge \vartheta \\ \mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega \end{cases} \tag{3.5}$$

Moreover, as a consequence of working with orthonormal frames, we have the skew–symmetry of the connection and curvature forms (Remark 1.4.5)

$$\begin{cases} \omega^i_j = -\omega^j_i \\ \Omega^i_j = -\Omega^j_i \end{cases} \tag{3.6}$$

and the following transformation formulae under a local change of frame $e_i \mapsto \widetilde{e}_i = f^j_i e_j$

$$\begin{cases} \widetilde{\omega} = f^{-1}\omega f + f^{-1}\mathrm{d}f \\ \widetilde{\Omega} = f^{-1}\Omega f \end{cases} \tag{3.7}$$

where at each point $p \in M$, f_p is an orthogonal linear map, i.e., $f^{-1} = f^T$ with $\det(f) = 1$.

Remark 3.2.1. We also remark that there exists an orthonormal frame in an open set U such that $\omega|_q = 0$ for a specific point $q \in U$. To construct it, we choose an orthonormal basis $(e_1|_q, \ldots, e_{2p}|_q)$ of T_qM and for every other point q' sufficiently close to q we consider the frame given by parallel transporting the frame at q through the geodesics from q to q'. As the parallel transport maintains both norms and angles, the frame is orthonormal; the parallel transport guarantees that the Christoffel symbols Γ^k_{ij} all vanish at q and so does ω, because of equation (1.35).

Theorem 3.2.2. *There exists a unique globally defined $2p$–differential form $\mathrm{Pf}(\Omega) \in \Omega^{2p}(M)$ with local expression*

$$\mathrm{Pf}(\Omega)\big|_U = \frac{1}{2^p p!} \sum_{\sigma \in \Sigma_{2p}} \mathrm{sgn}(\sigma) \bigwedge_{i=1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\,, \tag{3.8}$$

in any open set U with a local orthonormal frame $\{e_i\}_{i=1}^{2p}$.

Remark 3.2.3. By means of equality (1.35)

$$\Omega^i_j = \frac{1}{2} R^i_{\ jkl}\, \vartheta^k \wedge \vartheta^l\,,$$

formula (3.8) can be written in terms of the Riemann tensor as

$$\begin{aligned} \mathrm{Pf}(\Omega)\big|_U &= \frac{1}{2^p p!} \sum_{1 \le i_1, \ldots, i_{2p} \le n} \varepsilon^{i_1 \ldots i_{2p}} \Omega^{i_1}_{i_2} \wedge \Omega^{i_3}_{i_4} \wedge \cdots \wedge \Omega^{i_{2p-1}}_{i_{2p}} \\ &= \frac{1}{2^{2p} p!} \sum_{1 \le i_1, \ldots, i_{2p} \le n} \varepsilon^{i_1 \ldots i_{2p}} R^{i_1}_{\ \ i_2 j_1 j_2} \cdots R^{i_{2p-1}}_{\ \ i_{2p} j_{2p-1} j_{2p}} \vartheta^{j_1} \wedge \vartheta^{j_2} \wedge \cdots \wedge \vartheta^{j_{2p-1}} \wedge \vartheta^{j_{2p}} \\ &= \frac{1}{2^{2p} p!} \sum_{\substack{1 \le i_1, \ldots, i_{2p} \le n \\ 1 \le j_1, \ldots, j_{2p} \le n}} \varepsilon^{i_1 \ldots i_{2p}} \varepsilon^{j_1 \ldots j_{2p}} R_{i_1 i_2, j_1 j_2} \ldots R_{i_{2p-1} i_{2p}, j_{2p-1} j_{2p}}\, \mathrm{d}V_M\,, \end{aligned}$$

where we used the *Levi–Civita symbol* $\varepsilon^{i_1 \ldots i_{2p}}$, which is equal to 0 if any index is repeated and to $\mathrm{sgn}(\sigma)$ otherwise, with σ the permutation $(i_1 \ldots i_{2p}) \in \Sigma_{2p}$.

Proof of Theorem 3.2.2. Under a change of orthonormal local frames $\widetilde{e}_i = f_i^j e_j$, by using formula (1.34) and equation (3.3), one has

$$\mathrm{Pf}(\widetilde{\Omega})\big|_U = \mathrm{Pf}(f^{-1}\Omega f)\big|_U = \mathrm{Pf}(f^T\Omega f)\big|_U = \det(f)\mathrm{Pf}(\Omega)\big|_U = \mathrm{Pf}(\Omega)\big|_U\,,$$

being $\det(f) = 1$. Hence, the form $\mathrm{Pf}(\Omega)$ is independent of the choice of the local frame and all the local expressions can be pasted together into a (smooth) global section. □

Lemma 3.2.4 (Transgression lemma). *Let $\pi\colon \mathbb{S}M \to M$ be the unit tangent bundle of (M,g), then the form $\pi^*\,\mathrm{Pf}(\Omega)$ is exact. That is, there exists $\Pi \in \Omega^{2p-1}(\mathbb{S}M)$ such that*

$$\pi^*\,\mathrm{Pf}(\Omega) = \mathrm{d}\Pi\,.$$

Proof. To simplify the notation, in the following we identify forms on M with their pull-backs on $\mathbb{S}M$.
We consider on the open set $U \subseteq M$ an orthonormal frame $\{e_i\}_{i=1}^{2p}$ and on $\mathbb{S}U \subseteq \mathbb{S}M$ the coordinates $(v^1,\dots,v^{2p})$ given by such a frame, that is,

$$v = v^i e_i \qquad\qquad \text{for any } v \in \mathbb{S}M\,.$$

Clearly there holds

$$v_i v^i = 1\,, \tag{3.9}$$

and, applying the exterior differential to this relation, we have

$$v_i \mathrm{d}v^i = 0\,. \tag{3.10}$$

We then denote by η^i the 1–forms defined by

$$\eta^i = \mathrm{d}v^i + v^j\omega_j^i \tag{3.11}$$

and we observe that they play a similar role of the connection 1–form ω, compare indeed equation (3.4) to

$$\nabla v = \nabla v^i e_i = (\mathrm{d}v^i + v^j\omega_j^i)\otimes e_i = \eta^i \otimes e_i\,.$$

Moreover, they satisfy

$$\begin{cases} v_i\eta^i = 0 \\ \mathrm{d}\eta^i = \eta^j\wedge\omega_j^i + v^j\Omega_j^i \end{cases} \tag{3.12}$$

indeed, using equation (3.10) and the skew–symmetry of ω (see relations (3.6)), we have

$$v_i\eta^i = v_i\mathrm{d}v^i + v_i\omega_j^i v^j = 0\,.$$

and using Cartan second structural equation (3.5),

$$\begin{aligned}
\mathrm{d}\eta^i &= \mathrm{d}v^j\wedge\omega_j^i + v^j\mathrm{d}\omega_j^i \\
&= (\eta^j - v^k\omega_k^j)\wedge\omega_j^i + v^j(\omega_j^k\wedge\omega_k^i + \Omega_j^i) \\
&= \eta^j\wedge\omega_j^i - v^k\omega_k^j\wedge\omega_j^i + v^j\omega_j^k\wedge\omega_k^i + v^j\Omega_j^i \\
&= \eta^j\wedge\omega_j^i + v^j\Omega_j^i\,.
\end{aligned}$$

Let us define now, for $0 \le k \le p-1$, the differential forms

$$\Phi_k|_{\mathbb{S}U} = \sum_{\sigma\in\Sigma_{2p}} \operatorname{sgn}(\sigma)\, v^{\sigma(1)}\eta^{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\,, \tag{3.13}$$

$$\Psi_k|_{\mathbb{S}U} = \sum_{\sigma\in\Sigma_{2p}} \operatorname{sgn}(\sigma)\, \Omega^{\sigma(1)}_{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\,, \tag{3.14}$$

and $\Psi_{-1} := 0$.
For every k, Φ_k and Ψ_k are $(2p-1)$ and $2p$–differential forms, respectively. We underline that in the expression of Φ_k and Ψ_k, changing the index from k to $k+1$ amounts to substituting a pair $\eta^{\sigma(j)} \wedge \eta^{\sigma(j+1)}$ with $\Omega^{\sigma(j)}_{\sigma(j+1)}$. In particular, for $k = p-1$, the form Ψ_{p-1} reduces to the expression (3.8) of $\mathrm{Pf}(\Omega)$, up to a constant.
We now want to prove that these forms are globally well–defined (they are independent of the choice of the local orthonormal frame) and satisfy the recurrence relation

$$\mathrm{d}\Phi_k = \Psi_{k-1} + \frac{2(p-k)-1}{2(k+1)}\Psi_k\,, \qquad \text{for } k \in \{0,1,\dots,p-1\}\,. \tag{3.15}$$

As a consequence,

$$\begin{aligned}
\Psi_k &= \sum_{r=0}^{k} (-1)^r \prod_{i=0}^{r} \frac{2(k-i+1)}{2(p-k+i)-1} \mathrm{d}\Phi_{k-r} \\
&= \sum_{r=0}^{k} (-1)^r \frac{(2(k+1))!!}{(2(k-r))!!} \frac{(2(p-k-1)-1)!!}{(2(p-k+r)-1)!!} \mathrm{d}\Phi_{k-r} \\
&= \sum_{r=0}^{k} (-1)^r 2^{r+1} \frac{(k+1)!}{(k-r)!} \frac{(2(p-k-1)-1)!!}{(2(p-k+r)-1)!!} \mathrm{d}\Phi_{k-r}
\end{aligned}$$

and

$$\mathrm{Pf}(\Omega) = \frac{1}{2^p p!}\Psi_{p-1} = \mathrm{d}\Pi\,,$$

with

$$\Pi = \sum_{r=0}^{p-1} (-1)^r \frac{1}{2^r r! (2(p-r)-1)!!} \Phi_r\,. \tag{3.16}$$

To this aim, we observe that after a change of local frame $e_i \mapsto \widetilde{e}_i = f_i^j e_j$ the terms involved in equations (3.13) and (3.14) transform according to

$$\begin{cases} \widetilde{v}^i = f^i_j v^j \\ \widetilde{\eta}^i = f^i_j \eta^j \end{cases}$$

indeed,

$$\widetilde{v}^i = \widetilde{\vartheta}^i(\widetilde{v}^j \widetilde{e}_j) = f^i_k \vartheta^k (v^j e_j) = f^i_k v^k\,,$$

and by the transformation rule (3.7) for ω,

$$\begin{aligned}
\widetilde{\eta}^i &= \mathrm{d}\widetilde{v}^i + \widetilde{v}^j \widetilde{\omega}^i_j \\
&= \mathrm{d}f^i_j v^j + f^i_j \mathrm{d}v^j + f^j_s v^s (f^i_k \mathrm{d}f_j{}^k + f^i_k \omega^k_t f_j{}^t) \\
&= v^j \mathrm{d}f^i_j + f^i_j \mathrm{d}v^j + f^j_s v^s (\mathrm{d}(f^i_k f_j{}^k) - \mathrm{d}f^i_k f_j{}^k) + f^i_k \omega^k_t v^t \\
&= v^j \mathrm{d}f^i_j + f^i_j \mathrm{d}v^j - v^k \mathrm{d}f^i_k + f^i_j \omega^j_t v^t \\
&= f^i_j \eta^j\,.
\end{aligned}$$

Hence, we can see that in the expressions (3.13) and (3.14) for Φ_k and Ψ_k, respectively, the factors $v^i\eta^j$ and $\eta^i \wedge \eta^j$ transform, under a change of frame, according to the formulae

$$\begin{aligned}\widetilde{v}^i\widetilde{\eta}^j &= f^i_k v^k f^j_s \eta^s = f^i_k (v^k \eta_s) f^s_j \,,\\ \widetilde{\eta}^i \wedge \widetilde{\eta}^j &= f^i_k \eta^k \wedge f^j_s \eta^s = f^i_k (\eta^k \wedge \eta_s) f^s_j \,,\end{aligned}$$

which are the same transformation rules for Ω^i_j. Then, the very same computations used to show that $\mathrm{Pf}(\Omega)$ is globally well–defined (Theorem 3.2.2) prove that the same holds also for the forms Φ_k and Ψ_k.

To prove the recurrence relation (3.15), we fix $k \in \{0, 1, \ldots, p-1\}$ and compute using equations (3.11), (3.12) and the Bianchi identities (3.5) as follows,

$$\begin{aligned}\mathrm{d}\Phi_k &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, \mathrm{d}v^{\sigma(1)} \wedge \eta^{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\\ &\quad + (2p-2k-1) \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} \mathrm{d}\eta^{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\\ &\quad + k \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} \eta^{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \mathrm{d}\Omega^{\sigma(2p-2k+1)}_{\sigma(2p-2k+2)} \wedge \bigwedge_{i=p-k+2}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\\ &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, \eta^{\sigma(1)} \wedge \eta^{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\\ &\quad + (2p-2k-1) \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} v^t \Omega^{\sigma(2)}_t \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\\ &\quad + F(\omega)\\ &= \Psi_{k-1} + (2p-2k-1)\Lambda_k + F(\omega)\,,\end{aligned}$$

where we grouped together all the terms involving ω into $F(\omega)$ and called Λ_k the form (locally) defined as

$$\Lambda_k|_{\mathbb{S}U} = \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} v^t \Omega^{\sigma(2)}_t \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\,. \tag{3.17}$$

We now show that Λ_k is just a multiple of Ψ_k. This implies that also the term $F(\omega)$ has to be independent of the choice of the local frame. As such, taking a frame as in Remark 3.2.1 at a point q, we have $\omega|_q = 0$ and $F(\omega)|_q = 0$. As the values of $F(\omega)$ are independent of the choice of the frame, we must have $F(\omega) \equiv 0$ identically on $\mathbb{S}U$ and

$$\mathrm{d}\Phi_k = \Psi_{k-1} + (2p-2k-1)\Lambda_k\,. \tag{3.18}$$

To obtain Ψ_k from Λ_k we define the following auxiliary forms:

$$\begin{aligned}A_k|_{\mathbb{S}U} &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} v^{\sigma(1)} \Omega^{\sigma(1)}_{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\,;\\ B_k|_{\mathbb{S}U} &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} v_{\sigma(3)} \Omega^{\sigma(3)}_{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\,;\\ C_k|_{\mathbb{S}U} &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v_{\sigma(3)} v^{\sigma(3)} \Omega^{\sigma(1)}_{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)}\,,\end{aligned}$$

hence, we can rewrite Λ_k, as expressed in equation (3.17), as

$$\Lambda_k = A_k + 2(p-k-1)B_k \,. \tag{3.19}$$

Indeed, expanding the (implicit) summation over the index t in equation (3.17),

- when $t = \sigma(1)$, we get A_k;
- when $t = \sigma(2)$, we get zero, as $\Omega^{\sigma(2)}_{\sigma(2)} = 0$;
- when $t = \sigma(3), \dots, \sigma(2(p-k))$, we get $2(p-k-1)$–times the term B_k;
- when $t = \sigma(2(p-k)+1), \dots, \sigma(2p)$, we get zero, due to terms of the form (no sum intended) $\Omega^{\sigma(2)}_t \wedge \Omega^{\sigma(j)}_t$ and $\Omega^{\sigma(2)}_t \wedge \Omega^t_{\sigma(j)}$ which produce cancellations for the same argument used in the proof Theorem 3.1.8.

Then, by equation (3.9), we write

$$v_{\sigma(1)}v^{\sigma(1)} = 1 - \sum_{i=2}^{2p} v_{\sigma(i)}v^{\sigma(i)}$$

and we put it in the expression of A_k, as follows

$$\begin{aligned}
A_k &= \sum_{\sigma\in\Sigma_{2p}} \operatorname{sgn}(\sigma)\, v_{\sigma(1)}v^{\sigma(1)}\Omega^{\sigma(1)}_{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)} \\
&= \sum_{\sigma\in\Sigma_{2p}} \operatorname{sgn}(\sigma)\Big(1 - \sum_{i=2}^{2p} v_{\sigma(i)}v^{\sigma(i)}\Big)\Omega^{\sigma(1)}_{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)} \\
&= \Psi_k - A_k - 2(p-k-1)C_k - 2kA_k \,,
\end{aligned}$$

where at the last step we expanded the sum inside the parentheses taking into account that

- when $i = 1$, we get Ψ_k;
- when $i = 2$, we get $-A_k$;
- when $i = 3, \dots, 2(p-k)$, we get $2(p-k-1)$–times the term $-C_k$;
- when $i = 2(p-k)+1, \dots, 2p$, we get $2k$–times the term $-A_k$.

Hence, we conclude

$$\Psi_k = 2(k+1)A_k + 2(p-k-1)C_k \,. \tag{3.20}$$

Arguing similarly, using now relations (3.12), we write

$$v_{\sigma(3)}\eta^{\sigma(3)} = -\sum_{\substack{i=1\\ i\neq 3}}^{2p} v_{\sigma(i)}\eta^{\sigma(i)}$$

and we use it in order to deal with B_k, obtaining

$$\begin{aligned}
B_k &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} v_{\sigma(3)} \Omega^{\sigma(3)}_{\sigma(2)} \wedge \bigwedge_{i=3}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)} \\
&= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} \big(v_{\sigma(3)}\eta^{\sigma(3)}\big) \wedge \Omega^{\sigma(3)}_{\sigma(2)} \wedge \bigwedge_{i=4}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)} \\
&= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma)\, v^{\sigma(1)} \left(- \sum_{\substack{i=1\\ i\neq 3}}^{2p} v_{\sigma(i)}\eta^{\sigma(i)} \right) \wedge \Omega^{\sigma(3)}_{\sigma(2)} \wedge \bigwedge_{i=4}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega^{\sigma(2i-1)}_{\sigma(2i)} \\
&= C_k - (1+2k)B_k\,,
\end{aligned}$$

where, in expanding the sum inside the parentheses at the last step, we took into account that

- when $i=1$, we get C_k;
- when $i=2$, we get $-B_k$;
- when $i=4,\dots,2(p-k)$, we get zero, as $\eta^{\sigma(i)}\wedge\eta^{\sigma(i)}=0$;
- when $i=2(p-k)+1,\dots,2p$, we get $2k$–times the term $-B_k$.

