A Note on the Lalescu Sequence

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1 The Lalescu sequence

The following sequence

$$a_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$$

is called *Lalescu sequence* after the Romanian mathematician Traian Lalescu (1882 – 1929, see [5]) who proposed it in [1], asking about its convergence. Possibly due to its indubitable elegance, on one hand and its not so straightforward analysis, on the other, it attracted various authors, who discussed its properties and generalizations (we underline the evident connection with Stirling's formula and Euler's Gamma function).

We review some basic facts.

If we suppose that the sequence converges, considering the two sequences $\sqrt[n]{n!}$ and n, by means of the Stolz–Cesaro theorem, we have

$$\lim_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = 1/e,$$

by the well–known limit $\sqrt[n]{n!}/n \to 1/e$.
Alternatively, still assuming that the sequence a_n converges to some limit, we have that the sequence given by its arithmetic means

$$\frac{\sum_{k=1}^{n} a_k}{n}$$

converges to the same limit. So we conclude

$$\lim_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n} = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!} - 1}{n} = 1/e\,.$$

Thus, the tricky part is actually showing that the Lalescu sequence converges. This can be shown by means of Stirling's formula [4]: we rewrite the sequence as

$$\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \sqrt[n]{n!}\left(e^{\log(n+1)!/(n+1) - \log n!/n} - 1\right)$$

and examine the exponent of e:

$$\begin{aligned}
\frac{\log(n+1)!}{n+1} - \frac{\log n!}{n} &= \frac{n\log(n+1) - \log n!}{n(n+1)} \\
&= \frac{n\log(n+1) + n\log n - n\log n - n\log\sqrt[n]{n!}}{n(n+1)} \\
&= \frac{n\log(1+1/n) + n\log(n/\sqrt[n]{n!})}{n(n+1)} \\
&= \frac{\log(1+1/n) + \log(n/\sqrt[n]{n!})}{(n+1)}\,.
\end{aligned}$$

Then, since $n/\sqrt[n]{n!} \to e$, we have that such exponent is equal to

$$1/(n+1) + o(1/n).$$

If we consider that, by Stirling's formula,

$$\sqrt[n]{n!} \approx n/e\,,$$

we get

$$\sqrt[n]{n!}\left(e^{\log(n+1)!/(n+1) - \log n!/n} - 1\right) \approx \frac{n/e}{n+1} + o(1) \to 1/e\,.$$

An alternative line, without the use of Stirling's formula, goes as follows: we rewrite the sequence as

$$\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \frac{\sqrt[n]{n!}}{n}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1\right)n$$

and we observe that

$$\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n = \left(\frac{(n+1)!^n}{n!^{n+1}}\right)^{\frac{1}{n+1}} = \left(\frac{(n+1)^n}{n!}\right)^{\frac{1}{n+1}} = \left(\frac{n+1}{\sqrt[n]{n!}}\right)^{\frac{n}{n+1}} \to e\,, \qquad (1.1)$$

which implies

$$\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \to 1\,. \qquad (1.2)$$

Then,

$$\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \frac{\sqrt[n]{n!}}{n} \cdot \frac{\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1}{\log\left(1 + \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1\right)\right)} \cdot \log\left(1 + \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1\right)\right)^n$$

$$= \frac{\sqrt[n]{n!}}{n} \cdot \frac{\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1}{\log\left(1 + \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1\right)\right)} \cdot \log\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n \to 1,$$

for $n \to \infty$. Indeed, the first factor tends to $1/e$, while the second and third go to 1, by the limits (1.2) and (1.1), respectively.

Another "natural" way to show the convergence of the sequence would be to prove that it is bounded and monotone. The boundedness from below is actually easy: the sequence a_n is positive, for every $n \in \mathbb{N}$. Indeed, when above we expressed the sequence as

$$\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \sqrt[n]{n!}\left(e^{\log(n+1)!/(n+1) - \log n!/n} - 1\right),$$

we have seen that the exponent of e is given by

$$\frac{\log(n+1)!}{n+1} - \frac{\log n!}{n} = \frac{\log(1+1/n) + \log(n/\sqrt[n]{n!})}{(n+1)},$$

that is positive, since $1 + 1/n$ and $n/\sqrt[n]{n!}$ are both greater than 1, hence the positivity of a_n. Unfortunately, the monotonicity, that is, the fact that a_n is decresing (as one could expect), is not present in literature, up to our knowledge and it turns out being absolutely non trivial.

