Stability for the Surface Diffusion Flow

Carlo Mantegazza

(Joint work with Antonia Diana and Nicola Fusco)

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The surface diffusion flow

Definition

Let $E \subseteq \mathbb{T}^n$ be a smooth set. We say that the family $E_t \subseteq \mathbb{T}^n$, for $t \in [0, T)$ with $E_0 = E$, is a **surface diffusion flow starting from** *E, if the map $t \mapsto \chi_{E_t}$ is continuous from $[0, T)$ to $L^1(\mathbb{T}^n)$ and the hypersurfaces ∂E_t move by surface diffusion, that is, there exists a smooth family of embeddings $\varphi_t : \partial E \to \mathbb{T}^n$, for $t \in [0, T)$, with $\varphi_0 = \mathrm{Id}$ and $\varphi_t(\partial E) = \partial E_t$, such that*

$$\frac{\partial \varphi_t}{\partial t} = (\Delta \mathrm{H})\nu$$

where, at every point and time, H and Δ are respectively the mean curvature and the Laplacian (with the Riemannian metric induced by $\mathbb{T}^n$) of the moving hypersurface ∂E_t, while ν is the "outer" normal to the smooth set E_t.

We want to study the long–time behavior of the surface diffusion flow of embedded smooth hypersurfaces which are boundaries of smooth sets, in the n–dimensional flat torus $\mathbb{T}^n$ (the so called "periodic case"). Then, clearly all the results also hold for compact sets in $\mathbb{R}^n$.

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- When it is restricted to closed embedded hypersurfaces which are boundaries of sets, the **enclosed volume is preserved**.
- Its a (sort of) gradient flow and the *Area functional*

$$\mathcal{A}(\partial E) = \int_{\partial E} d\mu$$

decreases along the flow.

Preliminaries

For every smooth set $E \subseteq \mathbb{T}^n$ and $\varepsilon > 0$, we consider the *tubular neighborhood* of ∂E

$$N_\varepsilon = \{x \in \mathbb{T}^n \,:\, d(x, \partial E) < \varepsilon\}$$

where d is the "Euclidean" distance on $\mathbb{T}^n$.

Definition

Given a smooth set $E \subseteq \mathbb{T}^n$ and a tubular neighborhood N_ε of ∂E, for every $k \in \mathbb{N}$ and for any $M \in (0, \varepsilon/2)$, we denote by $\mathfrak{C}^k_M(\partial E)$, the class of all sets $F \subseteq E \cup N_\varepsilon$ such that $\mathrm{Vol}(F \triangle E) \leqslant M$ and

$$\partial F = \{y + \psi_F(y)\nu_E(y) \,:\, y \in \partial E\},$$

for some function $\psi_F \in C^k(\partial E)$, with $\|\psi_F\|_{C^k(\partial E)} \leqslant M$.

We will call a set F as in this definition, the *normal deformation* of E induced by the function ψ_F.

Short time existence and uniqueness

Theorem (J. Escher, U. F. Mayer and G. Simonett, 1998)

Let $E \subseteq \mathbb{T}^n$ be a smooth set, N_ε a tubular neighborhood of ∂E and $M_E < \varepsilon/2$. For every $E_0 \subseteq \mathbb{T}^n$ smooth set in $\mathfrak{C}^1_{M_E}(E)$, which is a normal deformation of E induced by a smooth function $\psi_0 : \partial E \to \mathbb{R}$, there exists a unique surface diffusion flow E_t, starting from E_0, such that every E_t is a normal deformation of E induced by a smooth function $\psi_t : \partial E \to \mathbb{R}$, for $t \in [0, T(E_0))$, with $T(E_0)$ depending on the $C^{2,\alpha}$–norm of ψ_0.