Hence, we have the equality

$$C_k = 2(k+1)B_k\,. \tag{3.21}$$

Putting equations (3.19), (3.20) and (3.21) together, we finally obtain

$$\Psi_k = 2(k+1)\big(A_k + 2(p-k-1)B_k\big) = 2(k+1)\Lambda_k\,,$$

that, once plugged in equation (3.18), gives the relation (3.15), proving the lemma. □

We are now ready to state and show the Chern–Gauß–Bonnet theorem.

Theorem 3.2.5 (Chern [10]). *Let (M,g) be a compact oriented $2p$–dimensional Riemannian manifold, then*

$$\int_M \mathrm{Pf}(\Omega) = (2\pi)^p\chi(M)\,, \tag{3.22}$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M.

Proof. The proof of the theorem consists of the following series of steps:

- By Remark 3.1.2 we can consider on M a unit vector field X with finitely many singularities.
- We "isolate" the singularity points of X using small balls $\overline{B}$ such that $\mathbb{S}\overline{B}\cong\overline{B}\times\mathbb{S}^{2p-1}$ and we split the integral of $\mathrm{Pf}(\Omega)$ in its parts inside and outside these balls.
- Since sending the radii of such balls to zero, the contributions of the integrals on the insides go to zero, we can focus on computing the integral on the outside (showing that it is independent of such radii).

- Outside the balls we use the unit vector field X to express the integral of the Pfaffian on M as an integral on $\mathbb{S}M$, "pulling–back" the form $\mathrm{Pf}(\Omega)$ via the map X^{-1} defined on the image of X, where it coincides with the projection map $\pi\colon \mathbb{S}M \to M$ of the unit tangent bundle of (M, g).

- Thanks to the transgression Lemma 3.2.4 the above pullback $\pi^* \mathrm{Pf}(\Omega)$ is the differential of the form Π defined by equation (3.16), hence, we can apply Stokes Theorem (1.1.14) and obtain that the integral of the Pfaffian outside the balls is equal to the integral of Π over the (image by X of the) boundaries of such balls.

- We compute these integrals by making use of Definition 3.1.5 of the index of a vector field, which allows us to "substitute" each integral with one on the standard sphere $\mathbb{S}^{2p-1}$, for every singularity point of the field X.

- The proof is concluded by applying the Poincaré–Hopf index theorem 3.1.6.

As said above, let us consider a unit vector field $X \in \Gamma(TM)$ with finitely many isolated singularities $z_1, \dots, z_m$ (Remark 3.1.2) and define $\mathcal{S} = X(M \setminus \{z_1, \dots, z_m\})$ to be its image. We then choose a family of closed balls $\overline{B}_\varepsilon(z_i)$ of radius $\varepsilon > 0$ around each point z_i, such that they are mutually disjoint and call $\mathfrak{B}_\varepsilon$ their union. We let $\pi\colon \mathbb{S}M \to M$ be the unit tangent bundle of M and, since $\mathcal{S} \subseteq \mathbb{S}M$, we consider the restriction $\pi\big|_{\mathcal{S}} = X^{-1}\colon \mathcal{S} \to M$. Then, by means of the transgression Lemma 3.2.4 and Stokes theorem 1.1.14, we evaluate

$$\begin{aligned}
\int_M \mathrm{Pf}(\Omega) &= \int_{\mathfrak{B}_\varepsilon} \mathrm{Pf}(\Omega) + \int_{M\setminus\mathfrak{B}_\varepsilon} \mathrm{Pf}(\Omega) \\
&= \int_{\mathfrak{B}_\varepsilon} \mathrm{Pf}(\Omega) + \int_{X(M\setminus\mathfrak{B}_\varepsilon)} \pi\big|_{\mathcal{S}}^* \mathrm{Pf}(\Omega) \\
&= \int_{\mathfrak{B}_\varepsilon} \mathrm{Pf}(\Omega) + \int_{\mathcal{S}\setminus X(\mathfrak{B}_\varepsilon)} \mathrm{d}\Pi \\
&= \int_{\mathfrak{B}_\varepsilon} \mathrm{Pf}(\Omega) + \int_{\partial\left(\mathcal{S}\setminus X(\mathfrak{B}_\varepsilon)\right)} \Pi\,,
\end{aligned}$$

as $\pi\big|_{\mathcal{S}}^* \mathrm{Pf}(\Omega) = \pi^* \mathrm{Pf}(\Omega)\big|_{\mathcal{S}} = \mathrm{d}\Pi\big|_{\mathcal{S}}$.
Now, the first integral clearly goes to zero, as $\varepsilon \to 0$, hence it will give no contribution if we show that the second one is independent of ε. The domain of the integration of this latter is given by

$$\partial(\mathcal{S} \setminus X(\mathfrak{B}_\varepsilon)) = \partial\Big(\mathcal{S} \setminus \bigcup_{i=1}^m X\big(\overline{B}_\varepsilon(z_i)\big)\Big) = \bigcup_{i=1}^m X\big(\partial\overline{B}_\varepsilon(z_i)\big)\,,$$

hence, by formula (3.16) for Π, we have

$$\int_{M\setminus\mathfrak{B}_\varepsilon} \mathrm{Pf}(\Omega) = \sum_{i=1}^m \int_{X\left(\partial\overline{B}_\varepsilon(z_i)\right)} \Pi = \sum_{i=1}^m \sum_{r=0}^{p-1} (-1)^r \frac{1}{2^r r!(2(p-r)-1)!!} \int_{X\left(\partial\overline{B}_\varepsilon(z_i)\right)} \Phi_r$$

where, once chosen an orthonormal frame $\{e_i\}_{i=1}^{2p}$ on an open set $U \subseteq M$ and coordinates $(v_q^1, \dots, v_q^{2p})$ on $\mathbb{S}M_q$ at each point $q \in U$, locally

$$\Phi_r = \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma) v^{\sigma(1)} \bigwedge_{i=2}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p} \Omega_{\sigma(2i)}^{\sigma(2i-1)}$$

and $\eta^i = \mathrm{d}v^i + v^j\omega_j^i$.
We now fix a singular point z of X and we take the corresponding ball $\overline{B}$ around z small enough, such that $\mathbb{S}\overline{B} \cong \overline{B} \times \mathbb{S}^{2p-1}$, denoting by $\overline{\pi}\colon \mathbb{S}\overline{B} \to \mathbb{S}^{2p-1}$ the induced projection on the second factor. In order to compute the index of the vector field X at z and apply the Poincaré–Hopf index Theorem 3.1.6, we consider the map $\overline{\pi} \circ X\colon \overline{B} \to \mathbb{S}M_z$, where we are identifying $\mathbb{S}^{2p-1}$ with $\mathbb{S}M_z$. This map allows us, by Definition 3.1.5, to change the domain of integration to $\mathbb{S}M_z$ up to multiplying by the index of X at z. Then, by choosing a frame at z as in Remark 3.2.1, we obtain $\omega|_z = 0$, hence $\Omega|_z = \mathrm{d}\omega|_z$ (see equation (3.4)) and $\eta^i|_z = \mathrm{d}v^i|_z$. Consequently, if $r \geq 1$, by applying again Stokes theorem, we have

$$\begin{aligned}
\int_{\mathbb{S}M_z} \Phi_r &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma) \int_{\mathbb{S}M_z} v^{\sigma(1)} \bigwedge_{i=2}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p-1} \Omega_{\sigma(2i)}^{\sigma(2i-1)} \wedge \mathrm{d}\omega_{\sigma(2p)}^{\sigma(2p-1)} \\
&= - \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma) \int_{\mathbb{S}M_z} \mathrm{d}\left(v^{\sigma(1)} \bigwedge_{i=2}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p-1} \Omega_{\sigma(2i)}^{\sigma(2i-1)} \wedge \omega_{\sigma(2p)}^{\sigma(2p-1)} \right) \\
&= - \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma) \int_{\partial\mathbb{S}M_z} v^{\sigma(1)} \bigwedge_{i=2}^{2(p-k)} \eta^{\sigma(i)} \wedge \bigwedge_{i=p-k+1}^{p-1} \Omega_{\sigma(2i)}^{\sigma(2i-1)} \wedge \omega_{\sigma(2p)}^{\sigma(2p-1)} \\
&= 0\,,
\end{aligned}$$

and if $r = 0$,

$$\begin{aligned}
\int_{\mathbb{S}M_z} \Phi_0 &= \sum_{\sigma\in\Sigma_{2p}} \mathrm{sgn}(\sigma) \int_{\mathbb{S}M_z} v^{\sigma(1)} \bigwedge_{i=2}^{2p} \eta^{\sigma(i)} \\
&= (2p-1)! \sum_{j=1}^{n} (-1)^{j+1} \int_{\mathbb{S}M_z} v^j \bigwedge_{\substack{i=1 \\ i\neq j}}^{2p} \mathrm{d}v^i \\
&= (2p-1)! \int_{\mathbb{S}^{2p-1}} \mathrm{d}V_{\mathbb{S}^{2p-1}}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_M \mathrm{Pf}(\Omega) &= \sum_{i=1}^{m} \frac{1}{(2p-1)!!} \int_{X\left(\partial\overline{B}_\varepsilon(z_i)\right)} \Phi_0 \\
&= \sum_{i=1}^{m} \deg(S_{z_i}) \frac{1}{(2p-1)!!} \int_{\mathbb{S}M_{z_i}} \Phi_0 \\
&= \sum_{i=1}^{m} \mathrm{ind}_{z_i}(X) \frac{(2p-1)!}{(2p-1)!!} \int_{\mathbb{S}^{2p-1}} \mathrm{d}V_{\mathbb{S}^{2p-1}} \\
&= (2p-2)!!\,\mathrm{Vol}(\mathbb{S}^{2p-1}) \sum_{i=1}^{m} \mathrm{ind}_{z_i}(X) \\
&= (2\pi)^p \sum_{i=1}^{m} \mathrm{ind}_{z_i}(X)\,,
\end{aligned}$$

by the well–known formula $\mathrm{Vol}(\mathbb{S}^{2p-1}) = 2\pi^p/(p-1)!$.
The conclusion (3.22)

$$\int_M \mathrm{Pf}(\Omega) = (2\pi)^p \chi(M)$$

then follows by applying the Poincaré–Hopf index Theorem (3.1.6). □

Remark 3.2.6.

- In odd dimensions, the Pfaffian $\mathrm{Pf}(\Omega)$ is zero by definition, then the theorem still holds (even if trivial), being the Euler–Poincaré characteristic of every odd–dimensional manifold equal to zero (see Remark 1.1.20).
- If (M, g) is nonorientable, by "passing" to its canonical 2–sheets (orientable) Riemannian covering $\widetilde{M}$, satisfying $\chi(\widetilde{M}) = 2\chi(M)$ (see [16, Section 2.2], for instance) and applying the theorem to $\widetilde{M}$, we get
$$2(2\pi)^{n/2}\chi(M) = (2\pi)^{n/2}\chi(\widetilde{M}) = \int_{\widetilde{M}} \mathrm{Pf}(\widetilde{\Omega}) = 2\int_M \mathrm{Pf}(\Omega)$$
where the last integral is computed by using the canonical Riemannian measure of the manifold in place of the volume form (which is not defined for M). Hence, the conclusion holds also in the nonorientable case.
- If the manifold (M, g) has a boundary, the theorem takes the form
$$\int_M \mathrm{Pf}(\Omega) = (2\pi)^{n/2}\chi(M) + \int_{\partial M} \nu^*\Pi$$
where $\nu \colon \partial M \to \mathbb{S}M$ is the outward–pointing unit normal vector. We refer to [33, Chapter 13, Addendum 2], for further reading.

The Chern–Gauß–Bonnet theorem asserts that there exists an "intrinsic" quantity $\mathrm{Pf}(\Omega)$, related to the curvature of an even–dimensional Riemannian manifold (M, g), such that its integral is a multiple (depending on the dimension) of the Euler–Poincaré characteristic, which is a topological invariant of M. In the case (historically relevant, being studied earlier) of a closed, embedded hypersurface M in the Euclidean space $\mathbb{R}^{n+1}$, in trying to generalise the "classical" Gauß–Bonnet theorem for surfaces in $\mathbb{R}^3$, H. Hopf [19] in 1926 proved that
$$\int_M K \,\mathrm{d}V_M = \frac{\mathrm{Vol}(\mathbb{S}^n)}{2}\chi(M) = \frac{(2\pi)^{n/2}}{(n-1)!!}\chi(M)\,. \tag{3.23}$$
Here, $K = \det \mathrm{d}\nu$ is the *Gaußian curvature* of the manifold, where $\nu \colon M \to \mathbb{S}^n$ is a (smooth) pointwise choice of a unit normal vector field on M, i.e., the *Gauß map.*
In even dimension, K is independent of the choice (up to the sign) of the normal, moreover it is easy to see, by pointwise diagonalising the second fundamental form in an orthonormal basis $\{e_i\}_{i=1}^n$ that then the basis $\{(e_i \wedge e_j)/\sqrt{2}\}_{i<j=1}^n$ of $\Lambda^2(TM)$ diagonalises the curvature operator $\mathscr{R} : \Lambda^2(TM) \to \Lambda^2(TM)$, with eigenvalues $\lambda_i\lambda_j$, where λ_i are the eigenvalues of the second fundamental form. Hence,
$$K = \prod_{i=1}^n \lambda_i \qquad \text{and} \qquad \det\mathscr{R} = \prod_{\substack{i,j=1\\ i<j}}^n \lambda_i\lambda_j = \Big(\prod_{i=1}^n \lambda_i\Big)^{n-1},$$
implying
$$K = \big(\det\mathscr{R}\big)^{\frac{1}{n-1}}\,.$$
In particular, if $n = 2$, we recover
$$K = \lambda_1\lambda_2 = \det\mathscr{R} = \mathrm{Sec}(e_1, e_2) = \mathrm{R}/2\,.$$

Since the integral of $\mathrm{d}\nu$ is the area of the image of the Gauß map, *counted with multiplicity*, it follows that

$$\int_M K\,\mathrm{d}V_M = \int_M \nu^*\mathrm{d}V_{\mathbb{S}^n} = \mathrm{Vol}(\mathbb{S}^n)\deg(\nu)\,,$$

then, the conclusion follows from the relation

$$\deg(\nu) + (-1)^n\deg(\nu) = \chi(M)\,, \tag{3.24}$$

which gives equality (3.23) in even dimensions.
This can be seen by considering a closed *tubular neighbourhood* of M (which always exists, being M embedded – see [15], for instance),

$$N = \Big\{\, p\in\mathbb{R}^{n+1} \;\Big|\; \mathrm{dist}(p,M)\leq\varepsilon \,\Big\}\,,$$

for $\varepsilon > 0$ small enough such that ∂N is diffeomorphic to two copies of M and the (unique, orthogonal) projection map $\pi : N \to M$ is well defined. Then, if X is a tangent vector field to M with isolated zeroes $z_1,\dots,z_n$, we consider the field Y on N given by

$$Y(p) = \big(p-\pi(p)\big) + X(\pi(p))$$

and it is easy to see that Y has the same zeroes of X, each one with the same index, that is $\mathrm{ind}_{z_i}(Y) = \mathrm{ind}_{z_i}(X)$. Moreover, it is clear that Y is outward–pointing at the boundary of N, hence $\chi(M) = \chi(N)$ by the Poincaré–Hopf index theorem 3.1.6.
Then, considering the unit vector field $Z = Y/|Y|$ on N, with finitely many singularities, enclosed in a family of disjoint closed balls $\overline{B}_i \subseteq \overset{\circ}{N}$, whose union we denote by $\mathfrak{B}$, we have that the boundary of $N\setminus\mathfrak{B}$ is given by the union of ∂N and $\partial\mathfrak{B}$, endowed with opposite orientation (for how the boundary of a manifold is canonically oriented). Hence, $\deg(Z|_{\partial(N\setminus\mathfrak{B})}) = \deg(Z|_{\partial N}) - \deg(Z|_{\partial\mathfrak{B}})$ and for any form $\omega\in\Omega^n(\mathbb{S}^n)$, we have

$$\big(\deg(Z|_{\partial N}) - \deg(Z|_{\mathfrak{B}})\big)\int_{\mathbb{S}^n}\omega = \int_{\partial(N\setminus\mathfrak{B})} Z^*\omega = \int_{N\setminus\mathfrak{B}}\mathrm{d}Z^*\omega = \int_{N\setminus\mathfrak{B}} Z^*\mathrm{d}\omega = 0\,,$$

thus,

$$\deg(Z|_{\partial N}) = \deg(Z|_{\mathfrak{B}}) = \sum_{i=1}^m \deg(Z|_{B_i}) = \sum_{i=1}^m \mathrm{ind}_{z_i}(Z) = \chi(N)$$

again by the Poincaré–Hopf index Theorem (3.1.6).
Being the field $Z|_{\partial N}$ homotopic to the unit normal vector field $\overline{\nu}$ outward–pointing on ∂N, we conclude that $\deg(\overline{\nu}) = \chi(N) = \chi(M)$. Relation (3.24) then follows by the easy fact that $\deg(\overline{\nu}) = \deg(\nu) + (-1)^n\deg(\nu)$, as ∂N is diffeomorphic to two copies of M and we considered on one of them the same orientation of M (induced by $\overline{\nu}$ and ν, respectively) and on the other the opposite one (see [32, Chapter 6, Addendum 2], for more details).

The result was later generalised in 1940 with a quite more complex proof by C. B. Allendoerfer [2] and W. Fenchel [13] to an n–manifold M embedded in higher (odd) codimension $\mathbb{R}^{n+k}$. Their proof consisted in using again a closed tubular neighbourhood N of M in $\mathbb{R}^{n+k}$ (hence, with a hypersurface boundary ∂N of codimension 1) and applying to its boundary the previous result to get

$$\int_{\partial N} K_{\partial N}\,\mathrm{d}V_{\partial N} = \frac{\mathrm{Vol}(\mathbb{S}^{n+k-1})}{2}\chi(\partial N)\,.$$

Then, after showing that $\chi(\partial N) = \chi(S^{k-1})\chi(M) = 2\chi(M)$, they expressed the integral of the Gaußian curvature of ∂N in terms of an integral involving the curvature of M, getting formula (3.23) with

$$K = \frac{1}{2^{n/2}n!} \sum_{\substack{1\leq i_1,\dots,i_n\leq n\\ 1\leq j_1,\dots,j_n\leq n}} \varepsilon^{i_1\dots i_n}\varepsilon^{j_1\dots j_n} R_{i_1 i_2 j_1 j_2}\cdots R_{i_{n-1}i_n j_{n-1}j_n} \tag{3.25}$$

in an orthonormal frame, coinciding with $(\det \mathscr{R})^{\frac{1}{n-1}}$ and reducing to formula (3.23) in codimension one. Recalling Remark 3.2.3, notice that

$$K\,\mathrm{d}V_M = \frac{1}{(n-1)!!}\,\mathrm{Pf}(\Omega)\,.$$

They were later able to get the same conclusion also for abstract Riemannian manifolds, by means of a different (still quite involved) technique, see [33, Chapter 13] for more details. The *Nash embedding theorem* [27] in 1956 clearly simplified such second part of the proof, however, in the spirit of the classical theorem of Gauß and Bonnet, one would have liked to have a purely "intrinsic" proof, without involving any embedding in the Euclidean space.