Our contribution to the study of the Lalescu sequence is then to show such monotonicity, first eventually (from some $n \in \mathbb{N}$ on, which is clearly sufficient for the convergence) and then fully (for every $n \in \mathbb{N}$).

As we will see in the next section, our analysis requires a more refined version of Stirling's formula than the "standard" one (a "higher order" expansion of $n!$, formula (2.3)) and quite precise estimates from above and below on $n!$ (formula (2.8)). Moreover, to obtain the full monotonicity, some numerical check is also needed in order to deal with the "small" values of $n \in \mathbb{N}$.

Let us say that we think that what follows can be seen as an interesting (and tough) problem for undergraduate students (as the second author) about dealing with orders of infinitesimals by means of Taylor expansions and estimates.

2 Decreasing monotonicity

We set $\ell_n = \sqrt[n]{n!}$. Clearly, for every $n \in \mathbb{N}$, we have $\ell_n > 0$.
To see that the sequence

$$a_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$$

is decreasing, we are going to prove equivalently that

$$\frac{\ell_{n+2}}{\ell_{n+1}} + \frac{\ell_n}{\ell_{n+1}} < 2. \tag{2.1}$$

Defining

$$x_n = \log \frac{\ell_{n+1}}{\ell_n},$$

the inequality (2.1) can then be written as

$$\exp(x_{n+1}) + \exp(-x_n) = 2\exp\left(\frac{x_{n+1} - x_n}{2}\right)\cosh\left(\frac{x_{n+1} + x_n}{2}\right) < 2. \tag{2.2}$$

2.1 Eventual monotonicity

We are going to use the following "extended" Stirling's formula (see [4]),

$$\begin{aligned} n! &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + O\left(\frac{1}{n^3}\right)\right) \\ &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + o\left(\frac{1}{n^2}\right)\right). \end{aligned} \tag{2.3}$$

Then,

$$\log \ell_n = \frac{\log(2\pi n)}{2n} + \log n - 1 + \frac{1}{n}\log\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + o\left(\frac{1}{n^2}\right)\right).$$

Expanding in Taylor series up to $o(1/n^3)$, we obtain

$$\begin{aligned} \log \ell_n &= \log n - 1 + \frac{\log n}{2n} + \frac{\log(2\pi)}{2n} + \frac{1}{12n^2} - \frac{1}{360n^4} + o\left(\frac{1}{n^4}\right) \\ &= \log n - 1 + \frac{\log n}{2n} + \frac{\log(2\pi)}{2n} + \frac{1}{12n^2} + o\left(\frac{1}{n^3}\right). \end{aligned}$$

Hence,

$$\begin{aligned} x_n = \log\frac{\ell_{n+1}}{\ell_n} &= \log(n+1) - 1 + \frac{\log(n+1)}{2(n+1)} + \frac{\log(2\pi)}{2(n+1)} + \frac{1}{12(n+1)^2} + o\left(\frac{1}{n^3}\right) \\ &\quad - \log n + 1 - \frac{\log n}{2n} - \frac{\log(2\pi)}{2n} - \frac{1}{12n^2} + o\left(\frac{1}{n^3}\right) \\ &= \log(1+1/n) + \frac{n\log(1+1/n) - \log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{2n+1}{12n^2(n+1)^2} + o\left(\frac{1}{n^3}\right) \\ &= \log(1+1/n) + \frac{\log(1+1/n)}{2(n+1)} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{1}{6n^2(n+1)} + o\left(\frac{1}{n^3}\right) \\ &= \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{1}{2n^2} + \frac{1}{2n(n+1)} \\ &\quad + \frac{1}{3n^3} - \frac{1}{6n^2(n+1)} - \frac{1}{4n^2(n+1)} + o\left(\frac{1}{n^3}\right) \\ &= \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{7}{12n^3} + o\left(\frac{1}{n^3}\right). \end{aligned}$$

