A SHORT PROOF OF THE RIESZ–MARKOV REPRESENTATION THEOREM

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1. The Riesz–Markov representation theorem

We want to give a short proof of the following *Riesz–Markov representation theorem* in the case of a compact metric space (see, for example, the book of Rudin [4] for the more general case), based on Carathéodory extension theorem and the theorem of Fubini–Tonelli. It clearly applies to compact subset of $\mathbb{R}^n$, for instance.

Theorem 1.1 (Riesz–Markov representation theorem)**.** *Let (X,d) be a compact metric space and let Φ be a positive linear functional on $C(X)$, the vector space of the continuous functions. Then, there exists a unique regular Borel measure μ on X such that*

$$\Phi(f) = \int_X f\,d\mu,$$

for every $f \in C(X)$, where "regular" means that for every $E \in \mathscr{B}$, there holds

$$\mu(E) = \inf\{\mu(G) \mid G \supseteq E,\, G \in \mathscr{B} \text{ and } G \text{ open}\} = \sup\{\mu(K) \mid K \subseteq E,\, K \in \mathscr{B} \text{ and } K \text{ compact}\}. \tag{1.1}$$

Proof. Given two functions $u, v \in C(X)$ with $u \leqslant v$, we define

$$R_{u,v} = \{(x,t) \mid u(x) \leqslant t < v(x)\} \subseteq X \times \mathbb{R}$$

and we let $\mathscr{R}$ be the family of all these sets.
If Φ is a positive linear functional on $C(X)$, we define the set function $\rho : \mathscr{R} \to [0,+\infty)$ as

$$\rho(R_{u,v}) = \Phi(v-u).$$

It is easy to see that this is a good definition, as if $R_{u_1,v_1} = R_{u_2,v_2}$ we must have $v_1(x) - u_1(x) = v_2(x) - u_2(x)$ for every $x \in X$, hence $v_1 - u_1 = v_2 - u_2$ and

$$\rho(R_{u_1,v_1}) = \Phi(v_1 - u_1) = \Phi(v_2 - u_2) = \rho(R_{u_2,v_2}).$$

We want to see that ρ is a σ–finite *premeasure* on $\mathscr{R}$, meaning that:

- $\mathscr{R}$ is a *semiring*, that is, a family containing the empty set, closed under finite intersection and such that for any pair of sets $A, B \in \mathscr{R}$, the difference $A \setminus B$ can be written as a finite disjoint union of sets in $\mathscr{R}$.
- For every countable family of disjoint sets $R_n \in \mathscr{R}$ such that $\bigcup_{n=1}^{\infty} R_n \in \mathscr{R}$, we have
$$\rho\left(\bigcup_{n=1}^{\infty} R_n\right) = \sum_{n=1}^{\infty} \rho(R_n),$$
that is, ρ is σ–additive.
- There exists a countable family of sets $R_n \in \mathscr{R}$ such that $X \times \mathbb{R} = \bigcup_{n=1}^{\infty} R_n$ and $\rho(R_n) < +\infty$, for every $n \in \mathbb{N}$.

The fact that $\mathscr{R}$ is closed under finite intersections follows by the formula (with a little abuse of notation) $R_{u_1,v_1} \cap R_{u_2,v_2} = R_{\max\{u_1,u_2\},\min\{v_1,v_2\}}$, while letting $M = \max\{|u_1|,|u_2|,|v_1|,|v_2|\}$, there holds

$$R_{u_1,v_1} \setminus R_{u_2,v_2} = R_{u_1,v_1} \cap \mathscr{C}R_{u_2,v_2} R_{u_1,v_1} \cap (R_{-M,u_2} \cup R_{v_2,M}) = (R_{u_1,v_1} \cap R_{-M,u_2}) \cup (R_{u_1,v_1} \cup R_{v_2,M}),$$

where the unions are disjoint, showing that $\mathscr{R}$ is a semiring.
We clearly have $\varnothing \in \mathscr{R}$ and $X \times \mathbb{R} = \bigcup_{n \in \mathbb{Z}} R_{n,n+1}$, with $\rho(R_{n,n+1}) = \Phi(1)$, hence we only have to

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show the σ–additivity of ρ.
If $R_{u,v}$ is the disjoint union of R_{u_n,v_n} for $n \in \mathbb{N}$, it must be

$$v(x) - u(x) = \sum_{n=1}^{\infty} (v_n(x) - u_n(x)),$$

for every $x \in X$ and being this series with positive terms, the continuous functions $f_k = \sum_{n=1}^{k}(v_n - u_n)$ converge monotonically to $f = v - u$ which is also continuous. Hence, by Dini's theorem, this convergence is uniform and being Φ continuous (which is easily checked), we conclude

$$\rho(R_{u,v}) = \Phi(v-u) = \lim_{k\to\infty} \Phi\left(\sum_{n=1}^{k}(v_n - u_n)\right) = \lim_{k\to\infty}\sum_{n=1}^{k}\Phi(v_n - u_n) = \lim_{k\to\infty}\sum_{n=1}^{k}\rho(R_{u_n,v_n}) = \sum_{n=1}^{\infty}\rho(R_{u_n,v_n}).$$