In order to use formula (3.25) in practice, we actually want to express more explicitly K or $\mathrm{Pf}(\Omega)$ in terms of the Riemann tensor. Even if from such formula follows that it must be a homogenous polynomial of degree $n/2$ in the components of Riem, in general, a simple expression is not known: only in the "classic" case of dimensions $n = 2$, the Pfaffian is easy to compute, indeed

$$\mathrm{Pf}(\Omega) = \tfrac{1}{2}R_{12ij}\vartheta^i\wedge\vartheta^j = R_{1212}\vartheta^1\wedge\vartheta^2 = \tfrac{1}{2}\mathrm{R}\,\mathrm{d}V_M\,,$$

where R is the scalar curvature, hence,

$$\int_M \mathrm{R}\,\mathrm{d}V_M = 4\pi\chi(M)\,, \tag{3.26}$$

which implies the Gauß–Bonnet theorem for a surface in $\mathbb{R}^3$ (as the Gaußian curvature K of a surface satisfies $K = \mathrm{R}/2$).

We will analyse in detail the four–dimensional case in the next section, which will be fundamental for the discussion of the next chapter. About the higher dimensions, we only mention that in dimension 6 a computation by S. Takashi [30] yields

$$\begin{aligned}\mathrm{Pf}(\Omega) = \frac{1}{48}(&\mathrm{R}^3 - 12\mathrm{R}|\mathrm{Ric}|^2 + 3\mathrm{R}|\mathrm{Riem}|^2 + 16R^{ij}R_i^kR_{jk} + 24R^{ij}R^{kl}R_{ijkl}\\ &- 24R^{st}R_s^{ijk}R_{tijk} - 8R^{ijkl}R_{i\,k}^{\,s\,t}R_{jtls} + 2R^{ijkl}R_{ij}^{\;\;st}R_{klst})\,\mathrm{d}V_M\,,\end{aligned}$$

which, in the special case of an Einstein manifold (i.e., $\mathrm{Ric} = \mathrm{R}g/6$), becomes

$$\mathrm{Pf}(\Omega) = \frac{1}{48}((1/9)\mathrm{R}^3 - \mathrm{R}|\mathrm{Riem}|^2 - 8R^{ijkl}R_{i\,k}^{\,s\,t}R_{jtls} + 2R^{ijkl}R_{ij}^{\;\;st}R_{klst})\,\mathrm{d}V_M$$

and that, up to our knowledge, no formula is present in literature in dimension $n \geq 8$.

3.3 The Chern–Gauß–Bonnet theorem in dimension four

In order to write an integral formula involving the Riemann tensor, we start "expanding" the Pfaffian and expressing it in terms of the components R_{ijkl} of Riem, in a local orthonormal frame.

$$
\begin{aligned}
\mathrm{Pf}(\Omega) &= \frac{1}{4}(R_{12ij}R_{34kl} + R_{13ij}R_{24kl} + R_{14ij}R_{23kl})\,\vartheta^i\wedge\vartheta^j\wedge\vartheta^k\wedge\vartheta^l \\
&= \frac{1}{4}\{4(R_{1212}R_{3434} + R_{1312}R_{2434} + R_{1412}R_{2334}) \\
&\quad + 4(R_{1213}R_{3424} + R_{1313}R_{2424} + R_{1413}R_{2324}) \\
&\quad + 4(R_{1214}R_{3423} + R_{1314}R_{2423} + R_{1414}R_{2323}) \\
&\quad + 4(R_{1223}R_{3414} + R_{1323}R_{2414} + R_{1423}R_{2314}) \\
&\quad + 4(R_{1224}R_{3413} + R_{1324}R_{2413} + R_{1424}R_{2313}) \\
&\quad + 4(R_{1234}R_{3412} + R_{1334}R_{2412} + R_{1434}R_{2312})\}\,\vartheta^1\wedge\vartheta^2\wedge\vartheta^3\wedge\vartheta^4 \\
&= \{R_{1212}R_{3434} + R_{1313}R_{2424} + R_{1414}R_{2323} \\
&\quad + 2(R_{1213}R_{2434} + R_{1214}R_{2334} + R_{1314}R_{2324}) \\
&\quad + 2(R_{1223}R_{1434} + R_{1224}R_{1334} + R_{1323}R_{1424}) \\
&\quad + R^2_{1234} + R^2_{1324} + R^2_{1423}\}\,\mathrm{d}V_M\,. \qquad (3.27)
\end{aligned}
$$

We now prove in Lemma 3.3.1 below that there exists a local orthonormal frame (e_1, e_2, e_3, e_4) such that

$$R_{1213} = R_{1214} = R_{1314} = R_{1223} = R_{1224} = R_{1323} = 0\,,$$

hence, equality (3.27) reduces to

$$
\begin{aligned}
\mathrm{Pf}(\Omega) &= \Big(R_{1212}R_{3434} + R_{1313}R_{2424} + R_{1414}R_{2323} + R^2_{1234} + R^2_{1324} + R^2_{1423}\Big)\,\mathrm{d}V_M \\
&= \Big(K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23} + R^2_{1234} + R^2_{1324} + R^2_{1423}\Big)\,\mathrm{d}V_M\,, \qquad (3.28)
\end{aligned}
$$

where we are denoting with K_{ij} the sectional curvatures $\mathrm{Sec}(e_i, e_j)$.
From now on, we will work in this particular frame.

Lemma 3.3.1. *Let $p \in M$ and π a 2–plane in T_pM with maximal sectional curvature. We choose $e_1 \in \pi$ and $e_3 \in \pi^\perp$ orthonormal vectors such that $\mathrm{Sec}(e_1, e_3)$ is the maximal sectional curvature among all 2–planes $\langle v, w\rangle$, with $v \in \pi$ and $w \in \pi^\perp$. Finally, we choose the remaining unit vectors $e_2 \perp e_1$ and $e_4 \perp e_3$ such that $\pi = \langle e_1, e_2\rangle$ and $\pi^\perp = \langle e_3, e_4\rangle$. Then, the terms R_{1213}, R_{1214}, R_{1223}, R_{1224}, R_{1323}, R_{1314} all vanish at $p \in M$.*

Remark 3.3.2. The vanishing terms are precisely the ones of the form R_{ikjk}, where $i \neq j$ and k is the lowest of the remaining two indices.

Proof. Let for $\vartheta, \varphi \in \mathbb{R}$, $i, j \in \{1, 2\}$ and $k \in \{3, 4\}$,

$$
\begin{aligned}
f_{ijk}(\vartheta) &= \mathrm{Sec}\big(e_i, (\cos\vartheta)e_j + (\sin\vartheta)e_k\big)\,, \\
g(\vartheta, \varphi) &= \mathrm{Sec}\big((\cos\vartheta)e_1 + (\sin\vartheta)e_2, (\cos\varphi)e_3 + (\sin\varphi)e_4\big)\,.
\end{aligned}
$$

Since $\mathrm{Sec}(e_i, e_j)$ maximises f_{ijk} for $\vartheta = 0$ and $\mathrm{Sec}(e_1, e_3)$ maximises g for $(\vartheta, \varphi) = (0, 0)$,

$$\frac{\partial f_{ijk}(0)}{\partial\vartheta} = \frac{\partial g(0,0)}{\partial\vartheta} = \frac{\partial g(0,0)}{\partial\varphi} = 0\,.$$

For every mutually orthogonal unit vectors $u, v, w \in \mathbb{S}_pM$ and $\vartheta \in \mathbb{R}$, we have the formula

$$\mathrm{Sec}\big(u, (\cos\vartheta)v + (\sin\vartheta)w\big) = \cos^2(\vartheta)\,\mathrm{Sec}(u, v) + \sin^2(\vartheta)\,\mathrm{Sec}(u, w) + \sin(2\vartheta)R(u, v, u, w)\,,$$

hence,

$$\frac{\partial}{\partial\vartheta}\Big|_{\vartheta=0}\operatorname{Sec}(u,(\cos\vartheta)v+(\sin\vartheta)w)=2R(u,v,u,w)\,.$$

Therefore,

$$0=\frac{\partial f_{ijk}(0)}{\partial\vartheta}=2R_{ijik}$$

for any $i,j\in\{1,2\}$, $k\in\{3,4\}$ and

$$0=\frac{\partial g(0,\varphi)}{\partial\vartheta}\Big|_{\varphi=0}=2R(e_1,(\cos\varphi)e_3+(\sin\varphi)e_4,e_2,(\cos\varphi)e_3+(\sin\varphi)e_4)\big|_{\varphi=0}=2R_{1323}\,,$$

$$0=\frac{\partial g(\vartheta,0)}{\partial\varphi}\Big|_{\vartheta=0}=2R((\cos\vartheta)e_1+(\sin\vartheta)e_2,e_3,(\cos\vartheta)e_1+(\sin\vartheta)e_2,e_4)\big|_{\vartheta=0}=2R_{1314}\,,$$

proving the lemma. □

We want now to express the right–hand side of equation (3.28) in terms of $|\mathrm{Riem}|^2$, $|\mathrm{Ric}|^2$ and R^2. Computing in the orthonormal frame given by Lemma 3.3.1, by equation (1.19), we have

$$\begin{aligned}\mathrm{R}^2=\Big(2\sum_{1\le s<t\le 4}K_{st}\Big)^2&=4\big(K_{12}+K_{13}+K_{14}+K_{23}+K_{24}+K_{34}\big)^2\\&=4\big(K_{12}^2+K_{13}^2+K_{14}^2+K_{23}^2+K_{24}^2+K_{34}^2\\&\quad+2(K_{12}K_{13}+K_{12}K_{14}+K_{12}K_{23}+K_{12}K_{24}+K_{12}K_{34}\\&\quad+K_{13}K_{14}+K_{13}K_{23}+K_{13}K_{24}+K_{13}K_{34}+K_{14}K_{23}\\&\quad+K_{14}K_{24}+K_{14}K_{34}+K_{23}K_{24}+K_{23}K_{34}+K_{24}K_{34})\big)\end{aligned}\tag{3.29}$$

and

$$\begin{aligned}R_{ii}^2=\Big(\sum_{s=1}^4 R_{isis}\Big)^2&=(K_{ij}+K_{ik}+K_{il})^2\\&=K_{ij}^2+K_{ik}^2+K_{il}^2+2K_{ij}K_{ik}+2K_{ij}K_{il}+2K_{ik}K_{il}\,,\\R_{ij}^2=\Big(\sum_{s=1}^4 R_{isjs}\Big)^2&=(R_{ikjk}+R_{iljl})^2=R_{iljl}^2\,,\end{aligned}$$

for every $i<j$ and $k<l$ all different, as we recall that then $R_{ikjk}=0$ (Remark 3.3.2). Hence,

$$\begin{aligned}|\mathrm{Ric}|^2=\sum_{i,j=1}^4 R_{ij}^2=2\big(&K_{12}^2+K_{13}^2+K_{14}^2+K_{23}^2+K_{24}^2+K_{34}^2\\&+K_{12}K_{13}+K_{12}K_{14}+K_{13}K_{14}+K_{12}K_{23}\\&+K_{12}K_{24}+K_{23}K_{24}+K_{13}K_{23}+K_{13}K_{34}\\&+K_{23}K_{34}+K_{14}K_{24}+K_{14}K_{34}+K_{24}K_{34}\\&+R_{1334}^2+R_{1424}^2+R_{1434}^2+R_{2324}^2+R_{2334}^2+R_{2434}^2\big)\,.\end{aligned}\tag{3.30}$$

We now see that the addends which are products of different sectional curvatures in equations (3.28)–(3.30) can be adequately combined to cancel and we get (ignoring $\mathrm{d}V$ in the expression of $\mathrm{Pf}(\Omega)$, for simplicity)

$$
\begin{aligned}
8\,\mathrm{Pf}(\Omega) + 4|\mathrm{Ric}|^2 - \mathrm{R}^2 = {} & 8\Big(R_{1234}^2 + R_{1324}^2 + R_{1423}^2\Big) \\
& + 8\Big(R_{1334}^2 + R_{1424}^2 + R_{1434}^2 + R_{2324}^2 + R_{2334}^2 + R_{2434}^2\Big) \\
& + 8\Big(K_{12}^2 + K_{13}^2 + K_{14}^2 + K_{23}^2 + K_{24}^2 + K_{34}^2\Big) \\
& - 4\Big(K_{12}^2 + K_{13}^2 + K_{14}^2 + K_{23}^2 + K_{24}^2 + K_{34}^2\Big) \\
& + 8\Big(K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23}\Big) \\
& + 8\Big(K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14} + K_{12}K_{23} \\
& + K_{12}K_{24} + K_{23}K_{24} + K_{13}K_{23} + K_{13}K_{34} \\
& + K_{23}K_{34} + K_{14}K_{24} + K_{14}K_{34} + K_{24}K_{34}\Big) \\
& - 8\Big(K_{12}K_{13} + K_{12}K_{14} + K_{12}K_{23} + K_{12}K_{24} + K_{12}K_{34} \\
& + K_{13}K_{14} + K_{13}K_{23} + K_{13}K_{24} + K_{13}K_{34} + K_{14}K_{23} \\
& + K_{14}K_{24} + K_{14}K_{34} + K_{23}K_{24} + K_{23}K_{34} + K_{24}K_{34}\Big) \\
= {} & 8\Big(R_{1234}^2 + R_{1324}^2 + R_{1423}^2 \\
& + R_{1334}^2 + R_{1424}^2 + R_{1434}^2 + R_{2324}^2 + R_{2334}^2 + R_{2434}^2\Big) \\
& + 4\Big(K_{12}^2 + K_{13}^2 + K_{14}^2 + K_{23}^2 + K_{24}^2 + K_{34}^2\Big) \\
= {} & 8\Big(R_{1234}^2 + R_{1324}^2 + R_{1423}^2 \\
& + R_{1334}^2 + R_{1424}^2 + R_{1434}^2 + R_{2324}^2 + R_{2334}^2 + R_{2434}^2\Big) \\
& + 4\Big(R_{1212}^2 + R_{1313}^2 + R_{1414}^2 + R_{2323}^2 + R_{2424}^2 + R_{3434}^2\Big) \\
= {} & |\mathrm{Riem}|^2\,,
\end{aligned}
\tag{3.31}
$$

writing explicitly the norm of the Riemann tensor in the special orthonormal frame given by Lemma 3.3.1, where several components are zero. Indeed, considering its symmetries, except the Bianchi identity and taking into account the 6 zero–conditions given by such lemma, the Riemann tensor is determined by the 15 components appearing in the second–last line of the above computation (whose squares must be added with the appropriate multiplicities in order to give $|\mathrm{Riem}|^2$).
We then get

$$\mathrm{Pf}(\Omega) = \frac{1}{8}\Big(|\mathrm{Riem}|^2 - 4|\mathrm{Ric}|^2 + \mathrm{R}^2\Big)\,dV_M$$

and equation (3.22) becomes the following Chern–Gauß–Bonnet formula, when $n = 4$,

$$\frac{1}{32\pi^2}\int_M \Big(|\mathrm{Riem}|^2 - 4|\mathrm{Ric}|^2 + \mathrm{R}^2\Big)\,dV_M = \chi(M)\,.$$

Recalling equalities (1.26) and (1.27), that, when $n = 4$, become

$$|\mathrm{Riem}|^2 = 2|\mathrm{Ric}|^2 - \mathrm{R}^2/3 + |\mathrm{Weyl}|^2 = 2|\overset{\circ}{\mathrm{Ric}}|^2 + \mathrm{R}^2/6 + |\mathrm{Weyl}|^2$$
$$|\overset{\circ}{\mathrm{Ric}}|^2 = |\mathrm{Ric}|^2 - \mathrm{R}^2/4\,,$$

we have the following alternative expressions:

$$\begin{aligned}\chi(M) &= \frac{1}{32\pi^2}\int_M \Big(|\mathrm{Riem}|^2 - 4|\mathrm{Ric}|^2 + \mathrm{R}^2\Big)\,\mathrm{d}V_M \\ &= \frac{1}{32\pi^2}\int_M \Big(|\mathrm{Riem}|^2 - 4|\mathring{\mathrm{Ric}}|^2\Big)\,\mathrm{d}V_M \\ &= \frac{1}{96\pi^2}\int_M \Big(3|\mathrm{Weyl}|^2 - 6|\mathrm{Ric}|^2 + 2\mathrm{R}^2\Big)\,\mathrm{d}V_M \\ &= \frac{1}{192\pi^2}\int_M \Big(6|\mathrm{Weyl}|^2 - 12|\mathring{\mathrm{Ric}}|^2 + \mathrm{R}^2\Big)\,\mathrm{d}V_M \\ &= \frac{1}{32\pi^2}\int_M \Big(|\mathrm{Weyl}|^2 + 16\sigma_2(S)\Big)\,\mathrm{d}V_M\,, \end{aligned} \tag{3.32}$$

where in the last line $S = \frac{1}{12}(6\,\mathrm{Ric} - \mathrm{R}g)$ is the Schouten tensor and $\sigma_2(S)$ is the second elementary symmetric polynomial in its eigenvalues, i.e.,

$$\begin{aligned}\sigma_2(S) &= \frac{1}{2}\big((\mathrm{tr}\,S)^2 - |S|^2\big) \\ &= \frac{1}{2}\Big(\frac{1}{144}(6\mathrm{R} - 4\mathrm{R})^2 - \frac{1}{144}(36|\mathrm{Ric}|^2 - 12\mathrm{R}^2 + 4\mathrm{R}^2)\Big) \\ &= \frac{1}{288}(4\mathrm{R}^2 - 36|\mathrm{Ric}|^2 + 8\mathrm{R}^2) \\ &= \frac{1}{24}(\mathrm{R}^2 - 3|\mathrm{Ric}|^2) = \frac{1}{96}(\mathrm{R}^2 - 12|\mathring{\mathrm{Ric}}|^2)\,.\end{aligned}$$

As a consequence of the previous formulae, since the *Weyl functional* (i.e., the integral of the square norm of the Weyl tensor) is conformally invariant in dimension four (by direct computation – we will see that in equation (4.14)) and the Euler–Poincaré characteristic is a topological invariant, we get the following result.

