Li & Yau estimates for some semilinear heat equations and applications

CARLO MANTEGAZZA
(Joint work with Giacomo Ascione, Daniele Castorina & Giovanni Catino)

Introduction

Let (M, g) be an n–dimensional, complete and connected Riemannian manifold without boundary, let Δ be Laplace–Beltrami operator and $p > 1$. We call a (classical) positive solution of the semilinear heat equation

$$u_t = \Delta u + u^p$$

- *ancient* if it is defined in $M \times (-\infty, T)$ for some $T \in \mathbb{R}$,
- *eternal* if it is defined in $M \times \mathbb{R}$,
- *immortal* if it is defined in $M \times (T, +\infty)$ for some $T \in \mathbb{R}$,
- *static* if it is independent of time, hence it satisfies $\Delta u + u^p = 0$.

We call a solution u *trivial* if it is constant in space, that is, $u(x, t) = u(t)$ and u solves the ODE $u' = u^p$. We say that u is simply *constant* if it is constant in space and time, in such case, it must be clearly identically zero.

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Clearly, $M = \mathbb{R}^n$ with its standard metric is a very special and interesting case on which almost all the existing literature concentrated.

Introduction – Why ancient and eternal solutions are important?

Ancient or eternal solutions typically arise as blow–up limits (in space and time) when the solutions of

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in a general domain $\Omega \times [0, T)$, develop a singularity at a certain time $T \in \mathbb{R}$, i.e. the solution u becomes unbounded as $t \to T^-$ (here Ω is an open subset of $\mathbb{R}^n$ or of a Riemannian manifold).

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They also appear naturally and play a key role in the analysis of mean curvature flow and of Ricci flow (from which we get several suggestions), which are also described by (much more complicated systems of) parabolic PDEs. In such cases, the solutions are respectively, evolving hypersurfaces and abstract Riemannian manifolds.

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Analyzing their properties and eventually classifying them lead to understand the behavior of the solutions close to the singularity or even (in very lucky situations, notably the motion by curvature of embedded curves in the plane and the 2–dimensional Ricci flow) actually *to exclude* the formation (existence) of singularities.

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Moreover, this analysis can also be used to get uniform (universal) estimates on the “rate” a solution (or some related quantity) becomes unbounded at a singularity. Indeed (roughly speaking), typically, a “faster” rate implies that performing a blow–up at a singularity, following

- Poláčik–Quittner–Souplet – for equations in $\mathbb{R}^n$ (universal estimates)
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we obtain a bounded, nonzero, nonconstant, eternal solution, while with the slower “standard” natural rate, we get an ancient solution (immortal solutions are usually less significative).

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Hence, for instance, excluding the existence of bounded, positive, nonconstant, eternal solutions to equation

$$u_t = \Delta u + u^p$$

in $\mathbb{R}^n \times \mathbb{R}$, we have a (universal, up to a constant) L^∞ bound from above on every solution, approaching the singular time T,

$$u(x,t) \leq \frac{C}{(T-t)^{\frac{1}{p-1}}}$$

Poláčik–Quittner–Souplet, by means of estimates of Bidaut–Véron

Theorem

*Let u be an **eternal and nonnegative** solution to the semilinear heat equation $u_t = \Delta u + u^p$ in $\mathbb{R}^n \times \mathbb{R}$, with $n \geq 2$ and $1 < p < \frac{n(n+2)}{(n-1)^2}$.*
Then u is identically zero.
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Theorem (Quittner)

If $n = 2$ the same results holds for every $p > 1$.

Questions – Eternal solutions

Conjecture

*Let u be an **eternal and nonnegative** solution to the semilinear heat equation* $u_t = \Delta u + u^p$ *in* $\mathbb{R}^n \times \mathbb{R}$*, with* $n \geq 3$ *and*

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Notice that $p_S = \frac{n+2}{n-2}$ is the best exponent that we can hope for, since in $\mathbb{R}^n$, with $n \geq 3$, there are positive and radial "Talenti's functions" satisfying $\Delta u + u^p = 0$, for every $p \geq \frac{n+2}{n-2}$, which then are static (hence eternal) solutions.

What about ancient only solutions?

Theorem (Poláčik–Quittner–Souplet + Merle–Zaag)

Let u be a nonnegative, ancient solution to the semilinear heat equation $u_t = \Delta u + u^p$, with $1 < p < \frac{n(n+2)}{(n-1)^2}$ and $n \geq 2$, in $\mathbb{R}^n$. Then u is trivial.

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The previous theorem holds also when the "ambient" space is a Riemannian manifold, with suitable assumptions on the "geometry at infinity", possibly with nonnegative Ricci tensor.

Castorina–M (maximum principle/blow–up techniques)

Theorem

Let (M, g) *be an n–dimensional Riemannian manifold* with bounded geometry. *Every nonnegative, eternal solution of the equation* $u_t = \Delta u + u^p$ *in* $M \times \mathbb{R}$, *with* $1 < p < \frac{n(n+2)}{(n-1)^2}$, *is identically zero.*

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Theorem

Let (M, g) be an n–dimensional compact Riemannian manifold without boundary such that $Ric > 0$. Let u be a nonnegative, ancient solution to the semilinear heat equation $u_t = \Delta u + u^p$, with $1 < p < \frac{n(n+2)}{(n-1)^2}$. Then u is trivial.

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Proposition

Let the Ricci tensor of (M, g) *be uniformly bounded below. If u is an ancient solution of the equation* $u_t = \Delta u + |u|^p$ *with* $p > 1$ *in* $M \times (-\infty, T)$*, then either* $u \equiv 0$ *or* $u > 0$ *everywhere.*

This last proposition clearly extends all the results (also the known ones in $\mathbb{R}^n$) to the equation $u_t = \Delta u + |u|^p$.