So we have

$$
\begin{aligned}
x_{n+1}-x_n =& \frac{1}{n+1}-\frac{\log(n+1)}{2(n+1)(n+2)}-\frac{\log(2\pi)}{2(n+1)(n+2)}-\frac{7}{12(n+1)^3}\\
&-\frac{1}{n}+\frac{\log n}{2n(n+1)}+\frac{\log(2\pi)}{2n(n+1)}+\frac{7}{12n^3}+o\Big(\frac{1}{n^3}\Big)\\
=&-\frac{1}{n(n+1)}-\frac{n\log(n+1)-(n+2)\log n}{2n(n+1)(n+2)}+\frac{\log(2\pi)}{n(n+1)(n+2)}+o\Big(\frac{1}{n^3}\Big)\\
=&-\frac{1}{n(n+1)}-\frac{n\log(1+1/n)-2\log n}{2n(n+1)(n+2)}+\frac{\log(2\pi)}{n^3}+o\Big(\frac{1}{n^3}\Big)\\
=&-\frac{1}{n(n+1)}+\frac{\log n}{n^3}-\frac{1}{2n^3}+\frac{\log(2\pi)}{n^3}+o\Big(\frac{1}{n^3}\Big)\\
=&-\frac{1}{n^2}+\frac{\log n}{n^3}+\frac{1}{n^3}-\frac{1}{2n^3}+\frac{\log(2\pi)}{n^3}+o\Big(\frac{1}{n^3}\Big)\\
=&-\frac{1}{n^2}+\frac{\log n}{n^3}+\frac{1+2\log(2\pi)}{2n^3}+o\Big(\frac{1}{n^3}\Big).
\end{aligned}
$$

For the sum $x_{n+1}+x_n$ we expand in Taylor series up to the term n^{-2}.

$$
\begin{aligned}
x_{n+1}+x_n =& \frac{1}{n+1}-\frac{\log(n+1)}{2(n+1)(n+2)}-\frac{\log(2\pi)}{2(n+1)(n+2)}\\
&+\frac{1}{n}-\frac{\log n}{2n(n+1)}-\frac{\log(2\pi)}{2n(n+1)}+o\Big(\frac{1}{n^2}\Big)\\
=&\frac{2n+1}{n(n+1)}-\frac{n\log(n+1)+(n+2)\log n}{2n(n+1)(n+2)}-\frac{\log(2\pi)}{n(n+1)}+o\Big(\frac{1}{n^2}\Big)\\
=&\frac{2n+1}{n(n+1)}-\frac{\log[n(n+1)]}{2n(n+1)}-\frac{\log(2\pi)}{n(n+1)}+o\Big(\frac{1}{n^2}\Big)\\
=&\frac{2}{n}-\frac{\log[n(n+1)]}{2n(n+1)}-\frac{1+\log(2\pi)}{n^2}+o\Big(\frac{1}{n^2}\Big).
\end{aligned}
$$

It follows, expanding up to $o(1/n^3)$,

$$
\begin{aligned}
(x_{n+1}+x_n)^2 =& \left(\frac{2}{n}-\frac{\log[n(n+1)]}{2n(n+1)}-\frac{1+\log(2\pi)}{n^2}+o\Big(\frac{1}{n^2}\Big)\right)^2\\
=&\frac{4}{n^2}-2\frac{\log[n(n+1)]}{n^2(n+1)}-\frac{4(1+\log(2\pi))}{n^3}+o\Big(\frac{1}{n^3}\Big)\\
=&\frac{4}{n^2}-4\frac{\log n}{n^3}-\frac{4(1+\log(2\pi))}{n^3}+o\Big(\frac{1}{n^3}\Big).
\end{aligned}
$$

Developing then in Taylor series formula (2.2), we obtain

$$
\begin{aligned}
2\exp\left(\frac{x_{n+1}-x_n}{2}\right)&\cosh\left(\frac{x_{n+1}+x_n}{2}\right)\\
&=2\left(1+\frac{x_{n+1}-x_n}{2}+\dots\right)\left(1+\frac{1}{2}\left(\frac{x_{n+1}+x_n}{2}\right)^2+\dots\right)\\
&=2\left(1-\frac{1}{2n^2}+\frac{\log n}{2n^3}+\frac{1+2\log(2\pi)}{4n^3}+o\left(\frac{1}{n^3}\right)\right)\\
&\quad\cdot\left(1+\frac{1}{8}\left(\frac{4}{n^2}-4\frac{\log n}{n^3}-\frac{4(1+\log(2\pi))}{n^3}+o\left(\frac{1}{n^3}\right)\right)\right)\\
&=2\left(1-\frac{1}{2n^2}+\frac{\log n}{2n^3}+\frac{1+2\log(2\pi)}{4n^3}+o\left(\frac{1}{n^3}\right)\right)\\
&\quad\cdot\left(1+\frac{1}{2n^2}-\frac{\log n}{2n^3}-\frac{1+\log(2\pi)}{2n^3}+o\left(\frac{1}{n^3}\right)\right)\\
&=2-\frac{1}{2n^3}+o\left(\frac{1}{n^3}\right),
\end{aligned}
$$

which is clearly smaller than 2, for large $n \in \mathbb{N}$.