Corollary 3.3.3. *For a compact oriented 4–dimensional Riemannian manifold, the integral*

$$\int_M \sigma_2(S)\,\mathrm{d}V_M = \frac{1}{24}\int_M \Big(\mathrm{R}^2 - 3|\mathrm{Ric}|^2\Big)\,\mathrm{d}V_M = \frac{1}{96}\int_M \Big(\mathrm{R}^2 - 12|\mathring{\mathrm{Ric}}|^2\Big)\,\mathrm{d}V_M$$

is conformally invariant. In particular, a compact oriented 4–dimensional conformally flat manifold must have

$$3\int_M |\mathrm{Ric}|^2\,\mathrm{d}V_M = \int_M \mathrm{R}^2\,\mathrm{d}V_M = 12\int_M |\mathring{\mathrm{Ric}}|^2\,\mathrm{d}V_M\,.$$

In the next chapter we will be interested in results connecting the curvature and the topology of a manifold. We start by noticing that it follows immediately by the Chern–Gauß–Bonnet Theorem 3.2.5, in any dimension, that if (M, g) is flat, then its Euler–Poincaré characteristic must be zero. The following examples show that, in general, the converse does not hold.

Example 3.3.4. Consider the 4–dimensional manifold $M = \mathbb{S}^1 \times \mathbb{S}^3$. As $\mathbb{S}^1$, $\mathbb{S}^3$ are connected, compact and oriented, the same holds for M. We now recall that the Euler–Poincaré characteristic of the product is the product of the Euler–Poincaré characteristics of the factors (Remark 1.1.20), thus $\chi(M) = \chi(\mathbb{S}^1)\chi(\mathbb{S}^3) = 0$, as both $\mathbb{S}^1$ and $\mathbb{S}^3$ are odd-dimensional, then we notice that no metric can make $\mathbb{S}^1 \times \mathbb{S}^3$ flat, as then its universal Riemann covering would be $\mathbb{R}^4$ (Theorem 1.2.19) and that is not the case as $\mathbb{R}^4$ is not homeomorphic to $\mathbb{R} \times \mathbb{S}^3$.
The same considerations apply to $N = \mathbb{T}^2 \times \mathbb{S}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^2$ with $\chi(N) = 0$ and universal covering $\mathbb{R}^2 \times \mathbb{S}^2$.

Moreover, another result and a weaker converse can be obtained by means of expression (3.28), in dimension four.

Corollary 3.3.5. *A compact oriented* 4*–dimensional Riemannian manifold whose nonzero sectional curvatures all share the same sign, has nonnegative Euler–Poincaré characteristic. If in addition, no sectional curvature vanishes, then the Euler–Poincaré characteristic is positive.*

Proof. The first assumption clearly makes all terms in equation (3.28) nonnegative; the second assumption makes at least three of them positive. The conclusion follows from the Chern–Gauß–Bonnet Theorem 3.2.5. □

We notice that a similar result to Corollary 3.3.5 also holds trivially for 2–dimensional manifolds by the classical Gauß–Bonnet theorem (3.26), as the only sectional curvature is (half of) the scalar curvature. More precisely, the Euler–Poincaré characteristic is positive, negative or zero if and only if the scalar curvature R is positive, negative or zero, respectively. The general statement is known in literature as the *Hopf conjecture*.

Conjecture 3.3.6 (Hopf conjecture [7], [86, Problems 8 and 10])**.** *A compact even–dimensional Riemannian manifold with positive (respectively nonnegative) sectional curvature has positive (respectively nonnegative) Euler–Poincaré characteristic. A compact* $2p$*–dimensional Riemannian manifold with negative (respectively nonpositive) sectional curvature has Euler-Poincaré characteristic of sign* $(-1)^p$ *(respectively* $(-1)^p$ *or zero).*

To our knowledge, the conjecture is still open in dimension $n \geq 6$.

Chapter 4

Einstein manifolds

We are going to analyse *Einstein manifolds*, that is Riemannian manifolds such that the Ricci tensor is proportional to the metric, with special attention to the dimension four. Most of the material of this chapter is taken from [5, 6, 9].
Satisfying $\mathrm{Ric} = \lambda g$ for some constant $\lambda \in \mathbb{R}$, Einstein manifolds "stay in the middle" between *constant curvature manifolds* (with $\mathrm{Riem} = \frac{\lambda}{2(n-1)} g \owedge g$), which are completely classified and *constant scalar curvature manifolds* (with $\mathrm{R} = \lambda n$), hence they are neither "too" nor "too little" rigid.
We start by showing the computation of the first variation of the *Einstein–Hilbert action*, whose nullity characterises Einstein manifolds and we also discuss the relations with the general relativity, as the Euler–Lagrange equation of the Einstein–Hilbert action gives the *Einstein's field equation.*
Then, we deal with "improving" the orthogonal decomposition of the Weyl tensor in dimension $n = 4$. In dimension $n = 4$ the space of the Weyl tensors can be further decomposed by considering its irreducible orthogonal components under the action of $SO(4)$ and, once applied to a Riemannian manifold, this refined decomposition yields the so–called *self–dual* and *anti–self–dual* components $W^{\pm}$ of its Weyl tensor. One can then define *half–conformally flat manifolds*, i.e., manifolds for which either W^+ or W^- vanishes. As a consequence, the Chern–Gauß–Bonnet formula in dimension 4 can be rewritten as

$$\chi(M) = \frac{1}{192\pi^2} \int_M \Big(6|W^+|^2 + 6|W^-|^2 - 12|\mathring{\mathrm{Ric}}|^2 + \mathrm{R}^2 \Big) \, \mathrm{d}V_M \, .$$

In Section 4.3, we introduce the *signature* $\tau(M)$, which is another topological invariant, of a four–manifold M and present (without proof, for which we refer the reader to [28]) the *Hirzebruch theorem*, showing the equality

$$\tau(M) = \frac{1}{48\pi^2} \int_M \Big(|W^+|^2 - |W^-|^2 \Big) \, \mathrm{d}V_M \, .$$

We then proceed by studying the *Weyl functional*,

$$\mathfrak{W}(g) = \int_M |\mathrm{Weyl}_g|^{n/2} \, \mathrm{d}V_g \, ,$$

which is quadratic in dimension four. We compute its first variation and we show that *conformal Einstein metrics* (i.e., metrics having an Einstein metric in their conformal class) and half–conformally flat metrics are critical metrics, in dimension four.
In the last section, we finally discuss four–dimensional Einstein manifolds. By combining

the Hirzebruch and Chern–Gauß–Bonnet formulae, we will obtain the so–called *Hitchin–Thorpe inequality*

$$\chi(M) \geq \frac{3}{2}|\tau(M)|$$

which is a necessary (only) condition for a compact oriented 4–dimensional manifold to be Einstein. As in general no sufficient conditions are known in literature, we conclude the chapter (and the thesis) by providing some examples of manifolds which *do not admit* any Einstein metric. Being the converse problem very difficult, the readers of the book of A. L. Besse [6] "*[...] are offered a meal in a starred restaurant in exchange for a new example [of an Einstein manifold]*".

4.1 The Einstein–Hilbert action

Let M be a compact oriented n–dimensional differential manifold and g any Riemannian metric on M. We define the *Einstein–Hilbert action* as the *total curvature* of (M, g), i.e.

$$\mathfrak{S}(g) = \int_M \mathrm{R}_g \, \mathrm{d}V_g \, ,$$

where R_g and $\mathrm{d}V_g$ are, respectively, the scalar curvature and the volume form of (M, g). As for $n = 2$ the classical Gauß–Bonnet formula (3.26) gives

$$\mathfrak{S}(g) = \int_M \mathrm{R}_g \, \mathrm{d}V_g = 4\pi\chi(M) \, ,$$

the functional is constant on a fixed 2–dimensional differential manifold, then we will always assume that the dimension n of M is at least three.
We define the *gradient* of $\mathfrak{S}$ at g as the $(0, 2)$–tensor $\nabla\mathfrak{S}(g)$, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathfrak{S}(g + th) = \int_M g\big(\nabla\mathfrak{S}(g), h\big) \, \mathrm{d}V_g$$

for every symmetric bilinear form h (this equation express the *first variation* of the functional $\mathfrak{S}$ at g). In order to compute it, we need the following two equalities,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}V(g + th) = \frac{1}{2} \operatorname{tr} h \, \mathrm{d}V_g \, , \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{R}(g + th) = -\Delta_g \operatorname{tr} h + \nabla_g^{ij} h_{ij} - \mathrm{Ric}_g^{ij} \, h_{ij} \, . \tag{4.2}$$

We show in detail how to get the first formula, while for the second one (for which one has to compute the variations of the Christoffel symbols, the Riemann tensor and the Ricci tensor), we refer [9, Section 2.1]).
In a coordinate chart, we have

$$\begin{aligned}
\mathrm{d}V(g + th) &= \sqrt{\det(g + th)} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sqrt{\det(\mathrm{Id} + tg^{-1}h)} \sqrt{\det g} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= \sqrt{\det(\mathrm{Id} + tg^{-1}h)} \, \mathrm{d}V_g \, ,
\end{aligned}$$

therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{d}V(g+th) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \sqrt{\det(\mathrm{Id}+tg^{-1}h)}\,\mathrm{d}V_g \\ &= \left(\frac{\mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}tg^{-1}h\right)}{2\sqrt{\det(\mathrm{Id}+tg^{-1}h)}}\right)\bigg|_{t=0}\mathrm{d}V_g \\ &= \frac{1}{2}\,\mathrm{tr}\,h\,\mathrm{d}V_g\,,\end{aligned}$$

where we intended $g^{-1}h$ as a matrix multiplication, hence

$$\mathrm{tr}\,(g^{-1}h)^i_j = \mathrm{tr}\,g^{ik}h_{kj} = g^{ik}h_{ki} = \mathrm{tr}\,h$$

and we used the well–known *Jacobi identity*[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\big(\mathrm{Id}+A(t)\big) = \mathrm{tr}\,A'(t)\,.$$

By means of equations (4.1) and (4.2), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathfrak{S}(g+th) &= \int_M \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathrm{R}(g+th)\,\mathrm{d}V_g + \int \mathrm{R}_g\,\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathrm{d}V(g+th) \\ &= \int_M \Big(-\Delta_g\,\mathrm{tr}\,h + \nabla^{ij}_g h_{ij} - \mathrm{Ric}^{ij}_g\,h_{ij} + \frac{\mathrm{R}_g}{2}\,\mathrm{tr}\,h\Big)\,\mathrm{d}V_g \\ &= -\int_M \Big(\mathrm{div}_g\,\nabla_g\,\mathrm{tr}\,h - \mathrm{div}_g\,\mathrm{div}_g\,h + g\Big(\mathrm{Ric}_g - \frac{\mathrm{R}_g}{2}g, h\Big)\Big)\,\mathrm{d}V_g \\ &= -\int_M g\Big(\mathrm{Ric}_g - \frac{\mathrm{R}_g}{2}g, h\Big)\,\mathrm{d}V_g \qquad (4.3)\end{aligned}$$

where we used the divergence theorem (1.2.13). Hence, $\nabla\mathfrak{S}(g)$ is the opposite of

$$E_g \coloneqq \mathrm{Ric}_g - \mathrm{R}_g g/2$$

which is called (in every dimension) the *Einstein tensor.* We remark that the Einstein tensor is a divergence–free symmetric $(2,0)$–tensor due to Schur's lemma (2.3).

The normalised action. Of equal interest is the *normalised Einstein–Hilbert action*

$$\overline{\mathfrak{S}}(g) = \mathrm{Vol}_g^{-\frac{n-2}{n}}\,\mathfrak{S}(g)$$

which is scaling invariant.
Indeed, if $\widetilde{u} = \lambda g$ for $\lambda > 0$, then

$$\sqrt{\det\widetilde{g}} = \lambda^{n/2}\sqrt{\det g}\,, \qquad \mathrm{d}V(\widetilde{g}) = \lambda^{n/2}\,\mathrm{d}V_g\,, \qquad \mathrm{Vol}(\widetilde{g}) = \lambda^{n/2}\,\mathrm{Vol}_g\,,$$

and $\mathrm{R}(\widetilde{g}) = \lambda^{-1}\mathrm{R}_g$ (see Theorem 2.1.4), hence

$$\overline{\mathfrak{S}}(\widetilde{g}) = \mathrm{Vol}(\widetilde{g})^{-\frac{n-2}{n}}\int_M \mathrm{R}(\widetilde{g})\,\mathrm{d}V(\widetilde{g}) = \lambda^{-\frac{n}{2}+1}\,\mathrm{Vol}_g^{-\frac{n-2}{n}}\int_M \lambda^{-1}\mathrm{R}_g\,\lambda^{n/2}\,\mathrm{d}V_g = \overline{\mathfrak{S}}(g)\,.$$

[1] After the German mathematician Carl Gustav Jacob Jacobi (1804–1851) [76].

Using equation (4.3), the first variation of the normalised action is given by

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \overline{\mathfrak{S}}(g+th) \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Big(\mathrm{Vol}(g+th)^{-\frac{n-2}{n}} \int_M \mathrm{R}(g+th)\,\mathrm{d}V(g+th)\Big) \\
&\quad= -\frac{n-2}{n}\,\mathrm{Vol}_g^{-\frac{n-2}{n}-1} \int_M \frac{1}{2}\operatorname{tr} h\,\mathrm{d}V_g \int_M \mathrm{R}_g\,\mathrm{d}V_g - \mathrm{Vol}_g^{-\frac{n-2}{n}} \int_M g(E_g,h)\,\mathrm{d}V_g \\
&\quad= -\,\mathrm{Vol}_g^{-\frac{n-2}{n}} \int_M g\Big(E_g + \frac{n-2}{2n}\overline{\mathrm{R}}_g g, h\Big)\,\mathrm{d}V_g\,,
\end{aligned}
$$

and

$$
\nabla\overline{\mathfrak{S}}(g) = -\,\mathrm{Vol}_g^{-\frac{n-2}{n}}\Big(E_g + \frac{n-2}{2n}\overline{\mathrm{R}}_g g\Big)\,,
$$

where $\overline{\mathrm{R}}_g = \mathrm{Vol}_g^{-1}\int_M \mathrm{R}_g\,\mathrm{d}V_g$ is the average of the scalar curvature on (M,g).

The Einstein's field equation. The action considered in general relativity, on a four–manifold M and defined on Lorentzian metrics g, is of the form

$$
\mathfrak{S}_*(g) = \int_M \Big(\frac{1}{2\kappa}(\mathrm{R}_g - 2\Lambda) + \mathcal{L}_{\mathrm{M}}(g)\Big)\,\mathrm{d}V_g\,,
$$

where Λ is the so–called *cosmological constant*, κ is the *Einstein's gravitational constant* and $\mathcal{L}_{\mathrm{M}}(g)$ is a Lagrangian describing a possibly present matter field.
In this case the first variation is given by

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathfrak{S}_*(g+th) \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_M \Big(\frac{1}{2\kappa}(\mathrm{R}(g+th) - 2\Lambda) + \mathcal{L}_{\mathrm{M}}(g+th)\Big)\,\mathrm{d}V(g+th) \\
&\quad= -\frac{1}{2\kappa}\int_M g(E_g,h)\,\mathrm{d}V_g - \frac{\Lambda}{2\kappa}\int_M \operatorname{tr} h\,\mathrm{d}V_g \\
&\qquad+ \int_M \Big(\frac{\mathrm{d}\mathcal{L}_{\mathrm{M}}(g+th)}{\mathrm{d}t}\Big|_{t=0} + \frac{\operatorname{tr} h}{2}\mathcal{L}_{\mathrm{M}}(g)\Big)\,\mathrm{d}V_g \\
&\quad= -\frac{1}{2\kappa}\int_M g\Big(E_g + \Lambda g - 2\kappa\nabla\mathcal{L}_{\mathrm{M}}(g) - \kappa\mathcal{L}_{\mathrm{M}}(g)g, h\Big)\,\mathrm{d}V_g \\
&\quad= -\frac{1}{2\kappa}\int_M g(E_g + \Lambda g - \kappa T_g, h)\,\mathrm{d}V_g\,,
\end{aligned}
$$

hence,

$$
\nabla\mathfrak{S}_*(g) = -\frac{1}{2\kappa}(E_g + \Lambda g - \kappa T_g)\,,
$$

where

$$
T_g \coloneqq 2\nabla\mathcal{L}_{\mathrm{M}}(g) + \mathcal{L}_{\mathrm{M}}(g)g
$$

is called the *stress–momentum tensor*.

The equations we obtain when the gradient of these functionals vanishes (i.e., the *Euler–Lagrange equations* of the functionals) are

(i) $\nabla\mathfrak{S}(g) = 0 = E_g \implies \mathrm{Ric}_g = \mathrm{R}_g g/2\,,$

(ii) $\nabla\overline{\mathfrak{S}}(g) = 0 = E_g + \frac{n-2}{2n}\overline{\mathrm{R}}_g g \implies \mathrm{Ric}_g = (\mathrm{R}_g/2 - \frac{n-2}{2n}\overline{\mathrm{R}}_g)g\,,$

(iii) $\nabla \mathfrak{S}_*(g) = 0 = E_g + \Lambda g - \kappa T_g \implies \mathrm{Ric}_g = (\mathrm{R}_g/2 - \Lambda)g + \kappa T_g\,,$

(the last equation is the so–called *Einstein's field equation*).
In the vacuum (that is, in the absence of mass) one has $T_g = 0$ and the third equation becomes $\mathrm{Ric}_g = (\mathrm{R}_g/2 - \Lambda)g$, then all three equations can be written as $\mathrm{Ric}_g = \lambda g$, for some $\lambda \in C^\infty(M)$. After contracting with the (inverse of the) metric, one sees that such a λ has necessarily to be equal to R_g/n and the Ricci trace–free tensor $\mathring{\mathrm{Ric}}_g = \mathrm{Ric}_g - \mathrm{R}_g g/n$ must vanish identically for all such critical metrics, which are then all Einstein metrics on M (notice that the first equation, since we assumed $n \geq 3$, implies that g is a *Ricci–flat metric*, i.e., $\mathrm{Ric}_g = 0$).

