Take again the Jacobi fields Y_i of 3.96. From 3.67, we have the asymptotic expansion

$$Y_i(t) = tE_i - \frac{t^3}{6}R(c', E_i)c' + o(t^3).$$

The claimed result follows from the asymptotic expansion of $J(u,t)$. To get that expansion, we use the following lemmas.

3.99 **Lemma.** *Let $A(t)$ be a differentiable map from $I \subseteq \mathbf{R}$ into $Gl_n\mathbf{R}$. Then*

$$(\det A)' = (\det A)\mathrm{tr}(A^{-1}A').$$

The proof is left to the reader.

3.100 **Lemma.** *For any symmetric bilinear form ϕ on $\mathbf{R}^n$*

$$\int_{S^{n-1}} \phi(v,v)dv = \frac{1}{n}\mathrm{vol}(S^{n-1})\mathrm{tr}(\phi).$$

Proof. Just diagonalize ϕ with respect to an orthonormal basis. ■

Remarks. i) As a by-product of this proof, we get the asymptotic expansion

$$(\exp_m^* v_g)(u) = \big(1 - \frac{1}{6}Ric(u,u) + o(|\, u\,|^2)\big)v_{\mathrm{eucl}}.$$

ii) A more general asymptotic expansion has been given by A. Gray (cf. [Gy]). The existence of such an expansion is not unexpected in view of the following result of Elie Cartan (see [B-G-M] for a proof in the spirit of this book, using Jacobi fields) : the coefficients of the Taylor expansion at 0 of $\exp_m^* g$ are universal polynomials in the curvature tensor and its covariant derivatives.

3.H.5 Volume estimates

The proof of 3.98 suggests that suitable curvature assumptions could give volume estimates. Denote by $V^k(r)$ the volume of a ball of radius r in the complete simply connected Riemannian manifold with constant curvature k. The following comparison theorem is due to Bishop (case i)) and Gunther (case ii)).

3.101 **Theorem** (Bishop-Gunther). *Let (M,g) be a complete Riemannian manifold, and $B_m(r)$ be a ball which does not meet the cut-locus of m.*
i) *If there is a constant a such that $Ric \geq (n-1)ag$, then*

$$\mathrm{vol}(B_m(r)) \leq V^a(r).$$

ii) *If there is a constant b such that $K \leq b$, then*

$$\mathrm{vol}(B_m(r) \geq V^b(r).$$

Proof. Take a geodesic $c(t) = \exp_m tu$ from m, and an orthonormal basis $\{u, e_2, ..., e_n\}$ of the tangent space at m. Take also, as in the proof of Myers' theorem for example, the parallel vector fields E_i (with $2 \leq i \leq n$) along c such that $E_i(0) = e_i(0)$. Suppose that

$$0 \leq r \leq \rho(u).$$

For such an r, there exists a unique Jacobi field Y_i^r such that

$$Y_i^r(0) = 0 \quad \text{and} \quad Y_i^r(r) = E_i(r).$$

Indeed, since $T_{ru} \exp_m$ is an isomorphism from the tangent space at m onto the tangent space at $c(r)$, this Jacobi field is given by

$$Y_i^r(t) = T_{tu} \exp_m \cdot tv,$$

where v is the unique tangent vector at m such that

$$T_{ru} \exp_m \cdot rv = E_i(r).$$

Now,

$$J(u,t) = C_r t^{1-n} \det\left(Y_2^r(t), \cdots, Y_n^r(t)\right),$$
$$\text{where} \quad C_r^{-1} = \det\left({Y'}_2^r(0), \cdots, {Y'}_n^r(0)\right).$$

For given u, set $f(t) = J(u,t)$.

3.102 Lemma. *Denoting by I the index form of energy, we have*

$$\frac{f'(r)}{f(r)} = \sum_{i=2}^{n} I(Y_i^r, Y_i^r) - \frac{(n-1)}{r}.$$

Proof of the lemma. First remark that

$$| \det(Y_2^r, ... Y_n^r) | = (\det g(Y_i^r, Y_j^r))^{1/2}.$$

In other words, denoting this last determinant by $D(t)$, we have

$$\frac{f'(t)}{f(t)} = \frac{D'(t)}{2D(t)} - \frac{n-1}{t}.$$

For $t = r$, the matrix $\left[g(Y_i^r, Y_j^r)\right]$ is just the unit matrix, and lemma 3.99 shows that

$$D'(r) = 2\sum_{i=2}^{n} g((Y_i^r)', Y_i^r).$$

On the other hand, by the same argument as in 3.76, the second variation formula 3.34, when applied to a Jacobi field Y, gives

$$I(Y,Y) = \int_0^r \left(| Y' |^2 - R(Y, c', Y, c')\right) ds = [g(Y, Y')]_0^r.$$

