A QUICK PROOF OF THE RIESZ–MARKOV–KAKUTANI REPRESENTATION THEOREM

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ABSTRACT. We give a quick and elementary proof of the Riesz–Markov–Kakutani representation theorem for compact metric spaces, based on the Carathéodory extension theorem and inspired by the theorem of Fubini–Tonelli. We also provide a road to reach the full theorem, i.e. for compact Hausdorff spaces, breaking the proof into two independent parts. This also pin–points the different nature of the two steps in the construction.

1. INTRODUCTION

The by now classical fundamental *Riesz–Markov–Kakutani representation theorem* (RMK) for positive linear functionals on the space of the continuous functions on a compact space, is the following.[1]

Theorem 1.1 (RMK representation theorem). *Let X be a compact Hausdorff space and let Φ be a positive linear functional on $C(X)$. Then, there exists a unique regular Borel measure μ on X such that*

$$\Phi(f) = \int_X f \, d\mu \,,$$

for every $f \in C(X)$.

This theorem was first stated and proved in 1909 by Frygies Riesz in the case of a compact interval $[a, b]$. Then, other proofs and extensions were given, in particular by Helly, Radon, Hildebrandt and Shoenberg, culminating with the contribution of Stefan Banach in 1937, who showed the theorem for compact metric spaces. As we shall see, this extension does not require substantially different ideas and techniques with respect of the case of an interval. In comparison, the comprehension of the general case of non–metrizable spaces turns out to have another level of difficulty and requires a deeper insight into some topological–measurable facts. In 1939 Andrey Markov Jr. obtained the first results in this direction and in 1941 Shizuo Kakutani proved the full representation theorem for compact Hausdorff spaces.[2]

Nowadays, the RMK theorem is a milestone at the crossing between topology and measure theory and it is a fundamental tool in several branches of analysis and probability. However, despite its importance in modern mathematics, it appears that it has not yet completely entered the "basic package" of notions a student must know. Although several textbooks provided very clear expositions, starting already from the fifties (see e.g. Halmos' treatise [2], or Rudin's book [4]), as far as we know, still very few undergraduate programs cover this topic.

The main aim of this note is to give a short and elementary proof of the RMK Theorem 1.1 for the case of compact metric spaces based on *Carathéodory extension theorem* [3] (see [3, Theorem 1.53]

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[1] Following Halmos [2], the theorem is also usually stated in the slightly more general setting of the space $C_c(X)$ of continuous functions with compact support on a locally compact Hausdorff space X. Moreover, the same name is usually given also to the important consequence that the dual space of $C(X)$ is isomorphic to the space of the regular Borel signed measures on X.

[2] To illustrate the relevance of the general case of compact Hausdorff spaces, recall that every Banach space E isometrically embeds into $C(X)$, for a suitable compact space X. Then, the RMK theorem in principle allows to represent the dual of every Banach space by means of (signed) measures on X (and even isometrically, by taking a quotient).

[3] Carathéodory extension theorem has not, of course, a quick proof, but it is completely elementary, relying only on suprema, infima and basic operations of set theory. Moreover, it fits naturally at the beginning of every elementary course

and also [2], for instance) and clearly inspired by the Fubini–Tonelli theorem. Moreover, with the same line, we will also show it for open subsets of $\mathbb{R}^n$.

Our analysis is based on a version of RMK for general topological spaces (actually, even only for sets, see Remark 2.6). Then, adding topological assumptions, we "gradually" get the above versions of RMK theorem.

2. The RMK representation theorem

We are going to employ the Carathéodory construction of a measure and the following (version of) *Carathéodory extension theorem* (see [3, Theorem 1.53] and also [2], for instance).

Theorem 2.1 (Carathéodory extension theorem)**.** *Let $\rho : \mathscr{R} \to [0,+\infty)$ be a σ–additive set–function defined on a semiring $\mathscr{R} \subseteq \mathscr{P}(X)$. Then ρ extends to a positive measure on the σ–algebra generated by $\mathscr{R}$. Moreover, if ρ is σ–finite, the extension is unique.*

For a general topological space X, given $u, v \in C_c(X)$ with $u \leqslant v$, we define

$$R_{u,v} = \{(x,t) \mid u(x) \leqslant t < v(x)\} \subseteq X \times \mathbb{R}$$

and we let $\mathscr{R}$ be the family of all these sets.
If Φ is a positive linear functional on $C_c(X)$, we define the set function $\rho : \mathscr{R} \to [0,+\infty)$ as

$$\rho(R_{u,v}) = \Phi(v-u).$$

It is easy to see that this is a good definition, as if $R_{u_1,v_1} = R_{u_2,v_2}$ we must have $v_1 - u_1 = v_2 - u_2$[4], hence

$$\rho(R_{u_1,v_1}) = \Phi(v_1 - u_1) = \Phi(v_2 - u_2) = \rho(R_{u_2,v_2}).$$

Also notice that ρ is invariant under translations along the $\mathbb{R}$–factor, that is,

$$\rho(R+c) = \rho(R),$$

for every $R \in \mathscr{R}$ and $c \in \mathbb{R}$.

