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Quasi–convexity of Riemannian integrands

TESI DI LAUREA MAGISTRALE

RELATORI
PROF. SSA CHIARA LEONE
PROF. CARLO MANTEGAZZA

CANDIDATA
AURORA CORBISIERO
Matr. P62000173

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Introduction

A fundamental result in the *Calculus of Variations* in searching minimizers of variational functionals, is the equivalence between the quasi–convexity and the sequential weakly (or weakly*) lower semicontinuity in the setting of the spaces $W^{1,p}(\Omega, \mathbb{R}^m)$, for $1 \leqslant p \leqslant +\infty$, where Ω is an open bounded subset of $\mathbb{R}^n$.

Historically, this equivalence was established through several steps: in the pioneering paper [27] by Tonelli, it is shown that for a twice differentiable and continuous function

$$f : [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \to [0, +\infty)$$

and for $u \in W^{1,1}((a, b), \mathbb{R}^m)$, the functional

$$F(u) = \int_a^b f(x, u(x), u'(x))\, dx$$

is sequentially weakly lower semicontinuous in $W^{1,1}(a, b)$ if and only if the function f is convex in the third variable. In the scalar case $m = 1$, this result was later generalized to functions defined on bounded open sets of $\mathbb{R}^n$ by various authors and Serrin in [25] proved that the differentiability assumptions are not actually required. Improvements of Serrin's theorem have then been given by De Giorgi [6], Olech [23] and Ioffe [15].

The results in the scalar case extend easily to the vectorial case. However, while for $m = 1$ the semicontinuity theorem stated above is optimal, in the sense that the convexity assumption of $f(x, s, \xi)$ with respect to ξ is necessary for the lower semicontinuity of F, in the vectorial case for $m > 1$, there are functionals (of considerable interest in the theory of nonlinear elasticity, for instance) that are lower semicontinuous without f being convex with respect to the matrix $\xi = \xi_{ij}$.

In the case of $m > 1$, the condition on f that turned out to be necessary and, with additional assumptions, also sufficient for the lower semicontinuity of integral functionals, is the quasi–convexity, introduced by Morrey [22] in 1952. More precisely, Morrey showed that under some strong regularity assumptions on f, the equivalence between the quasi–convexity of f and the weakly* sequential lower semicontinuity in $W^{1,\infty}(\Omega, \mathbb{R}^m)$ of the functional

$$F(u) = \int_\Omega f(x, u(x), Du(x))\, dx$$

holds. Meyers then extended Morrey's result to the setting of $W^{k,p}(\Omega, \mathbb{R}^m)$ spaces in [20].

Acerbi and Fusco [2] obtained a significant improvement of this result, which will be the subject of the first chapter of the thesis. Indeed, they established such equivalence for Carathéodory integrands with appropriate growth conditions in $W^{1,p}(\Omega, \mathbb{R}^m)$, for $1 \leqslant p \leqslant +\infty$. We also mention that Marcellini in [19] presented an alternative proof of this fact.

The aim of this thesis is to begin to develop an analogue of this theory in the Riemannian setting. Precisely, in the last chapter of the thesis, we will consider a smooth, complete and connected Riemannian manifold (M, g) and a continuous function

$$f : \mathscr{L}(TM, \mathbb{R}^m) \to \mathbb{R},$$

where $\mathscr{L}(TM, \mathbb{R}^m)$ is the vector bundle of the linear maps between a tangent space of M and $\mathbb{R}^m$, that is,

$$\mathscr{L}(TM, \mathbb{R}^m) = \big\{\alpha : T_xM \to \mathbb{R}^m \;\big|\; x \in M \text{ and } \alpha \text{ is linear}\big\}.$$

Then, after introducing a generalization of the notion of quasi–convexity (extending the usual one in the case of the Euclidean space), we will show that f is quasi–convex in our sense if and only if for every open and bounded subset $\Omega \subseteq M$, the functional

$$u \mapsto F(u, \Omega) = \int_\Omega f(du)\, d\mu$$

(where μ is the canonical volume measure of (M, g)) is sequentially lower semicontinuous in the weak* topology of $W^{1,\infty}(\Omega, \mathbb{R}^m)$, analogously to the Euclidean case.

We conclude the thesis with some related open problems and possible future research directions.

CHAPTER 1

Quasi–convexity and semicontinuity

1.1. Preliminary results

DEFINITION 1.1. A continuous function $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ is *quasi–convex* if for every $\xi \in \mathbb{R}^{n\times m}$ and for every open subset Ω of $\mathbb{R}^n$ and for every function $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, we have

$$f(\xi) \leqslant \fint_\Omega f(\xi + D\varphi(x))\, d\mathscr{L}^n(x).$$

DEFINITION 1.2. A real function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ is *quasi–convex* if there exists a subset Z of $\mathbb{R}^n$ with $\mathscr{L}^n(Z) = 0$, such that for every $x \in \mathbb{R}^n \setminus Z$ and for every $s \in \mathbb{R}^m$ the function $\xi \mapsto f(x, s, \xi)$ is *quasi–convex*.

We list some lemmas and definitions that will be useful in the sequel. The first is a result of Meyers in [20].

LEMMA 1.3. *Let $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ be quasi–convex. For every bounded open set $\Omega \subseteq \mathbb{R}^n$ and every sequence $u_k \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ weakly* convergent to zero, we have*

$$f(\xi) \leqslant \liminf_{k\to\infty} \fint_\Omega f(\xi + Du_k(x))\, d\mathscr{L}^n(x),$$

for every $\xi \in \mathbb{R}^{n\times m}$.

PROOF. Let $u_k \overset{\star}{\rightharpoonup} 0$ in $W^{1,\infty}(\Omega)$ and let ψ be a function in $C_c^\infty(\Omega)$ with $0 \leqslant \psi(x) \leqslant 1$. We have

$$\begin{aligned}\int_\Omega f(\xi + Du_k(x))\, d\mathscr{L}^n(x) = &\int_\Omega f(\xi + Du_k(x))\, d\mathscr{L}^n(x) + \int_\Omega f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x)\\ &- \int_\Omega f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x).\end{aligned}$$

If we now set Γ as the largest open subset of Ω such that $\psi(x) = 1$, for every $x \in \Gamma$, we have

$$\begin{aligned}\int_\Omega f(\xi + Du_k(x))\, d\mathscr{L}^n(x) = &\int_\Gamma f(\xi + Du_k(x))\, d\mathscr{L}^n(x) - \int_\Gamma f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x)\\ &+ \int_{\Omega\setminus\Gamma} f(\xi + Du_k(x))\, d\mathscr{L}^n(x)\\ &- \int_{\Omega\setminus\Gamma} f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x)\\ &+ \int_\Omega f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x).\end{aligned}$$

Now we set

$$J_1 = \int_\Gamma f(\xi + Du_k(x))\, d\mathscr{L}^n(x) - \int_\Gamma f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x);$$
$$J_2 = \int_{\Omega\setminus\Gamma} f(\xi + Du_k(x))\, d\mathscr{L}^n(x) - \int_{\Omega\setminus\Gamma} f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x);$$
$$J_3 = \int_\Omega f(\xi + D(\psi(x)u_k(x)))\, d\mathscr{L}^n(x);$$

thus,

$$\int_\Omega f(\xi + Du_k(x))\, d\mathscr{L}^n(x) = J_1 + J_2 + J_3.$$

We observe immediately that $J_1 = 0$ since $\psi \equiv 1$ in Γ and $D\psi \equiv 0$ in Γ.
We now focus on J_2. We have

$$\begin{aligned}|\xi + Du_k(x) - \xi - D(\psi(x)u_k(x))| &= |D((1-\psi(x))u_k(x))| = |(1-\psi)Du_k + u_k D\psi| \\ &\leqslant (1-\psi)\|Du_k\|_{L^\infty(\Omega\setminus\Gamma)} + \|u_k\|_{L^\infty(\Omega\setminus\Gamma)}\|D\psi\|_{L^\infty(\Omega\setminus\Gamma)},\end{aligned}$$

but $u_k \overset{\star}{\rightharpoonup} 0$, $D\psi$ is bounded, Du_k is uniformly bounded with respect to k and $\psi < 1$, so we get

$$|Du_k - D(\psi u_k)| \leqslant C'(\|Du_k\|_{L^\infty}, \|D\psi\|_{L^\infty}),$$

for every $x \in \Omega \setminus \Gamma$. Moreover, f is continuous, so

$$|f(\xi + Du_k(x)) - f(\xi + D(\psi(x)u_k(x)))| \leqslant C$$

and consequently,

$$\limsup_{k\to\infty} J_2 \leqslant C\mathscr{L}^n(\Omega \setminus \Gamma).$$

Moreover, by the quasi–convexity of the function, we have

$$\int_\Omega f(\xi)\, d\mathscr{L}^n(x) \leqslant \liminf_{k\to\infty} J_3.$$

Therefore, for every $\varepsilon > 0$, there exists a suitable choice of $\varphi(x)$ such that

$$\liminf_{k\to\infty} \int_\Omega f(\xi + Du_k(x))\, d\mathscr{L}^n(x) \geqslant \int_\Omega f(\xi)\, d\mathscr{L}^n(x) - \varepsilon = f(\xi)\mathscr{L}^n(\Omega) - \varepsilon,$$

hence, the thesis follows by the arbitrariness of $\varepsilon > 0$. □

DEFINITION 1.4. A function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ is a *Carathéodory function* if the following two conditions are satisfied:

- for every $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times m}$, the map $x \mapsto f(x, s, \xi)$ is measurable;
- for almost all $x \in \mathbb{R}^n$, the map $(s, \xi) \mapsto f(x, s, \xi)$ is continuous.

We will now recall some definitions and facts from [8] and adapt them to our context.

DEFINITION 1.5. A function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ is a *normal integrand* if for every $\Omega \subseteq \mathbb{R}^n$, we have:

- for almost every $x \in \Omega$, the map $(s, \xi) \mapsto f(x, s, \xi)$ is lower semicontinuous;
- there exists a Borel function $\widetilde{f} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}^m$ such that $\widetilde{f}(x, \cdot, \cdot) = f(x, \cdot, \cdot)$ for almost every $x \in \Omega$.

An important property of Carathéodory function is that they are normal integrands (see [8, p. 234]), then the following lemma by Scorza Dragoni provides a characterization of such functions (see [8]).

LEMMA 1.6. *A map $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ is a Carathéodory function if and only if for every compact set $K \subseteq \mathbb{R}^n$ and for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq K$, with $\mathscr{L}^n(K \setminus K_\varepsilon) < \varepsilon$, such that the restriction of f to $K_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$ is continuous.*

PROOF. Let us consider a compact subset K of $\mathbb{R}^n$ and let us fix $\varepsilon > 0$.
Since f is a Carathéodory function, we have that f is a normal integrand, so, by Lusin's theorem (see [24], for instance), there exists a compact set $K_+ \subseteq K$ such that

$$\mathscr{L}^n(K \setminus K_+) \leqslant \frac{\varepsilon}{2}$$

and the restriction of f to $K_+ \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$ is lower semicontinuous.
Now we notice that $-f$ is also a normal integrand, thus, we can find a compact set $K_- \subseteq K$ such that

$$\mathscr{L}^n(K \setminus K_-) \leqslant \frac{\varepsilon}{2}$$

and for which the restriction of $-f$ to $K_- \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$ is lower semicontinuous.
If we consider $K_\varepsilon = K_+ \cap K_-$, we see that the restriction of f to K_ε is lower semicontinuous and upper semicontinuous, hence f is continuous on K_ε. It also holds $\mathscr{L}^n(K \setminus K_\varepsilon) \leqslant \varepsilon$.
The converse statement is again a consequence of Lusin's theorem. □

The reference for the following lemma is [7].

LEMMA 1.7. *Let $G \subseteq \mathbb{R}^n$ be measurable with $\mathscr{L}^n(G) < +\infty$. Assume that M_l is a sequence of measurable subsets of G such that, for some $\varepsilon > 0$, we have $\mathscr{L}^n(M_l) \geqslant \varepsilon$ for all $l \in \mathbb{N}$. Then, a subsequence M_{l_h} satisfies $\bigcap_{h\in\mathbb{N}} M_{l_h} \neq \varnothing$.*

PROOF. Let us consider the function $\chi : G \to \mathbb{R}$ defined as

$$\chi(x) = \lim_{s\to\infty} \sum_{k=1}^{s} \chi_{M_k}(x).$$

Arguing by contradiction, we may suppose that

$$\bigcap_{j=1}^{\infty} M_{l_j} = \varnothing$$

for every subsequence M_{l_j}, thus we have $\chi(x) < +\infty$ for every $x \in G$.
Applying Lusin's theorem, we have a compact $K \subseteq G$ such that

- $\mathscr{L}^n(G \setminus K) < \frac{\varepsilon}{2}$;
- the function χ evaluated on K is continuous and thus limited, therefore there exists $n_0 \in \mathbb{N}$ such that $\chi(x) \leqslant n_0$, for all $x \in K$.

Letting $M'_k = M_k \cap K$, we have $\mathscr{L}^n(M'_k) > \frac{\varepsilon}{2}$, therefore, by Beppo Levi's theorem,

$$n_0\mathscr{L}^n(K) \geqslant \int_K \chi(x)\, d\mathscr{L}^n(x) = \lim_{s\to\infty} \sum_{l=1}^{s} \int_K \chi_{M'_l}(x)\, d\mathscr{L}^n(x)$$

$$= \lim_{s\to\infty} \sum_{l=1}^{s} \mathscr{L}^n(M'_l) > \lim_{s\to\infty} s\,\frac{\varepsilon}{2} = +\infty.$$

Hence, we obtained a contradiction and this concludes the proof. □

LEMMA 1.8. *Let φ_k be a bounded sequence in $L^1(\mathbb{R}^n)$. Then, for each $\varepsilon > 0$ there exists a triple $(A_\varepsilon, \delta, S)$ where A_ε is a measurable subset of $\mathbb{R}^n$ with $\mathscr{L}^n(A_\varepsilon) < \varepsilon$, $\delta > 0$ and S is an infinite subset of $\mathbb{N}$ such that for all $k \in S$,*

$$\int_B |\varphi_k(x)|\, d\mathscr{L}^n(x) < \varepsilon$$

whenever B and A_ε are disjoint and $\mathscr{L}^n(B) < \delta$.