Before proceeding further we recall some facts about Einstein manifolds, always assuming to be in dimension $n \geq 3$ and that all the manifolds are connected.

Definition 4.1.1. An n–dimensional Riemannian manifold (M, g) is said to be Einstein if the tensor $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \mathrm{R}g/n$ is identically zero.

Proposition 4.1.2. *The manifold (M, g) is Einstein if and only if $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$, in which case $\mathrm{R} = \lambda n$, hence constant. Such λ is called the* Einstein constant *of the manifold.*

Proof. The fact that $\mathrm{R} = \lambda n$, immediately follows by contracting equation $\mathrm{Ric} = \lambda g$ with the metric g. Applying Schur's lemma (2.3) to the equation $\mathrm{Ric} = \mathrm{R}g/n$, we obtain

$$\mathrm{dR} = 2\,\mathrm{div}\,\mathrm{Ric} = \frac{2}{n}\,\mathrm{div}(\mathrm{R}g) = \frac{2}{n}\mathrm{dR}$$

and, as $n \geq 3$, this implies $\mathrm{dR} = 0$, hence R constant. Then $\lambda = \mathrm{R}/n$ is also a constant. ☐

Proposition 4.1.3. *The manifold (M, g) has constant curvature if and only if $\mathring{\mathrm{Ric}} = W = 0$. In particular, 3–dimensional Einstein manifolds have constant curvature and if $n \geq 4$, then (M, g) has constant curvature if and only if it is Einstein and LCF.*

Proof. If $\mathring{\mathrm{Ric}} = W = 0$ then the decomposition formula (1.25) reduces to

$$\mathrm{Riem} = \frac{\mathrm{R}}{2n(n-1)} g \owedge g\,,$$

moreover, being (M, g) is Einstein, the scalar curvature $\mathrm{R} = \lambda n$ is constant, hence $\mathrm{Riem} = \frac{K}{2} g \owedge g$ with $K = \lambda/(n-1) \in \mathbb{R}$. The converse statement is trivial. ☐

As three–dimensional Einstein manifolds coincide with the constant curvature ones, hence they are "classified" by Theorem 1.2.19, the first interesting (dimensional) case is when $n = 4$, which will be the subject of the next sections.

4.2 Algebraic curvature tensors in dimension four

In order to study four–dimensional (Einstein) manifolds, we discuss the special decomposition of the algebraic curvature tensors which holds only in this dimension. More precisely, in the case of a 4–dimensional vector space, the orthogonal decomposition of the space of the algebraic curvature tensors given by equation (1.24), which is obtained considering the irreducible components under the action of $O(4)$,

$$P = \frac{\mathrm{P}}{24} g \owedge g + \frac{1}{2}\mathring{\mathrm{Pic}} \owedge g + W^P\,, \tag{4.4}$$

can be refined, as the space of Weyl tensors can be further decomposed into its irreducible components under the action of $SO(4)$,

$$P = \frac{\mathrm{P}}{24} g \owedge g + \frac{1}{2} \mathring{\mathrm{Pic}} \owedge g + W^+ + W^-,$$

(if $n \neq 4$, the action of $SO(n)$ does not provide any new decomposition). For more information, we refer to [6, Theorem 1.114].
For a Riemannian manifold, such a decomposition yields the so–called *self–dual* and *anti–self–dual* components $W^\pm$of the Weyl tensor and one can define the *half–conformally flat manifolds* as the ones such that either W^+ or W^- vanishes. In Section 4.4, we will show that these half–conformally flat metrics, like the Einstein metrics (or, more in general, any metric conformal to an Einstein metric), are critical to the *Weyl functional*, in dimension 4. We refer to [9] for further reading.

We let (V, g) be a 4–dimensional vector space with g scalar product and recall the notation of Section 1.3 for algebraic curvature tensor $P \in C^4(V)$, namely, the use of Pic, P, $\mathring{\mathrm{Pic}}$ and Pec to denote, respectively, the $(1, 3)$–trace of P, the complete trace of P, the trace–free component of Pic and the sectional curvature of P.
We start by showing some relations for the sectional curvatures of a tensor P such that $\mathring{\mathrm{Pic}} = 0$ (which corresponds to the condition of being an Einstein manifold).

Proposition 4.2.1. *If (V, g) is like above and P an algebraic curvature tensor such that $\mathring{\mathrm{Pic}} = 0$, letting $K_{ij} = \mathrm{Pec}(e_i, e_j)$ be the sectional curvatures with respect to an orthonormal basis $\{e_i\}_{i=1}^n$, then $K_{ij} = K_{kl}$ whenever i, j, k, l are all different.*

Proof. Since $\mathrm{Pic}_{ii} = \mathrm{P}/4$ for every $i \in \{1, 2, 3, 4\}$, it follows

$$\mathrm{P}/4 = K_{12} + K_{13} + K_{14} = K_{12} + K_{23} + K_{24} = K_{13} + K_{23} + K_{34} = K_{14} + K_{24} + K_{34}\,.$$

Then,

$$(K_{12} + K_{13} + K_{14}) - (K_{12} + K_{23} + K_{24}) = (K_{13} + K_{23} + K_{34}) - (K_{14} + K_{24} + K_{34})$$

hence, $2K_{14} = 2K_{23}$. Arguing similarly, we get $K_{12} = K_{34}$, $K_{13} = K_{24}$ and $K_{14} = K_{23}$. $\square$

We recall (Remark 1.1.8) that in dimension 4 the Hodge operator on $\Lambda^2(V)$ is idempotent, i.e., $\star^2 = 1$, hence, letting $v \in \Lambda^2(V)$ be an eigenvector of $\star$, we have

$$v = \star^2 v = \lambda^2 v$$

and the associated eigenvalue λ must satisfy $\lambda = \pm 1$. Moreover, for every 2–vector v we have the decomposition

$$v = v^+ + v^- = \frac{v + \star v}{2} + \frac{v - \star v}{2} \tag{4.5}$$

with

$$\star v^+ = \star \frac{v + \star v}{2} = \frac{v + \star v}{2} = v^+ \qquad \text{and} \qquad \star\, v^- = \star \frac{v - \star v}{2} = -\frac{v - \star v}{2} = -v^-\,,$$

thus, $\Lambda^2(V)$ admits the decomposition $\Lambda^2(V) = \Lambda^2_+(V) \oplus \Lambda^2_-(V)$ into the eigenspaces $\Lambda^2_\pm(V)$ of $\star$ relative to ± 1, respectively.

Choosing an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of V, for every $v = \frac{1}{2}v^{ij}\, e_i \wedge e_j$ with $\star v = \pm v$, there holds

$$\pm\tfrac{1}{2}v^{kl}\, e_k \wedge e_l = \pm v = \star v = \star\big(\tfrac{1}{2}v^{ij}\, e_i \wedge e_j\big) = \tfrac{1}{4}v_{ij}\varepsilon^{ijkl}\, e_k \wedge e_l\,,$$

thus,

$$v^{kl} = \pm\tfrac{1}{2}v_{ij}\varepsilon^{ijkl}\,,$$

where ε^{ijkl} is the Levi–Civita symbol, i.e., $\varepsilon^{ijkl} = \operatorname{sgn}(ijkl)$ if $(ijkl)$ is a permutation of (1234) and zero otherwise. Explicitly,

$$v^{34} = \pm v^{12}\,, \qquad v^{24} = \mp v^{13}\,, \qquad v^{23} = \pm v^{14}\,.$$

Hence, both $\Lambda^2_\pm(V)$ are 3–dimensional (as $\dim \Lambda^2(V) = 6$), with respective eigenbases $\{\chi_1^\pm, \chi_2^\pm, \chi_3^\pm\}$, where

$$\chi_1^\pm = e_1 \wedge e_2 \pm e_3 \wedge e_4\,, \qquad \chi_2^\pm = e_1 \wedge e_3 \mp e_2 \wedge e_4\,, \qquad \chi_3^\pm = e_1 \wedge e_4 \pm e_2 \wedge e_3\,,$$

which are also pairwise orthogonal, so the decomposition is orthogonal

$$\Lambda^2(V) = \Lambda^2_+(V) \oplus^\perp \Lambda^2_-(V)\,.$$

If $h \in S^2(V^*)$ is a symmetric bilinear form with associated operator $\mathscr{H}$, defined by

$$(h \owedge g)(x, y, z, w) = \tfrac{1}{2}g(\mathscr{H}(x \wedge y), z \wedge w)\,,$$

then to get the components of $\mathscr{H}(\chi_i^\pm)$ with respect to the basis $\chi_i^\pm$, we compute for $i, j, k, l \in \{1, 2, 3, 4\}$ all distinct,

$$\begin{aligned}
&\tfrac{1}{2}g\big(\mathscr{H}(e_i \wedge e_j), e_k \wedge e_l\big) = (h \owedge g)_{ijkl} = h_{ik}g_{jl} + h_{jl}g_{ik} - h_{il}g_{jk} - h_{jk}g_{il} = 0\,,\\
&\tfrac{1}{2}g\big(\mathscr{H}(e_i \wedge e_j), e_i \wedge e_k\big) = (h \owedge g)_{ijik} = h_{jk}\,,\\
&\tfrac{1}{2}g\big(\mathscr{H}(e_i \wedge e_j), e_i \wedge e_j\big) = (h \owedge g)_{ijij} = h_{ii} + h_{jj}\,,
\end{aligned}$$

hence,

$$\begin{aligned}
\tfrac{1}{2}g\big(\mathscr{H}(e_i \wedge e_j + s_1\, e_k \wedge e_l), e_i \wedge e_k + s_2\, e_j \wedge e_l\big) &= h_{jk} - s_2 h_{il} - s_1 h_{il} + s_1 s_2 h_{jk}\\
&= (1 + s_1 s_2)h_{jk} - (s_1 + s_2)h_{il}\,.
\end{aligned}$$

This implies that for any two different indices $i, j \in \{1, 2, 3, 4\}$

$$\begin{aligned}
&\tfrac{1}{2}g\big(\mathscr{H}(\chi_i^\pm), \chi_j^\pm\big) = 0\,,\\
&\tfrac{1}{2}g\big(\mathscr{H}(\chi_i^\pm), \chi_i^\pm\big) = h_{11} + h_{22} + h_{33} + h_{44} = \operatorname{tr} h\,.
\end{aligned}$$

As a consequence, if $\operatorname{tr} h = 0$, the operator $\mathscr{H}$ "swaps" the two eigenspaces $\Lambda^2_\pm(V)$.
We now consider an algebraic curvature tensor P with associated operator $\mathscr{P}$ and compute

$$\begin{aligned}
&\tfrac{1}{2}g\big(\mathscr{P}(\chi_1^\pm), \chi_2^\mp\big) = P_{1213} \pm P_{1224} \pm P_{3413} + P_{3424} = \mathrm{Pic}_{23} \pm \mathrm{Pic}_{14}\,,\\
&\tfrac{1}{2}g\big(\mathscr{P}(\chi_1^\pm), \chi_3^\mp\big) = P_{1214} \mp P_{1223} \pm P_{3414} - P_{3423} = \mathrm{Pic}_{24} \pm \mathrm{Pic}_{13}\,,\\
&\tfrac{1}{2}g\big(\mathscr{P}(\chi_2^\pm), \chi_3^\mp\big) = P_{1314} \mp P_{1323} \mp P_{2414} + P_{2423} = \mathrm{Pic}_{34} \mp \mathrm{Pic}_{12}\,,\\
&\tfrac{1}{2}g\big(\mathscr{P}(\chi_1^\pm), \chi_1^\mp\big) = P_{1212} \mp P_{1234} \pm P_{3412} - P_{3434} = K_{12} - K_{34}\,,\\
&\tfrac{1}{2}g\big(\mathscr{P}(\chi_2^\pm), \chi_2^\mp\big) = P_{1313} \pm P_{1324} \mp P_{2413} - P_{2424} = K_{13} - K_{24}\,,\\
&\tfrac{1}{2}g\big(\mathscr{P}(\chi_3^\pm), \chi_3^\mp\big) = P_{1414} \mp P_{1423} \pm P_{2314} - P_{2323} = K_{14} - K_{23}\,,
\end{aligned}$$

where $K_{ij} = \mathrm{Pec}_{ij}$. Then, by Proposition 4.2.1, if $\overset{\circ}{\mathrm{Pic}} = 0$ all previous computations gives zero as a result, hence P preserves the eigenspaces $\Lambda^2_\pm(V)$.
In general, by the decomposition (4.4)

$$P = \frac{\mathrm{P}}{12} \cdot \frac{1}{2} g \owedge g + \frac{1}{2} \overset{\circ}{\mathrm{Pic}} \owedge g + W^P ,$$

the operator associated to $g \owedge g/2$ is the identity, hence it preserves the eigenspaces $\Lambda^2_\pm(V)$ and, for the previous observations, the same holds for the one associated to W^P, being trace–free, while the one associated to $\overset{\circ}{\mathrm{Pic}} \owedge g$ swaps the two eigenspaces. This implies that W^P decomposes into two operators

$$W^+ := W^P|_{\Lambda^2_+(V)} \colon \Lambda^2_+(V) \to \Lambda^2_+(V) \qquad \text{and} \qquad W^- := W^P|_{\Lambda^2_-(V)} \colon \Lambda^2_-(V) \to \Lambda^2_-(V) ,$$

refining the decomposition in the special case $n = 4$,

$$P = \frac{\mathrm{P}}{12} \cdot \frac{1}{2} g \owedge g + \frac{1}{2} \overset{\circ}{\mathrm{Pic}} \owedge g + W^+ + W^-$$

still orthogonal. In particular,

$$|W|^2 = |W^+|^2 + |W^-|^2 .$$

In matrix form with respect to the basis $\{\chi_1^\pm, \chi_2^\pm, \chi_3^\pm\}$, the operator $\mathscr{P}$ can then be written as

$$\mathscr{P} = \left(\begin{array}{c|c} \mathscr{W}^+ + \mathrm{PI}_3/12 & \mathscr{C}/2 \\ \hline \mathscr{C}/2 & \mathscr{W}^- + \mathrm{PI}_3/12 \end{array} \right)$$

with $\mathscr{W}^\pm$ and $\mathscr{C}$ the operators associated to $W^\pm$ and $\overset{\circ}{\mathrm{Pic}} \owedge g$, respectively and I_3 is the 3×3 identity matrix.

Remark 4.2.2. By the symmetries of the Weyl tensor and the trace–free component of Pic, each of the four blocks composing $\mathscr{P} - \mathrm{PI}_6/12$ is trace–free, i.e., $\operatorname{tr} \mathscr{W}^\pm = \operatorname{tr} \mathscr{C} = 0$.

Remark 4.2.3. An easy check shows that

$$\mathscr{P}(v)^{kl} = \frac{1}{2} v^{ij} P_{ij}{}^{kl} ,$$

for every 2–vector $v = \frac{1}{2} v^{ij} e_i \wedge e_j$ and tensor $P \in C^4(V)$. Hence, there holds

$$|\mathscr{P}|^2_{\Lambda^2(V)} = \frac{1}{4} |P|^2 .$$

4.3 The Hirzebruch theorem

In the same spirit of the Chern–Gauß–Bonnet theorem, the Hirzebruch theorem builds another bridge between the geometry of a manifold and its underlying topology. Here the topological invariant is the so–called *signature* of a manifold, which we are now going to define. Most of the material in this section is taken from [6, 28], to which we refer for more details.

Let M be a compact oriented n–dimensional differential manifold, with n even and consider the bilinear form

$$([\eta], [\zeta]) \in H^{n/2}(M) \times H^{n/2}(M) \mapsto \langle [\eta] \smile [\zeta], [M] \rangle := \int_M \eta \wedge \zeta \in \mathbb{R}\,. \tag{4.6}$$

This form is symmetric if and only if n is a multiple of 4 and is in general nondegenerate, as if η is such that $\int_M \eta \wedge \zeta = 0$ for all $\zeta \in \Omega^{n/2}(M)$, then also $0 = \int_M \eta \wedge \star\eta = \int_M |\eta|^2 \, \mathrm{d}V_M$, hence $\eta = 0$.

Definition 4.3.1. Let M be a compact oriented $4k$–dimensional differential manifold and $\beta^+_{2k}(M)$, $\beta^-_{2k}(M)$, respectively, the number of positive and negative eigenvalues of the form (4.6) on $H^{2k}(M)$. Then, the *signature* of M is defined as

$$\tau(M) = \beta^+_{2k}(M) - \beta^-_{2k}(M)\,.$$

As the form is nondegenerate and being $\dim H^{2k}(M) = \beta_{2k}(M)$, we have the relation

$$\beta_{2k}(M) = \beta^+_{2k}(M) + \beta^-_{2k}(M)\,,$$

then (taking into account Poincaré duality, see Remark 1.1.20) we can rewrite the expression of the Euler–Poincaré characteristic in terms of the Betti numbers as follows

$$\chi(M) = 2 + 2\sum_{m=1}^{2k-1} (-1)^m \beta_m(M) + \beta^+_{2k}(M) + \beta^-_{2k}(M)$$

hence,

$$\frac{\chi(M) \pm \tau(M)}{2} = 1 + \sum_{m=1}^{2k-1} (-1)^m \beta_m(M) \pm \beta^\pm_{2k}(M)\,. \tag{4.7}$$

In the simplest case of a simply connected 4–dimensional manifolds, we have

$$\frac{\chi(M) \pm \tau(M)}{2} = 1 \pm \beta^\pm_2(M)\,.$$

Remark 4.3.2. From equation (4.7) it follows that the Euler–Poincaré characteristic and the signature have the same parity.

We collect in the next theorem some properties of the signature (see [28, Chapter 2], for these results and the subsequent discussion).