The claimed formula is now straightforward. ∎

3.103 Lemma. *If $c : [a,b] \to M$ is a minimizing geodesic, Y is a Jacobi field and X is a vector field along c with the same values as Y at the ends, then $I(X,X) \geq I(Y,Y)$.*

Proof of the lemma. Since $X - Y$ vanishes at the ends, we have

$$I(X-Y, X-Y) \geq 0$$

because c is minimizing. On the other hand we have

$$I(Y,Y) = [g(Y',Y)]_a^b \quad \text{and} \quad I(X,Y) = [g(Y',X)]_a^b.$$

Therefore $I(X-Y,X-Y) = I(X,X) - I(Y,Y)$ and the result follows. ∎

End of the proof of the theorem. i) We shall apply the above lemma to Y_i^r and to the vector field X_i^r given by

$$X_i^r(t) = \frac{s(t)}{s(r)} E_i(t),$$

where

$$s(t) = \sin \sqrt{a}t \quad \text{if} \quad a > 0$$

$$s(t) = t \quad \text{if} \quad a = 0$$

$$s(t) = \sinh \sqrt{-a}t \quad \text{if} \quad a < 0.$$

Lemma 3.103 gives

$$\sum_{i=2}^{n} I(Y_i^r, Y_i^r) \leq \sum_{i=2}^{n} I(X_i^r, X_i^r).$$

The right member of this inequality is just

$$\int_0^r \left(\frac{s(t)}{s(r)}\right)^2 \big((n-1)a - Ric(c',c')\big) ds + \sum_{i=2}^{n} g\big(X_i^r, (X_i^r)'\big)(r).$$

The assumption made on the curvature yields that the integral is negative. Then, using lemma 3.102 and the definition of X_i^r, we see that

$$\frac{f'(r)}{f(r)} \leq (n-1)\big(\sqrt{a}\,\mathrm{cotan}\sqrt{a}r - \frac{1}{r}\big) \quad \text{if} \quad a > 0$$

$$\frac{f'(r)}{f(r)} \leq 0 \quad \text{if} \quad a = 0$$

$$\frac{f'(r)}{f(r)} \leq (n-1)\big(\sqrt{-a}\,\mathrm{cotanh}\sqrt{-a}r - \frac{1}{r}\big) \quad \text{if} \quad a < 0.$$

In any case, if $f_a(r)$ denotes the function $J(u,r)$ for the "model space" with constant curvature a (recall that J does not depend on u in that case), we have

$$\frac{f'(r)}{f(r)} \leq \frac{f'_a(r)}{f_a(r)}.$$

By integrating, we get $f(r) \leq f_a(r)$, and the claimed inequality follows from a further integration, using 3.97. ∎

ii) Denoting by Y one of the Jacobi fields Y_i^r, we have (cf. the proof of lemma 3.102)

$$g(Y(r),Y'(r)) = \int_0^r (g(Y',Y') - R(Y,c',Y,c'))ds$$

$$\geq \int_0^r (g(Y',Y') - bg(Y,Y))ds.$$

Write

$$Y(t) = \sum_{i=2}^n y^i(t)E_i(t).$$

On the simply connected manifold with constant curvature b, take a geodesic $\tilde{c}$ of length r, and define vector fields $\tilde{E}_i$ along $\tilde{c}$ in the same way as the vectors E_i. Set

$$\tilde{Y}(t) = \sum_{i=2}^n y^i(t)\tilde{E}_i(t).$$

Then

$$\int_0^r \left(|\tilde{Y}'|^2 - b|\tilde{Y}|^2\right) dt = \int_0^r \left(|Y'|^2 - b|Y|^2\right) dt = I(\tilde{Y},\tilde{Y}).$$

Lemma 3.103, when applied to the simply connected manifold with constant curvature b, gives

$$I(\tilde{Y}_i^r,\tilde{Y}_i^r) \geq I(\tilde{X}_i^r,\tilde{X}_i^r),$$

where $\tilde{X}_i^r(t) = \frac{s(t)}{s(r)}\tilde{E}_i(t)$ is the Jacobi field which takes at the ends of $\tilde{c}$ the same values as $\tilde{Y}_i^r$. Using lemma 3.102, we see that

$$\frac{f'(r)}{f(r)} \geq \frac{f'_b(r)}{f_b(r)},$$

and the claim follows by integration. ∎

3.I Curvature and growth of the fundamental group

3.I.1 Growth of finite type groups

Let Γ be a group of finite type, and $S = \{a_1, ..., a_k\}$ be a system of generators of Γ. Any element s of Γ can be written as

$$s = \prod_i a_{k_i}^{r_i} \quad (r_i \in \mathbf{Z}),$$

with possible repetitions of the generators a_{k_i}. Such a representation is called a *word* with respect to the generators, and the integer

$$\sum_i | r_i |$$

is by definition the *length* of that word. For any positive integer s, the number of elements of Γ which can be represented by words whose length is not greater than s will be denoted by $\phi_S(s)$.