PROOF. We may assume, by contradiction, that there exists $\varepsilon > 0$ such that for every $(A_\varepsilon, \delta, S)$ as in the statement, we may choose a measurable set B such that:

- $\mathscr{L}^n(B) < \delta$;
- $B \cap A_\varepsilon = \varnothing$;
- there exists an index $k \in \mathbb{N}$ such that:
$$\int_B |\varphi_k(x)|\, d\mathscr{L}^n(x) \geqslant \varepsilon.$$

Let A and S be as in the statement, we set $S_1 = S$ and $\delta_1 = \frac{\varepsilon - \mathscr{L}^n(A)}{2}$. Then, there exists a set B_1 such that:

- $\mathscr{L}^n(B_1) < \delta_1$;
- $A \cap B_1 = \varnothing$
- there exists an index $k_1 \in \mathbb{N}$ such that
$$\int_{B_1} |\varphi_{k_1}(x)|\, d\mathscr{L}^n(x) \geqslant \varepsilon.$$

Arguing by induction, if we set

$$\delta_n = \frac{1}{2}\delta_{n-1} \quad \text{and} \quad S_n = \{k \in S_{n-1} \,|\, k > k_{n-1}\}$$

and let

$$C = \bigcup_{h \in \mathbb{N}} B_h \quad \text{and} \quad T = \{k_h \,|\, h \in \mathbb{N}\},$$

we obtain a couple of sets which satisfies our requests, indeed

- $\mathscr{L}^n(C) + \mathscr{L}^n(A) < \varepsilon$;
- $C \cap A = \varnothing$;
- $T \subseteq S$ is infinite and
$$\int_C |\varphi_k(x)|\, d\mathscr{L}^n(x) \geqslant \varepsilon,$$
for every $k \in T$.

As a consequence, we have that for every set A with $\mathscr{L}^n(A) < \varepsilon$ and every infinite set $S \subseteq \mathbb{N}$, there exists a set C such that

- $\mathscr{L}^n(C \cup A) < \varepsilon$;
- $C \cap A = \varnothing$;
- there exists an infinite subset T of S such that
$$\int_C |\varphi_k(x)|\, d\mathscr{L}^n(x) \geqslant \varepsilon \quad \text{for every } k \in T.$$

Now we show that if we consider

$$N > \varepsilon^{-1} \sup_{k\in\mathbb{N}} \|\varphi_k\|_{L^1(\mathbb{R}^n)}, \tag{1.1}$$

we have a contradiction. Indeed, let $A = \varnothing$ and $S = \mathbb{N}$ and C_1 and T_1 as above.
Starting from $A = C_1$ and $S = T_1$, we proceed to find C_2 and T_2 with $C_1 \cap C_2 = \varnothing$ and

$$\int_{C_1\cup C_2} |\varphi_k(x)|\, d\mathscr{L}^n(x) = \int_{C_1} |\varphi_k(x)|\, d\mathscr{L}^n(x) + \int_{C_2} |\varphi_k(x)|\, d\mathscr{L}^n(x) \geqslant 2\varepsilon$$

for all $k \in T_2$. Since $\mathscr{L}^n(C_1 \cup C_2) < \varepsilon$, we define $A = C_1 \cup C_2$ and $S = T_2$ and we continue iteratively. Then, after N iterations, by the assumption (1.1), we obtain a contradiction and the proof is complete. □

DEFINITION 1.9. Let $u \in C_c^\infty(\mathbb{R}^n)$. We define

$$(M^*u)(x) = (Mu)(x) + \sum_{i=1}^{n} (MD_i u)(x),$$

where we set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B_r(x)} |f(y)|\, dy$$

for every locally summable function f (ω_n is the volume of the n–dimensional unit ball of $\mathbb{R}^n$).

The reference for the following lemma is [26].

LEMMA 1.10. *If $u \in C_c^\infty(\mathbb{R}^n)$, then $M^*u \in C^0(\mathbb{R}^n)$ and*

$$|u(x)| + \sum_{i=1}^{n} |D_i u(x)| \leqslant (M^*u)(x),$$

for all $x \in \mathbb{R}^n$. Moreover, if $p > 1$, then

$$\|M^*u\|_{L^p(\mathbb{R}^n)} \leqslant c(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

and if $p = 1$, then

$$\mathscr{L}^n(\{x \in \mathbb{R}^n \mid (M^*u) \geqslant \lambda\}) \leqslant \frac{c(n)}{\lambda}\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

for all $\lambda > 0$.

PROOF. We give a sketch of the proof.
Let $E_\lambda = \{x : Mu(x) > \lambda\}$. We aim to estimate the measure of E_λ in terms of $\|u\|_{L^1(\mathbb{R}^n)}$.
To simplify the notation, we will sometimes denote by $|B_x|$ the n–dimensional Lebesgue measure of B_x.

(1) For each $x \in E_\lambda$, there exists a ball B_x centered at x, such that

$$\frac{1}{|B_x|} \int_{B_x} |u(y)|\, dy > \alpha\lambda,$$

for some fixed constant $\alpha > 0$.

(2) Using a Vitali–type covering lemma, we extract a disjoint countable subcollection of balls B_{x_k} such that

$$\sum_{k\in\mathbb{N}} |B_{x_k}| \geqslant c|E_\lambda|$$

and the union of dilations of these balls by a fixed factor covers E_λ.

(3) Then, we obtain:

$$\int_{\bigcup_{k\in\mathbb{N}} B_{x_k}} |u(y)|\,dy \geqslant \sum_{k\in\mathbb{N}} \int_{B_{x_k}} |u(y)|\,dy > \alpha\lambda \sum_{k\in\mathbb{N}} |B_{x_k}|.$$

(4) Thus,

$$\lambda|E_\lambda| \leqslant \frac{1}{\alpha}\int_{\mathbb{R}^n} |u(y)|\,dy = \frac{1}{\alpha}\|u\|_{L^1(\mathbb{R}^n)}.$$

(5) This gives us a L^1 estimate:

$$\mathscr{L}^n(\{x : Mu(x) > \lambda\}) \leqslant \frac{C}{\lambda}\|u\|_{L^1(\mathbb{R}^n)},$$

where C is a constant depending only on the dimension.

(6) The other cases are reduced to the case $p = 1$, the case $p = \infty$ is handled more easily.

In this way we have obtained a control on $\mathscr{L}^n(E_\lambda)$ by $\|u\|_{L^1(\mathbb{R}^n)}$, this can be extended to a control by $\|u\|_{W^{1,p}(\mathbb{R}^n)}$ just involving weak derivatives, which appear in the definition of M^*u. □

The following lemma can be found in [18, Lemma 2].

LEMMA 1.11. *Let $u \in C_c^\infty(\mathbb{R}^n)$ and let*

$$U(x,y) = \frac{\left|u(y) - u(x) - \sum_{i=1}^n D_i u(x)(y_i - x_i)\right|}{|y-x|}.$$

Then, for every $x \in \mathbb{R}^n$ and $r > 0$, there holds

$$\int_{B_r(x)} U(x,y)\,dy \leqslant 2\omega_n r^n (M^*u)(x).$$

PROOF. Let $F(t) = u(x + t(y-x))$. Then,

$$u(y) = F(1) = F(0) + \int_0^1 F'(t)\,dt = u(x) + \int_0^1 \sum_{i=1}^n D_i u(x + t(y-x))(y_i - x_i)\,dt.$$

Subtracting the linear part,

$$u(y) - u(x) - Du(x)\cdot(y-x) = \int_0^1 [Du(x + t(y-x)) - Du(x)]\cdot(y-x)\,dt$$

and taking the absolute value and dividing by $|y-x|$, we get

$$U(x,y) \leqslant \int_0^1 |Du(x+t(y-x)) - Du(x)|\,dt.$$

Then, integrating and using Fubini's theorem, we obtain

$$\begin{aligned}\int_{B_r(x)} U(x,y)\,dy &\leqslant \int_{B_r(x)} \int_0^1 |Du(x+t(y-x)) - Du(x)|\,dt\,dy \\ &= \int_0^1 \int_{B_r(x)} |Du(x+t(y-x)) - Du(x)|\,dy\,dt.\end{aligned}$$

We set now $z = x+t(y-x)$, therefore $y = x+\frac{1}{t}(z-x)$. Then, $z \in B_{tr}(x)$ and since $dy = t^{-n}dz$,

$$\int_{B_r(x)} |Du(x+t(y-x)) - Du(x)|\,dy = t^{-n} \int_{B_{tr}(x)} |Du(z) - Du(x)|\,dz,$$

consequently,

$$\int_{B_r(x)} U(x,y)\,dy \leqslant \int_0^1 t^{-n} \int_{B_{tr}(x)} |Du(z) - Du(x)|\,dz\,dt.$$

We bound the inner integral using the maximal function as follows:

$$\frac{1}{|B_{tr}(x)|} \int_{B_{tr}(x)} |Du(z) - Du(x)|\,dz \leqslant 2 \sum_{|\alpha|=1} M(D^\alpha u)(x).$$

Thus,

$$\int_{B_r(x)} U(x,y)\,dy \leqslant 2 \sum_{|\alpha|=1} M(D^\alpha u)(x) \int_0^1 |B_{tr}| t^{-n} dt = 2\omega_n r^n (M^* u)(x).$$

and the proof is completed. □

The lemma below, which comes from [2], is a slightly modified version of the original lemma in [18].

LEMMA 1.12. *Let $u \in C_c^\infty(\mathbb{R}^n)$ and $\lambda > 0$ and set*

$$H^\lambda = \{x \in \mathbb{R}^n \,|\, (M^* u)(x) < \lambda\}\,.$$

Then, for every $x, y \in H^\lambda$ we have

$$\frac{|u(y) - u(x)|}{|y-x|} \leqslant c(n)\lambda.$$

PROOF. We consider $c'(n)$ such that for every $x, y \in \mathbb{R}^n$ with $|x-y| = r$ there holds

$$\mathscr{L}^n(B_r(x) \cap B_r(y)) \geqslant \frac{2}{c'(n)} \omega_n r^n. \tag{1.2}$$

For a $\delta > 0$ let $W_r(x,\delta) = \{y \in B_r(x) \,|\, U(x,y) < \delta\}$, then, by Lemma 1.11, we have

$$2\omega_n r^n (M^* u)(x) \geqslant \int_{B_r(x)} U(x,y)\,dy \geqslant \int_{B_r(x)\setminus W_r(x,\delta)} U(x,y)\,dy \geqslant \delta \mathscr{L}^n(B_r(x) \setminus W_r(x,\delta)).$$

Let $z \in H^\lambda$, therefore

$$\mathscr{L}^n(B_r(z) \setminus W_r(z, 2c'(n)\lambda)) \leqslant \frac{2\omega_n r^n}{2c'(n)\lambda} (M^* u)(z) < \frac{\omega_n r^n}{c'(n)}. \tag{1.3}$$

We consider $x, y \in H^\lambda$, with $|x-y| = r$, then, using inequalities (1.2) and (1.3),

$$W_r(x, 2c'(n)\lambda) \cap W_r(y, 2c'(n)\lambda) \neq \varnothing.$$

Hence, letting $\widetilde{z}$ belonging in this intersection, so $|x-\widetilde{z}|<r$ and $|y-\widetilde{z}|<r$, we obtain

$$\begin{aligned}\frac{|u(y)-u(x)|}{|x-y|} &\leqslant \frac{|u(y)-u(\widetilde{z})|}{|x-y|}+\frac{|u(\widetilde{z})-u(x)|}{|x-y|} \leqslant \frac{|u(y)-u(\widetilde{z})|}{|y-\widetilde{z}|}+\frac{|u(\widetilde{z})-u(x)|}{|\widetilde{z}-x|} \\ &\leqslant U(y,\widetilde{z})+U(\widetilde{z},x)+\sum_{i=1}^{n}\big[|D_i u(x)|+|D_i u(y)|\big] \leqslant 4(c'(n)+2)\lambda,\end{aligned}$$

hence, we have the thesis. □

Finally, we need one last lemma that can be found in [8] and it is known as *McShane lemma*.

LEMMA 1.13 (McShane lemma). *Let X be a metric space and E a subset of X. Then, any k–Lipschitz mapping from E to $\mathbb{R}$ can be extended to a k–Lipschitz mapping defined on the whole X.*

PROOF. Let u be k–Lipschitz mapping from E to $\mathbb{R}$. For every $x\in X$ we can set

$$\widetilde{u}(x)=\sup_{e\in E}\{u(e)-kd(x,e)\}\in\mathbb{R}\cup\{\infty\}.$$

If we consider $\widetilde{e}\in E$, for the Lipschitz condition, we have

$$u(e)-kd(\widetilde{e},e)\leqslant u(\widetilde{e}),$$

for every $e\in E$, hence, $\widetilde{u}(\widetilde{e})=u(\widetilde{e})$ and thus $\widetilde{u}$ is an extension of u.
Let $x,y\in X$ and without loss of generality, we assume that $\widetilde{u}(y)\leqslant\widetilde{u}(x)$. Therefore,

$$\begin{aligned}0\leqslant\widetilde{u}(y)-\widetilde{u}(x)&=\sup_{e\in E}\{u(e)-kd(y,e)\}-\sup_{e\in E}\{u(e)-kd(x,e)\}\\ &\leqslant\sup_{e\in E}\{u(e)-kd(y,e)-u(e)+kd(x,e)\}\leqslant\sup_{e\in E}\{kd(x,e)-kd(y,e)\}\end{aligned}$$

and using the triangular inequality, we obtain

$$0\leqslant\widetilde{u}(y)-\widetilde{u}(x)\leqslant kd(y,x).$$

In particular, if we consider $x\in E$, we have

$$\widetilde{u}(y)\leqslant u(x)+kd(y,x)<+\infty\quad\text{for every } y\in X,$$

thus, $\widetilde{u}$ is a k–Lipschitz mapping from X to $\mathbb{R}$. □

1.2. Semicontinuity theorems

If f is a real function defined on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{n\times m}$, under the hypotheses that justify the integration and ensure the validity of the following formula, for every measurable set $\Omega\subseteq\mathbb{R}^n$, we define

$$F(u,\Omega)=\int_\Omega f(x,u(x),Du(x))\,d\mathscr{L}^n(x).$$

We can now state the first of the main results from [2].

THEOREM 1.14. *Let $f:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{n\times m}\to\mathbb{R}$ satisfy:*

(1) *f is a Carathéodory function;*
(2) *f is quasi–convex in ξ;*
(3) *$0\leqslant f(x,s,\xi)\leqslant a(x)+b(s,\xi)$, for every $x\in\mathbb{R}^n$, $s\in\mathbb{R}^m$, $\xi\in\mathbb{R}^{n\times m}$, where a is a non–negative locally summable function on $\mathbb{R}^n$ and $b\geqslant 0$ is locally bounded on $\mathbb{R}^n\times\mathbb{R}^{n\times m}$.*

Then, for every open bounded set Ω in $\mathbb{R}^n$ the functional $u\mapsto F(u,\Omega)$ is sequentially weakly lower semicontinuous in $W^{1,\infty}(\Omega,\mathbb{R}^m)$.*

PROOF. First, let us suppose that $\Omega = (0,1)^n$ and let us fix

- $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$;
- $z_k \in W^{1,\infty}(\Omega, \mathbb{R}^m)$, such that $z_k \overset{\star}{\rightharpoonup} 0$ in $W^{1,\infty}(\Omega, \mathbb{R}^m)$.

