# Quasiconvexity in the Riemannian Setting

Carlo Mantegazza

*Joint work (in progress) with Aurora Corbisiero and Chiara Leone*

# Quasiconvexity in $\mathbb{R}^n$

A fundamental result in the calculus of variations in order to apply the so–called *direct methods* in the search for minimizers of integral functionals defined on Sobolev spaces $W^{1,p}(\Omega,\mathbb{R}^m)$ is the characterization of their sequential lower semicontinuity with respect to the weak topology.

For scalar problems, the convexity of the integrand is enough (Tonelli, Serrin, De Giorgi, Olech, Ioffe...). However, in the vector–valued case, ordinary convexity is far too restrictive.

The breakthrough came in 1952, when Morrey introduced the notion of *quasiconvexity*. More precisely, Morrey showed that under some strong regularity assumptions on the function $f$, the equivalence between its quasiconvexity and the weakly$^*$ sequential lower semicontinuity in $W^{1,\infty}(\Omega,\mathbb{R}^m)$ of the functional

$$u \mapsto F(u,\Omega) = \int_\Omega f(x,u(x),Du(x))\, d\mathscr{L}^n(x)$$

holds. Meyers then extended Morrey's result to $W^{k,p}(\Omega,\mathbb{R}^m)$ spaces in 1965. Acerbi and Fusco in 1984 obtained a significant improvement of this result: they indeed established such equivalence for Carathéodory integrands with appropriate growth conditions in $W^{1,p}(\Omega,\mathbb{R}^m)$, for $1 \leqslant p \leqslant +\infty$.

# Quasiconvexity in $\mathbb{R}^n$ – Acerbi & Fusco results

We will focus on the following theorem of Acerbi and Fusco, where all the previous regularity hypotheses on $f$ were dropped. We will come back at the end to their results when $p < +\infty$.

## Theorem (Acerbi–Fusco, 1984)

*Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$0 \leqslant f(x, s, \xi) \leqslant a(x) + b(s, \xi),$$

*for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^{m\times n}$, where $a : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and locally summable and $b : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is nonnegative and locally bounded.*

*Then, $f$ is quasiconvex in $\xi$ if and only if for every bounded open set $\Omega$ in $\mathbb{R}^n$ the functional $u \mapsto F(u, \Omega)$ is sequentially weakly$^*$ lower semicontinuous on $W^{1,\infty}(\Omega, \mathbb{R}^m)$.*

# Quasiconvexity in $\mathbb{R}^n$ – The definition

The definition of quasiconvexity is not very intuitive!

## Definition (Morrey, 1952)

*A continuous function* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *is* quasiconvex *if for every* $\xi \in \mathbb{R}^{m\times n}$ *and for every bounded open subset* $\Omega$ *of* $\mathbb{R}^n$*, there holds*

$$f(\xi) \leqslant \fint_{\Omega} f(\xi + D\varphi(x))\, d\mathscr{L}^n(x),$$

*for every function* $\varphi \in C^\infty_c(\Omega, \mathbb{R}^m)$.

*A real function* $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *is* quasiconvex in $\xi \in \mathbb{R}^n$ *if there exists a subset* $Z$ *of* $\mathbb{R}^n$ *with* $\mathscr{L}^n(Z) = 0$*, such that for every* $x \in \mathbb{R}^n \setminus Z$ *and for every* $s \in \mathbb{R}^m$ *the function* $\xi \mapsto f(x, s, \xi)$ *is* quasiconvex.

In particular, it is not a "pointwise" definition like convexity (or polyconvexity or rank–one convexity, that we will discuss later), given by some "algebraic" conditions. It says that *affine* functions are minimizers of the functional among the functions with the same boundary data.

# Quasiconvexity in the Riemannian context

Our aim is to develop a generalization of this theory to the Riemannian setting, that is for functionals defined on Sobolev functions between two Riemannian manifolds (the physical motivations are clear: for instance, elasticity on membranes, liquid crystals on curved surfaces, etc...).