3.104 **Exercise.** i) Show that if Γ is the free Abelian group generated by the a_i,

$$\phi_S(s) = \sum_{i=0}^{k} 2^i \binom{k}{i}\binom{s}{i}.$$

In particular, $\phi_S(s) = O(s^k)$.
ii) Show that if Γ is the free group generated by the a_i, then

$$\phi_S(s) = \frac{k(2k-1)^s - 1}{k-1}.$$

In view of these examples, we are lead to introduce the following definitions.

3.105 **Definitions.** i) A group Γ of finite type is said to have *exponential growth* if for any system of generators S there is a constant $a > 0$ such that $\phi_S(s) \geq \exp(as)$.
ii) Γ is said to have *polynomial growth of degree* $\leq n$ if for any system of generators S there is an $a > 0$ such that $\phi_S(s) \leq as^n$.
iii) Γ is said to have *polynomial growth of degree* n if the growth is polynomial of degree $\leq n$ without being of degree $\leq n-1$.
It is not difficult to show (cf. [VCN] for instance, and [Wo 2] for more details) that Γ has exponential, or polynomial growth of degree n, as soon as the above properties hold for *some* system of generators.

3.106 **Theorem** (Milnor-Wolf, cf. [Mi3] and [Wo2]). *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature. Then any subgroup of $\pi_1(M)$ with finite type has polynomial growth whose degree is at most* $\dim M$. *The same property holds for $\pi_1(M)$ if M is compact.*

Proof. The fundamental group acts isometrically on the universal Riemannian cover $(\tilde{M}, \tilde{g})$. Take $a \in \tilde{M}$. From the very definition of covering maps, we can find $r > 0$ such that the balls $B(\gamma(a), r)$ are pairwise disjoint. Take a finite system S of generators of the subgroup we consider, and set

$$L = \max d(a, \gamma_i(a)), \quad \gamma_i \in S.$$

Now, if $\gamma \in \pi_1(M)$ can be represented as a word of length not greater than s with respect to the γ_i, clearly

$$d(a, \gamma(a)) \leq Ls.$$

Taking all such γ's, we obtain $\phi_S(s)$ disjoint balls $B(\gamma(a), r)$, such that

$$B(\gamma(a), r) \subseteq B(a, Ls + r).$$

Therefore

$$\phi_S(s) \leq \frac{\mathrm{vol}(B(a, Ls + r))}{\mathrm{vol}(B(a, r))} \leq C_M(Ls + r)^n$$

in view of Bishop's theorem 3.101.
The last claim is straightforward, since the fundamental group of a compact manifold has finite type (see [My]).■

3.107 **Example.** Take the Heisenberg group (cf. 2.90 bis) H, and the subgroup $H_{\mathbf{Z}}$ of H obtained by taking integer parameters. It can be proved (cf. [Wo 3]) that $H_{\mathbf{Z}}$ has polynomial growth of degree 4. Therefore, the compact manifold $H/H_{\mathbf{Z}}$ carries no metric with nonnegative Ricci curvature.

3.I.2 Growth of the fundamental group of compact manifolds with negative curvature

First recall some standard properties of the action of $\pi_1(M)$ on the universal covering $\tilde{M}$. First of all, if M is compact, there exists a compact K of $\tilde{M}$ whose translated $\gamma(K)$ cover $\tilde{M}$.

3.108 **Proposition.** *The covering $(\gamma(K)), \gamma \in \pi_1(M)$, is locally finite.*

Proof. Equip M with a Riemannian metric, and take the universal Riemannian covering $(\tilde{M}, \tilde{g})$ of (M, g). Let d be the distance which is given by $\tilde{g}$. As a consequence of the Lebesgue property for the compact K, there exists some $r > 0$ such that, for any ball B of radius r whose center lies in K, the balls $\gamma(B)$ are pairwise disjoint when γ goes through $\pi_1(M)$.
Now, we are going to show that for any x in $\tilde{M}$, the ball $B(x, \frac{r}{2})$ only meets a finite number of $\gamma(K)$. Since the γ's are isometries, we can suppose that x lies in K. Suppose there exists a sequence γ_n of distinct elements of Γ, and a sequence y_n of points of K, such that for any n

$$\gamma_n(y_n) \in B(x, \frac{r}{2}).$$

After taking a subsequence if necessary, we can suppose that y_n converges in K. Let y be the limit. Then, since the γ_n are isometries, $\gamma_n(y)$ belongs to $B(x,r)$ for n big enough, a contradiction. ∎

A direct consequence of this lemma is the following: for a given $D > 0$, the set

$$S = \{\gamma \in \pi_1(M), d(K, \gamma(K)) < D\}$$

is finite. Take D strictly bigger than the diameter δ of M.