Let us set:

$$\lambda = \|u\|_{W^{1,\infty}(\Omega,\mathbb{R}^m)} + \sup_{k\in\mathbb{N}} \|z_k\|_{W^{1,\infty}(\Omega,\mathbb{R}^m)}$$

and

$$M = \sup \{ b(s,\xi) : |s| \leqslant \lambda, |\xi| \leqslant \lambda \} .$$

Without loss of generality we may assume that $a(x) < +\infty$ for every $x \in \Omega$.
We now consider $\varepsilon > 0$ and let $\alpha \geqslant 1$ be large enough such that if

$$E = \{ x \in \Omega \,|\, a(x) \leqslant \alpha \} \setminus Z,$$

then,

$$\mathscr{L}^n(\Omega \setminus E) < \frac{\varepsilon}{M}, \quad \int_{\Omega\setminus E} a(x)\, d\mathscr{L}^n(x) < \varepsilon. \tag{1.4}$$

We want to show that

$$F(u,\Omega) \leqslant \liminf_{k\to\infty} F(u+z_k,\Omega).$$

Let define the sets

$$G^\nu = \left\{ 2^{-\nu}(x + (0,1)^n) \,|\, x \in \mathbb{Z}^n \right\}, \tag{1.5}$$

for any $\nu \in \mathbb{N}$. If we now neglect the sets of measure zero, then, for all $\nu \in \mathbb{N}$ we have

$$\Omega = \bigcup_{h=1}^{2^{n\nu}} Q_h^\nu, \quad \text{with } Q_h^\nu \in G^\nu.$$

From now on, till the end of the proof, we will assume $1 \leqslant h \leqslant 2^{n\nu}$.
We consider

$$u_h^\nu = 2^{-n\nu} \int_{Q_h^\nu} u(y)\, dy; \qquad u^\nu(x) = \sum_h u_h^\nu \chi_{Q_h^\nu}(x);$$

$$Du_h^\nu = 2^{-n\nu} \int_{Q_h^\nu} Du(y)\, dy; \quad Du^\nu(x) = \sum_{h\in\mathbb{N}} Du_h^\nu \chi_{Q_h^\nu}(x)$$

and we observe that

$$\|u^\nu\|_{L^\infty(\Omega,\mathbb{R}^m)} + \|Du^\nu\|_{L^\infty(\Omega,\mathbb{R}^m)} \leqslant \|u\|_{W^{1,\infty}(\Omega,\mathbb{R}^m)}$$

and that both sequences u^ν and Du^ν converge pointwise almost everywhere. The first converges to u, the second one to Du.
Recalling Lemma 1.6, we know that there exists a compact set $K \subseteq \Omega$ such that:

- f is continuous on $K \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$;
- there holds
$$\mathscr{L}^n(\Omega \setminus K) < \frac{\varepsilon}{\alpha + M}.$$

For every ν and h in $\mathbb{N}$, we fix $x_h^\nu \in Q_h^\nu \cap K \cap E$, when this intersection in not empty. We have then

$$F(u+z_k,\Omega) \geqslant F(u+z_k, K\cap E) = a_k + b_k^\nu + c_k^\nu + d^\nu + e,$$

with

$$\begin{aligned}
a_k &= \int_{K\cap E} [f(x,(u+z_k)(x),(Du+Dz_k)(x)) - f(x,u(x),(Du+Dz_k)(x))]\, d\mathscr{L}^n(x);\\
b_k^\nu &= \sum_{h\in\mathbb{N}} \int_{Q_h^\nu\cap K\cap E} [f(x,u(x),(Du+Dz_k)(x)) - f(x_h^\nu,u_h^\nu(x),Du_h^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x);\\
c_k^\nu &= \sum_{h\in\mathbb{N}} \int_{Q_h^\nu\cap K\cap E} [f(x_h^\nu,u_h^\nu(x),Du_h^\nu(x)+Dz_k(x)) - f(x_h^\nu,u_h^\nu(x),Du_h^\nu(x)]\, d\mathscr{L}^n(x);\\
d^\nu &= \sum_{h\in\mathbb{N}} \int_{Q_h^\nu\cap K\cap E} [f(x_h^\nu,u_h^\nu(x),Du_h^\nu) - f(x,u(x),Du(x)]\, d\mathscr{L}^n(x);\\
e &= \int_{K\cap E} [f(x,u(x),Du(x)]\, d\mathscr{L}^n(x).
\end{aligned}$$

Since f is uniformly continuous on the bounded set $K\times\mathbb{R}^m\times\mathbb{R}^{n\times m}$, there holds $\lim_{k\to\infty} a_k = 0$. Likewise, the uniform continuity of f and the pointwise convergence of u^ν and Du^ν ensure that

$$\lim_{\nu\to\infty} d^\nu = 0, \qquad \lim_{\nu\to\infty} b_k^\nu = 0 \text{ uniformly with respect to } k.$$

Indeed, we have

$$\begin{aligned}
b_k^\nu =& \sum_{h=1}^{2^{n\nu}} \int_{Q_h^\nu} [\, f(x,u(x),Du(x)+Dz_k(x)) - f(x,u_h^\nu(x),Du_h^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x)\\
&+ \sum_{h=1}^{2^{n\nu}} \int_{Q_h^\nu} [\, f(x,u_h^\nu(x),Du_h^\nu(x)+Dz_k(x)) - f(x_h^\nu,u_h^\nu(x),Du_h^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x)\\
\leqslant& \sum_{h=1}^{2^{n\nu}} \int_{Q_h^\nu} [\, f(x,u(x),Du(x)+Dz_k(x)) - f(x,u_h^\nu(x),Du_h^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x) + \varepsilon
\end{aligned}$$

by the uniform continuity of f.
Notice now that the first integral can be written as

$$\int_{K\cap E} [f(x,u(x),Du(x)+Dz_k(x)) - f(x,u^\nu(x),Du^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x).$$

Now we can use the Egorov theorem to deduce the existence of a compact set $\overline{K}$ such that $\mathscr{L}^n(\Omega\setminus\overline{K}) < \varepsilon$ on which u^ν and Du^ν converge uniformly to u and Du. So we obtain that

$$\begin{aligned}
&\int_{K\cap E} [f(x,u(x),Du(x)+Dz_k(x)) - f(x,u^\nu(x),Du^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x)\\
&\quad= \int_{(K\cap E)\cap\overline{K}} [f(x,u(x),Du(x)+Dz_k(x)) - f(x,u^\nu(x),Du^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x)\\
&\qquad+ \int_{(K\cap E)\setminus\overline{K}} [f(x,u(x),Du(x)+Dz_k(x)) - f(x,u^\nu(x),Du^\nu(x)+Dz_k(x))]\, d\mathscr{L}^n(x).
\end{aligned}$$

Now we have that the first integral tends to zero uniformly if ν tends to $+\infty$, due to the uniform continuity of f, while the second integral is controlled by ε.

Thus, we can assume that ν is large enough such that $|b_k^\nu| + |d^\nu| < \varepsilon$ for every k.
Notice that, by using (1.4), we have

$$\left| \sum_{h\in\mathbb{N}} \int_{Q_h^\nu \setminus (K\cap E)} [f(x_h^\nu, u_h^\nu, Du_h^\nu + Dz_k(x)) - f(x_h^\nu, u_h^\nu, Du_h^\nu)] \right| d\mathscr{L}^n(x)$$

$$\leqslant 2 \sum_h \int_{Q_h^\nu \setminus (K\cap E)} [a(x_h^\nu) + M]\, d\mathscr{L}^n(x)$$

$$\leqslant 2 \left[(\alpha + M)\mathscr{L}^n(\Omega \setminus K) + M\mathscr{L}^n(\Omega \setminus E) + \int_{\Omega\setminus E} \alpha\, d\mathscr{L}^n(x) \right]$$

$$\leqslant 4\varepsilon + 2 \int_{\Omega\setminus E} a(x)\, d\mathscr{L}^n(x) \leqslant 6\varepsilon.$$

Recalling then Lemma 1.3 and applying it in our setting to each Q_h^ν, we obtain

$$\liminf_{k\to\infty} c_k^\nu \geqslant -6\varepsilon.$$

On the other hand, we have

$$e = F(u, K\cap E) \geqslant F(u, \Omega) - 3\varepsilon.$$

Then,

$$\liminf_{k\to\infty} F(u + z_k, \Omega) \geqslant F(u, \Omega) - 10\varepsilon.$$

As ε is arbitrarily chosen, our result is proved when $\Omega = (0,1)^n$.
The same argument can be applied to every cube Ω with edges parallel to the coordinate axes and the general thesis follows from the fact that the sup of a family of lower semicontinuous functions is lower semicontinuous. □

REMARK 1.15. The theorem holds even if f satisfies the first and the second conditions of the statement and $|f|$ satisfies the third condition, the proof can be found in [11] and [20].

REMARK 1.16. If f is defined on the set $\Omega \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{n\times m} \,|\, |\xi| < r\}$ for some $r > 0$ and f satisfy the hypotheses of Theorem 1.14, then the functional $u \mapsto F(u, \Omega)$ is sequentially weakly lower semicontinuous on the space of functions $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ such that $\|Du\|_{L^\infty(\Omega;\mathbb{R}^{n\times m})} < r$.

Another relevant result is the converse of the previous theorem.

THEOREM 1.17. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ satisfy:*

(1) *f is a Carathéodory function;*
(2) *$0 \leqslant f(x, s, \xi) \leqslant a(x) + b(s, \xi)$, for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{n\times m}$, where a is a non–negative locally summable function on $\mathbb{R}^n$ and $b \geqslant 0$ is locally bounded on $\mathbb{R}^n \times \mathbb{R}^{n\times m}$.*

If the functional $u \mapsto F(u, \Omega)$ is sequentially weakly lower semicontinuous on $W^{1,\infty}(\Omega, \mathbb{R}^m)$, for every open bounded set $\Omega \subseteq \mathbb{R}^n$, then f is quasi–convex in ξ.*

PROOF. The thesis means that for every open set $\Omega \subseteq \mathbb{R}^n$, there exists a set $Z \subseteq \Omega$, with $\mathscr{L}^n(Z) = 0$, such that the function $\xi \mapsto f(x, s, \xi)$ is quasi–convex for every $x \in \Omega \setminus Z$ and $s \in \mathbb{R}^m$. To this purpose, we will use only the fact that $u \mapsto F(u, \Omega)$ is lower semicontinuous for a particular Ω. First, using Lemma 1.6, we are able to choose a nondecreasing sequence K_i of compact sets, such that for each $i \in \mathbb{N}$:

- $\mathscr{L}^n(\Omega \setminus K_i) < 1/i$;

- f is continuous on $K_i \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$.

Let us now define a set Z such that $x \in \Omega \setminus Z$ if:

- $x \in \bigcup_{i\in\mathbb{N}} K_i$;
- $a(x) < +\infty$;
- x is a Lebesgue point for χ_{K_i}, for every $i \in \mathbb{N}$, that is,
$$\lim_{r\to 0} \frac{1}{[\mathscr{L}^n(B_r(x))]} \int_{B_r(x)} \chi_{K_i}(y)dy = 1;$$
- x is a Lebesgue point for $a \cdot \chi_{\Omega\setminus K_i}$ for every i.

Fix $x_0 \in \Omega \setminus Z$, $s_0 \in \mathbb{R}^m$, $\xi_0 \in \mathbb{R}^{n\times m}$ and without losing generality, we may assume $x_0 = 0$, then set
$$u(x) = s_0 + \xi_0 \cdot x,$$
where ξ_0 is a matrix $m \times n$.
Let $z \in C_c^\infty(Y, \mathbb{R}^m)$ and put
$$\lambda = \|u\|_{W^{1,\infty}(\Omega,\mathbb{R}^m)} + \|z\|_{W^{1,\infty}(Y,\mathbb{R}^m)}.$$
We define z periodically on $\mathbb{R}^n$, setting $z(x) = z(x + y)$ for every $y \in \mathbb{Z}^n$.
We choose now $\widetilde{k}$ large enough such that
$$2^{-\widetilde{k}}Y \subseteq \Omega.$$
If $k \geqslant \widetilde{k}$ and $\nu \in \mathbb{N}$, we consider
$$z_k^\nu(x) = \begin{cases} 2^{-k\nu} z(2^{k\nu}x) & \text{if } x \in 2^{-k}Y \\ 0 & \text{otherwise} \end{cases}$$
hence, $\|z_k^\nu\|_{W^{1,\infty}(\Omega,\mathbb{R}^m)} \leqslant \lambda$.
For every k, $z_k^\nu \overset{\star}{\rightharpoonup} 0$ in $W^{1,\infty}(\Omega, \mathbb{R}^m)$ if $\nu \to +\infty$, thus $z_k^\nu \to 0$ in $L^\infty(\Omega, \mathbb{R}^m)$.
Notice now that for any fixed k, if we neglect the sets of measure 0, there holds
$$2^{-k}Y = \bigcup_{h=1}^{2^{n\nu}} Q_h^{k\nu}, \quad Q_h^{k\nu} \in G^{k\nu},$$
using $G^{k\nu}$ as defined in the equation (1.5).
We denote by x_h^ν the corner of $Q_h^{k\nu}$ nearest to the origin, so that
$$Q_h^{k\nu} = x_h^\nu + 2^{-k\nu}Y.$$
We may suppose also that $0 \in K_i$, for all i.
Let us choose $\varepsilon > 0$, then there exists a $\widetilde{i}$ such that if

- $i \geqslant \widetilde{i}$;
- we set
$$M = \sup\left\{b(s,\xi) \,|\, |s| + |\xi| \leqslant 2\lambda\right\},$$

then, we have
$$\int_{\Omega\setminus K_i} [a(x) + M] \, d\mathscr{L}^n(x) < \varepsilon. \tag{1.6}$$
Let $\widetilde{f}_i$ be a continuous function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$ such that

- $\widetilde{f}_i$ coincides with f on $K_i \times \{(s,\xi) : |s| + |\xi| \leqslant 2\lambda\} = K_i \times B_{2\lambda}$;

- there holds $0 \leqslant \widetilde{f}_i \leqslant \max_{K_i \times B_{2\lambda}} f$.