Therefore, the Lalescu sequence is *eventually* decreasing, by formula (2.2).

2.2 Full monotonicity

We will use the following "standard" inequalities

$$x - x^2/2 \leqslant \log(1+x) \leqslant x \tag{2.4}$$

$$x - x^2/2 + x^3/3 - x^4/4 \leqslant \log(1+x) \leqslant x - x^2/2 + x^3/3, \tag{2.5}$$

$$1 - x \leqslant \frac{1}{1+x} \leqslant 1 - x + x^2, \tag{2.6}$$

which are valid for $x > 0$.
Furthermore,

$$e^x \leqslant \frac{1}{1-x} \qquad \text{and} \qquad \cosh x \leqslant \frac{1}{1-x^2/2}, \tag{2.7}$$

for $x \in (-1,1)$. The second inequality above can be shown by comparing the Taylor series which converge uniformly in the interval $[-1,1]$,

$$\cosh x = \sum_{i=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \text{and} \qquad \frac{1}{1-x^2/2} = \sum_{i=0}^{\infty} \frac{x^{2n}}{2^n},$$

noticing that $2^n \leqslant (2n)!$, for every $n \in \mathbb{N}$.
From the following estimates due to Robbins [2],

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leqslant n! \leqslant \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}, \tag{2.8}$$

holding for every $n \in \mathbb{N}$, it follows that

$$\frac{\log(2\pi n)}{2n} + \log n - 1 + \frac{1}{(12n+1)n} \leqslant \log \ell_n \leqslant \frac{\log(2\pi n)}{2n} + \log n - 1 + \frac{1}{12n^2}.$$

Thus,

$$\begin{aligned}
x_n = \log\frac{\ell_{n+1}}{\ell_n} &\leqslant \log(n+1) - 1 + \frac{\log(n+1)}{2(n+1)} + \frac{\log(2\pi)}{2(n+1)} + \frac{1}{12(n+1)^2} \\
&\quad - \log n + 1 - \frac{\log n}{2n} - \frac{\log(2\pi)}{2n} - \frac{1}{(12n+1)n} \\
&= \log(1+1/n) + \frac{n\log(1+1/n) - \log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{23n+12}{12(12n+1)(n+1)^2 n} \\
&= \log(1+1/n) + \frac{\log(1+1/n)}{2(n+1)} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{23n+12}{12(12n+1)(n+1)^2 n}.
\end{aligned}$$

Then, applying the inequality at the right side of formula (2.5) to $\log(1+1/n)$, we have

$$\begin{aligned}
x_n &\leqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{23n+12}{12(12n+1)(n+1)^2 n} - \frac{1}{2n^2} + \frac{1}{2n(n+1)} \\
&\quad + \frac{1}{3n^3} - \frac{1}{4n^2(n+1)} + \frac{1}{6n^3(n+1)} \\
&= \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{23n+12}{12(12n+1)(n+1)^2 n} - \frac{1}{2n^2(n+1)} \\
&\quad + \frac{1}{12n^3} + \frac{5}{12n^3(n+1)} \\
&= \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{23n+12}{12(12n+1)(n+1)^2 n} - \frac{5}{12n^3} + \frac{11}{12n^3(n+1)} \\
&\leqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{23n+12}{12(12n+1)(n+1)^2 n} - \frac{5}{12n^3} + \frac{11}{12n^4},
\end{aligned}$$

where in the last step, we estimated $\frac{1}{12n^3(n+1)} \leqslant \frac{1}{12n^4}$.
By the inequality at the left side of formula (2.6), we have

$$\begin{aligned}
\frac{23}{12(12n+1)(n+1)^2} &= \frac{23}{144n^3\left(1 + \frac{25}{12n} + \frac{7}{6n^2} + \frac{1}{12n^3}\right)} \\
&\geqslant \frac{23}{144n^3}\left(1 - \frac{25}{12n} - \frac{7}{6n^2} - \frac{1}{12n^3}\right) \\
&\geqslant \frac{23}{144n^3} - \frac{1}{2n^4},
\end{aligned}$$

for every $n \geqslant 2$, therefore

$$\begin{aligned}
x_n &\leqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{23}{144n^3} + \frac{1}{2n^4} - \frac{5}{12n^3} + \frac{11}{12n^4} \\
&\leqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{83}{144n^3} + \frac{3}{2n^4}.
\end{aligned} \tag{2.9}$$

