Smooth geometric evolutions of hypersurfaces and singular approximation of mean curvature flow

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In the last 30 years, a large interest has grown in connection with geometric evolutions of submanifolds, also with motivations coming from mathematical physics (phase transitions, Stefan problem). A model problem is the evolution of surfaces by mean curvature, which can be considered as the gradient flow of the *Area* functional.

The mathematical problem is intriguing for several reasons related to the appearance of singularities during the flow (with the exceptions of planar Jordan curves, convex shapes, codimension one graphs), in particular, it is necessary some weak approach in order to get a global (in time) solution of the evolution problem.

Starting from the pioneering work of Brakke, a large literature is by now available on this subject (Chen–Giga–Goto, Evans–Spruck, Huisken, Ilmanen, Ambrosio–Soner, Evans–Soner-Souganidis). The weak formulations are mainly based either on geometric measure theory (currents, varifolds), or on the theory of viscosity solutions. In the latter approach, a crucial role is played by the analytical properties of the distance function $d^M(x)$ from the submanifold.
For instance, in the codimension one case, it turns out that the boundary $M_t = \partial U_t$ of a family of domains U_t flows by mean curvature if and only if

$$\partial_t \, d(x,t) = \Delta d(x,t) \qquad \text{for every } x \in M_t$$

where $d(x,t)$ is equal to the *signed* distance function from M_t.

Since the signed distance function makes no sense in higher codimension problems, De Giorgi suggested to work with the *squared distance function* $\eta^M(x) = [d^M(x)]^2/2$.
Setting $\eta(x,t) = \eta^{M_t}(x)$, it turns out (Ambrosio–Soner) that the mean curvature flow is characterized by the equation

$$\partial_t \nabla\eta(x,t) = \Delta\nabla\eta(x,t) \qquad \text{for } x \in M_t\,.$$

One of the parts of this thesis was a systematic study of the connections between the analytical properties of η^M and the geometric invariants of the submanifold *M*, like the second fundamental form and its covariant derivatives.

- L. Ambrosio and C. Mantegazza, *Curvature and distance function from a manifold. Dedicated to the memory of Fred Almgren*. J. Geom. Anal. **8** (1998), no. 5, 723–744.
- C. Mantegazza and A. C. Mennucci, *Hamilton–Jacobi equations and distance functions on Riemannian manifolds*. Appl. Math. Opt. **47** (2003), 1–25.
- M. Eminenti and C. Mantegazza, *Some properties of the distance function and a conjecture of De Giorgi*. J. Geom. Anal. **14** (2004), no. 2, 267–279.

Proposition (Manolo Eminenti, CM)

For every $m \geq 3$ there is a one–to–one correspondence between the covariant derivatives of the second fundamental form of a submanifold $M \subset \mathbb{R}^n$ up to the order m and the ordinary derivatives in $\mathbb{R}^n$ of the squared distance function η^M up to the order $m - 3$.

It follows that any autonomous “geometric” functional can be written as

$$\mathcal{F}(M) = \int_M f\left(\nabla^2_{i_1 i_2}\eta^M, \dots, \nabla^\gamma_{j_1 \dots j_\gamma}\eta^M\right)\, d\mathcal{H}^n$$

for some function f depending on the standard derivatives in $\mathbb{R}^n$ of η^M up to a given order γ. One of the results of this work is a constructive algorithm for the computation of the first variation of such general functional (Luigi Ambrosio–CM).

Two particular functionals on hypersurfaces of $\mathbb{R}^{n+1}$ are the following:

$$\mathcal{G}_\gamma(M) = \int_M |\nabla^\gamma \eta^M|^2 \, d\mathcal{H}^n \,,$$

where ∇^γ is the standard (iterated) derivative in $\mathbb{R}^{n+1}$, and

$$\mathcal{C}_m(\varphi) = \int_M |\nabla^m \nu|^2 \, d\mu \,,$$

representing hypersurfaces in $\mathbb{R}^{n+1}$ as immersions $\varphi : M \to \mathbb{R}^{n+1}$. Here μ and ∇ are respectively the canonical measure and the Levi–Civita connection on the Riemannian manifold (M, g), where the metric g is obtained by pulling back on M the usual metric of $\mathbb{R}^{n+1}$ via the map φ. The symbol ∇^m denotes the m–th iterated covariant derivative and ν a unit normal local vector field to the hypersurface.