We now select a function $\psi \in C_c^\infty(\Omega)$ such that

- $0 \leqslant \psi(x) \leqslant 1$ for all $x \in \Omega$;
- $\psi(x) = 1$ for all $x \in K_i$;
- $$\int_{\Omega \setminus K_i} \psi(x)\, d\mathscr{L}^n(x) < \frac{\varepsilon}{\max_{K_i \times B_{2\lambda}} f}. \tag{1.7}$$

Therefore, the function $f_i = \psi \widetilde{f}_i$ is another continuous extension of f outside $K_i \times B_{2\lambda}$. Let us consider the functional $F(u + z_k^\nu, 2^{-k}Y)$. We will use a splitting technique also in this situation: we can write

$$F(u + z_k^\nu, 2^{-k}Y) = a^\nu + b^\nu + c^\nu,$$

where

$$\begin{aligned}
a^\nu &= \int_{2^{-k}Y} [f(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x)) - f_i(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))]\, d\mathscr{L}^n(x);\\
b^\nu &= \sum_{h \in \mathbb{N}} \int_{Q_h^{k\nu}} [f_i(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x)) - f_i(x_h^\nu, u(x_h^\nu), Du(x_h^\nu) + Dz_k^\nu(x))]\, d\mathscr{L}^n(x);\\
c^\nu &= \sum_{h \in \mathbb{N}} \int_{Q_h^{k\nu}} f_i(x_h^\nu, u(x_h^\nu), Du(x_h^\nu) + Dz(2^{k\nu}x))\, d\mathscr{L}^n(x)\\
&= \sum_{h \in \mathbb{N}} 2^{-nk\nu} \int_Y f_i(x_h^\nu, u(x_h^\nu), Du(x_h^\nu) + Dz(y))\, dy.
\end{aligned}$$

Notice that $|a^\nu| < \varepsilon$, indeed, for the properties of $\widetilde{f}$ and f itself and by inequalities (1.6) and (1.7), we have

$$\begin{aligned}
|a^\nu| = \Bigg| \int_{2^{-k}Y} &[f(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))\\
&- f_i(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))]\, d\mathscr{L}^n(x) \Bigg|\\
\leqslant \Bigg| \int_{K_i} &[f(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))\\
&- f_i(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))]\, d\mathscr{L}^n(x) \Bigg|\\
+ \Bigg| \int_{2^{-k}Y \setminus K_i} &[f(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))\\
&- f_i(x, (u + z_k^\nu)(x), (Du + Dz_k^\nu)(x))]\, d\mathscr{L}^n(x) \Bigg|.
\end{aligned}$$

The first integral of the last sum is zero, thus,

$$\begin{aligned}|a^\nu| &\leqslant \bigg| \int_{2^{-k}Y\setminus K_i} [f(x,(u+z_k^\nu)(x),(Du+Dz_k^\nu)(x)) \\ &\qquad\qquad - f_i(x,(u+z_k^\nu)(x),(Du+Dz_k^\nu)(x))]\, d\mathscr{L}^n(x) \bigg| \\ &= \bigg| \int_{2^{-k}Y\setminus K_i} [f(x,(u+z_k^\nu)(x),(Du+Dz_k^\nu)(x)) \\ &\qquad\qquad - \psi(x)\widetilde{f}_i(x,(u+z_k^\nu)(x),(Du+Dz_k^\nu)(x))]\, d\mathscr{L}^n(x) \bigg| \\ &\leqslant \int_{\Omega\setminus K_i} |f(x,(u+z_k^\nu)(x),(Du+Dz_k^\nu)(x))| + \varepsilon < 2\varepsilon.\end{aligned}$$

So we have $|a^\nu| < 2\varepsilon$ for every ν and $i \geqslant \widetilde{i}$.
Moreover, we have $\lim_{\nu\to\infty} b^\nu = 0$ because $u \in C^1(\overline{\Omega},\mathbb{R}^m)$, $z_k^\nu \to 0$ uniformly and f_i is uniformly continuous.
About c^ν, we notice that it is a sum over the cube $2^{-k}Y$, of the continuous function

$$x \mapsto \int_Y f_i(x,u(x),Du(x)+Dz(y))\, dy,$$

therefore,

$$\lim_{\nu\to\infty} c^\nu = \int_{2^{-k}Y} \left[\int_Y f_i(x,u(x),Du(x)+Dz(y))dy\right] d\mathscr{L}^n(x)$$

and we immediately deduce that

$$\limsup_{\nu\to+\infty} F(u+z_k^\nu, 2^{-k}Y) \leqslant 2\varepsilon + \int_{2^{-k}Y} \left[\int_Y f_i(x,u(x),Du(x)+Dz(y))dy\right] d\mathscr{L}^n(x).$$

If ψ tends to χ_{K_i}, since $f = f_i$ on $K_i \times B_{2\lambda}$, from the dominated convergence theorem we have

$$\limsup_{\nu\to+\infty} F(u+z_k^\nu, 2^{-k}Y) \leqslant 2\varepsilon + \int_{K_i\cap 2^{-k}Y} \left[\int_Y f_i(x,u(x),Du(x)+Dz(y))dy\right] d\mathscr{L}^n(x).$$

Due to the fact that $u \mapsto F(u,\Omega)$ is semicontinuous and $z_k^\nu \equiv 0$ on $\Omega\setminus 2^{-k}Y$, there holds

$$\begin{aligned}F(u,\Omega) &= F(u,2^{-k}Y) + F(u,\Omega\setminus 2^{-k}Y) \\ &\leqslant \liminf_{\nu\to\infty}\left[F(u+z_k^\nu,2^{-k}Y) + F(u,\Omega\setminus 2^{-k}Y)\right],\end{aligned}$$

consequently, with an algebraic step, for $i \geqslant \widetilde{i}$,

$$\begin{aligned}F(u,2^{-k}Y) &\leqslant 2\varepsilon + \int_{2^{-k}Y\cap K_i}\int_Y f_i(x,u(x),Du(x)+Dz(y))\, dy\, d\mathscr{L}^n(x) \\ &= 2\varepsilon + \iint_{2^{-k}Y\times Y} \chi_{K_i}(x) f(x,u(x),Du(x)+Dz(y))\, d\mathscr{L}^n(x)\, dy.\end{aligned}$$

Now, if $i \to \infty$, since ε is arbitrary,

$$F(u,2^{-k}Y) \leqslant \int_{2^{-k}Y}\left[\int_Y f(x,u(x),Du(x)+Dz(y))\, dy\right] d\mathscr{L}^n(x),$$

hence,

$$2^{nk}\int_{2^{-k}Y}\left[f(x,u(x),Du(x))-\int_Y f(x,u(x),Du(x)+Dz(y))\,dy\right]\,d\mathscr{L}^n(x)\leqslant 0.$$

Denoting by h the integrand in the square brackets and using the hypotheses on the set Z and the continuity of f on $K_{\tilde{i}}\times B_{2\lambda}$, we get

$$\begin{aligned}
&\lim_{k\to\infty}2^{nk}\int_{2^{-k}Y\cap K_{\tilde{i}}}\left[f(x,u(x),Du(x))-\int_Y f(x,u(x),Du(x)+Dz(y))\,dy\right]\,d\mathscr{L}^n(x)\\
&\quad=\lim_{k\to\infty}2^{nk}\int_{2^{-k}Y\cap K_{\tilde{i}}}h(x;u,z)\,d\mathscr{L}^n(x)\\
&\quad=\lim_{k\to\infty}\left(2^{nk}\int_{2^{-k}Y}\chi_{K_{\tilde{i}}}(x)\,d\mathscr{L}^n(x)\right)\left[\mathscr{L}^n(2^{-k}Y\cap K_{\tilde{i}})\right]^{-1}\int_{2^{-k}Y\cap K_{\tilde{i}}}h(x;u,z)\,d\mathscr{L}^n(x).
\end{aligned}$$

Notice now that, by equation (1.2), we have

$$\lim_{k\to\infty}2^{nk}\int_{2^{-k}Y}\chi_{K_{\tilde{i}}}(x)\,d\mathscr{L}^n(x)=1,$$

then, for the mean value theorem, we get

$$\begin{aligned}
&\lim_{k\to\infty}2^{nk}\int_{2^{-k}Y\cap K_{\tilde{i}}}\left[f(x,u(x),Du(x))-\int_Y f(x,u(x),Du(x)+Dz(y))\,dy\right]\,d\mathscr{L}^n(x)\\
&\qquad=f(0,s,\xi)-\int_Y f(0,s,\xi+Dz(y))\,dy
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|2^{nk}\int_{2^{-k}Y\setminus K_{\tilde{i}}}h(x;u,z)\,d\mathscr{L}^n(x)\right|&\leqslant 2^{nk}\int_{2^{-k}Y\setminus K_{\tilde{i}}}[a(x)+M]\,d\mathscr{L}^n(x)\\
&=2^{nk}\int_{2^{-k}Y}[a(x)+M]\chi_{\Omega\setminus K_{\tilde{i}}}(x)\,d\mathscr{L}^n(x)
\end{aligned}$$

and the last term tends to 0 when $k\to\infty$.
Notice that we have obtained that the behaviour on $2^{-k}Y\cap K_{\tilde{i}}$ is what we hoped for and since we have the convergence to zero on $2^{-k}Y\setminus K_{\tilde{i}}$, our thesis is demonstrated on the open set Y. We can now extend it to every open set $\Omega\subseteq\mathbb{R}^n$. □

REMARK 1.18. Notice that the proof remains almost the same also if we assume that f is a Carathéodory function, $|f|$ satisfies the third condition of Theorem 1.14 and the functional $u\mapsto F(u,\Omega)$ is sequentially weakly* lower semicontinuous on each Dirichlet class $\widetilde{u}+W_0^{1,\infty}(\Omega,\mathbb{R}^m)$, with u a polynomial of degree one.

REMARK 1.19. Let f satisfy the first and third conditions of Theorem 1.14 and assume that the functional $u\mapsto F(u,\Omega)$ is weakly* lower semicontinuous on the space of functions u in $W^{1,\infty}(\Omega,\mathbb{R}^m)$ such that $\|Du\|_{L^\infty(\Omega,\mathbb{R}^{n\times m})}<r$, where $r<0$.
Then, there exists a set $Z\subseteq\Omega$ with $\mathscr{L}^n(Z)=0$, such that for every $x\in\Omega\setminus Z$, $s\in\mathbb{R}^m$ and $\xi\in W_c^{1,\infty}(\Omega,\mathbb{R}^m)$ which $\|\xi+Dz\|_{L^\infty(\Omega,\mathbb{R}^m)}<r$, we have

$$f(x,s,\xi)\leqslant\int_\Omega f(x,s,\xi+Dz(x))\,d\mathscr{L}^n(x).$$

REMARK 1.20. We mention that Theorems 1.14 and 1.17 are generalization of previous results in [11, 20].

Now deal with semicontinuity in $W^{1,p}$, when $p \geqslant 1$.
Before stating the next theorem, we recall the following result (see [3, Theorem 4.26]).

THEOREM 1.21. *Let Ω be either a half–space in $\mathbb{R}^n$ or a domain in $\mathbb{R}^n$ having the uniform C^m–regularity property and a bounded boundary. Then, for any $m \in \mathbb{N}$, there exists a strong m–extension operator E for Ω. Moreover, if α and γ are multi–indices with $|\gamma| \leqslant |\alpha| \leqslant m$, there exists a continuous linear operator $E_{\alpha\gamma}$ from $W^{j,p}(\Omega)$ to $W^{j,p}(\mathbb{R}^n)$, for $1 \leqslant j \leqslant m - |\alpha|$, such that if $u \in W^{|\alpha|,p}(\Omega)$, then*

$$D^\alpha Eu(x) = \sum_{|\gamma| \leqslant |\alpha|} E_{\alpha\gamma} D^\gamma u(x)$$

for almost every $\in \mathbb{R}^n$.

We then show the last theorem of this chapter.

THEOREM 1.22. *Let $p \in [1, +\infty)$ and assume that $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ is a Carathéodory function quasi–convex in ξ, such that*

$$0 \leqslant f(x, s, \xi) \leqslant a(x) + C(|s|^p + |\xi|^p), \tag{1.8}$$

for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{n\times m}$, where C is a non–negative constant and a is a non–negative locally summable function on $\mathbb{R}^n$.
Then, for every open bounded set $\Omega \subseteq \mathbb{R}^n$, the functional $u \mapsto F(u, \Omega)$ is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$.

PROOF. As in Theorem 1.14, without loss of generality, we may confine our proof to a particular set Ω; in this proof it is a ball.
We consider

- $u \in W^{1,p}(\Omega, \mathbb{R}^m)$;
- $z_k \in W^{1,p}(\Omega, \mathbb{R}^m)$, with $z_k \rightharpoonup 0$ in $W^{1,p}(\Omega, \mathbb{R}^m)$.

We may suppose

$$\liminf_{k\to\infty} F(u + z_k, \Omega) = \lim_{k\to\infty} F(u + z_k, \Omega),$$

hence, we will be able to select subsequences without altering $\liminf_{k\to\infty} F(u + z_k, \Omega)$ and it will not be necessary to denote subsequences differently.
Using Theorem 1.21, we may assume each function z_k to be defined on the whole $\mathbb{R}^n$, with $\|z_k\|_{W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)}$ uniformly bounded with respect to $k \in \mathbb{N}$.
Since $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ is dense in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$ and $u \mapsto F(u, \Omega)$ is continuous in the strong topology of $W^{1,p}(\Omega, \mathbb{R}^m)$, there exists a sequence $w_k \subseteq C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$ such that

- $\|w_k - z_k\|_{W^{1,p}(\mathbb{R}^n,\mathbb{R}^m)} < 1/k$;
- $|F(u + w_k, \Omega) - F(u + z_k, \Omega)| < 1/k$.

Therefore we can assume that the sequence z_k belongs to $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and it is bounded in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^m)$.
We set now a continuous and increasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\eta(0) = 0$, such that for every measurable set $B \subseteq \Omega$, there holds

$$\int_B [a(x) + C(|u(x)|^p + |Du(x)|^p)]\, d\mathscr{L}^n(x) < \eta(\mathscr{L}^n(B)). \tag{1.9}$$

Let $\varepsilon > 0$ be fixed, we now apply Lemma 1.8 to each of the m sequences $\left(M^* z_k^{(i)}\right)^p$, with $1 \leqslant i \leqslant m$. Thus, we obtain a subsequence z_k, a set $A_\varepsilon \subseteq \Omega$ with $\mathscr{L}^n(A_\varepsilon) < \varepsilon$ and a real number $\delta > 0$ such that

$$\int_B \left[\left(M^* z_k^{(i)}\right)(x)\right]^p d\mathscr{L}^n(x) < \varepsilon,$$

for all $k \in \mathbb{N}$, for $1 \leqslant i \leqslant m$ and for every $B \subseteq \Omega \setminus A_\varepsilon$, with $\mathscr{L}^n(B) < \delta$.
Recalling Lemma 1.10, we may take $\lambda > 0$ large enough such that, for all i, k, we have

$$\mathscr{L}^n\left(\left\{x \in \mathbb{R}^n \mid \left(M^* z_k^{(i)}\right)(x) \geqslant \lambda\right\}\right) < \min(\varepsilon, \delta). \tag{1.10}$$

For each $1 \leqslant i \leqslant m$ and $k \in \mathbb{N}$, we may choose

$$H_{i,k}^\lambda = \left\{x \in \mathbb{R}^n \mid \left((M^* z_k^{(i)}\right)(x)) < \lambda\right\}, \quad H_k^\lambda = \bigcap_{i=1}^m H_{i,k}^\lambda.$$

Thanks to Lemma 1.12 we know that for all $x, y \in H_k^\lambda$ and $1 \leqslant i \leqslant m$, there holds

$$\frac{\left|z_k^{(i)}(y) - z_k^{(i)}(x)\right|}{|y - x|} \leqslant c(n)\lambda.$$

By applying Lemma 1.13 we can choose $g_k^{(i)}$, a Lipschitz function extending $z_k^{(i)}$ outside H_k^λ, with a Lipschitz constant not greater than $c(n)\lambda$. Then, considering that H_k^λ is an open set, we have

$$g_k^{(i)}(x) = z_k^{(i)}(x), \quad \text{and} \quad Dg_k^{(i)}(x) = Dz_k^{(i)}(x) \quad \text{for all } x \in H_k^\lambda$$

and also that

$$\|Dg_k^{(i)}\|_{L^\infty(\mathbb{R}^n)} \leqslant c(n)\lambda.$$

We may also assume that

$$\|g_k^{(i)}\|_{L^\infty(\mathbb{R}^n)} \leqslant \|z_k^{(i)}\|_{L^\infty(H_k^\lambda)} \leqslant \lambda.$$