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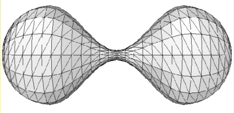
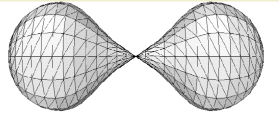
Unfortunately the surface diffusion flow could develop singularities in finite time, even in low dimensions. For instance, like for the mean curvature flow, a dumbbell surface with a thin neck should become singular during the evolution, but also embedded curves in the plane do not necessarily evolve smoothly (numerical evidences).





A smooth set E is a *critical set* (that is, with zero *constrained first variation*) for the volume–constrained Area functional $\mathcal{A}$ if and only if its boundary satisfy $\mathrm{H} = \lambda$, for some constant $\lambda \in \mathbb{R}$ (E is a *constant mean curvature hypersurface*).

Then, at a critical set E, the second variation of the Area functional is given by

$$\frac{d^2}{dt^2}\mathcal{A}(\partial E_t)\Big|_{t=0} = \int_{\partial E} \big(|\nabla\psi|^2 - \psi^2|\mathrm{B}|^2\big)\,d\mu$$

where $\partial E_t = \varphi_t(\partial E)$ is a volume–preserving variation of E and $\psi \in C^\infty(\partial E)$ (which satisfies $\int_{\partial E}\psi\,d\mu = 0$) is the normal component of its infinitesimal generator (B is the second fundamental form of ∂E).

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This motivates the following definition.

Definition

We say that a critical set $E \subseteq \mathbb{T}^n$ for $\mathcal{A}$ under a volume constraint is strictly stable *if*

$$\Pi_E(\psi) = \int_{\partial E} \big(|\nabla\psi|^2 - \psi^2|\mathrm{B}|^2\big)\,d\mu > 0$$

for all $\psi \in \widetilde{H}^1(\partial E) = \Big\{\psi \in H^1(\partial E)\,:\,\int_{\partial E}\psi\,d\mu = 0\Big\}$ *such that* $\int_{\partial E}\psi\nu\,d\mu = 0$.

Morally, in this definition we are “excluding” the translations of the set E.

- If $n = 2$, the only bounded strictly stable critical sets for the *Length* functional (that is, the Area functional in the 2–dimensional ambient space) in the plane are the disks, in $\mathbb{T}^2$ they are the disks and the “strips” with straight borders.

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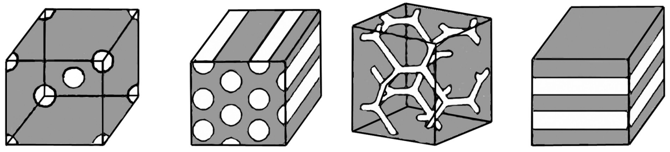
- If $n = 3$, the stable "periodic" critical sets for the volume–constrained Area functional, are *balls*, 2–*tori*, *gyroids* or *lamellae* (Ros, 2007).



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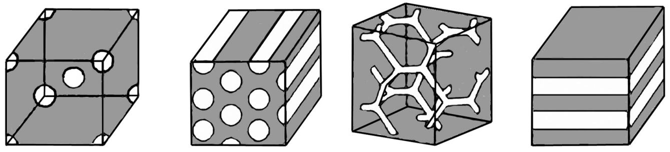

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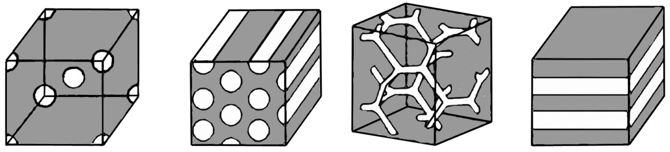

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Our aim was to show that the surface diffusion flow of an initial set close enough to a strictly stable critical set, exists for all positive times and asymptotically converges to a translation of it.

Previous stability results

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- $W^{n+1,2}$–closedness to an n–dimensional strictly stable hypersurface.

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- $W^{n+1,2}$–closedness to an n–dimensional strictly stable hypersurface.
- Very similar result, with different methods, by D. De Gennaro, A. Diana, A. Kubin and A. Kubin, 2023.