We want to see that ρ is a *premeasure* on $\mathscr{R}$, meaning that:

- $\mathscr{R}$ is a *semiring*, that is, a family containing the empty set, closed under finite intersection and such that for any pair of sets $A, B \in \mathscr{R}$, the difference $A \setminus B$ can be written as a finite disjoint union of sets in $\mathscr{R}$.
- For every countable family of disjoint sets $R_n \in \mathscr{R}$ such that $\bigcup_{n=1}^{\infty} R_n \in \mathscr{R}$, we have
$$\rho\left(\bigcup_{n=1}^{\infty} R_n\right) = \sum_{n=1}^{\infty} \rho(R_n),$$
that is, ρ is σ–additive.

We have that $\emptyset \in \mathscr{R}$, since $R_{0,0} = \emptyset$, while the formulas

$$R_{u_1,v_1} \cap R_{u_2,v_2} = R_{(u_1 \wedge v_2) \vee (u_2 \wedge v_1),\, v_1 \wedge v_2}$$
$$R_{u_1,v_1} \setminus R_{u_2,v_2} = R_{u_1 \wedge u_2,\, v_1 \wedge u_2} \cup R_{u_1 \vee v_2,\, v_1 \vee v_2}$$

show that $\mathscr{R}$ is a semiring. Thus, we only have to check the σ–additivity of ρ.
If $R_{u,v}$ is the disjoint union of R_{u_n,v_n} for $n \in \mathbb{N}$, then for every $\in X$, the interval $[u(x), v(x))$ is the disjoint union of the intervals $[u_n(x), v_n(x))$, so by the σ-additivity of the Lebesgue measure

$$v(x) - u(x) = \sum_{n=1}^{\infty} (v_n(x) - u_n(x)).$$

Since this is a series with nonnegative terms, the continuous functions $f_k = \sum_{n=1}^{k} (v_n - u_n)$ converge monotonically to $f = v - u$, a continuous function with compact support K. Then, K clearly

on abstract measure theory, as an useful tool to construct e.g. the Lebesgue measure, the Borel measures on $\mathbb{R}$, the product measures, etc.

[4] Note however that $R_{u_1,v_1} = R_{u_2,v_2}$ can occur for $u_1 \neq u_2$ and $v_1 \neq v_2$, e.g. whenever $u_1 = v_1 \neq u_2 = v_2$.

contains all the supports of f_k, hence, by Dini's theorem, this convergence is uniform on K. Since $0 \leqslant f - f_k \leqslant \sqrt{f - f_1}\,\|f - f_k\|_{\infty,K}^{1/2}$ and Φ is a positive functional, we have

$$0 \leqslant \Phi(f - f_k) \leqslant \|f - f_k\|_{\infty,K}^{1/2}\, \Phi(\sqrt{f - f_1}),$$

so $\Phi(f_k) \to \Phi(f)$, as $k \to \infty$ and we conclude

$$\rho(R_{u,v}) = \Phi(v-u) = \lim_{k\to\infty} \Phi\bigg(\sum_{n=1}^{k}(v_n - u_n)\bigg) = \lim_{k\to\infty}\sum_{n=1}^{k}\Phi(v_n - u_n) = \lim_{k\to\infty}\sum_{n=1}^{k}\rho(R_{u_n,v_n}) = \sum_{n=1}^{\infty}\rho(R_{u_n,v_n}).$$

Thus, by the *Carathéodory extension theorem* 2.1 ρ can be uniquely extended to a positive measure ν on the σ–algebra $\mathscr{N}$ generated by $\mathscr{R}$, which is invariant under translations along the $\mathbb{R}$–factor, as ρ. Then, we define the measure μ on X as

$$\mu(E) = \nu(E \times [0,1)),$$

for every set $E \subseteq X$ belonging to the σ–algebra

$$\mathscr{M} = \{E \subseteq X \mid E \times [0,1) \in \mathscr{N}\}.$$

Considering $f \in C_c(X)$ and letting $E = \{x \in X \mid f(x) \neq 0\}$, we have $\bigcup_{n\in\mathbb{N}} R_n = E \times [0,1)$, where $R_n = R_{0,\min\{n|f|,1\}} \in \mathscr{R} \in \mathscr{N}$, hence the σ–algebra $\mathscr{M}$ contains every open set $E \subseteq X$ which is the *non–zero–set* of a real valued continuous function with compact support. We recall that the *Baire σ–algebra $\mathscr{Ba}$* of a topological space X is the one generated by the counterimages of the open sets via the real valued continuous functions on X, so it is clearly contained in the *Borel σ–algebra $\mathscr{Bo}$* of X. A measure μ is then called *Baire measure* if it is defined on $\mathscr{Ba}$ which is the minimal σ–algebra such that all the continuous functions are measurable. Then, $\mathscr{M}$ contains the σ–algebra $\mathscr{Ba}_c \subseteq \mathscr{Ba}$ generated by this family of special open subsets of X, which actually coincides with the σ–algebra generated by all the counterimages of the open sets of $\mathbb{R}$ via the continuous functions *with compact support*. Indeed, $\mathscr{Ba}_c$ must contain all the zero–sets $\{f = 0\}$ of all $f \in C_c(X)$ (by complement) and if an interval (a,b) does not contain zero, we have