Thus, we may assume, at least for a subsequence, that

$$g_k^{(i)} \overset{\star}{\rightharpoonup} v^{(i)} \quad \text{in } W^{1,\infty}(\Omega), \quad 1 \leqslant i \leqslant m.$$

Set now $\left(g_k^{(1)}, \ldots, g_k^{(m)}\right) = g_k$ and $\left(v^{(1)}, \ldots, v^{(m)}\right) = v$. There holds

$$F(u + z_k, \Omega) \geqslant F(u + g_k, (\Omega \setminus A_\varepsilon) \cap H_k^\lambda) = F(u + g_k, \Omega \setminus A_\varepsilon) - F(u + g_k, (\Omega \setminus A_\varepsilon) \setminus H_k^\lambda).$$

Notice that

$$\mathscr{L}^n\left[(\Omega \setminus A_\varepsilon) \setminus H_k^\lambda\right] \leqslant \sum_{i=1}^m \mathscr{L}^n\left[(\Omega \setminus A_\varepsilon) \setminus H_{i,k}^\lambda\right] < m \min(\varepsilon, \delta). \tag{1.11}$$

Thanks to the equation (1.8), we get

$$\begin{aligned} F(u + g_k, (\Omega \setminus A_\varepsilon) \setminus H_k^\lambda) &\leqslant \int_{(\Omega \setminus A_\varepsilon) \setminus H_k^\lambda} \left[a(x) + C\left(|u + g_k|^p + |Du + Dg_k|^p\right)\right] d\mathscr{L}^n(x) \\ &\leqslant 2^{p-1} \int_{(\Omega \setminus A_\varepsilon) \setminus H_k^\lambda} a(x) + C\left(|u|^p + |Du|^p\right) d\mathscr{L}^n(x) \\ &\quad + 2^{p-1} \int_{(\Omega \setminus A_\varepsilon) \setminus H_k^\lambda} C\left(|g_k|^p + |Dg_k|^p\right) d\mathscr{L}^n(x). \end{aligned}$$

Consequently, by our choice of A_ε (as explained in equation (1.11)), equation (1.9) and the properties of g_k, we get

$$\begin{aligned}F(u+g_k,(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda) \leqslant 2^{p-1}&\int_{(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda} a(x)+C\big(|u|^p+|Du|^p\big)\,d\mathscr{L}^n(x)\\ &+2^{p-1}\int_{(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda} C\big(|g_k|^p+|Dg_k|^p\big)\,d\mathscr{L}^n(x)\\ &\leqslant 2^{p-1}\left\{\eta(m\varepsilon)+c(n,\Omega)\lambda^p\mathscr{L}^n\left[(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda\right]\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}F(u+g_k,(\Omega\setminus A_\varepsilon)\setminus H_k^\lambda) \leqslant & 2^{p-1}\left\{\eta(m\varepsilon)+c(n,\Omega)\sum_{i=1}^m\int_{((\Omega\setminus A_\varepsilon)\setminus H_k^\lambda)}\left[\left(M^*z_k^{(i)}(x)\right)\right]^p\,d\mathscr{L}^n(x)\right\}\\ &\leqslant 2^{p-1}\Big\{\eta(m\varepsilon)+m\,c(n,\Omega)\varepsilon\Big\}=O(\varepsilon).\end{aligned}$$

Therefore,

$$F(u+z_k,\Omega)\geqslant F(u+g_k,(\Omega\setminus A_\varepsilon))-O(\varepsilon).$$

As the functions g_k are uniformly bounded in $W^{1,\infty}(\Omega,\mathbb{R}^n)$, we can select an open set $\Omega'\subseteq\Omega$ containing $\Omega\setminus A_\varepsilon$ such that

$$|F(u+g_k,\Omega')-F(u+g_k,\Omega\setminus A_\varepsilon)|<\varepsilon.$$

We now apply Theorem 1.14 to the functional

$$\Gamma(w,S)=\int_S f(x,u(x)+w(x),Du(x)+Dw(x))\,d\mathscr{L}^n(x),$$

obtaining

$$\lim_{k\to\infty}F(u+z_k,\Omega)\geqslant\liminf_{k\to\infty}F(u+g_k,\Omega')-\varepsilon-O(\varepsilon)\geqslant F(u+v,\Omega')-\varepsilon-O(\varepsilon).$$

Passing to a subsequence, if necessary, we may assume that $z_k(x)\to 0$ for almost every $x\in\Omega$. We can put

$$G=\{x\in\Omega\,|\,v(x)\neq 0\}\quad\text{and}\quad\widetilde{G}=G\cap\{x\in\Omega\,|\,z_k(x)\to 0\},$$

then, $\mathscr{L}^n(\widetilde{G})=\mathscr{L}^n(G)$.
As the functions g_k are continuous and convergent to v in L^∞, we have

$$g_k(x)\to v(x)$$

for all $x\in\Omega$ and consequently, for all $x\in G$. Then, if we now assume

$$\mathscr{L}^n(G)>(m+1)\varepsilon,$$

we are led to a contradiction. Indeed, from equation (1.10), we have

$$\mathscr{L}^n(\widetilde{G}\cap H_k^\lambda)>\varepsilon,$$

for all $k\in\mathbb{N}$ and using Lemma 1.7, we get

$$\bigcap_{h\in\mathbb{N}}H_{k_h}^\lambda\cap\widetilde{G}\neq\varnothing.$$

If x belongs to this set, there holds

$$v(x) = \lim_{h\to\infty} g_{k_h}(x) = \lim_{h\to\infty} z_{k_h}(x) = 0$$

and this is in contrasts the definition of G.
Hence, we can finally write, by the positivity of f,

$$\lim_{k\to\infty} F(u + z_k, \Omega) \geqslant F(u, \Omega' \setminus G) - O(\varepsilon) - \varepsilon \geqslant F(u, \Omega) - O(\varepsilon) - \varepsilon - \eta[(m + 2)\varepsilon]$$

and the proof is concluded, as $\varepsilon > 0$ is arbitrary. □

REMARK 1.23. The hypothesis about the non–negativity of f is fundamental, indeed, if we assume the same hypotheses with the only difference that

$$|f(x, s, \xi)| \leqslant a(x) + C(|s|^p + |\xi|^p)$$

the theorem is false, at least for $n > 2$; but for all $\varepsilon > 0$ the functional $u \mapsto F(u, \Omega)$ is sequentially weakly lower semicontinuous on $W^{1,p+\varepsilon}(\Omega; \mathbb{R}^m)$. The proof of this fact can be found in [11].

We conclude this chapter "condensing" the previous theorems in a single one.

THEOREM 1.24. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ be a Carathéodory function such that*

$$0 \leqslant f(x, s, \xi) \leqslant a(x) + C(|s|^p + |\xi|^p),$$

for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{n\times m}$, where C is a non–negative constant and a is a non–negative locally summable function on $\mathbb{R}^n$, for some $p \geqslant 1$ (or alternately it satisfies the third condition of Theorem 1.14).
Then, the functional $u \mapsto F(u, \Omega)$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$, or weakly lower semicontinuous on $W^{1,\infty}(\Omega, \mathbb{R}^m)$ if and only if f is quasi–convex in ξ.*

REMARK 1.25. We underline that all the previous theorems remain valid for any open set $\Omega \subseteq \mathbb{R}^n$.

CHAPTER 2

Sobolev spaces on Riemannian manifolds

In this chapter, we define and discuss some properties of Sobolev spaces on Riemannian manifolds, with the aim to extend the results of the previous chapter to the Riemannian setting.

2.1. Preliminaries of Riemannian geometry

We briefly recall some basic facts of Riemannian geometry. Possible references for this section are the books [1, 12, 16], for instance.

Given a differentiable manifold M, we let TM the tangent bundle of M, whose section are the vector fields and TM^* the cotangent bundle, whose section are the differential 1–forms. We denote with $T^r_s M$ the vector bundle given by the union of the vector spaces $T^r_s M_p = \otimes^s T^*_p M \otimes^r T_p M$, for $p \in M$ and its sections $\Gamma(T^r_s M)$ are the tensor of type (r, s). In local coordinates, if $T \in \Gamma(T^r_s M)$, we have

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} \, dx^{j_1} \otimes \dots dx^{j_s} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}},$$

with

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} = T\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}}, \, dx^{i_1}, \dots, \, dx^{i_r} \right)$$

(with the Einstein convention of summing over repeated indexes, that we will adopt from now on). Moreover, since there exists a natural linear isomorphism between $T^r_s M_p$ and the space of multilinear maps from $\oplus^s T_p M \oplus^r T_p M^*$ to $\mathbb{R}$ for every $p \in M$, a tensor $T \in \Gamma(T^r_s M)$ can be seen as a $C^\infty(M)$–linear function from the $C^\infty(M)$–module $\Gamma(\oplus^s TM \oplus^r TM^*)$ to $C^\infty(M)$.

DEFINITION 2.1. A *Riemannian manifold* (M, g) is a differential manifold M endowed with a *metric tensor* $g \in \Gamma(T^0_2 M)$, which is a symmetric bilinear form positive definite at every point $p \in M$.

By means of the metric tensor, we are then able to measure the *length* of a C^1 curve $\gamma : I \to M$ in (M, g), as follows

$$\mathscr{L}(\gamma) = \int_I |\dot{\gamma}(t)|_{g_{\gamma(t)}} \, dt,$$

where, in general $|v|_{g_p} = \sqrt{g_p(v, v)}$, for every vector $v \in T_p M$. Hence, we can define the *Riemannian distance* d^M as

$$d^M(p, q) = \inf \big\{ \mathscr{L}(\gamma) \,\big|\, \gamma : I \to \mathbb{R} \text{ is a } C^1 \text{ curve between } p \text{ and } q \big\},$$

turning M in a metric space whose topology coincides with the original topology of M as a differential manifold.

The metric g induces on M a canonical volume measure μ which in a local coordinate chart can be written as $d\mu = \sqrt{\det g_{ij}}\, d\mathscr{L}^n$, where $\mathscr{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$.

THEOREM 2.2. *If (M, g) is a Riemannian manifold, there exists a unique affine connection ∇, called* Levi–Civita connection, *which is symmetric and compatible with the metric, that is, an application*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(X, Y) \mapsto \nabla_X Y$$

such that:

- *it is $C^\infty(M)$–linear in X and $\mathbb{R}$–linear in Y,*
- *the following formula holds*

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y,$$

  for every $X, Y \in \Gamma(TM)$ and every $f \in C^\infty(M)$,
- *it satisfies (*symmetry*)*

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

  for every $X, Y \in \Gamma(TM)$,
- *it satisfies (*compatibility with the metric*)*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

  for every $X, Y, Z \in \Gamma(TM)$.

DEFINITION 2.3. Given a smooth function $u : M \to \mathbb{R}$, its *gradient* is the unique vector field ∇u such that at every point $p \in M$ there holds $g_p(\nabla u(p), v) = du_p(v)$, for every $y \in T_pM$. Given a smooth vector field X on M, its *divergence* is defined as

$$\mathrm{div} X = \sum_{i=1}^{n} \left(\nabla_i X\right)^i .$$

The Levi–Civita connection induces an associated *covariant derivative along curves* which we will denote with $\frac{d}{dt}$. Then, a curve $\gamma : I \to M$ is a *geodesic* in M if

$$\frac{d}{dt}\dot\gamma(t) = 0,$$

for every $t \in I$.

By the theorem by Hopf–Rinow (see [12], for instance), if M with the Riemannian distance is a complete metric space, for every point $p \in M$ and vector $v \in T_pM$, there exists a unique geodesic $\gamma_v : \mathbb{R} \to M$ such that $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. Then, for every $p \in M$, the *exponential map* $\exp_p : T_pM \to M$, is defined as

$$\exp_p(tv) = \gamma_v(t).$$

In the whole thesis, (M, g) will be a smooth, connected and complete (as a metric space) Riemannian manifold without boundary, unless otherwise specified.

DEFINITION 2.4. The *injectivity radius at $p \in M$* is defined as

$$\mathrm{inj}(p) = \sup\left\{\varepsilon > 0 \,\middle|\, \exp_p \text{ is a diffeomorphism between } B_\varepsilon(O_p) \text{ and an open set of } M\right\}$$

where $B_\varepsilon(O_p) \subseteq T_pM$ is the open ball of center in O_p and radius $\varepsilon > 0$:

$$B_\varepsilon(O_p) = \{v \in T_pM \,|\, |v|_p < \varepsilon\}$$

and it is well known to be positive for every $p \in M$.
The *injectivity radius of M* is then $\inf_{p \in M} \mathrm{inj}(p)$.

We now introduce the notion of curvature of a Riemannian manifold.

DEFINITION 2.5. The *Riemann operator* is the application

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

defined by

$$R(X,Y)Z = \nabla^2_{Y,X}Z - \nabla^2_{X,Y}Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z,$$

for every X, Y, Z vector fields on M.

This operator measures the "error" in interchanging the covariant derivatives with respect to X and Y of a vector field.

DEFINITION 2.6. The *Riemann tensor* is the $(0,4)$ tensor defined as

$$R(X,Y,Z,W) = g(R(X,Y)Z, W),$$

for every X, Y, Z, W vector fields on M, that is, the $(0,4)$–version of the Riemann operator. We will denote it with Riem.

In local coordinates, we have

$$R_{ijkl} = \left(\frac{\partial \Gamma^m_{ij}}{\partial x^j} - \frac{\partial \Gamma^m_{jk}}{\partial x^j} + \Gamma^s_{ij}\Gamma^m_{js} - \Gamma^s_{jk}\Gamma^m_{is}\right) g_{ml},$$

where $\Gamma^k_{ij} = \left(\nabla_{\partial/\partial x_j}\partial/\partial x_j\right)^k$ are the *Christoffel symbols of the Levi–Civita connection*.

DEFINITION 2.7. The *Ricci tensor* Ric is the $(0,2)$–tensor defined by $(X,Y) \mapsto R(X,Y)$ equal to the trace of the endomorphism $Z \to R(X,Z)Y$, for every pair of vector fields X, Y on M.

In local coordinates, there holds

$$R_{ik} = g^{jl}R_{ijkl} = g^{jl}R_{jilk} = -g^{jl}R_{ijkl} = -g^{jl}R_{jikl},$$

by the symmetries of the Riemann tensor, moreover, it also follows that Ric is symmetric, that is $R_{ik} = R_{ki}$.

DEFINITION 2.8. The *scalar curvature* $R \in C^\infty(M)$ is the trace of the Ricci tensor

$$\mathrm{R} = \mathrm{tr}\,\mathrm{Ric} = g^{ij}R_{ij}.$$

2.2. Sobolev spaces

We will follow the book by Hebey [13] in defining and studying the Sobolev spaces on a Riemannian manifold. Moreover, we will refer to the book of Evans [9], for the standard material.

Let (M, g) be a smooth n–dimensional, connected, complete and oriented Riemannian manifold without boundary.