Theorem 4.3.3. *Let M, N and L be compact oriented differential manifolds with dimension $4k$, $4k$ and $4k+1$ respectively. Then*

(i) $\tau(-M) = -\tau(M)$, *where $-M$ is M with the reverse orientation;*

(ii) $\tau(M \sqcup N) = \tau(M) + \tau(N)$, *where $\sqcup$ is the disjoint union;*

(iii) $\tau(M \# N) = \tau(M) + \tau(N)$, *where $\#$ is the connected sum;*

(iv) $\tau(M \times N) = \tau(M)\tau(N)$;

(v) $\tau(\partial L) = 0$;

(vi) $\tau(\mathbb{CP}^{2m}) = 1$ *for all $m \in \mathbb{N}$.*

Points (i), (ii) and (v) imply that if two manifolds M_1 and M_2 are *oriented cobordant* (that is, their union is the oriented boundary of another manifold), they have the same signature. Hence, the signature is what is called a *genus operator*, that is, a ring homomorphism

$$\tau\colon (\Omega^+, \sqcup, \times) \to (\mathbb{Z}, +, \cdot)$$

from the *oriented cobordism ring* $\Omega^+ = \oplus_{m=0}^{\infty}\Omega^+_{4m}$ to the integers $\mathbb{Z}$, where $\sqcup$ denotes the disjoint union, $\times$ the Cartesian product and Ω^+_{4m} is the set of *oriented cobordism classes* of compact oriented $4m$–dimensional manifolds. By its algebraic properties, as the oriented cobordism ring is generated by $\{[\mathbb{CP}^{2m}]\}_{m=0}^{\infty}$, it is then sufficient to prove any integral formula expressing the signature for these particular manifolds. This is what F. Hirzebruch [17] did in order to prove his theorem in 1954:

$$\tau(M) = \int_M L_k(p_1, \dots, p_k)\,,$$

where L_k is a particular polynomial of degree k called the L–genus and $p_1, \dots, p_k$ are invariants built from the 2–form $\Omega/2\pi$, similarly to what we saw for the Chern–Gauß–Bonnet theorem.

Definition 4.3.4. We denote by $\mathcal{P}(k)$ the set of all partitions of the number $k \in \mathbb{N}$. If $I = (i_1, \dots, i_m)$ is one of such partitions and $\{\ell_i\}_{i=1}^{\infty}$ any sequence, we denote by ℓ_I the number

$$\ell_I = \prod_{j=1}^{m} \ell_{i_j} = \ell_{i_1}\ell_{i_2}\cdots\ell_{i_m}$$

and by $s_I = s_I(\sigma_1, \dots, \sigma_k)$ the polynomial such that, when σ_i is taken to be the i–th elementary symmetric polynomial in the k variables $r_1, \dots, r_k$, then s_I has the expression

$$s_I(\sigma_1, \dots, \sigma_k) = \sum_{\tau\in\Sigma_I}\prod_{j=1}^{m} r_{\tau(j)}^{i_j} = \sum_{\tau\in\Sigma_I} r_{\tau(1)}^{i_1} r_{\tau(2)}^{i_2}\cdots r_{\tau(m)}^{i_m}\,,$$

where Σ_I is any maximal subset of injective functions $\tau\colon \{1, \dots, m\} \to \{1, \dots, k\}$ with the following property: if τ and η are two distinct elements in Σ_I and there exists a permutation ϑ of $\{1, \dots, m\}$ such that $\tau = \eta\circ\vartheta$, then there exists at least one $j \in \{1, \dots, m\}$ for which $i_j \neq i_{\vartheta(j)}$.

Remark 4.3.5.

- For $k = 1$, $\mathcal{P}(1) = \{1\}$, the only elementary symmetric polynomial in $r_1, \dots, r_k$ is

  $$\sigma_1 = r_1\,,$$

  then s_1 has to satisfies

  $$s_1(\sigma_1) = \sum_{\tau\in\Sigma_1} r_{\tau(1)}^{1} = r_1 = \sigma_1\,.$$

- For $k = 2$, $\mathcal{P}(2) = \{(1,1), 2\}$, the elementary symmetric polynomials are

  $$\sigma_1 = r_1 + r_2 \qquad \text{and} \qquad \sigma_2 = r_1 r_2\,,$$

  then $s_{1,1}$ and s_2 have to satisfy

  $$s_{1,1}(\sigma_1, \sigma_2) = \sum_{\tau\in\Sigma_{1,1}} r_{\tau(1)}^{1} r_{\tau(2)}^{1} = r_1 r_2 = \sigma_2\,,$$
  $$s_2(\sigma_1, \sigma_2) = \sum_{\tau\in\Sigma_2} r_{\tau(1)}^{2} = r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 = \sigma_1^2 - 2\sigma_2\,.$$

- For $k=3$, $\mathcal{P}(3)=\{(1,1,1),(2,1),3\}$, the elementary symmetric polynomials are

$$\sigma_1 = r_1+r_2+r_3\,, \qquad \sigma_2 = r_1r_2+r_1r_3+r_2r_3 \qquad \text{and} \qquad \sigma_3 = r_1r_2r_3\,,$$

  then $s_{1,1,1}$, $s_{2,1}$ and s_3 have to satisfy

$$\begin{aligned} s_{1,1,1}(\sigma_1,\sigma_2,\sigma_3) &= \sum_{\tau\in\Sigma_{1,1,1}} r^1_{\tau(1)}r^1_{\tau(2)}r^1_{\tau(3)} = r_1r_2r_3\,,\\ s_{2,1}(\sigma_1,\sigma_2,\sigma_3) &= \sum_{\tau\in\Sigma_{2,1}} r^2_{\tau(1)}r^1_{\tau(2)} = r_1^2r_2+r_1^2r_3+r_2^2r_1+r_2^2r_3+r_3^2r_1+r_3^2r_2\,,\\ s_3(\sigma_1,\sigma_2,\sigma_3) &= \sum_{\tau\in\Sigma_3} r^3_{\tau(1)} = r_1^3+r_2^3+r_3^3\,. \end{aligned}$$

  An easy computation then gives

$$\begin{aligned} s_{1,1,1}(\sigma_1,\sigma_2,\sigma_3) &= \sigma_3\,,\\ s_{2,1}(\sigma_1,\sigma_2,\sigma_3) &= \sigma_1\sigma_2-3\sigma_3\,,\\ s_3(\sigma_1,\sigma_2,\sigma_3) &= \sigma_1^3-3\sigma_1\sigma_2+3\sigma_3\,. \end{aligned}$$

We observe that if every σ_i is taken to have degree i, then each s_I, for $I\in\mathcal{P}(k)$, is a homogenous polynomial of degree k.

Definition 4.3.6. Let B_i be the *i–th Bernoulli number*[2], that is, the number defined by the recurrence relation

$$\sum_{j=0}^{n}\binom{n+1}{j}B_j=0$$

with $B_0=1$.

All the odd Bernoulli numbers but $B_1=-1/2$ are zero, while the even Bernoulli numbers after $B_0=1$ have alternate signs and the first ones are

$$\frac{1}{6}\,,-\frac{1}{30}\,,\frac{1}{42}\,,-\frac{1}{30}\,,\frac{5}{66}\,,-\frac{691}{2730}\,,\frac{7}{6}\,,-\frac{3615}{510}\,,\frac{43867}{798}\,,\ldots\,.$$

Remark 4.3.7. We recall that for a given square matrix $A\in\mathbb{R}^{n\times n}$ its characteristic polynomial can be written as

$$p(t)=\det(A-t\mathrm{I}_n)=\sum_{r=0}^{n}(-1)^r t^r\sigma_{n-r}(A)\,,$$

where $\sigma_r(A)$ is the evaluation of the r–th elementary symmetric polynomial in the eigenvalues of A, which can be computed using the following $r\times r$ matrix

$$\sigma_r(A)=\frac{1}{r!}\det\begin{pmatrix} \mathrm{tr}\,A & r-1 & 0 & 0 & \ldots & 0\\ \mathrm{tr}\,A^2 & \mathrm{tr}\,A & r-2 & 0 & \ldots & 0\\ \vdots & \vdots & \ddots & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & \ddots & \vdots\\ \vdots & \vdots & & & \ddots & 1\\ \mathrm{tr}\,A^r & \mathrm{tr}\,A^{r-1} & \ldots & \ldots & \ldots & \mathrm{tr}\,A \end{pmatrix}\,. \tag{4.8}$$

[2] After the Swiss mathematician Jacob Bernoulli (1655–1705) [77].

Moreover, we observe that if A is skew–symmetric and r is odd, then $\sigma_r(A) = 0$. Indeed, the characteristic polynomial satisfies

$$p(-t) = \det(A + t\mathrm{I}_n) = \det\big((A + t\mathrm{I}_n)^T\big) = \det(-A + t\mathrm{I}_n) = (-1)^n p(t)\,,$$

that is, it is has the same parity as the dimension. Then $\sigma_r(A)$ is zero if r is odd, as it is the coefficient of a term whose power is of parity opposite to the dimension.

Theorem 4.3.8 (Hirzebruch [17], see also [28] for a proof)**.** *Let (M, g) be a compact oriented $4k$–dimensional Riemannian manifold, then*

$$\tau(M) = \int_M L_k(p_1, \dots, p_k)\,, \tag{4.9}$$

where p_i is the so–called i–th Pontryagin class[3]*, that is, the $4i$–differential form appearing in the expansion of the characteristic polynomial of $\Omega/2\pi$, (see Remark 4.3.7)*

$$\det\Big(\frac{\Omega}{2\pi} - t\mathrm{I}_{4k}\Big) = \sum_{i=0}^{2k} t^{2i} p_{4k-2i}\,,$$

*and L_k is the L–*genus of M*, that is,*

$$L_k = \sum_{I\in\mathcal{P}(k)} \ell_I s_I$$

with $(\ell_i)_{i=0}^{\infty}$ the sequence of coefficients in the Taylor expansion[4] of

$$\frac{\sqrt{x}}{\tanh\sqrt{x}} = \sum_{i=0}^{\infty} \frac{2^{2i}B_{2i}}{(2i)!}x^i = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + \frac{2}{945}x^3 + \cdots\,.$$

By means of Remark 4.3.5, we obtain the following first values of L_k,

$$\begin{aligned}
L_1 &= \ell_1 s_1 = \frac{1}{3}\sigma_1\,,\\
L_2 &= \ell_{1,1}s_{1,1} + \ell_2 s_2 = \frac{1}{9}\sigma_2 - \frac{1}{45}(\sigma_1^2 - 2\sigma_2)\,,\\
L_3 &= \ell_{1,1,1}s_{1,1,1} + \ell_{2,1}s_{2,1} + \ell_3 s_3\\
&= \frac{1}{27}\sigma_3 - \frac{1}{135}(\sigma_1\sigma_2 - 3\sigma_3) + \frac{2}{945}(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3)
\end{aligned}$$

and the Hirzebruch formula in dimension $n = 4, 8, 12$ in terms of the Pontryagin classes is given by

$$\begin{aligned}
\tau(M) &= \frac{1}{3}\int_M p_1\,,\\
\tau(M) &= -\frac{1}{45}\int_M (p_1\wedge p_1 - 3p_2)\,,\\
\tau(M) &= \frac{1}{945}\int_M (2p_1\wedge p_1\wedge p_1 - 13p_1\wedge p_2 + 62p_3)\,.
\end{aligned}$$

[3]After the Soviet mathematician Lev Semenovich Pontryagin (Лев Семёнович Понтрягин, 1908–1988) [78].

[4]After the English mathematician Brook Taylor (1685–1731) [79].

Using expression (4.8) we can expand the Pontryagin classes p_1, p_2, p_3 in these formulae as follows, denoting by $\widehat{\Omega} = \Omega/2\pi$ and by Ω^r and $\operatorname{tr}^r \Omega$ the r–times wedge product of Ω and $\operatorname{tr}\Omega$ with themselves, respectively,

$$p_1 = \frac{1}{2!}\det\begin{pmatrix} 0 & 1 \\ \operatorname{tr}\widehat{\Omega}^2 & 0 \end{pmatrix} = -\frac{1}{2}\operatorname{tr}\widehat{\Omega}^2 = -\frac{1}{8\pi^2}\operatorname{tr}\Omega^2\,,$$

$$p_2 = \frac{1}{4!}\det\begin{pmatrix} 0 & 3 & 0 & 0 \\ \operatorname{tr}\widehat{\Omega}^2 & 0 & 2 & 0 \\ 0 & \operatorname{tr}\widehat{\Omega}^2 & 0 & 1 \\ \operatorname{tr}\widehat{\Omega}^4 & 0 & \operatorname{tr}\widehat{\Omega}^2 & 0 \end{pmatrix} = \frac{3}{4!}(\operatorname{tr}^2\widehat{\Omega}^2 - 2\operatorname{tr}\widehat{\Omega}^4) = \frac{1}{128\pi^4}(\operatorname{tr}^2\Omega^2 - 2\operatorname{tr}\Omega^4)\,,$$

$$\begin{aligned} p_3 &= \frac{1}{6!}\det\begin{pmatrix} 0 & 5 & 0 & 0 & 0 & 0 \\ \operatorname{tr}\widehat{\Omega}^2 & 0 & 4 & 0 & 0 & 0 \\ 0 & \operatorname{tr}\widehat{\Omega}^2 & 0 & 3 & 0 & 0 \\ \operatorname{tr}\widehat{\Omega}^4 & 0 & \operatorname{tr}\widehat{\Omega}^2 & 0 & 2 & 0 \\ 0 & \operatorname{tr}\widehat{\Omega}^4 & 0 & \operatorname{tr}\widehat{\Omega}^2 & 0 & 1 \\ \operatorname{tr}\widehat{\Omega}^6 & 0 & \operatorname{tr}\widehat{\Omega}^4 & 0 & \operatorname{tr}\widehat{\Omega}^2 & 0 \end{pmatrix} \\ &= -\frac{15}{6!}(\operatorname{tr}^3\widehat{\Omega}^2 - 6\operatorname{tr}\widehat{\Omega}^2\wedge\operatorname{tr}\widehat{\Omega}^4 + 8\operatorname{tr}\widehat{\Omega}^6) \\ &= -\frac{1}{3072\pi^6}(\operatorname{tr}^3\Omega^2 - 6\operatorname{tr}\Omega^2\wedge\operatorname{tr}\Omega^4 + 8\operatorname{tr}\Omega^6)\,. \end{aligned}$$

Hence, we can rewrite the Hirzebruch formula (4.9) in dimension $n = 4, 8$ and 12 in terms of the curvature form Ω:

- when $n = 4$,
$$\tau(M) = -\frac{1}{24\pi^2}\int_M \operatorname{tr}(\Omega\wedge\Omega)\,, \tag{4.10}$$
- when $n = 8$,
$$\tau(M) = \frac{1}{5760\pi^4}\int_M \big(\operatorname{tr}(\Omega\wedge\Omega)\wedge\operatorname{tr}(\Omega\wedge\Omega) + 6\operatorname{tr}(\Omega\wedge\Omega\wedge\Omega\wedge\Omega)\big)\,,$$
- when $n = 12$,
$$\tau(M) = -\frac{1}{2903040\pi^6}\int_M \big(113\operatorname{tr}^3(\Omega^2) + 411\operatorname{tr}(\Omega^2)\wedge\operatorname{tr}(\Omega^4) - 496\operatorname{tr}(\Omega^6)\big)\,.$$

In dimension 4, we are going to express the integrand in terms of the Riemann tensor getting the commonly known formula, while, up to our knowledge, no more explicit expressions are present in literature in dimension $n \geq 8$.

Remark 4.3.9. We underline the analogy between the Chern–Gauß–Bonnet and Hirzebruch theorems, where in both cases an integral of a (polynomial) function of some so–called *characteristic classes* of a manifold (which are some particular differential forms constructed from the curvature form Ω), gives a topological invariant. For Hirzebruch theorem this is given by the polynomial L_k of the Pontryagin classes, while for the Chern–Gauß–Bonnet theorem it is the Pfaffian which, in this framework, is called the *Euler class* of the manifold. We also mention that these two theorems can be seen as special cases of a more general result, namely the *Atiyah—Singer index theorem*[5] (see [3]), which states that for

[5] After the British–Lebanese mathematician Sir Michael Francis Atiyah (1929–2019) [80] and the American mathematician Isadore Manuel Singer (1924–2021) [81].

an elliptic differential operator on a compact manifold, the *analytical index* (defined as the difference in dimension between the kernel and the cokernel of the operator) is equal to the *topological index* (defined in terms of *characteristic classes*, see [3, 26, 28]). The elliptic operator, the analytical index and the topological index for the Chern–Gauß–Bonnet theorem and the Hirzebruch theorem are respectively: $d+d^*$ and $\Delta = (d+d^*)^2$; the Euler–Poincaré characteristic and the signature; the integral of the *Euler class* and the integral of the L*–genus*, where $d^* = \star d\star$ is the formal adjoint to the exterior differential d with respect to the scalar product between differential forms and Δ is the *Hodge–Laplacian*.