$$f^{-1}(a,b) = \{x \in X \mid g(f(x)) \neq 0\}$$

where $g : \mathbb{R} \to \mathbb{R}$ is any continuous function which is positive exactly on (a,b) (notice that the support of $g \circ f$ is contained in the support of f, hence $g \circ f \in C_c(X)$). If instead (a,b) contains zero, we clearly have

$$f^{-1}(a,b) = f^{-1}(a,0) \cup f^{-1}(0) \cup f^{-1}(0,b),$$

so our claim is proved and $\mathscr{Ba}_c$ is the minimal σ–algebra such that all the functions in $C_c(X)$ are measurable.
Moreover, by the $\mathbb{R}$–translation invariance of ν, it is easy to see that

$$\nu(E \times I) = \mu(E)\mathscr{L}^1(I)$$

for every $E \in \mathscr{Ba}_c$ and I an interval of $\mathbb{R}$, where $\mathscr{L}^1$ is the Lebesgue measure.
We now show that for every function $f \in C_c(X)$, we have

$$\Phi(f) = \int_X f\, d\mu. \tag{2.1}$$

Indeed, for every nonnegative function f in $C_c(X)$, we have a sequence of $\mathscr{Ba}_c$–measurable non-negative simple functions $s_n = \sum_{i=1}^{k_n} \alpha_{n,i}\chi_{E_{n,i}}$ monotonically converging to f, then (for every $n \in \mathbb{N}$, the sets $E_{n,i} \in \mathscr{Ba}_c$ are mutually disjoint, for $i = 1, \dots, k_n$)

$$\Phi(f) = \nu(R_{0,f}) = \lim_{n\to\infty}\sum_{i=1}^{k_n}\nu(E_{n,i} \times [0,\alpha_{n,i})) = \lim_{n\to\infty}\sum_{i=1}^{k_n}\alpha_{n,i}\mu(E_{n,i}) = \lim_{n\to\infty}\int_X s_n\, d\mu = \int_X f\, d\mu,$$

by the monotone convergence theorem. The equality (2.1) then clearly follows for a general function $f \in C_c(X)$, by decomposing it into its positive and negative parts, by the linearity of Φ.

Hence, we have the following *measure–theoretic–only* version of the RMK theorem, without any topological assumption.

Theorem 2.2. *Let X be a topological space and let Φ be a positive linear functional on $C_c(X)$. Then, there exists a positive measure μ on $\mathscr{B}a_c$ such that*

$$\Phi(f) = \int_X f\, d\mu\,,$$

for every $f \in C_c(X)$.

2.1. **Uniqueness and regularity.**
G is an open non–zero–set of any function $f \in C_c(X)$, then for every μ satisfying the conclusion of this theorem, there holds

$$\mu(G) = \sup_{n\in\mathbb{N}} \int_X \min\{n|f|, 1\}\, d\mu = \sup_{n\in\mathbb{N}} \Phi(\min\{n|f|, 1\}),$$

hence, on the family $\mathscr{G}$ of such open sets, the measure μ is univocally determined. Let $\mathscr{F}$ be the minimal family containing $\mathscr{G}$, closed under monotone increasing countable unions and difference of two sets when one is included in the other. Then, if we have another measure η on $\mathscr{B}a_c$ representing the functional Φ as above and we consider $\mathscr{A} = \{E \in \mathscr{B}a_c \mid \eta(E) = \mu(E)\}$, this family shares the same properties and contains $\mathscr{G}$, hence $\mathscr{F} \subseteq \mathscr{A}$, by the minimality of $\mathscr{F}$.
It is then easy to see that if we show that for every $A, B \in \mathscr{F}$, also $A \cap B \in \mathscr{F}$, it follows that $\mathscr{F}$ contains the σ–ring generated by $\mathscr{G}$. Hence, μ is uniquely determined on such σ–ring.
This can be shown as follows: let $A \in \mathscr{G}$ and define

$$\mathscr{F}_A = \{B \in \mathscr{F} \mid A \cap B \in \mathscr{F}\} \subseteq \mathscr{F},$$

then, also $\mathscr{F}_A$ is closed for monotone increasing countable union, contains $\mathscr{G}$ (being closed for finite intersection) and if $B, C \in \mathscr{F}_A$ are such that $B \supseteq C$, we have $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C) \in \mathscr{F}$, as $A \cap B \subseteq A \cap C$, hence $B \setminus C \in \mathscr{F}_A$. It follows that $\mathscr{F}_A$ contains $\mathscr{F}$, by minimality, thus $\mathscr{F}_A = \mathscr{F}$.
If now $A \in \mathscr{F}$ is a general set, we have the same conclusion $\mathscr{F}_A = \mathscr{F}$, since we just showed that $\mathscr{G} \subseteq \mathscr{F}_A$ and this proves that $\mathscr{F}$ is closed under finite intersections.

We notice that a natural condition assuring that the σ–ring generated by $\mathscr{G}$ coincides with the whole σ–algebra $\mathscr{B}a_c$ (also generated by $\mathscr{G}$), hence implying that the measure μ in Theorem 2.2 is unique, is the existence of a countable family of functions $f_k \in C_c(X)$ such that the family of open sets $\{|f_n| \neq 0\}$ covers all X.

