1.2. First Variation of the Area Functional

Given an immersion $\varphi : M \to \mathbb{R}^{n+1}$ of a smooth hypersurface in $\mathbb{R}^{n+1}$, we consider the Area functional

$$\mathrm{Area}(\varphi) = \int_M d\mu$$

where μ is the canonical measure associated to the metric g induced by the immersion.

In this section we are going to analyze the first variation of such functional which is nothing else than the volume of the hypersurface.

We consider a smooth one parameter family of immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, with $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = \varphi$, such that, outside of a compact set $K \subset M$, we have $\varphi_t(p) = \varphi(p)$ for every $t \in (-\varepsilon, \varepsilon)$.
Defining the field $X = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ along M (as a submanifold of $\mathbb{R}^{n+1}$) we see that X is zero outside K, we call such field the *infinitesimal generator* of the variation φ_t.
We compute

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \varphi_t}{\partial x_i} \,\Big|\, \frac{\partial \varphi_t}{\partial x_j} \right\rangle\Big|_{t=0} \\
&= \left\langle \frac{\partial X}{\partial x_i} \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \left\langle \frac{\partial X}{\partial x_j} \,\Big|\, \frac{\partial \varphi}{\partial x_i} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \frac{\partial}{\partial x_j} \left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_i} \right\rangle - 2 \left\langle X \,\Big|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \frac{\partial}{\partial x_j} \left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_i} \right\rangle - 2\Gamma_{ij}^k \left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_k} \right\rangle - 2h_{ij} \langle X \,|\, \nu \rangle \,,
\end{aligned}$$

where X^M is the tangent component of the field X and we used the Gauss–Weingarten relations (1.1.1) in the last step.
Letting ω be the 1–form defined by $\omega(Y) = g(d\varphi^*(X^M), Y)$, this formula can be rewritten as

$$\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} = \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} - 2\Gamma_{ij}^k \omega_k - 2h_{ij} \langle X \,|\, \nu \rangle = \nabla_i \omega_j + \nabla_j \omega_i - 2h_{ij} \langle X \,|\, \nu \rangle \,.$$

Hence, using the formula $\partial_t \det A(t) = \det A(t) \mathrm{Trace}[A^{-1}(t) \partial_t A(t)]$, we get

$$\begin{aligned}
\frac{\partial}{\partial t} \sqrt{\det(g_{ij})}\Big|_{t=0} &= \frac{\sqrt{\det(g_{ij})}\, g^{ij} \frac{\partial}{\partial t} g_{ij}\big|_{t=0}}{2} \\
&= \frac{\sqrt{\det(g_{ij})}\, g^{ij} \big(\nabla_i \omega_j + \nabla_j \omega_i - 2h_{ij} \langle X \,|\, \nu \rangle\big)}{2} \\
&= \sqrt{\det(g_{ij})} \big(\operatorname{div} d\varphi^*(X^M) - \mathrm{H} \langle X \,|\, \nu \rangle\big) \,.
\end{aligned}$$

If the Area of the immersion φ is finite, the same holds for all the maps φ_t, as they are compact deformations of φ. Assuming that the compact K is contained in a single coordinate chart, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} &= \frac{\partial}{\partial t}\int_K d\mu_t\Big|_{t=0} \\
&= \frac{\partial}{\partial t}\int_K \sqrt{\det(g_{ij})}\,d\mathcal{L}^n\Big|_{t=0} \\
&= \int_K \frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\Big|_{t=0}\,d\mathcal{L}^n \\
&= \int_K \big(\mathrm{div}\,d\varphi^*(X^M) - \mathrm{H}\langle X\,|\,\nu\rangle\big)\sqrt{\det(g_{ij})}\,d\mathcal{L}^n \\
&= \int_M \big(\mathrm{div}\,d\varphi^*(X^M) - \mathrm{H}\langle X\,|\,\nu\rangle\big)\,d\mu \\
&= -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu
\end{aligned}$$

where we used the fact that X is zero outside K and in the last step we applied the divergence theorem. Notice that all the integrals are well defined because we are actually integrating only on the compact set K.
In the case that K is contained in several charts, the same conclusion follows from a standard argument using a partition of unity.

PROPOSITION 1.2.1. *The first variation of the* Area *functional depends only on the normal component of the infinitesimal generator* $X = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ *of the variation* φ_t*, precisely*

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} = -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu\,.$$

Clearly, such dependence is linear.

Given any immersion $\varphi : M \to \mathbb{R}^{n+1}$ and any vector field X with compact support along M, we can always construct a variation with infinitesimal generator X as $\varphi_t(p) = \varphi(p) + tX(p)$. It is easy to see that for $|t|$ small the map φ_t is still a smooth immersion.
Hence, as the hypersurfaces which are critical points of the Area functional must satisfy

$$\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu = 0$$

for every field X with compact support, they must have $\mathrm{H} = 0$ everywhere, that is, zero mean curvature (and conversely). This is the well known definition of the so called *minimal surfaces*.

