ON THE PARALLELIZABILITY OF THE SPHERES

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(The following note consists of excerpts from two letters.)

(*Milnor to Bott*; December 23, 1957.)

. . . Hirzebruch tells me that you have a proof of his conjecture that the Pontrjagin class p_k of a GL_m-bundle over the sphere S^{4k} is always divisible by $(2k-1)!$. I wonder if you have noted the connection of this result with classical problems, such as the existence of division algebras, and the parallelizability of spheres.

According to Wu the Pontrjagin classes of any GL_m-bundle, reduced modulo 4, are determined by the Stiefel-Whitney classes of the bundle. (See *On the Pontrjagin classes* III, Acta Math. Sinica vol. 4 (1954) in Chinese.) The proof makes use of the Pontrjagin squaring operation, together with the coefficient homomorphism $i: Z_2 \rightarrow Z_4$. Although I do not know the exact formula which Wu obtains, the following special case is not hard to prove:

LEMMA. *If the Stiefel-Whitney classes $w_1, w_2, \cdots, w_{4k-1}$ of a GL_m-bundle are zero then the Pontrjagin class p_k, reduced modulo* 4, *is equal to $i_* w_{4k}$.*

For a bundle over S^{4k} this means that w_{4k} is zero if and only if p_k is divisible by 4. Now if you can prove that p_k is divisible by $(2k-1)!$ it will follow that w_{4k} must be zero, whenever $k \geqq 3$.

THEOREM. *There exists a GL_m-bundle over S^n with $w_n \neq 0$ only if n equals* 1, 2, 4 *or* 8.

PROOF. Wu has shown that such a bundle can only exist if n is a power of 2. But the above remarks show that the cases $n=16, 32, \cdots$ cannot occur.

COROLLARY 1. *The vector space R^n possesses a bilinear product operation without zero divisors only for n equal to* 1, 2, 4 *or* 8.

PROOF. Given such a product operation the map $S^{n-1} \rightarrow GL_n$ defined by $x \rightarrow$(left multiplication by x) gives rise to a GL_n-bundle over S^n for which it can be shown that $w_n \neq 0$.

COROLLARY 2. *The sphere S^{n-1} is parallelizable only for $n-1$ equal to* 1, 3 *or* 7.

PROOF. Given linearly independent vector fields $v_1(x), \cdots, v_{n-1}(x)$, on S^{n-1}, the correspondence

$$x \to (x, v_1(x), \cdots, v_{n-1}(x))$$

carries S^{n-1} into the Stiefel manifold of n-frames in R^n. Identifying this space with GL_n, we again obtain a GL_n-bundle over S^n with $w_n \neq 0 \cdots$

(*Bott to Milnor*, January 6, 1958.)

... Here is what I can show:

THEOREM. *Let $B = B_U$ be the universal base-space of the infinite unitary group. Then the image of $\pi_{2n}(B)$ in $H_{2n}(B)$ is divisible by precisely $(n-1)!$.*

This then refines the result of Borel-Hirzebruch that these classes are divisible by $(n-1)!$ except for the prime 2, [3], and confirms their conjecture. Because the Pontryagin classes are in the last analysis pre-images of classes in B_U, it follows that for any $GL_n(R)$ bundle over S^{4k}, p_k is divisible by $(2k-1)!$. This is all you needed.

The precise divisibility of p_k, for a real bundle over S^{4k}, is actually given by:

$$p_k = 0, \quad \begin{matrix} \text{mod } (2k-1)!, & k \text{ even}, \\ \text{mod } (2k-1)!2, & k \text{ odd}. \end{matrix}$$

This is seen by considering the fibering $U/O \to B_O \to B_U$.

The theorem follows from the fact, that if $\Omega = \Omega SU$ is the loopspace on SU, then there exists a homotopy equivalence $f: B \to \Omega$, as was announced in [1] and is proved in [2]. By standard theory the double suspension, S, from Ω into B, defines a homomorphism $\pi_k(\Omega) \to \pi_{k+2}(B)$ which is bijective for dimensions $\geqq 1$.

Let $\lambda = f_* \circ S$. It is then clear that:

$$\pi_{2k}(\Omega) = \lambda^{k-1}(\pi_2(\Omega)). \tag{1}$$

Now in [2] the Hopf algebra $H_*(\Omega)$ is described. It turns out that $H_*(\Omega) = Z[\sigma_1, \sigma_2, \cdots]$, $\dim \sigma_i = 2i$, the diagonal map being: $\Delta_* \sigma_i = \sum \sigma_s \otimes \sigma_t$; $s+t=i$; $\sigma_0 = 1$. Hence the primitive subspace, P_*, is generated by elements $\{p_n\}$, $n = 1, 2, \cdots$, which are inductively determined by the relation:

$$p_n - p_{n-1} \cdot \sigma_1 + p_{n-2} \cdot \sigma_2 - \cdots \pm n\sigma_n = 0, \qquad n = 1, 2, \cdots. \tag{2}$$

Let λ_* be the homomorphism corresponding to λ in homology. It will preserve spherical classes, and annihilate decomposable elements. It therefore follows from (2) that $\lambda_* p_i = \pm i \lambda_* \sigma_i$. As the spherical classes generate P_* (over the rationals, SU is a product of odd spheres!) this relation implies that $\lambda_*^{k-1}(H_2(\Omega))$ is divisible by at least $(k-1)!$. By (1) it follows that the spherical classes in dimension $2k$ are divisible by at least $(k-1)!$. This is the best bound because it is not hard to see that λ_* is *not* divisible on all of $H_*(\Omega)$.

An easy corollary of the theorem is that $\pi_{2n}(U_n) = Z/n!Z$. Kervaire also has decided the parallelizability question. He uses this formula as his starting point . . . .

References

1. R. Bott, *The stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 43 (1951) pp. 933–935.

2. ———, *The Pontryagin ring of* ΩG (to be published in Michigan Math. J.).

3. A. Borel and F. Hirzebruch: *Characteristic classes and homogeneous spaces* (to be published in Amer. J. Math.).

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