We now turn our attention to the *regularity* properties of μ.
We define the families $\mathscr{A} \subseteq \mathscr{B}a_c$ of the countable unions of open sets in $\mathscr{G}$ and $\mathscr{K} \subseteq \mathscr{B}a_c$ of the closed and compact sets of X in $\mathscr{B}a_c$, noticing that every $K = \{x \in X \mid |f(x)| \geqslant 1/n\}$, for any $f \in C_c(X)$ and $n \in \mathbb{N}$ belongs to $\mathscr{K}$ (these are closed sets in $\mathscr{B}a_c$ contained in the compact support of f, hence compact).
Then, we consider the family $\mathscr{E}$ of the sets $E \in \mathscr{B}a_c$ such that

$$\mu(E) = \inf\{\mu(G) \mid G \supseteq E \text{ and } G \in \mathscr{A}\} = \sup\{\mu(K) \mid K \subseteq E \text{ and } K \in \mathscr{K}\}$$

and we notice that for every $G = \{|f| \neq 0\} \in \mathscr{G}$ open non–zero–set of a function $f \in C_c(X)$, being the countable union of the sets $\{x \in X \mid |f(x)| \geqslant 1/n\} \in \mathscr{K}$, it clearly belongs to $\mathscr{E}$, hence $\mathscr{G} \subseteq \mathscr{E}$.
Let $E_n \in \mathscr{E}$ all with finite measure and let $E = \bigcup_{n=1}^\infty E_n$. Then, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, we have $K_n \in \mathscr{K}$ and $G_n \in \mathscr{A}$ with $K_n \subseteq E_n \subseteq G_n$, such that $\mu(G_n) < \mu(E_n) + \varepsilon/2^n$ and $\mu(K_n) > \mu(E_n) - \varepsilon/2^n$.
Hence, $F = \bigcup_{n=1}^\infty K_n \subseteq E \subseteq \bigcup_{n=1}^\infty G_n = G$ and if $\mu(E) < +\infty$,

$$\mu(G) = \mu\Big(\bigcup_{n=1}^\infty G_n \setminus \bigcup_{n=1}^\infty E_n\Big) + \mu(E) \leqslant \mu\Big(\bigcup_{n=1}^\infty (G_n \setminus E_n\Big) + \mu(E) \leqslant \sum_{n=1}^\infty \mu(G_n \setminus E_n) + \mu(E) < +\mu(E) + \varepsilon,$$

while,

$$\mu(E) = \mu\Big(\bigcup_{n=1}^\infty E_n \setminus \bigcup_{n=1}^\infty K_n\Big) + \mu(F) \leqslant \mu\Big(\bigcup_{n=1}^\infty (E_n \setminus K_n\Big) + \mu(F) \leqslant \sum_{n=1}^\infty \mu(E_n \setminus K_n) + \mu(F) < +\mu(F) + \varepsilon.$$

Then, since

$$\mu(F) = \mu\Big(\bigcup_{n=1}^\infty K_n\Big) = \lim_{m\to\infty} \mu\Big(\bigcup_{n=1}^m K_n\Big),$$

there exists $m \in \mathbb{N}$ such that

$$\mu(F) \leqslant \mu\Big(\bigcup_{n=1}^{m} K_n\Big) + \varepsilon,$$

hence, setting $K = \bigcup_{n=1}^{m} K_n$, we have $\mu(F) \leqslant \mu(K) + \varepsilon$ and $\mu(E) < \mu(K) + 2\varepsilon$.
Then, since it is easy to see that $G \in \mathscr{A}$ and $K \in \mathscr{K}$, we conclude that $E \in \mathscr{E}$.
If $\mu(E) = +\infty$, then $\mu(G) = +\infty$ and clearly the same holds for every other open set containing E, moreover, by the estimates above, $\mu(F) = +\infty$ also, hence, by the same argument, for every $M > 0$, we can find $K \in \mathscr{K}$ such that $\mu(K) > M$. Thus, also in this case $E \in \mathscr{E}$.
If at least one of the sets $E_n \in \mathscr{E}$ has infinite measure, then $\mu(E) = +\infty$, again every open set containing E has infinite measure and there is $K \in \mathscr{K}$ contained in E_n with measure large as we want. All this shows that the family $\mathscr{E}$ is closed under countable union.
Assume now that $A, B \in \mathscr{E}$ and $A \supseteq B$. If A has finite measure, the same holds for B, then let $K \subseteq A \subseteq G$ and $H \subseteq A \subseteq U$ with $K, H \in \mathscr{K}$ such that $\mu(A) < \mu(K) + \varepsilon$ and $\mu(B) > \mu(H) + \varepsilon$, $G, U \in \mathscr{A}$ such that $\mu(A) > \mu(G) - \varepsilon$ and $\mu(B) > \mu(U) - \varepsilon$. Then, since

$$A \setminus B \subseteq (A \setminus K) \cup (K \setminus U) \cup (U \setminus B),$$

we have

$$\mu(A \setminus B) \leqslant \mu(A \setminus K) + \mu(K \setminus U) + \mu(U \setminus B) < \mu(K \setminus U) + 2\varepsilon,$$