DEFINITION 2.9. Let $u \in L^1_{\mathrm{loc}}(M)$. A *weak gradient of* u is a vector field $\nabla u \in L^1_{\mathrm{loc}}(M)$ such that the following relation holds:

$$\int_M u \,\mathrm{div} X \, d\mu = -\int_M g(\nabla u, X)\, d\mu,$$

for every smooth vector field X on M with compact support (see Definition 2.3 for the divergence of X). If such a vector field exists, it is easy to see that it is uniquely determined μ–almost everywhere on M.

When a scalar product is present, we have a canonical isomorphism between a vector space and its dual, hence, we can also define the *weak differential of* u as the 1–form du on M such that $du(X) = g(\nabla u, X)$ for every smooth vector field X. Hence, it follows that

$$\int_M u \,\mathrm{div} X \, d\mu = -\int_M du(X)\, d\mu.$$

Since the above association $\nabla u \leftrightarrow du$ is an isometry, all the definitions, convergences and estimates in the sequel can be equivalently stated for du.

As the metric tensor and the Levi–Civita connection can be both extended to all the tensor spaces $T^r_s M$ (see [12] for instance), one can define the iterated covariant derivatives $\nabla^k u$ of a smooth function u, for every $k \in \mathbb{N}$ and with a similar definition to the "distributional" one above, their weak versions. Anyway, in order to keep things simple, an equivalent definition of the weak k–times derivative of u is obtained working in coordinates and computing the weak k–times covariant derivative as if one is in the Euclidean space (with all the needed "correction terms" given by the Christoffel symbols and their smooth derivatives). Hence, we can speak of the *weak covariant derivatives* $\nabla^k u \in L^1(M)$ of order $k \in \mathbb{N}$.

We are then ready to define the Sobolev spaces on a Riemannian manifold.

DEFINITION 2.10. Let $1 \leqslant p < +\infty$, then we define

$$W^{1,p}(M) = \big\{u : M \to \mathbb{R} \,\big|\, u \in L^p(M),\ \nabla u \in L^p(M)\big\}$$

with the norm

$$\|u\|_{W^{1,p}(M)} = \left(\int_M |u|^p \, d\mu\right)^{1/p} + \left(\int_M |\nabla u|^p \, d\mu\right)^{1/p}$$

and

$$W^{1,\infty}(M) = \big\{u : M \to \mathbb{R} \,\big|\, u \in L^\infty(M), \nabla u \in L^\infty(M)\big\}$$

with the norm

$$\|u\|_{W^{1,\infty}(M)} = \operatorname*{ess\,sup}_M |u| + \operatorname*{ess\,sup}_M |\nabla u|,$$

where ∇u is the weak gradient of u.
Moreover, (keeping onto account what we said above), we let

$$W^{k,p}(M) = \{u : M \to \mathbb{R} \mid u, \nabla^\ell u \in L^p(M) \text{ for every } \ell \in \{1, ..., k\}\}$$

and

$$W^{k,\infty}(M) = \{u : M \to \mathbb{R} \mid u, \nabla^\ell u \in L^\infty(M) \text{ for every } \ell \in \{1, ..., k\}\}.$$

where $\nabla^\ell u$ for $\ell \in \{1, ..., k\}$, denotes the weak covariant derivative of order $i \in \mathbb{N}$.

The following propositions are easy consequences of the definition above.

PROPOSITION 2.11. *If M is a compact Riemannian manifold, the space $W^{k,p}(M)$ does not depend on the Riemannian metric.*

REMARK 2.12. We easily observe that this property holds only for compact manifolds. Indeed, consider a non–compact manifold M endowed with two metrics, one for which the volume of is finite and one for which the volume is infinite. Then, the constant functions belong to the Sobolev spaces associated with the finite–volume metric, but they does not belong to the other.

PROPOSITION 2.13. *If $1 < p < +\infty$, the space $W^{k,p}(M)$ is a reflexive Banach space.*

PROPOSITION 2.14. *The space $W^{k,2}(M)$ is a Hilbert space if we consider the equivalent norm*

$$\|u\| = \sqrt{\sum_{\ell=0}^{k} \int_M |\nabla^\ell u|^2 \, d\mu},$$

associated to the scalar product

$$\langle u, v \rangle = \sum_{\ell=0}^{k} \int_M g^{i_1 j_1} ... g^{i_\ell j_\ell} (\nabla^\ell u)_{i_1 \dots i_\ell} (\nabla^\ell v)_{j_1 \dots j_\ell} \, d\mu.$$

Now we want to discuss the density properties of the smooth maps in these spaces. We let

$$C^{k,p}(M) = \left\{ u \in C^\infty(M) \,\middle|\, \int_M |\nabla^j u|^p \, d\mu < +\infty \text{ for every } \ell \in \{1, ..., k\} \right\}$$

and we define $H^{k,p}(M)$ as the closure of $C^{k,p}(M)$ with respect to the norm

$$\|u\|_{H^{k,p}(M)} = \sum_{\ell=0}^{k} \left(\int_M |\nabla^\ell u|^p \, d\mu \right)^{1/p}.$$

Then, the following analogue of Meyers–Serrin's theorem in [21] holds (see [5], for instance).

THEOREM 2.15. *Let $k \in \mathbb{N}$ and $p \in [1, +\infty)$. Let $u \in W^{k,p}(M)$, then there exists a sequence $u_n \in C^\infty(M)$ such that $u_n \to u$ in $W^{k,p}(M)$. It follows that $W^{k,p}(M)$ is separable,*

We let $\mathcal{D}(M)$ denote the space of the C^∞ functions with compact support in M.

DEFINITION 2.16. The Sobolev space $W_0^{k,p}(M)$ is the closure of $\mathcal{D}(M)$ in $W^{k,p}(M)$.

The following two theorems gives conditions in order that these two spaces actually coincide.

THEOREM 2.17 ([13, Theorem 2.7]). *If* (M, g) *is a complete Riemannian manifold, then, for every* $p \in [1, +\infty)$*, we have* $W^{1,p}(M) = W_0^{1,p}(M)$.

THEOREM 2.18 ([13, Theorem 2.8]). *Let* (M, g) *be a complete Riemannian manifold with positive injectivity radius and let* $k \geqslant 2$ *If there exist constants* C_ℓ *such that* $|\nabla^\ell \mathrm{Ric}| \leqslant C_\ell$ *for any* $\ell \in \{0, ..., k-2\}$*, then, for any* $p \in [1, +\infty)$*, we have* $W^{k,p}(M) = W_0^{k,p}(M)$.

CHAPTER 3

Quasi–convexity in the Riemannian setting

In this chapter, after introducing a suitable "Riemannian" definition of quasi–convexity generalizing the "classical" Euclidean one, we will prove that the weak* sequential lower semicontinuity in $W^{1,\infty}$ of a functional holds if and only if the integrand is quasi–convex, according to such definition.

Let (M, g) be a smoooth, connected and complete Riemannian manifold of dimension n. We fix some notation for this chapter.

For every $x \in M$ and $r > 0$ smaller than the injectivity radius at x, we define the cube

$$\widetilde{Q}_x^r = r\left(-\frac{1}{2}, \frac{1}{2}\right)^n \subseteq T_xM \approx \mathbb{R}^n$$

(where the identification of T_xM with $\mathbb{R}^n$ is done via an othornormal basis of T_xM) and its diffeomorphic image

$$Q_x^r = \exp_x\left(\widetilde{Q}_x^r\right) \subseteq M$$

by means of the exponential map

$$\exp_x : T_xM \to M.$$

We now define $\mathscr{L}(TM, \mathbb{R}^m)$ as the space of the linear maps between a tangent space of M and $\mathbb{R}^m$, that is,

$$\mathscr{L}(TM, \mathbb{R}^m) = \left\{\alpha : T_xM \to \mathbb{R}^m \;\middle|\; x \in M \text{ and } \alpha \text{ is linear}\right\}.$$

This space has a natural structure of vector bundle via the local parametrizations of M, for instance if $m = 1$ it coincides with T^*M and in general it is the union of all the spaces $(T_x^*M)^m$ for $x \in M$ (which are the fibers on the points of M). It is clearly locally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^{n\times m}$, indeed, choosing a local orthonormal frame $E_1, \dots, E_n$ in a neighbourhood $U \subseteq M$ (diffeomorphic to $\mathbb{R}^n$) of a point $x_0 \in M$, in order to locally "trivialize" the vector bundle, we define the map $I : \mathscr{L}(TU, \mathbb{R}^m) \to \mathbb{R}^{n\times m}$ as

$$\alpha \mapsto I\alpha = \big(\alpha(E_1), \dots, \alpha(E_n)\big) \in (\mathbb{R}^m)^n, \tag{3.1}$$

for every $\alpha \in \mathscr{L}(TU, \mathbb{R}^m)$, hence the map sending any $\alpha : T_xM \to \mathbb{R}^m$ to

$$(x, I\alpha) = \big(x, \alpha(E_1), \dots, \alpha(E_n)\big) \in U \times (\mathbb{R}^m)^n \approx \mathbb{R}^n \times \mathbb{R}^{n\times m}$$

is a diffeomorphism.

Then, considering the standard quadratic norm $\|\cdot\|$ on $(\mathbb{R}^m)^n \approx \mathbb{R}^{n\times m}$, we have a distance δ on $\mathscr{L}(TU, \mathbb{R}^m)$ defined as follows:

$$\delta(\alpha, \beta) = d^M(x, y) + \|I\alpha - I\beta\| = d^M(x, y) + \sqrt{\sum_{i=1}^n \sum_{j=1}^m \left|\alpha^j(E_i) - \beta^j(E_i)\right|^2} \tag{3.2}$$

for every $\alpha, \beta \in \mathscr{L}(TU, \mathbb{R}^m)$ such that $\alpha \in (T_x^*M)^m$ and $\beta \in (T_y^*M)^m$, where d^M is the Riemannian distance on M. It is then easy to see that such distance, restricted to every fiber $(T_x^*M)^m$ coincides with the one associated to the "quadratic" norm (equivalent to the operatorial one) induced by the metric tensor g of M, that we will denote with $|\cdot|_{g_x}$.
Clearly, if we have a function $u : M \to \mathbb{R}^m$, its differential map $x \mapsto du[x] : T_xM \to \mathbb{R}^m$ is a section of $\mathscr{L}(TM, \mathbb{R}^m)$.

We chose to adopt the notation $du[x]$ for the differential of a map $u : M \to \mathbb{R}^m$ at a point $x \in M$, for sake of clarity in the computations that follow.

DEFINITION 3.1. Let (M, g) be a smoooth, connected and complete Riemannian manifold and μ its canonical volume measure. A continuous function $f : \mathscr{L}(TM, \mathbb{R}^m) \to \mathbb{R}$ is called *quasi–convex* if for every $x_0 \in M$, $\alpha_{x_0} \in (T_{x_0}M)^m \subseteq \mathscr{L}(TM, \mathbb{R}^m)$ and $\varphi \in C_c^\infty(Q_{x_0}^r, \mathbb{R}^m)$, there holds

$$f(\alpha_{x_0}) \leqslant \fint_{Q_{x_0}^r} f\big(\alpha_{x_0} + d\varphi[x] \circ d\exp_{x_0}[\exp_{x_0}^{-1}(x)]\big)\, d\mu(x) + o(1), \tag{3.3}$$

where $o(1)$ is a function which goes to zero as $r \to 0$ and depends in a monotonically non-decreasing way only on the L^∞ norm of $d\varphi$ (for fixed x_0 and α_{x_0}).

REMARK 3.2. We mention that another definition of quasi–convexity for maps defined on manifolds was given in [17].

REMARK 3.3. The main obstacle in generalizing to the Riemannian context the usual definition of quasi–convexity in the Euclidean ambient, is due to the fact that we cannot identify all the tangent spaces of a manifold as in $\mathbb{R}^n$, hence two differentials at two different points cannot be added together. Hence, in the definition above we needed to morally "carry" the differential of the perturbation at every point of the manifold to be an element of $T_{x_0}^*M$, via the differential of the exponential map at $x_0 \in M$ (exponential map that in the Euclidean case would simply be the identity, under the identification mentioned above – see also the following discussion). This forces the introduction of a "correction term" in the quasi–convexity inequality that one reasonably expects it is going to zero as we get closer and closer to the point x_0, since the differential of the exponential map then tends to be the identity.

If (M, g) is $\mathbb{R}^n$ with its standard metric, for every point $x \in \mathbb{R}^n$ we have a standard identification $T_x^*\mathbb{R}^n \approx T_x\mathbb{R}^n \approx \mathbb{R}^n$, hence a function $f : \mathscr{L}(T\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ can be one–to–one associated with a function $\widetilde{f} : \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}$ as follows: if $A = A_i^j \in \mathbb{R}^{n\times m}$, for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, then,

$$\widetilde{f}(x_0, A) = f(\alpha_{x_0}), \tag{3.4}$$

where $\alpha_{x_0} \in (T_{x_0}^*\mathbb{R}^n)^m \approx \mathbb{R}^{n\times m}$ is given by

$$\alpha_{x_0}^j(v) = \sum_{i=1}^n A_i^j v^j,$$

for every vector $v = (v^1, \dots, v^n) \in T_{x_0}\mathbb{R}^n \approx \mathbb{R}^n$ and viceversa, if α_{x_0} is operating as in this formula, then f is defined by equality (3.4).
Then, with this one–to–one correspondence, our "Riemannian" definition of quasi–convexity in the case of the Euclidean space, is equivalent to

$$\widetilde{f}(x_0, A) \leqslant \fint_{Q_{x_0}^r} \widetilde{f}\big(x, A + D\varphi(x)\big)\, d\mathscr{L}^n(x) + o(1),$$

for every $x_0 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times m}$ and $\varphi \in C_c^\infty(Q_{x_0}^r, \mathbb{R}^m)$, since, under the identification $T_x^*\mathbb{R}^n \approx T_x\mathbb{R}^n \approx \mathbb{R}^n$, all the exponential maps are the identity and it is clearly satisfied by a continuous quasi–convex function $\widetilde{f} : \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}$, actually without the "correction term" $o(1)$, according to the usual Definition 1.2 of quasi–convexity.
Hence, a "classical" quasi–convex function $\widetilde{f}$ in the Euclidean space, having as arguments only x and A, is also quasi–convex in the sense of Definition 3.1 (with the above identification of $\widetilde{f}$ with f). Viceversa, if the function f satisfies Definition 3.1, then, by Theorem 3.4 that we are going to show in the next section and the identification of $\widetilde{f}$ with f, it follows that the functional

$$u \mapsto F(u,\Omega) = \int_\Omega \widetilde{f}\big(x, Du(x)\big)\, d\mathscr{L}^n(x)$$

is sequential lower semicontinuous in the weak* topology of $W^{1,\infty}(\Omega,\mathbb{R}^m)$, hence $\widetilde{f}$ is quasi–convex according to the Definition 1.2, by the Acerbi–Fusco Theorem 1.17.
Thus, our Definition 3.1 is an extension to Riemannian manifolds the usual definition of quasi–convexity for continuous integrands.