Precisely, we will consider two smooth, complete and connected Riemannian manifolds $(M, g)$ and $(N, h)$ and a function

$$f : \mathscr{L}(TM, TN) \to \mathbb{R},$$

where

$$\mathscr{L}(TM, TN) = \left\{\alpha : T_xM \to T_yN \;\middle|\; x \in M, y \in N \text{ and } \alpha \text{ is linear}\right\}.$$

Then, after introducing a generalization of the Euclidean notion of quasiconvexity, we will show that $f$ is quasiconvex in this sense if and only if for every open and bounded $\Omega \subseteq M$, the functional

$$u \mapsto F(u, \Omega) = \int_\Omega f(du)\, d\mu$$

(where $\mu$ is the canonical measure of $(M, g)$) is sequentially lower semicontinuous in the weak* topology of $W^{1,\infty}(\Omega, \mathbb{R}^m)$.

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- On a manifold, in general there are no *affine* functions.
- In the "classical" definition by Morrey, (considering differentials instead of gradients) the map $\xi$ actually goes from $T_{x_0}\mathbb{R}^n \approx \mathbb{R}^n$ to $T_{y_0}\mathbb{R}^m \approx \mathbb{R}^m$ and it is "perturbed" by a map $D\varphi(x) : T_x\mathbb{R}^n \to T_{\varphi(x)}\mathbb{R}^m$. These two linear maps can then be added together since we can identify all the tangent spaces of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself (and the same for $\mathbb{R}^m$). If the domain or the target space is a manifold, this identification is not possible, hence such sum is not possible and we need a different way to "perturb" the map $\xi$.

# Riemannian quasiconvexity

Let $(M, g)$ and $(N, h)$ be a pair of smooth, connected and complete Riemannian manifolds of dimension $n$ and $m$, respectively and let $\mu$ be the canonical volume measure of $(M, g)$.

## Definition

*We call a real function* $f : \mathscr{L}(TM, TN) \to \mathbb{R}$ Riemannian–quasiconvex, *if for* $\mu$*–almost every* $x_0 \in M$ *and every* $y_0 \in N$*, the "restricted" map*

$$f_{x_0}^{y_0} = f|_{\mathscr{L}(T_{x_0}M, T_{y_0}N)} : \mathscr{L}(T_{x_0}M, T_{y_0}N) \to \mathbb{R}$$

*is continuous and for every* $\alpha_{x_0}^{y_0} \in \mathscr{L}(T_{x_0}M, T_{y_0}N)$ *and* $\varphi \in C^\infty(B_r(x_0), B_s(y_0))$ *which is equal to* $y_0$ *outside a compact subset of* $B_r(x_0)$*, for every* $r > 0$ *small enough, there holds*

$$f(\alpha_{x_0}^{y_0}) \leqslant \fint_{B_r(x_0)} f\left(\alpha_{x_0}^{y_0} + d\exp_{y_0}^{-1}[\varphi(x)] \circ d\varphi[x] \circ d\exp_{x_0}[\exp_{x_0}^{-1}(x)]\right) d\mu(x) + \mathbf{o}(\mathbf{1})$$

*where* $\mathbf{o}(\mathbf{1})$ *is a function which goes to zero as* $r \to 0$ *and depends in a monotonically nondecreasing way, only on the* $L^\infty$ *norm of* $d\varphi$*.*

# Riemannian quasiconvexity

The following equivalent definition, clarifies the presence of the "error term" $\mathbf{o}(\mathbf{1})$ in the previous one.

## Definition (Equivalent – I)

*We call a real function* $f : \mathscr{L}(TM, TN) \to \mathbb{R}$ Riemannian–quasiconvex, *if for* $\mu$*–almost every* $x_0 \in M$ *and every* $y_0 \in N$*, the "restricted" map*

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$$f(\alpha_{x_0}^{y_0}) \leqslant \fint_{B_r(x_0)} f\big(\alpha_{x_0}^{y_0} + d\exp_{y_0}^{-1}[\varphi(x)] \circ d\varphi[x] \circ d\exp_{x_0}[\exp_{x_0}^{-1}(x)]\big)\, \mathbf{J}(x)\, d\mu(x)$$

*where* $\mathbf{J}(x)$ *is the Jacobian of the map* $\exp_{x_0}^{-1}$ *at the point* $x \in B_r(x_0)$.