where $K \setminus U \in \mathscr{Ba}_c$ is closed and compact, hence belonging to $\mathscr{K}$.
Analogously,

$$G \setminus H \subseteq (G \setminus A) \cup (A \setminus B) \cup (B \setminus H)$$

and we have

$$\mu(G \setminus H) \leqslant \mu(G \setminus A) + \mu(A \setminus B) + \mu(B \setminus H) < \mu(A \setminus B) + 2\varepsilon,$$

with $G \setminus H \in \mathscr{Ba}_c$ open, hence belonging to $\mathscr{A}$.
This shows that $A \setminus B$ belongs to $\mathscr{E}$ in this case. The other cases, with $\mu(A) = +\infty$ are analogous.
Thus, we conclude that the family $\mathscr{E}$ is closed under countable unions and difference of two sets when one is included in the other, moreover, it contains $\mathscr{G}$. Then, by arguing as above for the uniqueness, $\mathscr{E}$ contains the σ–ring generated by $\mathscr{G}$.

Proposition 2.3. *Any measure satisfying the conclusion of Theorem 2.2 is uniquely determined on the σ–ring generated by the family $\mathscr{G} \subseteq \mathscr{Ba}_c$ of the non–zero–open sets of the functions in $C_c(X)$. Moreover, it is* regular*, meaning that*

$$\mu(E) = \inf\{\mu(G) \mid G \supseteq E \text{ and } G \in \mathscr{Ba}_c \text{ open}\} = \sup\{\mu(K) \mid K \subseteq E \text{ and } K \in \mathscr{Ba}_c \text{ closed and compact}\},$$

for every E in such σ–ring.
If there exists a countable family of functions $f_k \in C_c(X)$ such that the family of open sets $\{|f_n| \neq 0\}$ covers all X, these properties hold for the whole σ–algebra $\mathscr{Ba}_c$.

If X is compact, we have $C(X)_c = C(X)$ and $\mathscr{Ba}_c = \mathscr{Ba}$, hence we have the following conclusion.

Theorem 2.4. *Let X be a compact space and let Φ be a positive linear functional on $C(X)$. Then, there exists a unique positive Baire measure μ on X such that*

$$\Phi(f) = \int_X f \, d\mu\,,$$

for every $f \in C(X)$. Moreover, μ is finite and regular*, that is,*

$$\mu(E) = \inf\{\mu(G) \mid G \supseteq E \text{ and } G \in \mathscr{Ba} \text{ open}\} = \sup\{\mu(K) \mid K \subseteq E \text{ and } K \in \mathscr{Ba} \text{ compact}\},$$

for every $E \in \mathscr{Ba}$.

It is easy to see that in a metric space the zero–sets coincide with the whole family of the closed sets, then also the σ–algebras $\mathscr{Ba}$ and $\mathscr{Bo}$ coincide, hence Baire and Borel measures are the same, as we said in the introduction. Thus, all the previous discussion about the uniqueness and the regularity simplifies considerably (several steps are trivial and can be skipped) and we have a quick proof of the following theorem, or more in general, of the RMK theorem in any compact space such that $\mathscr{Ba} = \mathscr{Bo}$ (for instance, any *perfectly normal* Hausdorff space – see [1, Chapter VII, Section 4]). A compact subset of $\mathbb{R}^n$ is clearly a special case.

Theorem 2.5. *Let (X,d) be a compact metric space and let Φ be a positive linear functional on $C(X)$. Then, there exists a unique finite regular positive Borel measure μ on X such that*

$$\Phi(f) = \int_X f\,d\mu\,,$$

for every $f \in C(X)$.

However, dealing with the "classical version" Theorem 1.1, a general compact Hausdorff space is *normal* but unfortunately, not necessarily *perfectly normal*, hence it could happen that $\mathscr{Ba} \neq \mathscr{Bo}$, as in the example in the Appendix. Then, one may ask whether with our construction, the Carathéodory measurable sets already include the Borel σ–algebra. In general the answer is negative: there are Baire measures on compact Hausdorff spaces whose induced σ–algebra of Carathéodory measurable sets coincides with the Baire σ–algebra itself and this latter is strictly contained in the Borel σ–algebra (see the example in the Appendix). Thus, neither a completion, nor a Carathéodory extension, produce any proper extension. Contrary to the first step, from the functional to a measure on Baire sets, which can be performed in an abstract measure setting (see the following remark, where such point of view is pushed even further), such extension of the measure from Baire to Borel sets, is actually topological in nature. In the next section we will show that in such case of a general compact Hausdorff space, we are able to extend the above Baire measure μ to a regular Borel measure (theorem 3.1, thus getting Theorem 1.1. This can be done also if the space is only *locally compact* but not compact (as for instance in the book of Rudin [4]), but the arguments are quite more technical (ultimately, due to the possible lack of the σ–finiteness of the premeasure ρ and of the measures ν and μ).

Remark 2.6. We mention that with the very same proof, following it step by step, one can obtain a very abstract version of the representation theorem, called *Daniell–Stone theorem*.