3.1. Quasi–convexity and semicontinuity

We are going to show the analogues of some special cases of Theorems 1.14 and 1.17 of Acerbi and Fusco in our context, following the line of [4] (we mention that the main argument in the next Theorem 3.4 is due to Fonseca–Muller [14]). We underline (see the discussion above) that we are generalizing the special case of integrands which are continuous and depends only on x and Du (*but not on* u – see the final section), from the Euclidean to the Riemannian setting.

The weak* convergence of a sequence $u_j \overset{\star}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega,\mathbb{R}^m)$ is defined in the usual way: the sequences of integrals of u_j and du_j "against" fixed functions and 1–forms in $L^1(\Omega,\mathbb{R}^m)$, respectively, converge to the analogous integrals relative to u and du. Moreover, we notice that by the weak* sequential compactness of the closed unit ball of $L^\infty(\Omega,\mathbb{R}^m)$ given by the Banach–Alaoglu–Bourbaki theorem, we can simply ask that the sequence of differentials du_j is bounded and that u_j weakly* converges to u in $L^\infty(\Omega,\mathbb{R}^m)$.

THEOREM 3.4. *Let (M,g) be a smooth, connected and complete Riemannian manifold and μ its canonical volume measure. Let $f : \mathscr{L}(TM,\mathbb{R}^m) \to \mathbb{R}$ be continuous and quasi–convex in the sense of Definition 3.1. Then, for every open and bounded subset $\Omega \subseteq M$, the functional*

$$u \mapsto F(u,\Omega) = \int_\Omega f(du)\, d\mu$$

is sequentially lower semicontinuous in the weak topology of $W^{1,\infty}(\Omega,\mathbb{R}^m)$, that is,*

$$F(u,\Omega) = \int_\Omega f\big(du\big)\, d\mu \leqslant \liminf_{j\to\infty} \int_\Omega f\big(du_j\big)\, d\mu = \liminf_{j\to\infty} F(u_j,\Omega),$$

for every sequence $u_j \overset{\star}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega,\mathbb{R}^m)$.

PROOF. Let $n \in \mathbb{N}$ be the dimension of M. Let $x_0 \in \Omega$ and we assume that $r > 0$ is small enough such that $Q_{x_0}^r$ is contained in a neighbourhood $U \subseteq \Omega$ of x_0 such that the vector bundle $W^{1,\infty}(\Omega,\mathbb{R}^m)$ can be "trivialized" as we discussed above at the beginning of this chapter, by choosing an orthonormal frame $E_1,\dots,E_n$ in U, with an associated map I and distance δ as in formulas (3.1) and (3.2). Moreover, since in what follows all the arguments

of the continuous function f will be bounded (since we are working in $W^{1,\infty}(\Omega, \mathbb{R}^m)$ and u_j is a bounded sequence in $W^{1,\infty}$, being weakly* convergent), we can assume that f is bounded and has a uniform modulus of continuity ω in $\mathscr{L}(TU, \mathbb{R}^m)$ (with respect to the distance δ) which is continuous, bounded and concave, hence subadditive. Moreover, since it is bounded, by possibly adding a constant to f, we can also assume that it is positive.
We consider a smooth function $\psi : Q^r_{x_0} \to [0, 1]$ with compact support and we set, for every $j \in \mathbb{N}$,

$$\varphi_j = \psi(u_j - u) \in W^{1,\infty}_0(Q^r_{x_0}, \mathbb{R}^m),$$

hence,

$$d\varphi_j = \psi\, d(u_j - u) + d\psi \otimes (u_j - u) \tag{3.5}$$

To simplify the notation in the computations below, we define

$$L_x = d \exp_{x_0}[\exp^{-1}_{x_0}(x)],$$

for every $x \in Q^r_{x_0}$ and we start by applying the hypothesis of quasi–convexity for f, observing that inequality (3.3) also holds for maps in $W^{1,\infty}_0(Q^r_{x_0}, \mathbb{R}^m)$, by approximating them with a sequence in $C^\infty_c(Q^r_{x_0}, \mathbb{R}^m)$, as the function $o(1)$ depends, by hypothesis, only on the L^∞ norm of the gradient of such maps.
Hence, setting $\alpha_{x_0} = du[x_0]$ in inequality (3.3), for every $j \in \mathbb{N}$, we have

$$\begin{aligned}
f\big(du[x_0]\big) &\leqslant \fint_{Q^r_{x_0}} f\big(du[x_0] + d\varphi_j[x] \circ L_x\big)\, d\mu(x) + o_j(1)\\
&= \fint_{Q^r_{x_0}} f\big(du[x_0] + \psi\, d(u_j - u)[x] \circ L_x + d\psi[x] \otimes (u_j - u) \circ L_x\big)\, d\mu(x) + o_j(1)\\
&= \fint_{Q^r_{x_0}} f\big(du[x_0] - \psi\, du[x] \circ L_x + \psi\, du_j[x] \circ L_x + d\psi[x] \otimes (u_j - u) \circ L_x\big)\, d\mu(x) + o_j(1),
\end{aligned}$$

where the function $o_j(1)$ depends monotonically on $\|d\varphi_j\|_\infty$, as in Definition 3.1.
Then, by the properties of the modulus of continuity ω of f, there holds

$$\begin{aligned}
&f\big(du[x_0] - \psi\, du[x] \circ L_x + \psi\, du_j[x] \circ L_x + d\psi[x] \otimes (u_j - u) \circ L_x\big) - f\big(du_j[x] \circ L_x\big)\\
&\quad\leqslant \omega\big(\big|du[x_0] - \psi\, du[x] \circ L_x + (\psi - 1)du_j[x] \circ L_x + d\psi[x] \otimes (u_j - u) \circ L_x\big|_{g_{x_0}}\big)\\
&\quad\leqslant \omega\big(\big|du[x_0] - \psi\, du[x] \circ L_x|_{g_{x_0}}\big) + \omega\big(\big|(\psi - 1)du_j[x] \circ L_x\big|_{g_{x_0}}\big)\\
&\qquad + \omega\big(\big|d\psi[x] \otimes (u_j - u) \circ L_x\big|_{g_{x_0}}\big)\\
&\quad\leqslant \omega\big(\big|du[x_0] - \psi\, du[x] \circ L_x|_{g_{x_0}}\big) + \omega\big(C|\psi - 1|\big) + \omega\big(\big|d\psi[x] \otimes (u_j - u) \circ L_x\big|_{g_{x_0}}\big),
\end{aligned}$$

where we kept into account that the distance δ on $\mathscr{L}(TU, \mathbb{R}^m)$, restricted to the fibers coincides with the one induced by the metric g of M. Thus, we obtain

$$\begin{aligned}
f\big(du[x_0]\big) \leqslant{}& \fint_{Q^r_{x_0}} f\big(du_j[x] \circ L_x\big)\, d\mu(x) + \fint_{Q^r_{x_0}} \omega\big(\big|du[x_0] - \psi\, du[x] \circ L_x|_{g_{x_0}}\big)\, d\mu(x)\\
&+ \fint_{Q^r_{x_0}} \omega\big(C|\psi - 1|\big)\, d\mu(x) + \fint_{Q^r_{x_0}} \omega\big(\big|d\psi[x] \otimes (u_j - u) \circ L_x\big|_{g_{x_0}}\big)\, d\mu(x) + o_j(1)\,.
\end{aligned} \tag{3.6}$$

We then deal with the term

$$\fint_{Q^r_{x_0}} f\big(du_j[x]\circ L_x\big)\,d\mu(x) \leqslant \fint_{Q^r_{x_0}} f\big(du_j[x]\big)\,d\mu(x) + \fint_{Q^r_{x_0}} \omega\big(\delta\big(du_j[x]\circ L_x, du_j[x]\big)\big)\,d\mu(x),$$

showing that the last integral is bounded by a function $o(1)$ independent of $j \in \mathbb{N}$.
Indeed, recalling formula (3.2), we have

$$\begin{aligned}\omega\big(\delta\big(du_j[x]\circ L_x, du_j[x]\big)\big) &= \omega\big(d^M(x_0,x) + \|\, Idu_j[x]\circ L_x - Idu_j[x]\,\|\big)\\ &\leqslant \omega\big(d^M(x_0,x)\big) + \omega\big(\|\, Idu_j[x]\circ L_x - Idu_j[x]\,\|\big)\\ &\leqslant \omega(r) + \omega\big(\|\, Idu_j[x]\circ L_x - Idu_j[x]\,\|\big)\end{aligned}$$

and setting $\alpha_k = \big(Idu_j[x]\big)_k = du_j[x](E_k) \in \mathbb{R}^m$, for every $k \in \{1,\dots,n\}$, we have

$$\begin{aligned}\big\|Idu_j[x]\circ L_x - Idu_j[x]\big\| &= \big\|I\big(du_j[x]\circ d\exp_{x_0}[\exp^{-1}_{x_0}(x)]\big) - Idu_j[x]\big\|\\ &= \Big\|\sum_{k=1}^n \alpha_k\big(d\exp_{x_0}[\exp^{-1}_{x_0}(x)](E_i)\big)^k - \alpha_i\Big\|\\ &= \Big\|\sum_{k=1}^n \alpha_k\Big\{\big(d\exp_{x_0}[\exp^{-1}_{x_0}(x)](E_i)\big)^k - \delta^k_i\Big\}\Big\|\\ &\leqslant C\|\alpha\|\,\big\|d\exp_{x_0}[\exp^{-1}_{x_0}(x)] - \mathrm{Id}_{T_{x_0}M}\big\|\\ &= C\big\|Idu_j[x]\big\|\,\big\|d\exp_{x_0}[\exp^{-1}_{x_0}(x)] - \mathrm{Id}_{T_{x_0}M}\big\|\\ &= \|du_j\|_\infty\, o(1) = o(1)\end{aligned} \tag{3.7}$$

for some constant C, as the norms $\|du_j\|_\infty$ are uniformly bounded.
Hence, inequality (3.6) becomes

$$\begin{aligned}f\big(du[x_0]\big) \leqslant{}& \fint_{Q^r_{x_0}} f\big(du_j[x]\big)\,d\mu(x) + \fint_{Q^r_{x_0}} \omega\big(\big|du[x_0] - \psi\, du[x]\circ L_x|_{g_{x_0}}\big)\,d\mu(x) + o(1)\\ &+ \fint_{Q^r_{x_0}} \omega\big(C|\psi-1|\big)\,d\mu(x) + \fint_{Q^r_{x_0}} \omega\big(\big|d\psi[x]\otimes(u_j-u)\circ L_x\big|_{g_{x_0}}\big)\,d\mu(x) + o_j(1)\end{aligned} \tag{3.8}$$

where the function $o(1)$ is independent by $j \in \mathbb{N}$, while the functions $o_j(1)$ are equal to $\eta(r, \|d\varphi_j\|)$ for some function η going to zero as $r \to 0$ and monotone nondecreasing (and we can also clearly assume continuous from the right) in its second argument.
Observing now that, by equation (3.5), there holds

$$\|d\varphi_j\|_\infty \leqslant \|\psi\|_\infty\,\|du_j - du\|_\infty + \|d\psi\|_\infty\,\|u_j - u\|_\infty,$$

hence, we have

$$\begin{aligned}o_j(1) = \eta(r, \|d\varphi_j\|_\infty) &\leqslant \eta(r, \|\psi\|_\infty\,\|du_j - du\|_\infty + \|d\psi\|_\infty\,\|u_j-u\|_\infty)\\ &\leqslant \eta(r, C + \|d\psi\|_\infty\,\|u_j - u\|_\infty),\end{aligned}$$

for some constant C uniformly bounding from above $\|\psi\|_\infty\,\|du_j - du\|_\infty$. It follows that

$$\limsup_{j\to\infty}\, o_j(1) \leqslant \limsup_{j\to\infty} \eta(r, C + \|d\psi\|_\infty\,\|u_j - u\|_\infty) = \eta(r, C) = o(1),$$

with a function $o(1)$ independent of $j \in \mathbb{N}$, by the properties of the function η and the fact that $\|u_j - u\|_\infty \to 0$ (by the theorem of Ascoli–Arzelà, being all the functions u_j equibounded and equicontinuous). Then, passing to the liminf as $j \to \infty$ in formula (3.8), we obtain

$$\begin{aligned} f\big(du[x_0]\big) \leqslant \liminf_{j\to\infty} \fint_{Q^r_{x_0}} f\big(du_j[x]\big)\, d\mu(x) + \fint_{Q^r_{x_0}} \omega\big(\big|du[x_0] - \psi\, du[x] \circ L_x|_{g_{x_0}}\big)\, d\mu(x) \\ + \fint_{Q^r_{x_0}} \omega\big(C|\psi - 1|\big)\, d\mu(x) + o(1)\,, \end{aligned}$$

as the last integral in such formula goes to zero, by the dominated convergence theorem. Now letting ψ go to the characteristic function of $Q^r_{x_0}$, the last integral in this formula vanishes (again by the dominated convergence theorem), hence

$$f\big(du[x_0]\big) \leqslant \liminf_{j\to\infty} \fint_{Q^r_{x_0}} f\big(du_j[x]\big)\, d\mu(x) + \fint_{Q^r_{x_0}} \omega\big(\big|du[x_0] - du[x] \circ L_x|_{g_{x_0}}\big)\, d\mu(x) + o(1)\,.$$

Finally, we want to show that the second integral on the right is bounded by a function $o(1)$. By arguing as above, when we dealt with du_j, we have

$$\begin{aligned} \omega\big(\big|du[x_0] - du[x] \circ L_x|_{g_{x_0}}\big) &\leqslant \omega\big(\delta\big(du[x_0], du[x]\big) + \delta\big(du[x], du[x] \circ L_x|_{g_{x_0}}\big)\big) \\ &\leqslant \omega\big(\delta\big(du[x_0], du[x]\big)\big) + \omega\big(\delta\big(du[x], du[x] \circ L_x|_{g_{x_0}}\big)\big) \\ &= d^M(x_0, x) + \omega\big(\big\|Idu[x_0] - Idu[x]\big\|\big) + o(1) \\ &= \omega\big(\big\|Idu[x_0] - Idu[x]\big\|\big) + o(1), \end{aligned}$$

hence,

$$\fint_{Q^r_{x_0}} \omega\big(\big|du[x_0] - du[x] \circ L_x|_{g_{x_0}}\big)\, d\mu(x) \leqslant \fint_{Q^r_{x_0}} \omega\big(\big\|Idu[x_0] - Idu[x]\big\|\big)\, d\mu(x) + o(1).$$

If now x_0 is a Lebesgue point for the function Idu, it is easy to show that the integral on the right goes to zero as $r \to 0$, being the function $t \mapsto \omega(t)$ continuous and bounded (and going to zero as $t \to 0$). Thus, we conclude that at μ–almost every point x_0 of Ω, there holds

$$f\big(du[x_0]\big) \leqslant \liminf_{j\to\infty} \fint_{Q^r_{x_0}} f\big(du_j\big)\, d\mu + o(1)\,. \tag{3.9}$$

As f and Ω are bounded, the following integral is finite,

$$\liminf_{j\to\infty} \int_\Omega f\big(du_j\big) d\mu = m < +\infty$$

and we can define the finite Radon measures ν_j on Ω, for every $j \in \mathbb{N}$, given by

$$d\nu_j = f\big(du_j\big)\, d\mu,$$

satisfying the uniform bound $\|\nu_j\| = \int_\Omega f\big(du_j\big)\, d\mu \leqslant C$. Hence, by the Banach–Alaoglu–Bourbaki theorem and without loss of generality, we may assume that ν_j weakly* converges to some limit Radon measure ν.