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We consider a suitable "energy" depending on (iterated) covariant derivatives of the mean curvature, that is,

$$\mathcal{E}(t) = \int_{\partial E_t} |\nabla^2 \mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \, .$$

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Its relevance is expressed by the following lemma.

Lemma

Let $E \subseteq \mathbb{T}^4$ be a smooth set and N_ε be a tubular neighborhood of ∂E. For M_E small enough and $\delta > 0$, there exists a constant $C = C(E, M_E, \delta)$ such that if $F \in \mathfrak{C}^1_{M_E}(E)$ is a normal deformation of E induced by a smooth function $\psi_F : \partial E \to \mathbb{R}$ and

$$\int_{\partial F} |\nabla^2 \mathrm{H}|^2 \, d\mu + \int_{\partial F} |\nabla \mathrm{H}|^2 \, d\mu \leqslant \delta \qquad \text{and} \qquad \mathrm{Vol}(F \triangle E) \leqslant \delta \, ,$$

there hold

$$\|\mathrm{B}\|_{L^\infty(\partial F)} + \|\nabla \mathrm{B}\|_{L^6(\partial F)} \leqslant C \qquad \text{and} \qquad \|\psi_F\|_{W^{4,2}(\partial E)} \leqslant C \, .$$

Moreover, for every $1 \leqslant p < 6$, we have $\|\psi_F\|_{W^{3,p}(\partial E)} \to 0$, as $\delta \to 0$.

Compactness

Corollary

As a consequence, if $E_i \subseteq \mathfrak{C}^1_{M_E}(E)$ is a sequence of smooth sets such that

$$\sup_{i\in\mathbb{N}} \int_{\partial E_i} |\nabla^2 \mathrm{H}|^2 \, d\mu_i + \int_{\partial E_i} |\nabla \mathrm{H}|^2 \, d\mu_i < +\infty\,,$$

then there exists a (non necessarily smooth) set $E' \in \mathfrak{C}^1_{M_E}(E)$ such that, up to a (non relabeled) subsequence, $E_i \to E'$ in $W^{3,p}$ as $i \to \infty$, for all $1 \leqslant p < 6$.
Moreover, if

$$\int_{\partial E_i} |\nabla^2 \mathrm{H}|^2 \, d\mu_i + \int_{\partial E_i} |\nabla \mathrm{H}|^2 \, d\mu_i \to 0\,,$$

as $i \to \infty$, the set E' is critical for the volume–constrained Area functional, that is, its mean curvature is constant.

Evolution of the energy

Proposition

Let $E_t \subseteq \mathbb{T}^4$ be a surface diffusion flow such that $E_t \in \mathfrak{C}^1_M(E)$, for some smooth set E. Then,

$$\frac{d}{dt}\int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \leqslant -2\Pi_{E_t}(\Delta \mathrm{H}) + C_1 \int_{\partial E_t} |\mathrm{B}||\nabla \mathrm{H}|^2 |\Delta \mathrm{H}| \, d\mu_t \,,$$

$$\frac{d}{dt}\int_{\partial E_t} |\nabla^2 \mathrm{H}|^2 \, d\mu_t \leqslant -\|\nabla^4 \mathrm{H}\|^2_{L^2(\partial E_t)} + C_2\big(1 + \|\nabla \mathrm{H}\|^\tau_{L^2(\partial E_t)}\big)\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \,,$$

for some exponent $\tau > 0$ and constant C_1, C_2 such that C_1 depends only on E and M, while C_2 depends also on $\|\mathrm{B}\|_{L^\infty(\partial E_t)}$ and $\|\nabla \mathrm{B}\|_{L^6(\partial E_t)}$.