When $\gamma = m+2$, the first variations of these two functionals have the same leading terms, up to the constant $m+2$, that is

$$E_{\mathcal{G}_\gamma}(M) = 2\gamma(-1)^{\gamma-1}\Big(\overbrace{\Delta\Delta\dots\Delta}^{(\gamma-2)\text{–times}}H\Big)\,\nu + LOT$$

and

$$E_{\mathcal{C}_m}(M) = 2(-1)^{m+1}\Big(\overbrace{\Delta\Delta\dots\Delta}^{m\text{–times}}H\Big)\,\nu + LOT$$

where Δ is the *Laplace–Beltrami* operator of the hypersurface and H is the (scalar) mean curvature of M.

In one of his last papers De Giorgi stated the following conjecture:

Conjecture (Ennio De Giorgi)

Any compact, n–dimensional, smooth submanifold M of $\mathbb{R}^{n+k}$ without boundary, moving by the gradient of the functional

$$\mathcal{DG}_k(M) = \int_M 1 + |\nabla^m \eta^M|^2 \, d\mathcal{H}^n \,,$$

where η^M is the square of the distance function from M and $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+k}$, does not develop singularities, if $m > n + 1$.

E. De Giorgi, *Congetture riguardanti alcuni problemi di evoluzione. A celebration of John F. Nash, Jr.*, Duke Math. J. **81** (1996), no. 2, 255–268.

In the codimension one case, that is for hypersurfaces, instead of dealing directly with the functionals $\mathcal{DG}_k$, we analyzed the gradient flow associated to the functionals

$$\mathcal{F}_m(\varphi) = \int_M 1 + |\nabla^m \nu|^2 \, d\mu$$

and then we deduced the same conclusion for the original functionals of De Giorgi, thanks to their close connection (Eminenti–CM).

- C. Mantegazza, *Smooth geometric evolutions of hypersurfaces*. Geom. Funct. Anal. **12** (2002), no. 1, 138–182.

Theorem (CM)

If the differentiation order m is strictly larger than $[n/2]$*, then the flows by the gradient of* $\mathcal{DG}_{m+2}$ *and* $\mathcal{F}_m$ *of any initial, smooth, compact, n–dimensional, immersed hypersurface of* $\mathbb{R}^{n+1}$ *exist, are unique and smooth for every positive time (*$[n/2]$ *means the integer part of* $n/2$*).*
Moreover, as $t \to +\infty$*, the evolving hypersurface* φ_t *sub–converges (up to reparametrization and translation) to a smooth critical point of the respective functional.*

Notice that the hypothesis $m > [n/2]$ in general is weaker than the original one in De Giorgi's conjecture.

This evolution problem turns out to be a higher order quasilinear parabolic systems of PDE's on the manifold M. The very first step of the analysis is showing the short time existence and uniqueness of a smooth flow.
This is a particular case of a result of Polden. His proof was anyway flawed and we corrected and generalized it.

- C. Mantegazza and L. Martinazzi, *A note on quasilinear parabolic equations on manifolds*. Ann. Sc. Norm. Super. Pisa **9** (2012), 857–874.

Then, the long time existence is guaranteed as soon as one has suitable a priori estimates on the flow.

In the study of the *mean curvature flow* of a hypersurface (which gives rise to a second order quasilinear parabolic system of PDE's) by means of techniques such as varifolds, level sets, viscosity solutions, the maximum principle is the key tool to get comparison results and estimates on solutions.
In our case, even if $m = 1$, the first variation and hence the corresponding parabolic problem turns out to be of order higher than two, precisely of order $2m + 2$, so we have to deal with equations of fourth order at least.

This fact has the relevant consequence that we cannot employ the maximum principle to get pointwise estimates and to compare two solutions, thus losing a whole bunch of geometric results holding for the mean curvature flow. In particular, it cannot be expected that an initially embedded hypersurface remains embedded during the flow, since self–intersections could actually develop.