For every open subset G of Ω, by the properties of the weak* convergence, we have

$$\nu(G) \leqslant \liminf_{j\to\infty} \nu_j(G) = \liminf_{j\to\infty} \int_G f\big(du_j\big)\, d\mu \leqslant \liminf_{j\to\infty} \int_\Omega f\big(du_j\big)\, d\mu = m,$$

hence, by the outer regularity of the measure μ, it follows that $\nu \ll \mu$. Thus, we may apply the Radon–Nikodym theorem, obtaining a function $h : \Omega \to [0, +\infty]$ such that $d\nu = h\, d\mu$ and

$$h(x) = \lim_{r\to 0} \frac{\nu\big(\overline{Q}^r_x\big)}{\mu\big(\overline{Q}^r_x\big)},$$

for μ–almost every $x \in \Omega$.
Then, at the points $x_0 \in \Omega$ where inequality (3.9) is satisfied and the above limit holds, we have

$$\begin{aligned} f\big(du[x_0]\big) &\leqslant \liminf_{r\to 0}\left(\liminf_{j\to\infty} \fint_{Q^r_{x_0}} f\big(du_j\big)\, d\mu + o(1)\right) = \liminf_{r\to 0}\liminf_{j\to\infty} \frac{\nu_j\big(Q^r_{x_0}\big)}{\mu\big(Q^r_{x_0}\big)} \\ &= \liminf_{r\to 0}\liminf_{j\to\infty} \frac{\nu_j\big(\overline{Q}^r_{x_0}\big)}{\mu\big(\overline{Q}^r_{x_0}\big)} \leqslant \liminf_{r\to 0} \frac{\nu\big(\overline{Q}^r_{x_0}\big)}{\mu\big(\overline{Q}^r_{x_0}\big)} = h(x_0), \end{aligned}$$

as $\liminf_{j\to\infty} \nu_j\big(\overline{Q}^r_{x_0}\big) \leqslant \nu\big(\overline{Q}^r_{x_0}\big)$, being $\overline{Q}^r_{x_0}$ closed sets. Hence, $f\big(du[x]\big) \leqslant h(x)$ μ–almost everywhere in Ω, thus

$$F(u,\Omega) = \fint_\Omega f\big(du\big)\, d\mu \leqslant \fint_\Omega h\, d\mu = \nu(\Omega) \leqslant \liminf_{j\to\infty} \nu_j(\Omega) = \liminf_{j\to\infty} \fint_\Omega f\big(du_j\big)\, d\mu = \liminf_{j\to\infty} F(u_j,\Omega)$$

and the proof is complete. □

The following theorem is the converse of the previous one.

THEOREM 3.5. *Let (M, g) be a smooth, connected and complete Riemannian manifold and μ its canonical volume measure. Let $f : \mathscr{L}(TM, \mathbb{R}^m) \to \mathbb{R}$ be a continuous function. If for every open and bounded subset $\Omega \subseteq M$, the functional*

$$F(u,\Omega) = \int_\Omega f\big(du\big)\, d\mu$$

is sequentially lower semicontinuous in the weak topology of $W^{1,\infty}(\Omega, \mathbb{R}^m)$, then the function f is quasi–convex in the sense of Definition 3.1.*

PROOF. Let $n \in \mathbb{N}$ be the dimension of M. We adopt the same setting and notation as in the proof of Theorem 3.4, in particular, fixed $x_0 \in M$ and $\alpha_{x_0} \in (T^*_{x_0}M)^m$, we work in a neighbourhood $U \subseteq M$ of x_0 such that the vector bundle $\mathscr{L}(TU, \mathbb{R}^m)$ can be "trivialized" and $Q^r_{x_0} \subseteq U$.
We consider a smooth function $u : M \to \mathbb{R}^m$ such that $du[x_0] = \alpha_{x_0}$ (which clearly exists) and let $\varphi \in C^\infty_c\big(Q^r_{x_0}, \mathbb{R}^m\big)$, then the function $\varphi \circ \exp_{x_0}$ is defined on the open cube $r\left(-\frac{1}{2}, \frac{1}{2}\right)^n \subseteq T_{x_0}M$, smooth and with compact support. We define the extension by periodicity $\psi : T_{x_0}M \to \mathbb{R}^m$ of $\varphi \circ \exp_{x_0}$ to the whole $T_{x_0}M$ (defined zero on the boundaries of the cubes) and we set

$$\varphi_h(x) = \frac{1}{h}\psi(h \exp^{-1}_{x_0}(x))$$

for every $x \in Q^r_{x_0}$ and $h \in \mathbb{N}$. Then, the functions $\varphi_h : Q^r_{x_0} \to \mathbb{R}^m$ are smooth with compact support and

$$\varphi_h \overset{\star}{\rightharpoonup} 0 \quad \text{in } W^{1,\infty}_0(Q^r_{x_0}, \mathbb{R}^m)$$

as $h \to \infty$, indeed, clearly $\varphi_h \to 0$ and the differentials $d\varphi_h$ are uniformly bounded, holding

$$d\varphi_h[x] = \frac{1}{h} d\psi\big[h \exp^{-1}_{x_0}(x)\big] h \circ d\exp^{-1}_{x_0}[x] = d\psi\big[h \exp^{-1}_{x_0}(x)\big] \circ d\exp^{-1}_{x_0}[x].$$

Thus, since $u + \varphi_j \overset{\star}{\rightharpoonup} u$ in $W^{1,\infty}(Q^r_{x_0}, \mathbb{R}^m)$, by the hypothesis of lower semicontinuity of the functional F, there holds

$$\begin{aligned}\int_{Q^r_{x_0}} f\big(du[x]\big)\,d\mu(x) &\leqslant \liminf_{h\to\infty} \int_{Q^r_{x_0}} f\big(du[x] + d\varphi_h[x]\big)\,d\mu(x)\\ &= \liminf_{h\to\infty} \int_{Q^r_{x_0}} f\big(du[x] + d\psi\big[h\exp^{-1}_{x_0}(x)\big]\circ d\exp^{-1}_{x_0}[x]\big)\,d\mu(x),\end{aligned}$$

for every $r > 0$ small enough. Then, arguing as in the proof of Theorem 3.4 (more precisely, following the same argument leading to estimate (3.7)), considering a suitable modulus of continuiy ω of f, we have

$$\begin{aligned}&\big|f\big(du\,[x] + d\psi\big[h\exp^{-1}_{x_0}(x)\big]\circ d\exp^{-1}_{x_0}[x]\big) - f\big(du[x_0] + d\psi\big[h\exp^{-1}_{x_0}(x)\big]\circ \mathrm{Id}_{T_{x_0}M}\big)\big|\\ &\quad\leqslant \omega\big(\delta\big(du[x], du[x_0]\big)\big) + \omega\big(\delta\big(d\psi\big[h\exp^{-1}_{x_0}(x)\big]\circ d\exp^{-1}_{x_0}[x], d\psi\big[h\exp^{-1}_{x_0}(x)\big]\big)\big)\\ &\quad= \omega\big(d^M(x,x_0)\big) + \omega\big(\big\|Idu[x] - Idu[x_0]\big\|\big) + o(1)\|d\psi\|_\infty\\ &\quad= o(1)(1 + \|d\psi\|_\infty)\end{aligned}\tag{3.10}$$

as $r \to 0$, since the function $x \mapsto Idu[x]$ is smooth. Hence,

$$\int_{Q^r_{x_0}} f\big(du[x]\big)\,d\mu(x) \leqslant \liminf_{h\to\infty} \int_{Q^r_{x_0}} f\big(du[x_0] + d\psi\big[h\exp^{-1}_{x_0}(x)\big]\big)\,d\mu(x) + o(1)\mu(Q^r_{x_0}).\tag{3.11}$$

Changing variables as $y = \exp^{-1}_{x_0}(x)$, so $y \in r\big(-\frac{1}{2},\frac{1}{2}\big)^n \subseteq T_{x_0}M$ and denoting with dy the Lebesgue measure on $T_{x_0}M$ relative to the metric tensor g_{x_0}, we have

$$\begin{aligned}\int_{Q^r_{x_0}} f\big(du[x_0] + d\psi\big[h\exp^{-1}_{x_0}(x)\big]\big)\,d\mu(x) &= \int_{r\left(-\frac{1}{2},\frac{1}{2}\right)^n} f\big(du[x_0] + d\psi[hy]\big)\,\mathrm{J}(y)\,dy\\ &\leqslant \int_{r\left(-\frac{1}{2},\frac{1}{2}\right)^n} f\big(du[x_0] + d\psi[hy]\big)\,dy + Cr^{n+1},\end{aligned}\tag{3.12}$$

for some constant C independent of $h \in \mathbb{N}$, since the Jacobian

$$\mathrm{J}(y) = \Big|\det_{ij}\big(g_{\exp_{x_0}(y)}(d\exp_{x_0}[y](E_i), E_j)\big)\Big|$$

is a smooth function and goes uniformly to 1 as $r \to 0$, being $|y| \leqslant r\sqrt{n}/2$ (we recall that $d\exp_{x_0}[0])$ is the identity of $T_{x_0}M$) and $f\big(du[x_0] + d\psi[hy]\big)$ is bounded. Then, changing the variables again as $w = hy$ in the last integral, we get

$$\int_{Q^r_{x_0}} f\big(du[x_0] + d\psi\big[h\exp^{-1}_{x_0}(x)\big]\big)\,d\mu(x) \leqslant \frac{1}{h^n}\int_{hr\left(-\frac{1}{2},\frac{1}{2}\right)^n} f\big(du[x_0] + d\psi[w]\big)\,dw + Cr^{n+1}.$$

Now, by the periodicity of ψ, there holds

$$\frac{1}{h^n}\int_{hr\left[-\frac{1}{2},\frac{1}{2}\right]^n} f\big(du[x_0]{+}d\psi[w]\big)\,dw = \frac{1}{h^n}h^n\int_{r\left[-\frac{1}{2},\frac{1}{2}\right]^n} f\big(du[x_0]{+}d\varphi[\exp_{x_0}(w)]{\circ}d\exp_{x_0}[w]\big)\,dw$$

and if we change (back) variables as $x = \exp_{x_0}(w)$, we obtain

$$\begin{aligned}\int_{r\left[-\frac{1}{2},\frac{1}{2}\right]^n} & f\big(du[x_0] + d\varphi(\exp_{x_0}(w)) \circ d\exp_{x_0}[w]\big)\, dw \\ &= \int_{Q^r_{x_0}} f\big(du[x_0] + d\varphi[x] \circ d\exp_{x_0}[\exp^{-1}_{x_0}(x)]\big) \mathrm{J}(x)^{-1}\, d\mu(x) \\ &\leqslant \int_{Q^r_{x_0}} f\big(du[x_0] + d\varphi[x] \circ d\exp_{x_0}[\exp^{-1}_{x_0}(x)]\big)\, d\mu(x) + Cr\mu(Q^r_{x_0}),\end{aligned}$$

arguing as above. Hence, by this inequality where $h \in \mathbb{N}$ is not present in the right hand side and formulas (3.11), (3.12), we conclude

$$\int_{Q^r_{x_0}} f\big(du[x]\big)\, d\mu(x) \leqslant \int_{Q^r_{x_0}} f\big(du[x_0] + d\varphi[x] \circ d\exp_{x_0}[\exp^{-1}_{x_0}(x)]\big)\, d\mu(x) + o(1)\mu(Q^r_{x_0}),$$

as $\mu(Q^r_{x_0}) \approx \omega_n r^n$, where ω_n is the measure of the unit ball of $\mathbb{R}^n$.
Since, by the continuity of du, it easily follows that

$$\int_{Q^r_{x_0}} f\big(du[x_0]\big)\, d\mu(x) \leqslant \int_{Q^r_{x_0}} f\big(du[x]\big)\, d\mu(x) + o(1)\mu(Q^r_{x_0})$$

and $du[x_0] = \alpha_{x_0}$, we finally have

$$f(\alpha_{x_0}) \leqslant \frac{1}{\mu(Q^r_{x_0})} \int_{Q^r_{x_0}} f\big(\alpha_{x_0} + d\varphi[x] \circ d\exp_{x_0}[\exp^{-1}_{x_0}(x)]\big)\, d\mu(x) + o(1)$$

for every $x_0 \in M$ and with $o(1)$ going to zero as $r \to 0$, depending only on the L^∞ norm of $d\varphi$. Moreover, by tracing back how we obtained such function $o(1)$, it is clear that it can be chosen in a way that such dependence is monotonically nondecreasing (see in particular, estimate (3.10)). Hence, the function f is quasi–convex according to Definition 3.1. □

Putting together the two theorems, we have the following one which generalizes (in our case of f continuous) the results of Acerbi and Fusco for $p = +\infty$.

THEOREM 3.6. *Let (M, g) be a smooth, connected and complete Riemannian manifold and μ its canonical volume measure. Let $f : \mathscr{L}(TM, \mathbb{R}^m) \to \mathbb{R}$ be continuous. Then, f is quasi–convex in the sense of Definition 3.1 if and only if for every open and bounded subset $\Omega \subseteq M$, the functional*

$$u \mapsto F(u, \Omega) = \int_\Omega f(du)\, d\mu$$

is sequentially lower semicontinuous in the weak topology of $W^{1,\infty}(\Omega, \mathbb{R}^m)$.*

3.2. Some remarks and possible research directions

A natural continuation of our work would consist in extending the previous results also to the $W^{1,p}(\Omega, \mathbb{R}^m)$ setting, with $p \in [1, +\infty)$, in order to obtain the analogue of Theorem 1.22 (the converse fact that the semicontinuity of the functional implies the quasi–convexity of f follows immediately by Theorem 3.5 in such setting, as in the Euclidean case).

Focardi e Spadaro in [10] showed an extension of the Acerbi–Fusco theorems (with f continuous, like us) in the case of Sobolev maps $u : \Omega \to M$, with Ω an open bounded subset of $\mathbb{R}^n$ and M a Riemannian manifold (this case actually "correspond" to the Euclidean situation with a continuous function f depending on u and Du, but not on x). Therefore, a possible future research line could be to "combine" Theorem 3.6 with the results of Focardi and Spadaro, thus obtaining a characterization of quasi–convexity in the completely Riemannian context of Sobolev maps between two Riemannian manifolds.

Moreover, the full generalization of the Acerbi–Fusco results would be to show them for the "naturally defined" Carathéodory functions on a Riemannian manifold, that is, functions which Carathédory in the usual way after expressing them by means of the local "trivializations" of the bundle $\mathscr{L}(TM, \mathbb{R}^m)$ (or of $\mathscr{L}(TM, TN)$ after "combining" our case with the one of Focardi–Spadaro), as we did at the beginning of this chapter.

Finally, we mention that since a quasi–convex function is actually locally Lipschitz in the Euclidean case, a natural open question if this is true also in the Riemannian context with our definition of quasi–convexity.

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