Then, by the *Carathéodory extension theorem* (see [2, Theorem 1.53] and also [1], for instance), ρ can be uniquely extended to a positive measure ν on the σ–algebra $\mathscr{N}$ generated by $\mathscr{R}$. We define the measure μ on X as

$$\mu(E) = \nu(E \times [0,1]),$$

for every set $E \subseteq X$ belonging to the σ–algebra

$$\mathscr{M} = \{E \subseteq X \mid E \times [0,1] \in \mathscr{N}\}.$$

Notice that $\mathscr{M}$ contains every closed set $C \subseteq X$, indeed considering the continuous function $f(x) = 1 - \min\{d_C(x), 1\}$ and setting $R_n = R_{0,f^n} \in \mathscr{N}$, we have that $C \times [0,1] = \bigcap_{n\in\mathbb{N}} R_n$. Hence, $\mathscr{M}$ contains the Borel σ–algebra $\mathscr{B}$ of X.

We now see that for every function $f \in C(X)$, we have

$$\Phi(f) = \int_X f\,d\mu. \tag{1.2}$$

Indeed, it is easy to see that the measure ν on $X \times \mathbb{R}$ satisfies

$$\nu(E \times I) = \mu(E)\mathscr{L}^1(I)$$

for every $E \in \mathscr{B}$ and I an interval of $\mathbb{R}$, where $\mathscr{L}^1$ is the Lebesgue measure, so $\nu = \mu \times \mathscr{L}^1$.
For every nonnegative continuous function f in $C(X)$, we then have

$$\Phi(f) = \nu(R_{0,f}) = \int_{X\times\mathbb{R}} \chi_{[0,f(x)]}(t)\,d\nu = \int_{X\times\mathbb{R}} \chi_{[0,f(x)]}(t)\,d(\mu\times\mathscr{L}^1) = \int_X \mathscr{L}^1(0,f(x))\,d\mu(x) = \int_X f\,d\mu,$$

by Fubini–Tonelli theorem. The equality (1.2) then follows for a general function $f \in C(X)$, decomposing it into its positive and negative parts, by the linearity of Φ. Notice that, it also follows that the measure μ is the "projection" of ν on the first factor of $X \times \mathbb{R}$.

The regularity of μ is actually Theorem G, Section 52, Chapter X in the book of Halmos [1], we give a sketch of the argument. The family $\mathscr{E}$ of Borel sets E such that (1.1) holds is easily seen to be a σ–algebra. Then, every zero–set E of X can be clearly approximated from inside, moreover, it is intersection of open set in $\mathscr{B}$, indeed if $E = f^{-1}(0)$, we have $E = \bigcap_{n\in\mathbb{N}} f^{-1}(-1/n, 1/n)$, hence $E \in \mathscr{E}$. This shows that $\mathscr{E} = \mathscr{B}$.
About the uniqueness of μ, by the regularity property, it is sufficient to show it only on the compact subsets of X in $\mathscr{B}$: assume that $\mu_1(K) = \mu_2(K)$ for all compact $K \subseteq X$, whenever μ_1 and μ_2 are measures for which the theorem holds. So, fixing $K \in \mathscr{B}$ and $\varepsilon > 0$, there exists an open set $G \in \mathscr{B}$ such that $G \supseteq K$ with $\mu_2(V) < \mu_2(K) + \varepsilon$, then, by Urysohn lemma, we have a continuous function $f : X \to [0,1]$ such that $f = 1$ on K and $f = 0$ on $\mathscr{C}G$, hence

$$\mu_1(K) = \int_X \chi_K\,d\mu_1 \leqslant \int_X f\,d\mu_1 = \Phi(f) = \int_X f\,d\mu_2 \leqslant \int_X \chi_G\,d\mu_2 = \mu_2(G) < \mu_2(K) + \varepsilon.$$

Thus, $\mu_1(K) \leqslant \mu_2(K)$ by the arbitrariness of $\varepsilon > 0$ and if we interchange the roles of μ_1 and μ_2, the opposite inequality is obtained, hence the uniqueness of μ is proved and we are done. □

The above line works analogously if the metric space X is only *locally compact* and the positive linear functional Φ is only defined on $C_c(X)$. Hence, we have the Riesz–Markov representation theorem in $\mathbb{R}^n$.

Remark 1.2. In order to get the "classical version" of the theorem in a general locally compact Hausdorff space (the one in [4], for instance), there is a "missing point" in the proof, due the fact that possibly not every closed set is the zero–set of a continuous function (in a metric space we got this by means of the function $f(x) = 1 - \min\{d_C(x), 1\}$), hence the measure μ obtained by the above argument is only defined on the so–called *Baire σ–algebra* of X, which is a (possibly proper) subfamily of the Borel σ–algebra (an example is given by the so–called *uncountable Fort space* [5, Example 24]). Nevertheless, in the case of a locally compact Hausdorff space, with some effort, it is anyway possible to extend the measure μ to a regular Borel measure on the whole Borel σ–algebra (see [1, Theorem D, Section 54, Chapter X] and [3]), getting the conclusion.

References

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