Definition 2.7. A linear functional Φ on a vector space V of real–valued functions on a set X satisfies the *Denjoy condition* if it is positive and, for every sequence of nonnegative functions f_k in V that converges monotonically to zero at each point, we have $\Phi(f_k) \to 0$.

Theorem 2.8 (Daniell–Stone theorem)**.** *If Φ satisfies the Denjoy condition on a lattice and vector space V of real–valued functions, then there exists a unique measure on the σ–ring generated by V that represents Φ by integration.*

We conclude this section with the case of an open set of $\mathbb{R}^n$ (which is a locally compact Hausdorff and σ–compact space), because of its clear relevance. As it is easy to see that $\mathscr{Ba}_c = \mathscr{Ba} = \mathscr{Bo}$, we do not need to deal with the Baire–Borel extension of the measure and we have immediately the following conclusion (the same holds for every perfectly normal and σ–compact Hausdorff space).

Theorem 2.9. *Let Ω be an open subset of $\mathbb{R}^n$ and let Φ be a positive linear functional on $C_c(\Omega)$. Then, there exists a unique regular Borel measure μ on Ω such that*

$$\Phi(f) = \int_\Omega f\,d\mu\,,$$

for every $f \in C_c(\Omega)$.

Also in this very special case, several of the above arguments can be simplified and we have an easy and direct proof, very suitable for undergraduate courses.

3. Extension of Baire measures

The following is actually Theorem D in Section 54, Chapter X of the book of Halmos [2], we give here a direct proof for the reader's convenience.

Theorem 3.1. *Let X be a compact Hausdorff topological space and let μ a regular Baire measure on X. Then, there exists a unique regular positive Borel measure ν on X that extends μ.*

If $G \subseteq X$ is open, we define

$$\rho(G) = \sup\{\mu(K) \mid K \subseteq G, K \in \mathscr{Ba} \text{ and } K \text{ compact}\}$$

and we notice that, by the regularity of μ, if G belongs to $\mathscr{Ba}$, we have $\rho(G) = \mu(G)$. Moreover, if $G_1 \subseteq G_2$, then $\rho(G_1) \leqslant \rho(G_2)$.
Then, for every $E \subseteq X$, we set

$$\nu^*(E) = \inf\{\rho(G) \mid G \supseteq E \text{ and } G \text{ open}\},$$

noticing that if G is open, $\mu^*(G) = \rho(G)$, by the monotonicity property of ρ above.
We let $\mathscr{M}$ be family of subsets E of X such that for every $\varepsilon > 0$ there exist a compact K and an open G with $K \subseteq E \subseteq G$ and $\nu^*(G \setminus K) < \varepsilon$. In such case, we set $\nu(E) = \nu^*(E)$. It clearly follows immediately that:

- ν^* is a monotone set function bounded above.
- If $E \in \mathscr{M}$ then $E^c \in \mathscr{M}$, as in general, for $K \subseteq E \subseteq G$ there holds $G^c \subseteq E^c \subseteq K^c$ and
$$\nu^*(K^c \setminus G^c) = \nu^*(K^c \cap G) = \nu^*(G \setminus K).$$
- By the regularity of the Baire measure μ, every open set $G \in \mathscr{Ba}$ belongs to $\mathscr{M}$ and $\nu(G) = \mu(G)$. Then, by the previous point, also every compact $K \in \mathscr{Ba}$ belongs to $\mathscr{M}$, moreover, $\nu(K) = \mu(K)$, again by the regularity of μ.

We want to show that:

- $\mathscr{M}$ is a σ–algebra containing the open sets of X.
- ν is a measure on $\mathscr{M}$.
- ν coincides with μ on $\mathscr{Ba}$.
- ν is a regular Borel measure.

We argue by steps.

(1) *Open and compact sets belong to $\mathscr{M}$.*
If G is open, there exists a sequence of compact sets $K_i \subseteq G$, $K_i \in \mathscr{Ba}$ with

$$\lim_{i\to\infty} \mu(K_i) = \lim_{i\to\infty} \nu(K_i) = \rho(G) = \nu(G).$$

If we consider $\nu^*(G \setminus K_i)$ and we suppose that

$$\nu^*(G \setminus K_i) \not\to 0,$$

eventually passing to a subsequence, we can assume that $\nu^*(G \setminus K_i) > \varepsilon > 0$, for every $i \in \mathbb{N}$. Then, being every $G \setminus K_i$ open, $\rho(G \setminus K_i) > \varepsilon$ for every $i \in \mathbb{N}$ and there exists a compact set $H_i \subseteq G \setminus K_i$ in $\mathscr{Ba}$ with $\mu(H_i) > \varepsilon$, hence $K_i \cup H_i$ is a compact set in $\mathscr{Ba}$ such that $K_i \cup H_i \subseteq G$ and there holds

$$\nu^*(G) \geqslant \nu^*(K_i \cup H_i) = \mu(K_i \cup H_i) = \mu(K_i) + \mu(H_i) = \mu(K_i) + \varepsilon \ \to\ \rho(G) + \varepsilon = \nu(G) + \varepsilon,$$