Furthermore, it is easily seen that, when $n \geqslant 3$, this inequality implies the "simpler" inequality

$$x_n \leqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)}, \tag{2.10}$$

which will be useful later.
Similarly, using the left inequalities in formulas (2.4) and (2.5) on $\log(1+1/n)$, we have

$$
\begin{aligned}
x_n = \log\frac{\ell_{n+1}}{\ell_n} &\geqslant \log(1+1/n) + \frac{n\log(1+1/n)-\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{25n+13}{12(12n+13)(n+1)n^2} \\
&\geqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{1}{2n^2} + \frac{1}{2n(n+1)} - \frac{25n+13}{12(12n+13)(n+1)n^2} - \frac{1}{4n^2(n+1)} \\
&\quad + \frac{1}{3n^3} - \frac{1}{4n^4} \\
&= \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{25n+13}{12(12n+13)(n+1)n^2} - \frac{3}{4n^2(n+1)} + \frac{1}{3n^3} - \frac{1}{4n^4} \\
&\geqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{25n+13}{12(12n+13)(n+1)n^2} - \frac{3}{4n^3} + \frac{1}{3n^3} - \frac{1}{4n^4} \\
&= \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{25n+13}{12(12n+13)(n+1)n^2} - \frac{5}{12n^3} - \frac{1}{4n^4}
\end{aligned}
$$

where, in the penultimate step, we estimated $\frac{3}{4n^2(n+1)} \leqslant \frac{3}{4n^3}$.
Since clearly

$$
\frac{25n+13}{12(12n+13)(n+1)n^2} \leqslant \frac{25}{144n^3} + \frac{13}{144n^4},
$$

we have

$$
\begin{aligned}
x_n &\geqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{25}{144n^3} - \frac{13}{144n^4} - \frac{5}{12n^3} - \frac{1}{4n^4} \\
&= \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{85}{144n^3} - \frac{49}{144n^4}, \qquad (2.11)
\end{aligned}
$$

which implies the "simpler" inequality

$$
x_n \geqslant \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} - \frac{1}{n^3}, \qquad (2.12)
$$

for each $n \geqslant 1$, which we will use later.

Let us therefore estimate the difference $x_{n+1} - x_n$ from above with the inequalities (2.9) and (2.11):

$$
\begin{aligned}
x_{n+1} - x_n &\leqslant \frac{1}{n+1} - \frac{\log(n+1)}{2(n+1)(n+2)} - \frac{\log(2\pi)}{2(n+1)(n+2)} - \frac{83}{144(n+1)^3} + \frac{3}{2(n+1)^4} \\
&\quad - \frac{1}{n} + \frac{\log n}{2n(n+1)} + \frac{\log(2\pi)}{2n(n+1)} + \frac{85}{144n^3} + \frac{49}{144n^4} \\
&\leqslant -\frac{1}{n(n+1)} - \frac{n\log(n+1)-(n+2)\log n}{2n(n+1)(n+2)} + \frac{\log(2\pi)}{n(n+1)(n+2)} \\
&\quad + \frac{85(1+3n+3n^2)+2n^3}{144n^3(n+1)^3} + \frac{49}{144n^4} + \frac{3}{2n^4} \\
&\leqslant -\frac{1}{n^2} - \frac{n\log(1+1/n)-2\log n}{2n(n+1)(n+2)} + \frac{\log(2\pi)}{n^3} + \frac{1}{n^3} \\
&\quad + \frac{85(1+3n+3n^2)}{144n^3(n+1)^3} + \frac{2}{144n^3} + \frac{265}{144n^4},
\end{aligned}
$$

where in the last step we used $-\frac{1}{n(n+1)} = -\frac{1}{n^2} + \frac{1}{n^2(n+1)} \leqslant -\frac{1}{n^2} + \frac{1}{n^3}$.
Since $1 + 3n + 3n^2 \leqslant 4n^2$, for $n \geqslant 4$, we have