To get such proposition, one first proves, by straightforward computations,

$$\frac{d}{dt}\int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t = -\, 2\Pi_{E_t}(\Delta \mathrm{H}) + \int_{\partial E_t} \mathrm{H}\Delta\mathrm{H}|\nabla \mathrm{H}|^2 \, d\mu_t$$

$$- \int_{\partial E_t} 2\mathrm{B}(\nabla \mathrm{H}, \nabla \mathrm{H})\Delta \mathrm{H} \, d\mu_t$$

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$$+ \int_{\partial E_t} \mathfrak{q}^{10}(\nabla(\mathrm{B}^2), \nabla^4 \mathrm{H}) \, d\mu_t$$

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Then, one has to carefully analyse the “algebraic structure” of the monomials in such polynomials in order to estimate the integrals by means of Gagliardo–Nirenberg, Hölder and Young “uniform” inequalities.

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Then, one has to carefully analyse the “algebraic structure” of the monomials in such polynomials in order to estimate the integrals by means of Gagliardo–Nirenberg, Hölder and Young “uniform” inequalities.

Here “uniform” means that the constants in such inequalities must be uniformly controlled, since these are inequalities in an ambient space (hypersurface) which is moving by the flow – We will get back on this point later.

Global existence and asymptotic behavior around a strictly stable critical set in $\mathbb{T}^4$

Theorem (A. Diana, N. Fusco and C. M., 2023)

Let $E \subseteq \mathbb{T}^4$ be a strictly stable critical set for the Area functional under a volume constraint. For M_E small enough, there exists $\delta > 0$ such that, if E_0 is a smooth set in $\mathfrak{C}^1_{M_E}(E)$ with $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$, satisfying

$$\mathrm{Vol}(E_0 \triangle E) \leqslant \delta \qquad \textit{and} \qquad \int_{\partial E_0} |\nabla^2 \mathrm{H}|^2 \, d\mu_0 + \int_{\partial E_0} |\nabla \mathrm{H}|^2 \, d\mu_0 \leqslant \delta \,,$$

then, the unique smooth surface diffusion flow E_t, starting from E_0, is defined for all times and converges smoothly to $E' = E + \eta$ exponentially fast, for some $\eta \in \mathbb{R}^4$.

More precisely, the sequence of smooth functions $\psi_t : \partial E \to \mathbb{R}$ representing ∂E_t as "normal graphs" on ∂E, satisfies, for every $k \in \mathbb{N}$,

$$\|\psi_t - \psi\|_{C^k(\partial E)} \leqslant C_k e^{-\beta_k t} \,,$$

for every $t \in [0, +\infty)$, for some positive constants C_k and β_k, where $\psi : \partial E \to \mathbb{R}$ represents $\partial E' = \partial E + \eta$ as a "normal graph" on ∂E.

Obtained by E. Acerbi, N. Fusco, M. Morini and V. Julin, for $n = 3$.

Line of proof

By choosing M_E small enough in order that all the constants in the inequalities are *uniform* and after choosing some small $\delta_0 > 0$, we consider the surface diffusion flow E_t starting from $E_0 \in \mathfrak{C}^1_{M_E}(E)$ satisfying $\mathrm{Vol}(E_0 \triangle E) \leqslant \delta_0$ and $\mathcal{E}(0) \leqslant \delta_0$.
We denote by $T(E_0) \in (0, +\infty]$ the maximal time such that the flow is defined for t in the interval $[0, T(E_0))$, $E_t \in \mathfrak{C}^1_{2M_E}(E)$,

$$\mathrm{Vol}(E_t \triangle E) \leqslant \delta_0 \qquad \text{and} \qquad \mathcal{E}(t) = \int_{\partial E_t} |\nabla^2 \mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \leqslant \delta_0 \,.$$

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If $T(E_0) < +\infty$, then at least one of the following must hold:

- $\limsup_{t \to T(E_0)} \|\psi_t\|_{C^1(\partial E)} = 2M_E$
- $\limsup_{t \to T(E_0)} \mathcal{E}(t) = \delta_0$
- $\limsup_{t \to T(E_0)} \mathrm{Vol}(E_t \triangle E) = \delta_0$