Castorina–Catino–M (integral estimates)

Theorem

Let (M, g) *be an n–dimensional Riemannian manifold* with bounded geometry *and nonnegative Ricci tensor. Every* monotone–in–time, *nonnegative, eternal solution of the equation* $u_t = \Delta u + u^p$ *in* $M \times \mathbb{R}$*, with* $1 < p < \frac{n+2}{n-2}$*, is identically zero.*

Li & Yau estimates (no blow–up techniques)

For a positive solution on $M \times [T_0, T)$ of the standard heat equation $u_t = \Delta u$, where (M, g), of dimension n, has nonnegative Ricci tensor, the following estimate holds

$$\Delta u = u_t \geq \frac{|\nabla u|^2}{u} - \frac{nu}{2(t - T_0)}$$

in $M \times (T_0, T)$.

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Hamilton's matrix extension under geometric (curvature) assumptions on (M, g)

$$\mathrm{Hess}_{ij} u - \frac{\nabla_i u \nabla_j u}{u} + \frac{u}{2(t - T_0)} g_{ij} \geq 0$$

Li & Yau estimates – Semilinear extension

Theorem

Let $u : M \times [T_0, T) \to \mathbb{R}$ a classical positive solution of the equation $u_t = \Delta u + u^p$ on an n–dimensional, complete Riemannian manifold (M, g) with nonnegative Ricci tensor. Then, for every pair $\alpha, \beta \in (0, 1)$ "admissible" for $p > 1$, there exists some $\varepsilon = \varepsilon(n, p, \alpha, \beta) > 0$ such that

$$u_t \geq \alpha \frac{|\nabla u|^2}{u} + \beta u^{p-1} - \frac{\varepsilon u}{t - T_0}$$

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A pair of constants $\alpha, \beta \in (0, 1)$ is "admissible" for $p > 1$ if

$$(p-1)(\beta p - \alpha) + \frac{4\alpha(1-\alpha)(1-\beta)}{n} \geq 0$$

$$p < 1 + \frac{8\alpha(1-\beta)}{n}$$

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By a straightforward computation or using Mathematica™ we have the following lemma.

Lemma

- *if* $n \leq 3$ *and* $p < 8/n$
- *if* $n \geq 4$ *and*

$$p < \frac{3n + 4 + 3\sqrt{n(n+4)}}{2(3n-4)} = \overline{p}_n$$

there exists at least one pair $\alpha, \beta \in (0,1)$ *"admissible" for* $p > 1$.

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Corollary

If u is an ancient (or eternal) and positive solution of the equation $u_t = \Delta u + u^p$ *on an n–dimensional, complete Riemannian manifold* (M, g) *with nonnegative Ricci tensor, with* $p < \overline{p}_n$*, there holds*

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are monotone nondecreasing in time. In particular, all the maps $t \mapsto u(x, t)$ are convex.

Li & Yau estimates – Harnack–type estimate

Arguing as Li & Yau, we then have the following (local) estimate.

Proposition

Let $u : M \times [T_0, T) \to \mathbb{R}$ *a classical positive solution of the equation* $u_t = \Delta u + u^p$ *with* $p \in (1, \overline{p}_n)$, *on an* n*–dimensional complete Riemannian manifold* (M, g) *with nonnegative Ricci tensor. Then, for every pair* (α, β) *admissible for* p *and* $\varepsilon = \varepsilon(n, p, \alpha, \beta) > 0$ *as above, given any* $T_0 < t_1 < t_2 < T$ *and* $x_1, x_2 \in M$, *the following inequality holds,*

$$u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2 - T_0}{t_1 - T_0} \right)^{1/\varepsilon} exp \left(\inf_{\gamma \in \Gamma(x_1, x_2)} \int_0^1 \frac{|\dot{\gamma}(s)|^2}{4\alpha(t_2 - t_1)} \, ds \right)$$

where $\Gamma(x_1, x_2)$ *denotes the set of all the paths in* M *parametrized by* $[0, 1]$, *joining* x_2 *to* x_1.

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where $\Gamma(x_1, x_2)$ denotes the set of all the paths in M parametrized by $[0, 1]$, joining x_2 to x_1.

Moreover, it is also possible to have a semilinear version of Hamilton's matrix Li & Yau estimate, under the same geometric quite restrictive hypotheses (and also a relative Harnack–type estimate).

Ancient and eternal solutions on Riemannian manifolds with nonnegative Ricci tensor

Corollary

Every eternal and nonnegative solution of the equation $u_t = \Delta u + u^p$ with $p < \overline{p}_n$, is identically zero (there are no positive eternal solutions).

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Every ancient and nonnegative solution of the equation $u_t = \Delta u + u^p$ with $p < \overline{p}_n$, is trivial (constant in space).

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The convexity in time of the solutions and the monotonicity of the modulus of their gradient seems very strongly restrictive properties, but still we are able to show this up–to–now–conjecture only in some very special cases.

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- $n \geq 3$ and $p < p_Q = \frac{n}{n-2}$
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- $n \geq 2$ and $p < p_B = \frac{n(n+2)}{(n-1)^2}$
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- $n \leq 2$ and every $p > 1$
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- $n \geq 3$ and $p < p_S = \frac{n+2}{n-2}$, if u is *radial*
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Conjecture

Every eternal and nonnegative solution of the equation $u_t = \Delta u + u^p$ with $p < p_S$, is identically zero (there are no positive eternal solutions).

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Theorem (Quittner – Duke 2021)

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Thanks for your attention