$$
\begin{aligned}
x_{n+1} - x_n \leqslant & -\frac{1}{n^2} - \frac{\log(1+1/n)}{2(n+1)(n+2)} + \frac{\log n}{n(n+1)(n+2)} + \frac{\log(2\pi)}{n^3} \\
& + \frac{340}{144n(n+1)^3} + \frac{146}{144n^3} + \frac{265}{144n^4} \\
\leqslant & -\frac{1}{n^2} - \frac{\log(1+1/n)}{2(n+1)(n+2)} + \frac{\log n}{n^3} + \frac{\log(2\pi)}{n^3} \\
& + \frac{340}{144n^4} + \frac{146}{144n^3} + \frac{265}{144n^4} \\
= & -\frac{1}{n^2} - \frac{\log(1+1/n)}{2(n+1)(n+2)} + \frac{\log n}{n^3} + \frac{\log(2\pi)}{n^3} + \frac{146}{144n^3} + \frac{605}{144n^4}
\end{aligned}
$$

and applying the left inequality in formula (2.4) to $\log(1+1/n)$, we conclude

$$
\begin{aligned}
x_{n+1} - x_n & \leqslant -\frac{1}{n^2} - \frac{1}{2n(n+1)(n+2)} + \frac{1}{4n^2(n+1)(n+2)} + \frac{\log n}{n^3} + \frac{\log(2\pi)}{n^3} + \frac{146}{144n^3} + \frac{605}{144n^4}. \\
& \leqslant -\frac{1}{n^2} - \frac{1}{2n^3} + \frac{3}{2n^4} + \frac{1}{4n^4} + \frac{\log n}{n^3} + \frac{\log(2\pi)}{n^3} + \frac{146}{144n^3} + \frac{605}{144n^4}. \\
& \leqslant -\frac{1}{n^2} + \frac{\log n}{n^3} + \frac{74/144 + \log(2\pi)}{n^3} + \frac{5}{n^4},
\end{aligned}
$$

for $n \geqslant 4$, where we used $-\frac{1}{2n(n+1)(n+2)} = -\frac{1}{2n^3} + \frac{3n+2}{2n^3(n+1)(n+2)} \leqslant -\frac{1}{2n^3} + \frac{3}{2n^4}$.
Instead, using the inequalities (2.10) and (2.12) to estimate $x_{n+1} - x_n$ from below, we have

$$
\begin{aligned}
x_{n+1} - x_n \geqslant & \frac{1}{n+1} - \frac{\log(n+1)}{2(n+1)(n+2)} - \frac{\log(2\pi)}{2(n+1)(n+2)} - \frac{1}{(n+1)^3} \\
& - \frac{1}{n} + \frac{\log n}{2n(n+1)} + \frac{\log(2\pi)}{2n(n+1)} \\
= & -\frac{1}{n(n+1)} - \frac{n\log(n+1) - (n+2)\log n}{2n(n+1)(n+2)} - \frac{1}{(n+1)^3} + \frac{\log(2\pi)}{n(n+1)(n+2)} \\
= & -\frac{1}{n^2} + \frac{1}{n^2(n+1)} - \frac{\log(1+1/n)}{2(n+1)(n+2)} + \frac{\log n}{n(n+1)(n+2)} - \frac{1}{(n+1)^3} + \frac{\log(2\pi)}{n(n+1)(n+2)} \\
\geqslant & -\frac{1}{n^2} - \frac{1}{2n(n+1)(n+2)} + \frac{\log n}{n(n+1)(n+2)} + \frac{\log(2\pi)}{n(n+1)(n+2)} \\
\geqslant & -\frac{1}{n^2},
\end{aligned}
$$

for every $n \in \mathbb{N}$.
Therefore, for $n \geqslant 4$, we conclude

$$
-\frac{1}{n^2} \leqslant x_{n+1} - x_n \leqslant -\frac{1}{n^2} + \frac{\log n}{n^3} + \frac{74/144 + \log(2\pi)}{n^3} + \frac{5}{n^4}. \tag{2.13}
$$