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If $T(E_0) < +\infty$, then at least one of the following must hold:

- $\limsup_{t \to T(E_0)} \|\psi_t\|_{C^1(\partial E)} = 2M_E$
- $\limsup_{t \to T(E_0)} \mathcal{E}(t) = \delta_0$
- $\limsup_{t \to T(E_0)} \mathrm{Vol}(E_t \triangle E) = \delta_0$

otherwise, by the previous lemma/corollary and Sobolev (uniform) embeddings, we can “restart” the flow from a time close enough to $T(E_0)$, we have the contradiction that $T(E_0)$ cannot be the maximal time satisfying the properties above.

Line of proof

By choosing M_E small enough in order that all the constants in the inequalities are *uniform* and after choosing some small $\delta_0 > 0$, we consider the surface diffusion flow E_t starting from $E_0 \in \mathfrak{C}^1_{M_E}(E)$ satisfying $\mathrm{Vol}(E_0 \triangle E) \leqslant \delta_0$ and $\mathcal{E}(0) \leqslant \delta_0$.
We denote by $T(E_0) \in (0, +\infty]$ the maximal time such that the flow is defined for t in the interval $[0, T(E_0))$, $E_t \in \mathfrak{C}^1_{2M_E}(E)$,

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Thus, we want to show that if δ_0 is chosen small enough, there exists $\delta > 0$ (as in the statement of the theorem) such that none of these conditions can occur, hence $T(E_0) = +\infty$.

Monotonicity of the energy

We define, for $K > 1$, the following modification on the energy function:

$$\mathcal{E}_K(t) = \int_{\partial E_t} |\nabla^2 \mathrm{H}|^2 \, d\mu_t + K \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t \geqslant \mathcal{E}(t)$$

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Then, by the previous Proposition and by the boundedness (from below) of Π_{E_t}, for a suitable $K > 1$, we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_K(t) \leqslant & -2K\Pi_{E_t}(\Delta \mathrm{H}) - \frac{1}{2}\|\nabla^4 \mathrm{H}\|^2_{L^2(\partial E_t)} + S_1(\mathcal{E}_K(t))\|\nabla \mathrm{H}\|^2_{L^2(\partial E_t)} \\
& + S_1(\mathcal{E}_K(t))\|\nabla \mathrm{H}\|^{2+\tau}_{L^2(\partial E_t)} + S_2(\mathcal{E}_K(t))K^2\|\nabla \mathrm{H}\|^{2+\tau'}_{L^2(\partial E_t)} \\
\leqslant & -\mathcal{E}_K(t)/K
\end{aligned}$$

for every $t \in [0, T(E_0))$, for some exponent $\tau, \tau' > 0$ and continuous, monotone nondecreasing functions $S_1, S_2 : [0, +\infty) \to \mathbb{R}^+$, depending on E and M_E.

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Hence, the function $\mathcal{E}_K$ is never increasing so it remains bounded by $\mathcal{E}_K(0) \leqslant K\mathcal{E}(0) < K\delta < \delta_0/2$ (moreover, it decreases exponentially and converges to zero, as $t \to +\infty$, if the flow is "eternal").

Global existence of the flow

It follows that $\mathcal{E}(t) \leqslant \mathcal{E}_K(t) \leqslant \delta e^{-t/K} \leqslant \delta$, for every $t \in [0, T(E_0))$, hence, by the previous Lemma, $\|\psi_t\|_{C^{2,\alpha}(\partial E)}$ cannot be unbounded, as $t \to T(E_0)$.

