STAR–SHAPEDNESS UNDER MEAN CURVATURE FLOW

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ABSTRACT. We show that a star–shaped curve in the plane remains star–shaped moving by curvature.

1. SIMPLE CURVES IN THE PLANE MOVING BY CURVATURE

Given a closed C^1 curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ we say that it is regular if $\gamma_\theta = \frac{d\gamma}{d\theta}$ is never zero. It is then well defined its unit tangent vector $\tau = \gamma_\theta/|\gamma_\theta|$. We define its unit normal vector as $\nu = R\tau = R\gamma_\theta/|\gamma_\theta|$, where $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation centered in the origin of angle $\pi/2$.
If the curve γ is C^2 and regular, its *curvature vector* is well defined as

$$\underline{k} = \tau_\theta/|\gamma_\theta| = \frac{d\tau}{d\theta}\Big/|\gamma_\theta|.$$

The curvature of γ is then given by $k = \langle \underline{k} \,|\, \nu\rangle$, as $\underline{k} = k\nu$.
The arclength parameter s of the curve γ is given by

$$s = s(\theta) = \int_0^\theta |\gamma_\theta(\xi)|\, d\xi\,.$$

Notice that $\partial_s = |\gamma_\theta|^{-1}\partial_\theta$, then $\tau = \partial_s\gamma$ and $\underline{k} = \partial_s\tau = \partial^2_{ss}\gamma$.

With these notations, a circle covered counterclockwise, has positive curvature and ν is the inner unit normal vector.

Defining the quantity

$$Q = \langle \gamma \,|\, \nu\rangle,$$

it is easy to see that a simple curve γ in the plane is star–shaped with respect to the origin of $\mathbb{R}^2$ if and only if $Q \leq 0$ at every point of the curve.

Consider then a smooth, regular, simple, closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, star–shaped with respect to the origin, and its motion by curvature, given by

$$\gamma_t = \underline{k} = k\nu.$$

It is well known that such a curve evolves smoothly by curvature, it remains embedded, becomes convex and shrinks to a point in finite time, becoming asymptotically rounder and rounder, see [1–7].

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We compute the evolution equation for the quantity Q during the flow. As $\partial_t \tau = k_s \nu$ and $\partial_t \nu = \partial_t R\tau = -k_s \tau$, we have

$$\begin{aligned}\partial_t Q - \partial^2_{ss} Q =& \langle \gamma_t \,|\, \nu\rangle + \langle \gamma \,|\, \partial_t \nu\rangle - \langle \gamma_{ss} \,|\, \nu\rangle - \langle \gamma \,|\, \partial^2_{ss}\nu\rangle - 2\langle \gamma_s \,|\, \partial_s \nu\rangle \\ =& k - k_s\langle \gamma \,|\, \tau\rangle - k + \langle \gamma \,|\, k_s\tau + k^2\nu\rangle + 2k \\ =& 2k + Qk^2\end{aligned}$$

and

$$\partial_s Q = \langle \gamma_s \,|\, \nu\rangle + \langle \gamma \,|\, \partial_s \nu\rangle = \langle \tau \,|\, \nu\rangle - k\langle \gamma \,|\, \tau\rangle = -k\langle \gamma \,|\, \tau\rangle.$$

If the initial curve γ is star–shaped with respect to the origin $O \in \mathbb{R}^2$, we have $Q \leq 0$, then, if at some time t and point $\gamma(\theta, t)$, we have $Q = 0$, at such point there holds

$$0 = \partial_s Q = -k\langle \gamma \,|\, \tau\rangle$$

and

$$0 \leq \partial_t Q - \partial^2_{ss} Q = 2k + Qk^2 = 2k.$$

If $O \neq \gamma(\theta, t)$, for instance if the open region bounded by the curve γ at time t contains the origin of $\mathbb{R}^2$, we have $\gamma(\theta, t) \neq 0$, and being $Q = \langle \gamma \,|\, \nu\rangle = 0$ at such point, there must be $\langle \gamma \,|\, \tau\rangle \neq 0$, hence, k is zero. It follows that $\partial_t Q - \partial^2_{ss} Q = 0$ and we can conclude by the maximum principle that Q stays nonpositive during the flow, hence the curve remains star–shaped with respect to the origin.

Since the flow by curvature is translation invariant, we have the following conclusion.

Theorem 1.1. *An initial smooth, regular, simple closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, star–shaped with respect to a point P, remains star–shaped under its motion by curvature, till the point P is contained in the open region of the plane bounded by the evolving curve.*

Problem. Despite this theorem, we do not know if the property of star–shapedness (in general, not *with respect to a fixed point*) is actually preserved under the motion by curvature of a simple closed curve in the plane.

2. Higher Dimension

By a little bit more involved, but straightforward, computation (see [8] for the formulas), it can be seen that for an n–dimensional hypersurface $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ moving by curvature, the analogous quantity

$$Q = \langle \varphi \,|\, \nu\rangle,$$

satisfies the parabolic equation

$$\partial_t Q - \Delta Q = 2H + Q|A|^2$$

and

$$\nabla Q = -G(\pi^M \varphi),$$

where A is the second fundamental form, G is the Gauss operator of M and π^M is the projection on the tangent space to M (thinking of $M_t = \varphi(M, t)$ as a submanifold of $\mathbb{R}^{n+1}$ and identifying any tangent space $T_p M_t$ with a hyperplane in $\mathbb{R}^{n+1}$).

Unfortunately, if $n \geq 2$, we cannot repeat the argument above, since the nullity of the gradient of Q at $p \in M_t$ only says that the second fundamental form of M_t has a null eigenvalue in the direction $\pi_t^M \varphi \in T_p M_t$, while we would need that $H \leq 0$ at such point $p \in M$ where $Q = 0$ (notice that $Q = 0$ implies that $\varphi \in T_p M_t$, hence $\pi^M \varphi = \varphi$).

Actually, even if we are not able to construct an explicit example where the star–shapedness of a surface is lost during its motion by mean curvature, it is very likely that this can happen.

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