Furthermore, noticing that the term on the right is certainly negative for $n \geqslant 4$, we have in such case

$$
(x_{n+1} - x_n)^2 \leqslant \frac{1}{n^4}. \tag{2.14}
$$

Let us now estimate the sum $x_{n+1} + x_n$ using the inequality (2.10):

$$
\begin{aligned}
x_{n+1} + x_n \leqslant \frac{1}{n+1} - \frac{\log(n+1)}{2(n+1)(n+2)} - \frac{\log(2\pi)}{2(n+1)(n+2)} + \frac{1}{n} - \frac{\log n}{2n(n+1)} - \frac{\log(2\pi)}{2n(n+1)} \\
= \frac{2n+1}{n(n+1)} - \frac{n\log(n+1) + (n+2)\log n}{2n(n+1)(n+2)} - \frac{\log(2\pi)}{n(n+2)} \\
= \frac{2n+1}{n(n+1)} - \frac{\log[n(n+1)]}{2n(n+1)} - \frac{\log(2\pi)}{n(n+2)} + \frac{\log(n+1)}{n(n+1)(n+2)}
\end{aligned}
$$

and applying inequalities

$$
\begin{aligned}
&\frac{2n+1}{n(n+1)} = \left(\frac{2}{n} + \frac{1}{n^2}\right)\left(\frac{1}{1+1/n}\right) \leqslant \frac{2}{n} - \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} \leqslant \frac{2}{n} - \frac{1}{n^2} + \frac{2}{n^3} \\
&\frac{\log[n(n+1)]}{2n(n+1)} \geqslant \frac{\log n}{n(n+1)} \geqslant \frac{\log n}{n^2} - \frac{\log n}{n^3} \\
&\frac{1}{n(n+1)} \geqslant \frac{1}{n^2} - \frac{1}{n^3} \\
&\frac{1}{n(n+2)} \geqslant \frac{1}{n^2} - \frac{2}{n^3}
\end{aligned}
$$

(where in the first one we used the inequality at the left side of formula (2.6)), we obtain

$$
x_{n+1} + x_n \leqslant \frac{2}{n} - \frac{\log n}{n^2} - \frac{1+\log(2\pi)}{n^2} + \frac{\log n}{n^3} + \frac{2+2\log(2\pi)}{n^3} + \frac{\log(n+1)}{n^3},
$$

Then, considering the inequality

$$
\log n + 2 + 2\log(2\pi) + \log(n+1) \leqslant \frac{n}{16},
$$

from which it obviously follows

$$
\frac{\log n}{n^3} + \frac{2+2\log(2\pi)}{n^3} + \frac{\log(n+1)}{n^3} \leqslant \frac{1}{16n^2}
$$

and which holds for $n \geqslant 271$ (keeping in mind the concavity of the left–hand side and checking numerically), we conclude

$$
0 \leqslant x_{n+1} + x_n \leqslant \frac{2}{n} - \frac{\log n}{n^2} - \frac{15/16 + \log(2\pi)}{n^2} \leqslant \frac{2}{n}, \tag{2.15}
$$

for every $n \geqslant 271$.

For $n \geqslant 271$, both $y = \left(\frac{x_{n+1}-x_n}{2}\right)$ and $z = \left(\frac{x_{n+1}+x_n}{2}\right)$ are smaller than 1, so we can use the inequalities in formula (2.7) and evaluate

$$
\exp\left(\frac{x_{n+1}-x_n}{2}\right) = e^y \leqslant \frac{1}{1-y} = 1 + y + \frac{y^2}{1-y} \leqslant 1 + y + y^2,
$$

being $y \leqslant 0$ and

$$
\cosh\left(\frac{x_{n+1}+x_n}{2}\right) = \cosh z \leqslant \frac{1}{1-z^2/2} = 1 + \frac{z^2}{2} + \frac{z^4}{4(1-z^2/2)} \leqslant 1 + \frac{z^2}{2} + \frac{z^4}{2},
$$

being $z^2 \leqslant 1$. Therefore, for the inequalities (2.13), (2.14) and (2.15), we get