The Hirzebruch formula in dimension four. In the simplest four–dimensional case, we have shown (equation (4.10))

$$\tau(M) = -\frac{1}{24\pi^2}\int_M \operatorname{tr}(\Omega\wedge\Omega)\,, \tag{4.11}$$

where

$$-\operatorname{tr}(\Omega\wedge\Omega) = -\operatorname{tr}(\Omega^i_j\wedge\Omega^j_k) = -\Omega^i_j\wedge\Omega^j_i = \sum_{i,j=1}^{4}\Omega^i_j\wedge\Omega^i_j = 2\sum_{1\leq i<j\leq 4}\Omega^i_j\wedge\Omega^i_j\,.$$

From now on, in order to use the curvature operator $\mathscr{R}$ in place of Ω, we identify 2–forms and 2–vectors, hence we can write

$$2\sum_{1\leq i<j\leq 4}\Omega^i_j\wedge\Omega^i_j = 2\sum_{1\leq i<j\leq 4}\mathscr{R}(e_i\wedge e_j)\wedge\mathscr{R}(e_i\wedge e_j) = \sum_{i^\pm=1}^{3}\mathscr{R}(\chi_i^\pm)\wedge\mathscr{R}(\chi_i^\pm)\,,$$

indeed, as $\chi_1^\pm = e_1\wedge e_2 \pm e_3\wedge e_4$, we have

$$e_1\wedge e_2 = (1/2)(\chi_1^+ + \chi_1^-) \qquad\text{and}\qquad e_3\wedge e_4 = (1/2)(\chi_1^+ - \chi_1^-)\,,$$

then,

$$\begin{aligned}2(\mathscr{R}(e_1\wedge\, e_2)&\wedge\mathscr{R}(e_1\wedge e_2)+\mathscr{R}(e_3\wedge e_4)\wedge\mathscr{R}(e_3\wedge e_4))\\ &= (1/2)(\mathscr{R}(\chi_1^+ + \chi_1^-)\wedge\mathscr{R}(\chi_1^+ + \chi_1^-) + \mathscr{R}(\chi_1^+ - \chi_1^-)\wedge\mathscr{R}(\chi_1^+ - \chi_1^-))\\ &= \mathscr{R}(\chi_1^+)\wedge\mathscr{R}(\chi_1^+) + \mathscr{R}(\chi_1^-)\wedge\mathscr{R}(\chi_1^-)\,.\end{aligned}$$

and similarly for $\chi_2^\pm$ and $\chi_3^\pm$.
If we write any 2–vector (or 2–form) v as $v = v^+ + v^-$, like in equation (4.5), then

$$\begin{aligned}v\wedge v &= v^+\wedge v^+ + v^-\wedge v^- + 2v^+\wedge v^-\\ &= v^+\wedge\star v^+ - v^-\wedge\star v^- - 2v^+\wedge\star v^-\\ &= (|v^+|^2 - |v^-|^2)\,\mathrm{d}V_M\,,\end{aligned}$$

where we used the definition (1.8) of the Hodge operator and the orthogonality between $\Lambda^2_+(TM)$ and $\Lambda^2_-(TM)$.
We now only have to compute $|\mathscr{R}(\chi_i^\pm)^\pm|^2$ for every $i\in\{1,2,3\}$ and subtract the value obtained for each sign. As $|\chi_i^\pm|^2 = 2$ for every $i\in\{1,2,3\}$, this just amounts to subtracting twice the square norm of the upper and lower part of the matrix

$$\left(\begin{array}{c|c}\mathscr{W}^+ + \mathrm{RI}_3/12 & \mathscr{C}/2\\ \hline \mathscr{C}/2 & \mathscr{W}^- + \mathrm{RI}_3/12\end{array}\right)$$

that is,

$$\begin{aligned} - \operatorname{tr}(\Omega \wedge \Omega) &= 2\Big(|\mathscr{W}^+ + \mathrm{R}\mathrm{I}_3/12|^2 + |\mathscr{C}/2|^2 - |\mathscr{C}/2|^2 - |\mathscr{W}^- + \mathrm{R}\mathrm{I}_3/12|^2\Big)\,\mathrm{d}V_M \\ &= 2(|\mathscr{W}^+| - |\mathscr{W}^-|^2)\,\mathrm{d}V_M\,, \end{aligned}$$

indeed,

$$\begin{aligned} \sum_{i=1}^{3}(\mathscr{W}^{\pm} + \mathrm{R}\mathrm{I}_3/12)_{ii}^2 &= \sum_{i=1}^{3}((\mathscr{W}_{ii}^{\pm})^2 + (\mathrm{R}/6)\mathscr{W}_{ii}^{\pm} + \mathrm{R}^2/144) \\ &= \sum_{i=1}^{3}(\mathscr{W}_{ii}^{\pm})^2 + (\mathrm{R}/6)\operatorname{tr}\mathscr{W}^{\pm} + \mathrm{R}^2/48 \\ &= \sum_{i=1}^{3}(\mathscr{W}_{ii}^{\pm})^2 + \mathrm{R}^2/48\,, \end{aligned}$$

being both $\mathscr{W}^{\pm}$ trace–free, and we obtain

$$|\mathscr{W}^+ + \mathrm{R}\mathrm{I}_3/12|^2 - |\mathscr{W}^- + \mathrm{R}\mathrm{I}_3/12|^2 = |\mathscr{W}^+|^2 - |\mathscr{W}^-|^2\,.$$

Hence, the Hirzebruch integrand in dimension 4 (taking into account Remark 4.2.3) is given by

$$-\frac{1}{24\pi^2}\operatorname{tr}(\Omega \wedge \Omega) = \frac{1}{12\pi^2}(|\mathscr{W}^+|^2 - |\mathscr{W}^-|^2)\,\mathrm{d}V_M = \frac{1}{48\pi^2}(|W^+|^2 - |W^-|^2)\,\mathrm{d}V_M\,,$$

leading to the *Hirzebruch formula* in dimension 4,

$$\tau(M) = \frac{1}{48\pi^2}\int_M \Big(|W^+|^2 - |W^-|^2\Big)\,\mathrm{d}V_M\,. \tag{4.12}$$

Since $|W|^2 = |W^+|^2 + |W^-|^2$, we can rewrite the Chern–Gauß–Bonnet formula (3.32) in dimension 4 as

$$\chi(M) = \frac{1}{32\pi^2}\int_M \Big(|W^+|^2 + |W^-|^2 + 16\sigma_2(S)\Big)\,\mathrm{d}V_M$$

and combining it with the Hirzebruch formula, we get

$$2\chi(M) \pm 3\tau(M) = \frac{1}{8\pi^2}\int_M \Big(|W^{\pm}|^2 + 8\sigma_2(S)\Big)\,\mathrm{d}V_M\,. \tag{4.13}$$

4.4 The Weyl functional

We introduce another physically–relevant functional, namely the *Weyl functional* and we discuss its critical metrics. We will show that the half–conformally flat metrics and the Einstein metrics (or more in general any metric conformal to an Einstein metric) are critical in dimension 4. We refer to [9] for further reading.
We define it, for any metric g on a compact oriented n–dimensional differential manifold M, as

$$\mathfrak{W}(g) = \int_M |\mathrm{Weyl}_g|_g^{n/2}\,\mathrm{d}V_g\,.$$

It is easy to check that $\mathfrak{W}$ is conformally invariant, indeed, if $\widetilde{g} = ug$, with $u > 0$, then $\sqrt{\det \widetilde{g}} = u^{n/2}\sqrt{\det g}$, $\mathrm{d}V(\widetilde{g}) = u^{n/2}\,\mathrm{d}V_g$ and (see Theorem 2.1.4)

$$|W_{\widetilde{g}}|_{\widetilde{g}}^2 = \widetilde{g}(W_{\widetilde{g}}, W_{\widetilde{g}}) = u^2\widetilde{g}(W_g, W_g) = u^{-2}g(W_g, W_g) = u^{-2}|W_g|_g^2\,,$$

hence,

$$\mathfrak{W}(\widetilde{g}) = \int_M |\mathrm{Weyl}_{\widetilde{g}}|_{\widetilde{g}}^{n/2} \,\mathrm{d}V(\widetilde{g}) = \int_M u^{-n/2} |\mathrm{Weyl}_g|_g^{n/2} \, u^{n/2} \,\mathrm{d}V_g = \mathfrak{W}(g)\,. \tag{4.14}$$

It is clear that it is quadratic only in dimension 4 and in such case, the Hirzebruch formula (4.12) gives

$$\mathfrak{W}(g) = 2\int_M |W^{\pm}|^2 \,\mathrm{d}V_M \mp 48\pi^2\tau(M)\,,$$

hence, the study of the Weyl functional is equivalent to the study of either

$$\mathfrak{W}^+(g) = \int_M |W_g^+|_g^2 \,\mathrm{d}V_g \qquad \text{or} \qquad \mathfrak{W}^-(g) = \int_M |W_g^-|_g^2 \,\mathrm{d}V_g\,.$$

Moreover, we have

$$\mathfrak{W}(g) \geq 48\pi^2 |\tau(M)| \tag{4.15}$$

with equality if and only if the manifold is half–conformally flat, i.e., either $W^{\pm} \equiv 0$ everywhere. This clearly implies that these metrics (and, a fortiori, the LCF ones) *minimize* the functional, hence they are trivially critical.

Remark 4.4.1. From equation (4.15) it also follows that in order that a compact oriented 4–dimensional manifold carry an LCF metric, it must have zero signature. In particular, from equation (4.7), any 4–manifold with odd Euler–Poincaré characteristic does not admit any LCF metric. An example is (see Remark 1.1.20) $\mathbb{CP}^2$, as $\chi(\mathbb{CP}^2) = 3$.

In order to see that another family of critical metrics is given by the Einstein metrics (or any metric conformal to an Einstein metric), we have to compute the first variation of $\mathfrak{W}$. Referring again to [9, Section 2.1], we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ric}(g+th)_{ij} = -\frac{1}{2}(&\Delta h_{ij} + 2R_{ikjl}h^{kl} - R_{ik}h_j^k - R_{jk}h_i^k \\ &+ \nabla_{ij}\operatorname{tr} h - \nabla_{ik}h_j^k - \nabla_{jk}h_i^k) \end{aligned}$$

and by means of this one and equation (4.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{R}^2(g+th) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} 2\mathrm{R}\mathrm{R}(g+th) = -2\mathrm{R}\Delta\operatorname{tr} h + 2\mathrm{R}\nabla^{ij}h_{ij} - 2\mathrm{R}R^{ij}h_{ij}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} |\mathrm{Ric}(g+th)|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (g+th)^{ik}(g+th)^{jl}R(g+th)_{ij}R(g+th)_{kl} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} 2(g+th)^{ik}R_{ij}R_k^j + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} 2R^{ij}R(g+th)_{ij} \\ &= -2h^{ik}R_{ij}R_k^j - R^{ij}\Delta h_{ij} - 2R^{ij}R_{ikjl}h^{kl} \\ &\quad + R^{ij}R_{ik}h_j^k + R^{ij}R_{jk}h_i^k - R^{ij}\nabla_{ij}\operatorname{tr} h + R^{ij}\nabla_{ik}h_j^k + R^{ij}\nabla_{jk}h_j^k \\ &= -R^{ij}\Delta h_{ij} - 2R_{kl}R^{ikjl}h_{ij} - R^{ij}\nabla_{ij}\operatorname{tr} h + 2R_k^i\nabla^{kj}h_{ij}\,. \end{aligned}$$

Then, by the equality at the third line of the Chern–Gauß–Bonnet formula (3.32), we can write

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathfrak{W}(g+th) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\frac{2}{3}\int_M \left(3|\mathrm{Ric}(g+th)|^2 - \mathrm{R}(g+th)^2\right)\mathrm{d}V_g + 32\pi^2\chi(M)\right)\\
&= \frac{2}{3}\int_M \Big(\left(-3R^{ij}\Delta h_{ij} - 6R_{kl}R^{ikjl}h_{ij} - 3R^{ij}\nabla_{ij}\operatorname{tr}h + 6R^i_k\nabla^{kj}h_{ij}\right)\\
&\quad + \left(2\mathrm{R}\Delta\operatorname{tr}h - 2\mathrm{R}\nabla^{ij}h_{ij} + 2\mathrm{R}R^{ij}h_{ij}\right)\\
&\quad + \left((3/2)|\mathrm{Ric}|^2\operatorname{tr}h - (1/2)\mathrm{R}^2\operatorname{tr}h\right)\Big)\,\mathrm{d}V_g
\end{aligned}
$$

and we observe that

- the terms $-6R_{kl}R^{ikjl}h_{ij}$, $2\mathrm{R}R^{ij}h_{ij}$, $(3/2)|\mathrm{Ric}|^2\operatorname{tr}h$ and $-(1/2)\mathrm{R}\operatorname{tr}h$ are ready to be written in the form $g(T,h)$,
- by the divergence theorem (1.2.13), in the remaining terms, $-3R^{ij}\Delta h_{ij}$, $-3R^{ij}\nabla_{ij}\operatorname{tr}h$, $6R^i_k\nabla^{kj}h_{ij}$, $2\mathrm{R}\Delta\operatorname{tr}h$ and $-2\mathrm{R}\nabla^{ij}h_{ij}$, we can switch where the second covariant derivative is applied, for instance (using Schur's lemma (2.3))

$$
\begin{aligned}
-3R^{ij}\Delta h_{ij} &= -3h_{ij}\Delta R^{ij} + \operatorname{div}(\text{other terms})\,,\\
-3R^{ij}\nabla_{ij}\operatorname{tr}h &= -3\operatorname{tr}h\nabla_{ij}R^{ij} + \operatorname{div}(\text{other terms})\\
&= -(3/2)\operatorname{tr}h\Delta\mathrm{R} + \operatorname{div}(\text{other terms})\,,\\
6R^i_k\nabla^{kj}h_{ij} &= 6h_{ij}\nabla^{ik}R^j_k + \operatorname{div}(\text{other terms})\\
&= 3h_{ij}\nabla^{ij}\mathrm{R} + \operatorname{div}(\text{other terms})\,,\\
2\mathrm{R}\Delta\operatorname{tr}h &= 2\operatorname{tr}h\Delta\mathrm{R} + \operatorname{div}(\text{other terms})\,,\\
-2\mathrm{R}\nabla^{ij}h_{ij} &= -2h_{ij}\nabla^{ij}\mathrm{R} + \operatorname{div}(\text{other terms})\,.
\end{aligned}
$$

Together, they give the following variation of the Weyl functional in gradient form,

$$
\begin{aligned}
\nabla\mathfrak{W}(g)_{ij} &= \frac{2}{3}\Big(\left(-6R^{kl}R_{ikjl} + 2\mathrm{R}R_{ij} + (3/2)|\mathrm{Ric}|^2 g_{ij} - (1/2)\mathrm{R}^2 g_{ij}\right)\\
&\quad + \left(-3\Delta R_{ij} - (3/2)\Delta\mathrm{R}g_{ij} + 3\nabla_{ij}\mathrm{R} + 2\Delta\mathrm{R}g_{ij} - 2\nabla_{ij}\mathrm{R}\right)\Big)\\
&= \frac{1}{3}\left(-12R^{kl}R_{ikjl} + 4\mathrm{R}R_{ij} + 3|\mathrm{Ric}|^2 g_{ij} - \mathrm{R}^2 g_{ij} - 6\Delta R_{ij} + \Delta\mathrm{R}g_{ij} + 2\nabla_{ij}\mathrm{R}\right).
\end{aligned}
$$

We now notice that the differential terms appearing above are exactly those of the divergence of the Cotton tensor in dimension 4 (see equation (2.7)),

$$
\operatorname{div}C_{ij} = \frac{1}{12}\left(6R^{kl}R_{ikjl} - 6R_{ik}R^k_j + 6\Delta R_{ij} - \Delta\mathrm{R}g_{ij} - 2\nabla_{ij}\mathrm{R}\right),
$$

hence,

$$
3\nabla\mathfrak{W}(g)_{ij} + 12\operatorname{div}C_{ij} = -6R^{kl}R_{ikjl} - 6R_{ik}R^k_j + 4\mathrm{R}R_{ij} + 3|\mathrm{Ric}|^2 g_{ij} - \mathrm{R}^2 g_{ij}\,.
$$

Finally, using the explicit decomposition formula (1.28) in dimension 4,

$$
6R_{ikjl} = 6W_{ikjl} + 3R_{ij}g_{kl} + 3R_{kl}g_{ij} - 3R_{il}g_{jk} - 3R_{jk}g_{il} - \mathrm{R}g_{ij}g_{kl} + \mathrm{R}g_{il}g_{jk}
$$

we can expand the term $R^{kl}R_{ikjl}$ as follows,

$$
\begin{aligned}
6R^{kl}R_{ijkl} &= 6R^{kl}W_{ikjl} + 3\mathrm{R}R_{ij} + 3|\mathrm{Ric}|^2 g_{ij} - 3R^j_l R_{il} - 3R^k_i R_{jk} - \mathrm{R}^2 g_{ij} + \mathrm{R}R_{ij} \\
&= 6R^{kl}W_{ikjl} - 6R_{ik}R^k_j + 4\mathrm{R}R_{ij} + 3|\mathrm{Ric}|^2 g_{ij} - \mathrm{R}^2 g_{ij}\,,
\end{aligned}
$$

to obtain

$$
3\nabla\mathfrak{W}(g)_{ij} + 12\,\mathrm{div}\,C_{ij} = -6R^{kl}W_{ikjl} = -12S^{kl}W_{ikjl}\,,
$$

that is,

$$
\nabla\mathfrak{W}(g)_{ij} = -4(\mathrm{div}\,C_{ij} + S^{kl}W_{ikjl}) = -4B_{ij}\,,
$$

where we call (in every dimension) $B_{ij} := \mathrm{div}\,C_{ij} + S^{kl}W_{ikjl}$ the *Bach tensor*[6]
It is easy to verify that the Bach tensor is a trace–free, symmetric $(0,2)$–tensor and that Einstein metrics are always also Bach–flat metrics, indeed, if g is Einstein and $\mathrm{Ric} = \lambda g$, then

$$
\begin{aligned}
B_{ij} &= S^{kl}W_{ikjl} + \mathrm{div}\,C_{ij} \\
&= \frac{1}{n-2}R^{kl}W_{ikjl} - \frac{\mathrm{R}}{2(n-1)(n-2)}g^{kl}W_{ikjl} + \frac{1}{n-2}(R_{ikjl}R^{kl} - R_{ik}R^k_j + \Delta R_{ij}) \\
&\quad - \frac{1}{2(n-1)(n-2)}\Delta\mathrm{R}g_{ij} - \frac{1}{n-1}\nabla_{ij}\mathrm{R} \\
&= \frac{\lambda}{n-2}g^{kl}W_{ikjl} + \frac{\lambda}{n-2}(R_{ikjl}g^{kl} - R_{ik}g^k_j) \\
&= 0\,.
\end{aligned}
$$

In particular, Einstein metrics (or any metric conformal to an Einstein metric) are critical to the Weyl functional in dimension 4.