As the quantity $-\mathrm{H}\nu$ can be interpreted as the *gradient* of the Area functional (be careful here, the measure μ is varying with the immersion, we are not computing the gradient with respect to some fixed L^2–structure on the space of immersions of M in $\mathbb{R}^{n+1}$), we can consider the motion of a hypersurface by minus this gradient, that is, the *mean curvature flow*. So, one looks for hypersurfaces moving with velocity $\mathrm{H}\nu$ at every point. This means choosing, among all the velocity functions with fixed $L^2(\mu)$–norm equal to $\big(\int_M \mathrm{H}^2\,d\mu\big)^{1/2}$, the one such that the Area of hypersurface decreases most rapidly.

This idea is quite natural and arises often in studying the dynamics of models of physical situations where the energy is given by the "Area" of the interfaces between the phases of a system. Moreover, as the Area functional is the simplest (in terms of derivatives of the parametrization) geometric functional, that is, invariant by isometries of $\mathbb{R}^{n+1}$ and diffeomorphisms of M, the motion by mean curvature is the simplest *variational* geometric flow for immersed hypersurfaces. Other geometric functionals (for instance, depending on the next simpler geometric invariant, the curvature) generally produce a first variation of order higher than two in the derivatives of the parametrization and a relative higher order PDE's system.

One can consider second order flows where the velocity of the motion is related to different functions of the curvature, like the Gauss flow of surfaces, for instance, where the velocity is given by $\mathrm{G}\nu$ (G is the Gauss curvature of M, that is, $\mathrm{G} = \det \mathrm{A}$) or more complicated flows, but these evolutions are usually not variational, they do not arise as "gradients" (in the above sense) of geometric functionals (see Section 1.6).

1.3. The Mean Curvature Flow

DEFINITION 1.3.1. Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an n–dimensional manifold. The mean curvature flow of φ_0 is a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$ such that setting $\varphi(p, t) = \varphi_t(p)$ the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ is a smooth solution of the following system of PDE's

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases} \tag{1.3.1}$$

where $\mathrm{H}(p, t)$ and $\nu(p, t)$ are respectively the mean curvature and the unit normal of the hypersurface φ_t at the point $p \in M$.

REMARK 1.3.2. Notice that even if the unit normal vector is defined up to a sign, the field $\mathrm{H}(p, t)\nu(p, t)$ is independent of such choice.

Using equation (1.1.2), this system can be rewritten in the appealing form

$$\frac{\partial \varphi}{\partial t} = \Delta \varphi$$

but, despite its formal analogy with the heat equation, actually, it is a second order, *quasilinear* and *degenerate*, parabolic system, as the Laplacian is the one associated to the evolving hypersurfaces at time t,

$$\Delta\varphi(p,t) = \Delta_{g(p,t)}\varphi(p,t) = g^{ij}(p,t)\nabla_i^{g(p,t)}\nabla_j^{g(p,t)}\varphi(p,t)$$

and its coefficients as second order partial differential operator depend on the first derivatives of φ. Moreover, this operator is degenerate, as its symbol (the symbol of the linearized operator) admits zero eigenvalues due to the invariance of the Laplacian by diffeomorphisms, see **[49]** for details.

Like the Area functional, the flow is obviously invariant by rotations and translations, or more generally under any isometry of $\mathbb{R}^{n+1}$. Moreover, if $\varphi(p, t)$ is a mean curvature flow and $\Psi : M \to M$ is a diffeomorphism, then the reparametrization $\widetilde{\varphi}(p, t) = \varphi(\Psi(p), t)$ is still a mean curvature flow. This last property can be reread as "the flow is invariant under reparametrization", suggesting that the important objects in the flow are actually the subsets $M_t = \varphi(M, t)$ of $\mathbb{R}^{n+1}$.

The problem also satisfies the following parabolic invariance under rescaling (consequence of the property $\mathrm{Area}(\lambda\varphi) = \lambda^n \mathrm{Area}(\varphi)$, for any n–dimensional immersion), if $\varphi(p, t)$ is a mean curvature flow of φ_0 and $\lambda > 0$, then $\widetilde{\varphi}(p, t) = \lambda\varphi(p, \lambda^{-2}t)$ is a mean curvature flow of the initial hypersurface $\lambda\varphi_0$.

During the flow the Area of the hypersurfaces (which is the natural energy of the problem) is nonincreasing, indeed, by the same computation for the first variation of such functional in the previous section, we have

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t) = \frac{\partial}{\partial t}\int_M d\mu_t = -\int_M \mathrm{H}^2\, d\mu_t\,.$$

This clearly implies the estimate

$$\int_0^{T_{\max}} \int_M \mathrm{H}^2\, d\mu_t \leq \mathrm{Area}(\varphi_0)$$

in the maximal time interval $[0, T_{\max})$ of smooth existence of the flow.

EXERCISE 1.3.3. By means of this last inequality, try to get an estimate from above for the maximal time of smooth existence $T_{\max}$ for closed curves in $\mathbb{R}^2$ and compact *surfaces* in $\mathbb{R}^3$.