$$\begin{aligned}
2\exp\left(\frac{x_{n+1}-x_n}{2}\right)\cosh\left(\frac{x_{n+1}+x_n}{2}\right) &\leqslant (1+y+y^2)\left(1+\frac{z^2}{2}+\frac{z^4}{2}\right)\\
&=2\left(1+\frac{x_{n+1}-x_n}{2}+\left(\frac{x_{n+1}-x_n}{2}\right)^2\right)\left(1+\frac{1}{2}\left(\frac{x_{n+1}+x_n}{2}\right)^2+\frac{1}{2}\left(\frac{x_{n+1}+x_n}{2}\right)^4\right)\\
&\leqslant 2\left(1-\frac{1}{2n^2}+\frac{\log n}{2n^3}+\frac{74/144+\log(2\pi)}{2n^3}+\frac{5}{2n^4}+\frac{1}{4n^4}\right)\\
&\quad\cdot\left(1+\frac{1}{2}\left(\frac{1}{n}-\frac{\log n}{2n^2}-\frac{15/16+\log(2\pi)}{2n^2}\right)^2+\frac{1}{2n^4}\right)\\
&=2\left(1-\frac{1}{2n^2}+\frac{\log n}{2n^3}+\frac{74/144+\log(2\pi)}{2n^3}+\frac{11}{4n^4}\right)\\
&\quad\cdot\left(1+\frac{1}{2n^2}-\frac{15/16+\log(2\pi)+\log n}{2n^3}+\frac{\left(16\log n+15+16\log(2\pi)\right)^2}{2048n^4}+\frac{1}{2n^4}\right)\\
&=2\left(1-\frac{1}{2n^2}+\frac{\log n}{2n^3}+\frac{74/144+\log(2\pi)}{2n^3}+\frac{11}{4n^4}\right)\\
&\quad\cdot\left(1+\frac{1}{2n^2}-\frac{15/16+\log(2\pi)+\log n}{2n^3}+\frac{\left(16\log n+15+16\log(2\pi)\right)^2+1024}{2048n^4}\right).
\end{aligned}$$

It is easy to show that

$$\frac{\left(16\log n+15+16\log(2\pi)\right)^2+1024}{2048n^4}\leqslant\frac{1}{32n^3},$$

for $n \geqslant 304$, hence

$$\begin{aligned}
&2\exp\left(\frac{x_{n+1}-x_n}{2}\right)\cosh\left(\frac{x_{n+1}+x_n}{2}\right)\\
&\quad\leqslant 2\left(1-\frac{1}{2n^2}+\frac{\log n}{2n^3}+\frac{74/144+\log(2\pi)}{2n^3}+\frac{11}{4n^4}\right)\left(1+\frac{1}{2n^2}-\frac{14/16+\log(2\pi)+\log n}{2n^3}\right)\\
&\quad\leqslant 2\left(1-\frac{1}{2n^2}+\frac{\log n}{2n^3}+\frac{76/144+\log(2\pi)}{2n^3}\right)\left(1+\frac{1}{2n^2}-\frac{\log n}{2n^3}-\frac{14/16+\log(2\pi)}{2n^3}\right),
\end{aligned}$$

for $n \geqslant 396$, since then $\frac{11}{4n^4} \leqslant \frac{1}{144n^3}$. We therefore finally conclude that for $n \geqslant 396$, we have (after a straightforward computation)

$$\begin{aligned}
2\exp\left(\frac{x_{n+1}-x_n}{2}\right)\cosh\left(\frac{x_{n+1}+x_n}{2}\right) &\leqslant 2\left(1-\frac{25}{144n^3}-\frac{1}{4n^4}+\frac{\log n/2+101/288+\log(2\pi)/2}{n^5}\right)\\
&\quad-2\frac{\log n+76/144+\log(2\pi)}{2n^3}\cdot\frac{\log n+14/16+\log(2\pi)}{2n^3}\\
&<2\left(1-\frac{25}{144n^3}-\frac{1}{4n^4}+\frac{\log n/2+101/288+\log(2\pi)/2}{n^5}\right)\\
&<2,
\end{aligned}$$

since, by a numerical check, there holds

$$-\frac{25}{144}-\frac{1}{4n}+\frac{\log n/2+101/288+\log(2\pi)/2}{n^2}<0$$

for each $n \geqslant 3$.

All previous estimates being valid for $n \geqslant 396$, for such $n \in \mathbb{N}$ the sequence is decreasing. By numerically checking the decreasing for $n = 1, \dots, 396$, we then obtain that the Lalescu sequence is *always* decreasing, remembering formula (2.2).

To numerically check the decreasing for $n = 1, \dots, 396$, we used the following code for the *Julia – Version 1.10.4* programming language, with the *IntervalArithmetic.jl* package [3]:

```
using IntervalArithmetic

function rootfactorial(n)
    I = @interval(1)
    exp = @interval(1) / @interval(n)
    for j = 1:n
        I = I * @interval(j)^exp
    end
    return I
end

function lalescu(n)
    return rootfactorial(n+1) - rootfactorial(n)
end

setdisplay(:full)
for k =  1:400
    println("a_$(k) = $(lalescu(k))")
    if !(precedes(lalescu(k), lalescu(k-1)))
        @error("lalescu($k) is not guaranteed to be smaller
        than lalescu($(k-1))")
    end
end
```

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