Remark 4.4.2. In dimension 4 the Bach tensor is conformally invariant, as in this dimension it is proportional to the gradient of the Weyl functional and divergence–free, by satisfying in general

$$
\mathrm{div}\,B_i = (n-4)S^{jk}C_{ijk}\,,
$$

which follows from

$$
\mathrm{div}\,\mathrm{div}\,C_i = R_{ijkl}\nabla^l S^{jk}
$$

by means of the following computations:

$$
\nabla^k C_{kij} = \Delta S_{ij} - \frac{1}{2(n-1)}\nabla_{ij}\mathrm{R} - R_{il}S^l_j + R_{ikjl}S^{kl}\,,
$$

[6] After the German physicist Rudolf Bach.

from equation (2.7), then

$$
\begin{aligned}
\nabla^i\nabla^k C_{kij} &= \nabla^i\Delta S_{ij} - \frac{1}{2(n-1)}\nabla^i_{\ ij}R - \nabla^i R_{il}S^l_j + \nabla^i R_{ikjl}S^{kl} \\
&= \nabla^{ki}{}_i S_{ij} - R_{ki}{}^{kl}\nabla_l S^i_j - R_{ki}{}^{il}\nabla^k S_{lj} - R_{kijl}\nabla^k S^{il} \\
&\quad - \frac{1}{2(n-1)}\Delta\nabla_j \mathrm{R} - \tfrac{1}{2}\nabla_l \mathrm{R} S^l_j - R_{il}\nabla^i S^l_j + S^{kl}\nabla^i R_{ikjl} + R_{ikjl}\nabla^i S^{kl} \\
&= \Delta\nabla^i S_{ij} + \nabla^k(-R_{ki}{}^{il}S_{jl} - R_{kijl}S^{il}) \\
&\quad - R_{il}\nabla^l S^i_j + R_{kl}\nabla^k S^l_j - R_{il}\nabla^i S^l_j + R_{ikjl}\nabla^k S^{il} + R_{ikjl}\nabla^i S^{kl} \\
&\quad - \frac{1}{2(n-1)}\Delta\nabla_j \mathrm{R} - \tfrac{1}{2}S^l_j\nabla_l \mathrm{R} + S^{kl}\nabla^i R_{ikjl} \\
&= \frac{1}{2(n-1)}\Delta\nabla_j \mathrm{R} + \tfrac{1}{2}S^l_j\nabla_l \mathrm{R} + R_{kl}\nabla^k S^l_j + R_{ikjl}\nabla^k S^{il} - S^{il}\nabla^k R_{kijl} \\
&\quad - R_{il}\nabla^l S^i_j + R_{kl}\nabla^k S^l_j - R_{il}\nabla^i S^l_j + R_{ikjl}\nabla^k S^{il} + R_{ikjl}\nabla^i S^{kl} \\
&\quad - \frac{1}{2(n-1)}\Delta\nabla_j \mathrm{R} - \tfrac{1}{2}S^l_j\nabla_l \mathrm{R} + S^{kl}\nabla^i R_{ikjl} \\
&= -R_{ikjl}\nabla^i S^{kl}
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla^i B_{ij} &= \nabla^i\nabla^k C_{kij} + \nabla^i S^{kl}W_{ikjl} \\
&= -R_{ikjl}\nabla^i S^{kl} + W_{ikjl}\nabla^i S^{kl} + (n-3)S^{kl}C_{jlk} \\
&= (n-3)S^{kl}C_{jlk} - (S \owedge g)_{ikjl}\nabla^i S^{kl} \\
&= (n-3)S^{kl}C_{jlk} - (S_{ij}g_{kl} + S_{kl}g_{ij} - S_{il}g_{kj} - S_{kj}g_{il})\nabla^i S^{kl} \\
&= (n-3)S^{kl}C_{jlk} - \frac{1}{2(n-1)}S_{ij}\nabla^i \mathrm{R} - S_{kl}\nabla_j S^{kl} + S_{il}\nabla^i S^l_j + S_{kj}\nabla^i S^k_i \\
&= (n-3)S^{kl}C_{jlk} - \frac{1}{2(n-1)}S^i_j\nabla_i \mathrm{R} - S^{kl}\nabla_j S_{kl} + S^{il}\nabla_i S_{jl} + \frac{1}{2(n-1)}S^k_j\nabla_k \mathrm{R} \\
&= (n-3)S^{kl}C_{jlk} - S^{kl}C_{jlk} \\
&= (n-4)S^{kl}C_{jlk}\,.
\end{aligned}
$$

4.5 Four–dimensional Einstein manifolds

As the Chern–Gauß–Bonnet theorem applies to $2k$–dimensional manifolds and the Hirzebruch one to $4k$–dimensional manifolds, 4 is the lowest dimension in which we can use both results and actually, the only dimension where it is easy to do so. We have already seen at the end of Section 4.3 some general consequences of their “combination”, now we concentrate on the special case of four–dimensional Einstein manifolds.

The Chern–Gauß–Bonnet formulae (3.32) for a compact oriented 4–dimensional Einstein manifold (M, g) of constant λ, that is $\mathrm{Ric} = \lambda g$, give

$$
\begin{aligned}
\chi(M) &= \frac{1}{32\pi^2}\int_M |\mathrm{Riem}|^2\,\mathrm{d}V_M \\
&= \frac{1}{32\pi^2}\int_M \Big(|W^+|^2 + |W^-|^2\Big)\,\mathrm{d}V_M + \frac{\lambda^2}{12\pi^2}\,\mathrm{Vol}(M) \\
&= \frac{1}{4\pi^2}\int_M \Big(K_{12}^2 + K_{13}^2 + K_{14}^2 + R_{1234}^2 + R_{1324}^2 + R_{1423}^2\Big)\,\mathrm{d}V_M\,, \qquad (4.16)
\end{aligned}
$$

where the last equality is true in the frame given by Lemma 3.3.1, and equation (4.13) becomes

$$2\chi(M) \pm 3\tau(M) = \frac{1}{8\pi^2}\int_M |W^\pm|^2 \,\mathrm{d}V_M + \frac{\lambda^2}{6\pi^2}\operatorname{Vol}(M)\,. \tag{4.17}$$

The following theorem is then immediate by the first equation for the Euler–Poincaré characteristic.

Theorem 4.5.1. *A compact oriented 4–dimensional differentiable manifold that admits an Einstein metric must have nonnegative Euler–Poincaré characteristic. If in addition, such metric is not flat, then the Euler–Poincaré characteristic is positive.*

Another easy consequence of the equations above is the following theorem.

Theorem 4.5.2. *If (M, g) is a compact oriented 4–dimensional Riemannian manifold with Einstein constant λ, then*

$$\chi(M) \geq \frac{\lambda^2}{12\pi^2}\operatorname{Vol}(M)\,.$$

with equality if and only if the manifold has constant curvature.

Proof. The inequality follows by the second line of equations (4.16). In the equality case (or if the manifold has constant curvature) we have $W = 0$, and the conclusion follows from Proposition 4.1.3. □

We now see the important *Hitchin–Thorpe inequality* for four–dimensional Einstein manifolds.

Theorem 4.5.3 (Hitchin–Thorpe inequality [18, 34])**.** *A compact oriented 4–dimensional Einstein manifold (M, g) must satisfy the inequality*

$$\chi(M) \geq \frac{3}{2}|\tau(M)|\,. \tag{4.18}$$

In the equality case, the manifold is Ricci–flat and half–conformally flat.

Proof. The inequality clearly follows from equation (4.17). In the equality case, $\lambda = 0$ must hold and either $W^\pm = 0$. □

Remark 4.5.4. More precisely, equality can only occur if M is either flat or a Riemannian quotient of a $K3$ *surface*[7]. These are complex surfaces diffeomorphic to quartic surfaces in $\mathbb{CP}^3$. For a proof we refer to [18, Theorem 1] and for more details about $K3$ surfaces to [20].

Remark 4.5.5. The nonsufficiency of the Hitchin–Thorpe inequality in order to admit an Einstein metric was shown (independently) in 1996 by C. LeBrun [21] and A. Sambusetti [31], who exhibited infinitely many nonhomeomorphic compact oriented 4–dimensional manifolds M (also simply connected in the case of LeBrun), that cannot carry any Einstein metrics, but satisfy nevertheless the even stronger inequality $2\chi(M) > 3|\tau(M)|$.

[7]Named by André Weil after the initials of the two German mathematicians Ernst Eduard Kummer (1810–1893) [82], Erich Kähler (1906–2000) [54], the Japanese mathematician Kunihiko Kodaira (小平邦彦, 1915–1997) [83] and "*[...] the beautiful mountain $K2$ in Kashmir*".

The Riemann tensor of an Einstein manifold in the special frame given in Lemma 3.3.1 is determined by the 6 components

$$K_{12}\,,\ K_{13}\,,\ K_{14}\,,\ \ R_{1234}\,,\ R_{1324}\,,\ R_{1423}\,.$$

This is easy to verify by direct computation. Anyway, we observe that the Einstein condition $\mathrm{Ric} = \lambda g$ defines, within the 21–dimensional space of the tensors satisfying the symmetries of the Riemann tensor, except for the Bianchi identity, a subspace of dimension 12; if in addition the tensor satisfies the 6 zero–conditions we have by choosing the orthonormal frame given by Lemma 3.3.1, then this latter reduces to a 6–dimensional subspace.
With these few components, we can easily compute for $i < j$

$$\mathscr{R}(e_i \wedge e_j) = \sum_{1\leq s<t\leq 4} R_{ij}{}^{st} e_s \wedge e_t = K_{ij}\, e_i \wedge e_j + R_{ijkl}\, e_k \wedge e_l$$

where $k < l$ are the remaining indices. Then, in the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_3 \wedge e_4, e_4 \wedge e_2, e_2 \wedge e_3\}$, the matrix associated to $\mathscr{R}$ takes the form

$$\mathscr{R} = \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right),$$

where

$$A = \begin{pmatrix} K_{12} & & \\ & K_{13} & \\ & & K_{14} \end{pmatrix} = \begin{pmatrix} \mu^1 & & \\ & \mu^2 & \\ & & \mu^3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} R_{1234} & & \\ & R_{1342} & \\ & & R_{1423} \end{pmatrix} = \begin{pmatrix} \nu^1 & & \\ & \nu^2 & \\ & & \nu^3 \end{pmatrix}.$$

In particular, letting $\mu = (\mu^1, \mu^2, \mu^3)$ and $\nu = (\nu^1, \nu^2, \nu^3)$, we can express as follows

- the Einstein condition,
$$\sum_{i=1}^{3} \mu^i = \lambda\,,$$
- the Bianchi identity,
$$\sum_{i=1}^{3} \nu^i = 0\,,$$
- the Chern–Gauß–Bonnet formula,
$$\int_M \Big(|\mu|^2 + |\nu|^2\Big)\, \mathrm{d}V_M = 4\pi^2\chi(M)$$
- and the Hirzebruch formula,
$$\int_M \langle \mu, \nu\rangle\, \mathrm{d}V_M = 3\pi^2\tau(M)\,.$$

Indeed, such expression of the Chern–Gauß–Bonnet formula comes from equation (4.16), whereas for the Hirzebruch formula, we write, for $i < j$ and $k < l$ all different

$$\mathscr{R}(e_i \wedge e_j) \wedge \mathscr{R}(e_i \wedge e_j) = \pm 2K_{ij}R_{ijkl}\,\mathrm{d}V_M\,,$$

where the minus sign is in the case of $(i, j) = (1, 3)$ or $(i, j) = (2, 4)$, then, by equation (4.11), we compute the Hirzebruch integrand as

$$\begin{aligned}
\frac{1}{3}p_1 &= \frac{1}{12\pi^2}\sum_{1\leq i<j\leq 4}\mathscr{R}(e_i \wedge e_j) \wedge \mathscr{R}(e_i \wedge e_j)\\
&= \frac{1}{6\pi^2}(K_{12}R_{1234} - K_{13}R_{1324} + K_{14}R_{1423} + K_{23}R_{2314} - K_{24}R_{2413} + K_{34}R_{3412})\,\mathrm{d}V_M\\
&= \frac{1}{3\pi^2}(\mu^1\nu^1 + \mu^2\nu^2 + \mu^3\nu^3)\,\mathrm{d}V_M\,.
\end{aligned}$$

In this setting, the Hitchin–Thorpe inequality (4.18) amounts to the simple observation

$$3\pi^2|\tau(M)| = \left|\int_M \langle\mu,\nu\rangle\,\mathrm{d}V_M\right| \leq \int_M |\mu||\nu|\,\mathrm{d}V_M \leq \frac{1}{2}\int_M \Big(|\mu|^2 + |\nu|^2\Big)\,\mathrm{d}V_M = 2\pi^2\chi(M)\,, \tag{4.19}$$

and in the equality case, it must be

$$\mathrm{R}/4 = \sum_{i=1}^{3}\mu^i = \text{constant}\cdot\sum_{i=1}^{3}\nu^i = 0\,,$$

for μ and ν have to be proportional and $\mathrm{Ric} = \mathrm{R}g/4 = 0$. We now improve it, in the case of nonnegative or nonpositive curvature.

Theorem 4.5.6 (Hitchin [18]). *Let (M, g) be a compact oriented 4–dimensional Einstein manifold whose nonzero sectional curvatures all share the same sign, then there holds*

$$\chi(M) \geq \left(\frac{3}{2}\right)^{\frac{3}{2}}|\tau(M)|\,,$$

where equality can occur if and only if (M, g) is flat.

Proof. Let $\{e_1, e_2, e_3, e_4\}$ a special orthonormal frame as above and observe that the angle ϑ between any two vectors

$$\mu \in A = \Big\{\, (x^1, x^2, x^3) \in \mathbb{R}^3 \;\Big|\; x^1, x^2, x^3 \geq 0 \text{ or } x^1, x^2, x^3 \leq 0 \,\Big\}$$

and

$$\nu \in B = \Big\{\, (x^1, x^2, x^3) \in \mathbb{R}^3 \;\Big|\; x^1 + x^2 + x^3 = 0 \,\Big\}$$

satisfies $\cos\vartheta \leq \sqrt{2/3}$. Indeed, for a fixed ν the angle ϑ between μ and ν clearly decreases as μ moves towards the boundary, hence (by symmetry), we can assume that μ is any unit vectors lying on the boundary of A. We let $\mu = (1, 0, 0)$, then by means of the Lagrange multiplier method applied to the function

$$\cos\vartheta = f(\nu) = \frac{\nu^1}{|\nu|}\,,$$

under the constraint $\nu_1 + \nu_2 + \nu_3 = 0$, we get

$$-\frac{\nu^1\nu^2}{|\nu|^3} = \Lambda = -\frac{\nu^1\nu^3}{|\nu|^3}$$

and $\nu^2 = \nu^3 = -\nu^1/2$, thus

$$\cos\vartheta_{\max} = \frac{\nu^1}{\sqrt{(\nu^1)^2 + 2(\nu^1/2)^2}} = \sqrt{2/3}\,.$$

Hence, we obtain

$$\langle\mu,\nu\rangle = |\mu||\nu|\cos\vartheta \leq \frac{|\mu|^2 + |\nu|^2}{2}\sqrt{2/3}\,,$$

and integrating as in formula (4.19), we conclude

$$3\pi^2\tau(M) \leq \frac{1}{2}\sqrt{\frac{2}{3}}\,4\pi^2\chi(M)\,.$$

Reversing the orientation, we get the inequality for $-\tau(M)$.
Since $(3/2)^{3/2}$ is irrational, in the equality case it must hold $|\tau(M)| = \chi(M) = 0$, then (M, g) is flat, by Theorem 4.5.1. □

We conclude with some examples of manifolds not admitting any Einstein metrics. We will see that the Einstein property does not behave well under common topological constructions. In particular, with respect to the connected sum which, for connected 4–manifolds, satisfies (see Remark 1.1.20)

$$\chi(M\#N) = \chi(M) + \chi(N) - 2\,. \tag{4.20}$$

Example 4.5.7. Recalling Example 3.3.4, the spaces $M = \mathbb{S}^1 \times \mathbb{S}^3$ and $N = \mathbb{T}^2 \times \mathbb{S}^2$ cannot be endowed with Einstein metrics due to Theorem 4.5.1, as they cannot carry a flat metric and $\chi(\mathbb{S}^1 \times \mathbb{S}^3) = \chi(\mathbb{T}^2 \times \mathbb{S}^2) = 0$. Then, by equation (4.20), for every positive integers m and n one has

$$\chi(M^{\#m}\#N^{\#n}) = -2(m + n - 1) < 0$$

and none of these manifolds admits an Einstein metric, by Theorem 4.5.1.

Example 4.5.8. The manifold $M = \mathbb{T}^4$ is flat with canonical metric, thus Einstein and $\chi(M) = 0$. The manifold $N = \mathbb{RP}^2 \times \mathbb{RP}^2$ with canonical metric satisfies $\chi(N) = 1$ (Remark 1.1.20) and is also Einstein, as it is the product of an Einstein manifolds with itself. Nonetheless, for $m > 1$ and $n > 2$,

$$\chi(M^{\#m}\#N^{\#n}) = -(2m + n - 2) < 0\,.$$

These manifolds, again by Theorem 4.5.1, cannot admit any Einstein metrics for every nonnegative integers such that $2m + n > 2$.
Actually, also $N^{\#2} = (\mathbb{RP}^2 \times \mathbb{RP}^2)^{\#2}$ admits no Einstein metrics: $\chi(N^{\#2}) = 0$ and it is easy to verify using the Seifert–van Kampen theorem[8] (see [16, Theorem 1.20]) that its fundamental group has a finite subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$, as such, it cannot carry a flat metric (or any metric of nonpositive curvature) due to a theorem of Cartan (see [22, Corollary 12.18]).
We remark that these results are true despite $\mathbb{RP}^2$ being nonorientable, due to Remark 3.2.6.

[8] After the German mathematician Herbert Karl Johannes Seifert (1897–1996) [84] and the Dutch mathematician Egbert Rudolf van Kampen (1908–1942) [85].

Example 4.5.9. We recall from Remark 1.1.20 that $\chi(\mathbb{CP}^2) = 3$, and denote by

$$M^{k,\ell} = \left(\mathbb{CP}^2\right)^{\#k} \# \left(-\mathbb{CP}^2\right)^{\#\ell} .$$

By means of Theorem 4.3.3, we compute

$$\chi(M^{k,\ell}) = k + \ell + 2 \,, \qquad \tau(M^{k,\ell}) = k - \ell \,.$$

Consequently, for $k \geq 5(\ell + 1)$ or $\ell \geq 5(k + 1)$, the manifold $M^{k,\ell}$ does not admit any Einstein metric by the Hitchin–Thorpe inequality (4.18).
Actually, if $k = 5\ell + 4$ or $\ell = 5k + 4$ the manifold $M^{k,\ell}$ still does not admit any Einstein metric, as it satisfies the equality in the Hitchin–Thorpe inequality, is simply connected and not flat (see Remarks 4.4.1 and 4.5.4).

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