(being H_i and K_i disjoint and μ a measure on $\mathscr{Ba}$) which is a contradiction.
Hence, $\nu^*(G \setminus K_i) \to 0$, showing that $G \in \mathscr{M}$. It follows that also every compact K belongs to $\mathscr{M}$.
Moreover, it is clear that if E is open or compact, then

$$\nu(E) = \inf\{\nu(G) \mid G \supseteq E \text{ and } G \text{ open}\} = \sup\{\nu(K) \mid K \subseteq E \text{ and } K \text{ compact}\},$$

by the monotonicity of ν and what we said above.
Finally, if K is compact and G is an open set containing K, by considering a continuous function "separating" the compact K and G^c, we have an open set $G' \subseteq G$, still containing K and clearly satisfying

$$\nu(G' \setminus K) \leqslant \nu(G \setminus K).$$

This implies that if K is compact,

$$\nu(K) = \inf\{\nu(G) \mid G \supseteq K, G \in \mathscr{Ba} \text{ and } G \text{ open}\}.$$

(2) *ν is σ–subadditive on $\mathscr{M}$.*
If A and B are open, we let K be a compact in $\mathscr{Ba}$ contained in $A \cup B$ and we consider a partition of

unity f, g associated to the two open sets A and B such that $f + g = 1$ on K, then $K_A = \{f \geqslant 1/2\} \subseteq A$ and $K_B = \{g \geqslant 1/2\} \subseteq B$ are a couple of compact sets in $\mathscr{B}a$ such that $K \subseteq K_A \cup K_B$. Hence, as μ is a measure on $\mathscr{B}a$, we have

$$\mu(K) \leqslant \mu(K_A \cup K_B) \leqslant \mu(K_A) + \mu(K_B) \leqslant \nu(A) + \nu(B)$$

and taking the supremum on K we conclude

$$\nu(A \cup B) = \rho(A \cup B) \leqslant \nu(A) + \nu(B).$$

Then, if G_i are open sets and $G = \bigcup_{i=1}^{\infty} G_i$, for every K compact in $\mathscr{B}a$ with $K \subseteq G$, the family G_i covers K, hence a finite subfamily $G_{i_1}, \dots, G_{i_m}$ covers K and we have

$$\mu(K) = \nu(K) \leqslant \nu\left(\bigcup_{k=1}^{m} G_{i_k}\right) \leqslant \sum_{k=1}^{m} \nu(G_{i_k}) \leqslant \sum_{i=1}^{\infty} \nu(G_i).$$

Taking the supremum on K, we have

$$\nu(G) = \rho(G) \leqslant \sum_{i=1}^{\infty} \nu(G_i),$$

that is, the function ν is σ–subadditive on open sets.
Then, if $A_i \in \mathscr{M}$ and $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{M}$, the fact that ν is σ–subadditive follows easily by approximating every A_i from outside with open sets up to $\varepsilon/2^i$, applying the above conclusion and sending $\varepsilon \to 0$.

(3) *$\mathscr{M}$ is an algebra.*
Since we already know that $\mathscr{M}$ is closed by taking complement, we only have to show that it is closed for finite unions. If $A, B \in \mathscr{M}$, then there exist compact K_A, K_B and open G_A, G_B with $K_A \subseteq A \subseteq G_A$, $K_B \subseteq B \subseteq G_B$ and $\nu(G_A \setminus K_A) < \varepsilon$, $\nu(G_A \setminus K_A) < \varepsilon$. We have (notice that all the sets involved are open)

$$\nu((G_A \cup G_B) \setminus (K_A \cup K_B)) \leqslant \nu((G_A \setminus K_A) \cup (G_B \setminus K_B)) \leqslant \nu(G_A \setminus K_A) + \nu(G_B \setminus K_B) < 2\varepsilon,$$

where we applied the previous point (2). Hence, $A \cup B$ belong to $\mathscr{M}$.

(4) *If $E \in \mathscr{M}$, then $\nu(E) = \sup\{\nu(K) \mid K \subseteq E$ and K compact$\}$.*
By point (2), if G is open, K compact with $K \subseteq A \subseteq G$ and $\nu(G \setminus K) < \varepsilon$, there holds

$$\nu(A) \leqslant \nu(G) \leqslant \nu(K) + \nu(G \setminus K) < \nu(K) + \varepsilon$$

and this clearly implies the claim.

(5) *ν is σ–additive on $\mathscr{M}$.*
We already showed the σ–subadditivity of ν at point (2).
If $A_i \in \mathscr{M}$ and $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{M}$, we can assume that the sets A_i are disjoint. Then, if ν is (finitely) superadditive, we have

$$\nu(A) \geqslant \left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \nu(A_i)$$

and sending $k \to \infty$, we have the σ–additivity, by the previous σ–subadditivity conclusion.
Thus, we have only to show that for two disjoint sets $A, B \in \mathscr{M}$, there holds

$$\nu(A \cup B) \geqslant \nu(A) + \nu(B).$$

For every couple of (disjoint) compact sets $K_A \subseteq A$ and $K_B \subseteq B$, being X normal, we can "separate" them with a couple of disjoint open sets $U \supseteq K_A$ and $V \supseteq K_B$ in $\mathscr{B}a$. Moreover, by point (1) we have an open set $G_{AB} \supseteq K_A \cup K_B$, belonging to $\mathscr{B}a$ with $\nu(G_{AB}) < \nu(K_A \cup K_B) + \varepsilon$. Hence,