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Moreover, with an appropriate choice of a small δ (again by such lemma), the condition

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By means of some manipulations, for all $\bar{t} \in [0, T(E_0))$ we can also show that

$$\mathrm{Vol}(E_{\bar{t}} \triangle E) \leqslant C\|\psi_{\bar{t}}\|_{L^2(\partial E)} \leqslant C\sqrt[4]{\delta}\,,$$

hence, also the condition

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Then, we conclude that the surface diffusion flow of E_0 exists smooth for every time, moreover $E_t \in \mathfrak{C}^1_{2M_E}(E)$ and

$$\mathcal{E}(t) \leqslant \delta e^{-t/K} \quad \text{for every } t \in [0, +\infty).$$

Exponential convergence to a translated of E

Let $t_i \to +\infty$, since $\mathcal{E}(t_i) \to 0$ as $i \to +\infty$, by the "compactness" corollary, up to a subsequence, there exists a critical set $E' \in \mathfrak{C}^1_{2M_E}(E)$ (actually E' is smooth, by standard regularity theory for quasilinear equations, having constant mean curvature and being a graph over ∂E of a C^1 function) such that

$$E_{t_i} \to E' \quad \text{in } W^{3,2}\,.$$

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$$E_{t_i} \to E' \quad \text{in } W^{3,2}\,.$$

Moreover, since we know that near a strictly stable critical set there are no other smooth critical sets, up to translations (E. Acerbi, N. Fusco, V. Julin and M. Morini, 2019), we have that it must be $E' = E + \eta$, for some (small) $\eta \in \mathbb{R}^4$.

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Then, arguing by contradiction, it can be seen easily that the full sequence converges (exponentially) to E'. Finally, standard parabolic estimates give the smooth exponential convergence of E_t to E'.

General dimension

With the same line (and a little more technical effort) effort, one can show the same result in any dimension $n \geqslant 3$, by considering the energy

$$\mathcal{E}_n(t) = \int_{\partial E_t} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_t + \int_{\partial E_t} |\nabla \mathrm{H}|^2 \, d\mu_t$$

and modifying the interpolation inequalities with some suitable exponents (depending on n).

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Theorem (A. Diana, N. Fusco and C. M., 2023)

Let $E \subseteq \mathbb{T}^n$ be a strictly stable critical set for the Area functional under a volume constraint. For M_E small enough, there exists $\delta > 0$ such that, if E_0 is a smooth set in $\mathfrak{C}^1_{M_E}(E)$ with $\mathrm{Vol}(E_0) = \mathrm{Vol}(E)$, satisfying

$$\mathrm{Vol}(E_0 \triangle E) \leqslant \delta \qquad \textit{and} \qquad \int_{\partial E_0} |\nabla^{n-2}\mathrm{H}|^2 \, d\mu_0 + \int_{\partial E_0} |\nabla \mathrm{H}|^2 \, d\mu_0 \leqslant \delta \,,$$

then, the unique smooth surface diffusion flow E_t, starting from E_0, is defined for all times and converges smoothly to $E' = E + \eta$ exponentially fast, for some $\eta \in \mathbb{R}^n$.

It should be possible to extend our result (as E. Acerbi, N. Fusco, V. Julin and M. Morini did, in dimension $n = 3$) to other flows, like the *Mullins–Sekerka flow*. Representing the moving hypersurfaces as smooth embeddings $\varphi_t : M \to \mathbb{T}^n$, it can be described as

$$\frac{\partial \varphi_t}{\partial t} = (\Delta^{1/2}\mathrm{H})\nu = -\Delta^{3/2}\varphi_t + \text{ lower order terms}$$

which is *nonlocal*, due to the presence of the fractional Laplacian.

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which is *nonlocal*, due to the presence of the fractional Laplacian.

It is then natural to try to generalize these results (Antonia Diana is working on that) to the nonlocal evolutions of hypersurfaces given by the laws

$$\frac{\partial \varphi_t}{\partial t} = (\Delta^{s}\mathrm{H})\nu = -\Delta^{s+1}\varphi_t + \text{ lower order terms},$$

for any $s \in (0, 1]$.

Up to our knowledge, these flows are not present in literature.