In order to show regularity, a good substitute of the pointwise estimates coming from the maximum principle, are suitable estimates on the second fundamental form in Sobolev spaces, using Gagliardo–Nirenberg interpolation type inequalities for tensors. Since the constants involved in these inequalities depends on the Sobolev constants and these latter on the geometry of the hypersurface where the tensors are defined, it is absolutely needed some uniform control independent of time on these constants.

In the works of Polden on curves, these controls are obvious as the constants depend only on the length, on the contrary, much more work is needed here because of the richer geometry of the hypersurfaces.

Despite the large number of papers on the mean curvature flow, the literature on fourth or even higher order flows is quite limited. Our work borrows from the papers of Chrusciel and Polden the basic idea of using interpolation inequalities as a tool to get a priori estimates.

We want to remark here that another strong motivation for the study of these flows is actually the fact that, in general, regularity is not shared by second order flows, with the notable exceptions of the evolution by mean curvature of embedded curves in the plane (Gage, Hamilton, Grayson) and of convex hypersurfaces (Huisken).

When m is large enough, the functionals $\mathcal{DG}_{m+2}$ and $\mathcal{F}_m$, which decrease during the flow, control the L^p norm of the second fundamental form for some exponent p larger than the dimension n of the hypersurface. This fact, combined with a universal Sobolev type inequality due to Michael and Simon, where the dependence of the constants on the curvature is made explicit, allows to get an uniform bound on the Sobolev constants of the evolving hypersurfaces and then to obtain time–independent estimates on curvature and all its derivatives in L^2. These bounds will imply in turn the desired pointwise estimates and the long time existence and regularity of the flows.

Pushing a little the analysis, it also follows that considering a general, positive, geometric functional

$$\mathcal{G}(\varphi) = \int_M f(\varphi, g, \mathrm{B}, \nu, \dots, \nabla^s \mathrm{B}, \nabla^l \nu)\, d\mathcal{H}^n\,,$$

such that the function f is smooth and has polynomial growth, choosing an integer m large enough, the gradient flow of the "perturbed" functionals, for any $\varepsilon > 0$,

$$\mathcal{G}^\varepsilon_m(\varphi) = \mathcal{G}(\varphi) + \varepsilon \mathcal{F}_m(\varphi)$$

does not develop singularities (the same holds if we perturb the functional $\mathcal{G}$ with $\varepsilon\mathcal{D}\mathcal{G}_{m+2}$).
We then say that $\mathcal{F}_m$ and $\mathcal{D}\mathcal{G}_{m+2}$ are *smoothing terms* for the functional $\mathcal{G}$, that possibly does not admit a gradient flow even for short time starting from a generic initial, smooth, compact, embedded hypersurface.

It is then natural to investigate what happens when the constant $\varepsilon > 0$ in front of these smoothing terms goes to zero.
This program, suggested by De Giorgi in the same paper mentioned before, can be described as follows: given a geometric functional $\mathcal{G}$ defined on submanifolds of the Euclidean space (or a more general ambient space),

- find a functional $\mathcal{F}$ such that the perturbed functionals $\mathcal{G}^\varepsilon = \mathcal{G} + \varepsilon\mathcal{F}$ give rise to globally smooth flows for every $\varepsilon > 0$;
- study what happens when $\varepsilon \to 0$, in particular, the existence of a limit flow and in this case its relation with the gradient flow of $\mathcal{G}$ (if it exists, smooth or singular).

If proved successful, this scheme would give a *singular approximation* procedure of the gradient flow associated to the functional $\mathcal{G}$ with a family of globally smooth flows.

The previous discussion shows that the functionals $\mathcal{F}_m$ and $\mathcal{DG}_{m+2}$ satisfy the first point for any geometric functional $\mathcal{G}$ with polynomial growth, defined on hypersurfaces in $\mathbb{R}^{n+1}$, provided we choose an order m large enough (depending on $\mathcal{G}$).

About the second point, the very first case is concerned with the possible limits when $\varepsilon \to 0$ of the gradient flows of $\int_M 1 + \varepsilon|\nabla^m \nu|^2 \, d\mu$ when $m > [n/2]$ and their relation with the mean curvature flow, which is the gradient flow of the *Area* functional, which is obtained if $\varepsilon = 0$.