3.109 Lemma. *Take $a \in K$, and $\gamma \in \pi_1(M)$ such that, for some integer s,*

$$d(a, \gamma(K)) \leq (D - \delta)s + \delta.$$

Then γ can be written as the product of s elements of S.

Proof. Take $y \in \gamma(K)$, a minimizing geodesic c from a to y, and points $y_1, y_2, \cdots, y_{s+1}$ such that

$$d(a, y_1) < \delta \quad \text{and} \quad d(y_i, y_{i+1}) \leq (D - \delta) \quad \text{for} \quad 1 \leq i \leq s.$$

Any y_i can be written as $\gamma_i(x_i)$, for some γ_i in $\pi_1(M)$ and some x_i in K, and we can take $\gamma_1 = Id$ and $\gamma_{s+1} = \gamma$. Then

$$\gamma = (\gamma_1^{-1}\gamma_2)(\gamma_2^{-1}\gamma_3)\dots(\gamma_s^{-1}\gamma_{s+1}).$$

On the other hand

$$d(x_i, \gamma_{i-1}^{-1}(\gamma_i(x_i)) = d(\gamma_{i-1}(x_i), y_i)$$

is smaller than

$$d(\gamma_{i-1}(x_i), \gamma_{i-1}(x_{i-1})) + d(\gamma_{i-1}(x_{i-1}), y_i)).$$

But this is just $d(x_{i-1}, x_i) + d(y_{i-1}, y_i)$, which is smaller than D, so that $\gamma_{i-1}^{-1}\gamma_i$ is in S.

3.110 Theorem (Milnor, cf. [Mi 3]). *If (M, g) is a compact manifold with strictly negative sectional curvature, then $\pi_1(M)$ has exponential growth.*

Proof. Take a system S of generators as in the preceding lemma. This lemma says that the ball

$$B(a, (D - \delta)s + \delta)$$

is covered by $\phi_S(s)$ compact sets $\gamma(K)$, so that

$$\text{vol}(B(a, (D - \delta)s + \delta) \leq \phi_S(s)\text{vol}(K).$$

On the other hand, if the sectional curvature is smaller than some $-b$, where $b > 0$, theorem 3.101 ii) gives

$$\text{vol}(B(a, (D - \delta)s + \delta) \geq V^{-b}((D - \delta)s + \delta) \approx c_n \exp((n - 1)Ds).$$

For s big enough, we get the lower bound we claimed for $\phi_S(s)$.

Theorem (Preissmann, [Pr]): *If (M, g) is a Riemannian manifold with strictly negative curvature, then $\pi_1(M)$ does not contain $\mathbf{Z}^2$.*

As a consequence of any of the two above theorems (Milnor or Preissmann), we obtain the following result.

Corollary. *The torus $\mathbf{T}^3$ does not carry metrics with strictly negative curvature.*

For a modern version of such results, using the notion of *simplicial volume*, see [Gr3].

Remarks. a) As soon as $n \geq 3$, there are n-dimensional compact manifold whose fundamental group has exponential growth and which carry no metric with negative sectional curvature (cf. [VCN]).

b) These results can be considered as the prehistory of a *Geometric Group Theory* which has known dramatic developments in the nineties of the last century, and is still very active today. See for instance [G-H] and [Gr6].

3.J Curvature and topology: some important results

This section is expository, no proofs are given.

3.J.1 Integral formulas

In dimension 2, the Gauss-Bonnet formula says everything about the relations between curvature and topology. Namely, if (M, g) is a compact Riemannian surface, its Euler-Poincaré characteristic, that is the alternate sum of Betti numbers, is given by

$$\chi(M) = \frac{1}{4\pi} \int_M Scal(g) v_g = \frac{1}{2\pi} \int_M K_g dv_g$$

(recall that the topological type of a compact surface, once known whether it is orientable or not, is entirely given by $\chi(M)$).

In higher dimension, the Gauss-Bonnet formula has been extended by Chern as follows. Suppose the dimension n is even, and take the $\frac{n}{2}$-th exterior power of the curvature tensor: we get a field of endomorphisms of $\Lambda^n TM$, therefore a scalar field. In that way, we have obtained a polynomial $P_n(R)$ of degree $\frac{n}{2}$ with respect to the curvature tensor. Then, for some universal constant c_n (which can be computed by taking the standard sphere) we have (see for example [Sp],t.V)

$$\chi(M) = c_n \int_M P_n(R) v_g.$$