$$\begin{aligned}\nu(A \cup B) &\geqslant \nu(K_A \cup K_B) > \nu(G_{AB}) - \varepsilon = \mu(G_{AB}) - \varepsilon \geqslant \mu((G_{AB} \cap U) \cup (G_{AB} \cap V)) - \varepsilon\\ &= \mu(G_{AB} \cap U) + \mu(G_{AB} \cap V) - \varepsilon = \nu(G_{AB} \cap U) + \nu(G_{AB} \cap V) - \varepsilon \geqslant \nu(K_A) + \nu(K_B) - \varepsilon\end{aligned}$$

and the conclusion follows by sending $\varepsilon \to 0$ and then taking the supremum on the compact K_A and K_B, by point (4).

(6) *$\mathscr{M}$ is a σ–algebra, hence ν is a regular Borel measure on $\mathscr{M}$ extending μ.*
Let $A_i \in \mathscr{M}$ and $A = \bigcup_{i=1}^{\infty} A_i$ and as before, by point (3), we can assume that the sets A_i are disjoint. Then we have compact K_i and open G_i with $K_i \subseteq A_i \subseteq G_i$ and $\nu(G_i \setminus K_i) < \varepsilon/2^i$.
By the additivity and being ν finite, it must be $\sum_{i=1}^{\infty} \nu(K_i) < +\infty$, hence there exists $n \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} \nu(K_i) < \varepsilon$. It follows that (by the σ–subadditivity)

$$\nu\biggl(\bigcup_{i=n+1}^{\infty} G_i\biggr) \leqslant \sum_{i=n+1}^{\infty} \nu(G_i) = \sum_{i=n+1}^{\infty} \nu(G_i \setminus K_i) + \nu(K_i) \leqslant \varepsilon + \sum_{i=n+1}^{\infty} \varepsilon/2^i \leqslant 2\varepsilon.$$

Then, if we consider the open set $G = \bigcup_{i=1}^{\infty} G_i \supseteq A$ and the compact set $K = \bigcup_{i=1}^{n} K_i \subseteq A$, there holds

$$\nu(G \setminus K) = \nu\biggl(\bigcup_{i=1}^{\infty} G_i \setminus \bigcup_{i=1}^{n} K_i\biggr) \leqslant \nu\biggl(\bigcup_{i=1}^{n} (G_i \setminus K_i) \cup \bigcup_{i=n+1}^{\infty} G_i\biggr) \leqslant \sum_{i=1}^{n} \nu(G_i \setminus K_i) + \sum_{i=n+1}^{\infty} \nu(G_i) \leqslant 3\varepsilon,$$

showing that $A \in \mathscr{M}$ and we are done.
The regularity of ν is given by point (4), while that ν and μ coincide on $\mathscr{Ba}$ follows by the regularity of both measures and by the fact that they coincide on the compact sets in $\mathscr{Ba}$ (keeping also into account their monotonicity).

Appendix A. An example of a compact Hausdorff space with $\mathscr{Ba} \neq \mathscr{Bo}$

Let $X = Y \cup \{\infty\}$ be the one–point compactification of an uncountable set Y endowed with the discrete topology. This space is Hausdorff and compact, it is called *uncountable Fort space* and gives an example of a completely normal, but not perfectly normal space (see [1, Chapter VII, Section 4] and [5, Example 24]).

Every subset of Y is open and the compact subsets of Y are the finite subsets. Then, the open sets of X are all the subsets of $Y \subseteq X$, plus all the cofinite subsets containing ∞, hence the G_δ subsets are all the subsets of Y, plus all the cocountable subsets. The closed sets of X are the union of ∞ with any subset of Y plus the finite subsets of Y and the F_σ subsets are the union of ∞ with any subset of Y plus any countable set.

It follows that the family of the open and closed subsets of X is the whole power set of X, which then coincides with the Borel σ–algebra $\mathscr{Bo}$ of X. As a consequence, every function $f : X \to \mathbb{R}$ is Borel.

If $f : X \to \mathbb{R}$ is continuous, it must be constant equal to $f(\infty)$ on X with the exception of at most a countable set x_i such that $f(x_i) \to f(\infty)$. Hence, the counterimages of the open sets of $\mathbb{R}$ are the countable subsets of Y and all the unions of ∞ with the cofinite subsets of Y. Then, the zero–sets of X are the unions of ∞ with any cocountable subset of Y and the finite subsets of Y.
The union of these two families is given by:

- the countable subsets of Y
- the unions of ∞ with the cofinite subsets of Y
- the unions of ∞ with any cocountable subset of Y
- the finite subsets of Y

that is, by the family containing all the countable subsets of Y and the unions of ∞ with any cocountable subset of Y.
This family is clearly closed for complement and any countable union of its sets still belongs to the family if they are all of the first or of the second type, while if at least one is of the second type, such union is also of second type.
Hence, this family is the Baire σ–algebra $\mathscr{Ba}$ of X which is clearly smaller that $\mathscr{Bo}$, as it does not contain $\{\infty\}$.

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