De Giorgi, in the same paper cited before, stated the following conjecture.

Conjecture (Ennio De Giorgi)

Let $m > [n/2]$, if the parameter $\varepsilon > 0$ goes to zero, the flows φ_t^ε associated to the functionals

$$\mathcal{DG}_m^\varepsilon(M) = \int_M 1 + \varepsilon |\nabla^{m+2}\eta^M|^2 \, d\mathcal{H}^n$$

and starting from a common initial, smooth, compact, immersed hypersurface $\varphi_0 : M \to \mathbb{R}^{n+1}$, converge in some sense *to the mean curvature flow of φ_0,*

$$\frac{\partial \varphi}{\partial t} = H\nu$$

(which is the gradient flow associated to the limit Area functional, as $\varepsilon \to 0$).

De Giorgi proposed this conjecture in general codimension. We discuss it only in the case of evolving hypersurfaces.

Clearly, an analogous conjecture can be stated for the ε–parametrized family of functionals

$$\mathcal{F}_m^\varepsilon(M) = \int_M 1 + \varepsilon |\nabla^m \nu|^2 \, d\mu \, .$$

- G. Bellettini, C. Mantegazza and M. Novaga, *Singular perturbations of mean curvature flow*. J. Diff. Geom. **75** (2007), no. 3, 403–431.

Theorem

Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth, compact, n–dimensional, immersed submanifold of $\mathbb{R}^{n+1}$. Let $T_{\mathrm{sing}} > 0$ be the first singularity time of the mean curvature flow $\varphi : M \times [0, T_{\mathrm{sing}}) \to \mathbb{R}^{n+1}$ of M. For any $\varepsilon > 0$ let $\varphi^\varepsilon : M \times [0, +\infty) \to \mathbb{R}^{n+1}$ be the flows associated to the functionals $\mathcal{DG}^\varepsilon_m$ (or $\mathcal{F}^\varepsilon_m$) with $m > [n/2]$, that is,

$$\frac{\partial \varphi^\varepsilon}{\partial t} = H\nu + 2\varepsilon(m+2)(-1)^m \Big(\overbrace{\Delta^{M_t}\Delta^{M_t}\dots\Delta^{M_t}}^{m\text{–times}} H\Big)\,\nu + \varepsilon\mathrm{LOT}\,\nu\,,$$

all starting from the same initial immersion φ_0.
Then, the maps φ^ε converge locally in $C^\infty(M \times [0, T_{\mathrm{sing}}))$ to the map φ, as $\varepsilon \to 0$.

The above regularization of mean curvature flow with a singular perturbation of higher order could lead to a new definition of generalized solution in any dimension and codimension.

At the moment we are not able to show the existence or characterize the limits of the approximating flows after the first singularity time, as the proof of the above theorem relies heavily on the smoothness of the mean curvature flow in the time interval of existence.

As an example, we mention the simplest open problem in defining a limit flow after the first singularity. It is well known (Gage–Hamilton and Huisken) that a convex curve in the plane (or hypersurface in $\mathbb{R}^{n+1}$) moving by mean curvature shrinks to a point in finite time, becoming exponentially round. In this case we expect that the approximating flows converge (in a way to be made precise) to such a point at every time after the "extinction" one.

We remark here that this method works in general for any geometric evolution of submanifolds in a Riemannian manifold till the first singularity time, even when the equations are of high order (like, for instance, the Willmore flow) choosing an appropriate regularizing term of higher order.

Finally, it should be noticed, comparing the evolution equations above with the one of the mean curvature flow, that these perturbations could be considered, in the framework of geometric evolution problems, as an analogue of the so–called *vanishing viscosity method* for PDE's. Indeed, we perturb the mean curvature flow equation with a regularizing higher order term multiplied by a small parameter $\varepsilon > 0$.

However, the analogy with the classical viscosity method cannot be pushed too far. For instance, because of the condition $m > [n/2] + 2$, the regularized equations are of order not less than four (precisely at least four for evolving curves, at least six for evolving surfaces). Moreover, as the Laplacians appearing in the evolution equation are relative to the induced metric, the system of PDE's is actually quasilinear and the lower order terms are nonlinear (polynomial).

Thanks for your attention

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