Uniform inequalities

In several step of our (and related) work, one need to apply inequalities (in particular, to control the behavior of the curvature – that is, morally the C^2–norm of the “representing maps”) to functions on the moving hypersurfaces. Hence, the constants in such inequalities must be uniformly controlled independently of the curvature.

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We then showed that families of smooth hypersurfaces all C^1–close enough to a fixed compact, embedded one, have uniformly bounded constants in some relevant inequalities for the analysis, like Poincaré, Sobolev, Gagliardo–Nirenberg and "geometric" Calderón–Zygmund inequalities.

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These technical results are clearly useful in general for the study of the geometric flows of hypersurfaces.

Uniform inequalities

Proposition

Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$. Then, there exist uniform bounds, depending only on M_0 and δ, for all the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ on:

- *the volume of M,*
- *the Sobolev constants for $p \in [1, n)$ of the embeddings $W^{1,p}(M) \hookrightarrow L^{p^*}(M)$,*
- *the Sobolev constants for $p \in (n, \infty]$ of the embeddings $W^{1,p}(M) \hookrightarrow C^{0,1-n/p}(M)$,*
- *the constants in the Poincaré–Wirtinger inequalities on M, for $p \in [1, +\infty]$,*
- *the Sobolev constant of the embedding $W^{1,n}(M) \hookrightarrow BMO(M)$,*
- *all the constants in the embeddings of the fractional Sobolev spaces $W^{s,p}(M)$,*
- *all the constants in the Gagliardo–Nirenberg interpolation inequalities on M.*

Moreover, all these bounds go to the corresponding constants for M_0, as $\delta \to 0$.

Uniform Calderón–Zygmund and Schauder inequalities

Proposition

Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$ and $1 < p < +\infty$. Then, if $\delta > 0$ is small enough, there exists a constant $C = C(M_0, p, \delta)$ such that the following geometric Calderón–Zygmund inequality holds,

$$\|\mathrm{B}\|_{L^p(M)} \leqslant C\big(1 + \|\mathrm{H}\|_{L^p(M)}\big)$$

for every $M \in \mathfrak{C}^1_\delta(M_0)$.

Proposition

Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$ and $\alpha \in (0,1]$. Then, if $\delta > 0$ is small enough, there exists a constant $C = C(M_0, \alpha, \delta)$ such that the following geometric (Schauder–type) estimate holds,

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant C\left(1 + \|\mathrm{H}\|_{C^{0,\alpha}(M)}\right)$$

for every $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$.

Uniform higher order Calderón–Zygmund inequalities

Proposition

Let M_0 as above, $k \in \mathbb{N}$ and $p > 1$. Assuming that we have a uniform bound

$$\|\mathrm{H}\|_{L^q(M)} \leqslant C_H$$

with $q > n$, there exists a constant $C = C(M_0, k, p, C_H, \delta)$ such that

$$\|\nabla^k \mathrm{B}\|_{L^p(M)} \leqslant C\big(1 + \|\nabla^k \mathrm{H}\|_{L^p(M)}\big)$$

and

$$\|\mathrm{B}\|_{W^{k,p}(M)} \leqslant C\big(1 + \|\mathrm{H}\|_{W^{k,p}(M)}\big)\,.$$

for any $M \in \mathfrak{C}^1_\delta(M_0)$, with $\delta > 0$ small enough.

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- All the constants depend on the geometric properties of M_0, in particular on the maximal width of a tubular neighborhood, its volume and its second fundamental form. Hence, uniformly controlling such quantities gives uniform estimates for larger families of hypersurfaces.
- All the inequalities holds uniformly also for families of immersed–only hypersurfaces (non necessarily embedded), if they can be expressed as graphs on a fixed compact, smooth hypersurface.
- Everything still works if the ambient space is any complete Riemannian manifold. Then, the constants also depends on the geometry (in particular, on the curvature) of such manifold.

Thanks for your attention