# Riemannian quasiconvexity

The next one (still equivalent) simply says that for $\mu$–almost every $x_0 \in M$ and every $y_0 \in N$, the "restricted" map

$$f_{x_0}^{y_0} : \mathscr{L}(T_{x_0}M, T_{y_0}N) \approx \mathbb{R}^{m \times n} \to \mathbb{R}$$

is quasiconvex according to the standard Euclidean definition.

## Definition (Equivalent – II)

*We call a real function $f : \mathscr{L}(TM, TN) \to \mathbb{R}$* Riemannian–quasiconvex, *if for $\mu$–almost every $x_0 \in M$ and every $y_0 \in N$, the "restricted" map*

$$f_{x_0}^{y_0} = f|_{\mathscr{L}(T_{x_0}M, T_{y_0}N)} : \mathscr{L}(T_{x_0}M, T_{y_0}N) \to \mathbb{R}$$

*is continuous and for every $\alpha_{x_0}^{y_0} \in \mathscr{L}(T_{x_0}M, T_{y_0}N)$ and $\psi \in C_c^\infty\left(\widetilde{B}_r^{x_0}, \widetilde{B}_s^{y_0}\right)$, for every $r > 0$ small enough, there holds*

$$f(\alpha_{x_0}^{y_0}) \leqslant \fint_{\widetilde{B}_r^{x_0}} f\left(\alpha_{x_0}^{y_0} + d\psi[z]\right) d\mathscr{L}^n(z)$$

*where $\widetilde{B}_r^{x_0}$ and $\widetilde{B}_s^{y_0}$ are the balls of radii $r, s > 0$ centered at the origins of $T_{x_0}M$ and $T_{y_0}N$, respectively.*

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# Riemannian quasiconvexity

- The first definition is very useful in dealing with the approximation arguments employed in the proof of the characterization of lower semicontinuous functionals.
- The second one is the “bridge” between the first (on the manifold) and the third one (on the tangent space).
- The third one will be relevant in comparing R–quasiconvexity with other notions of convexity.

# R-quasiconvexity and lower semicontinuity
## The analogues of Acerbi & Fusco results

**R–quasiconvexity** $\Longrightarrow$ **Lower semicontinuity**

### Theorem

*Let $(M, g)$ and $(N, h)$ be a pair of smooth, connected and complete Riemannian manifolds of dimension $n$ and $m$, respectively and let $\mu$ be the canonical volume measure of $(M, g)$. Let $f : \mathscr{L}(TM, TN) \to \mathbb{R}$ be Carathéodory and R–quasiconvex. Then, for every open and bounded subset $\Omega \subseteq M$, the functional*

$$u \mapsto F(u, \Omega) = \int_\Omega f(du)\, d\mu$$

*is sequentially lower semicontinuous in the weak* topology of $W^{1,\infty}(\Omega, N)$, that is,*

$$F(u, \Omega) = \int_\Omega f\big(du\big)\, d\mu \leqslant \liminf_{j\to\infty} \int_\Omega f\big(du_j\big)\, d\mu = \liminf_{j\to\infty} F(u_j, \Omega),$$

*for every sequence $u_j \overset{\star}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega, N)$.*

# R-quasiconvexity and lower semicontinuity
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- The weak* convergence of a sequence $u_j \overset{\star}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega, N)$ is defined in the usual way: the sequences of integrals of $u_j$ and $du_j$ "against" fixed functions and 1–forms in $L^1(\Omega, N)$, respectively, converge to the analogous integrals relative to $u$ and $du$.

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- The function $f : \mathscr{L}(TM, TN) \to \mathbb{R}$ is Carathéodory if once locally "trivialized" the vector bundle $\mathscr{L}(TM, TN)$, expressing $f$ in local coordinates, it is Carathéodory in the usual sense.

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The first definition combines perfectly with the very elegant proof in $\mathbb{R}^n$ by Fonseca & Muller, which consists in showing a sort of "asymptotic" lower semicontinuity (LSC in all small balls $B_r(x_0)$ with $r \to 0$) and then concluding (recovering the LSC for any domain $\Omega$) by means of the theorem of Radon–Nikodym. The "error term" $o(1)$ in such definition asymptotically vanishes, so it does not affect the argument of Fonseca & Muller and the conclusion follows as in the Euclidean case (with some technical details due to the approximations).

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**Lower semicontinuity** $\Longrightarrow$ **R–quasiconvexity**

### Theorem

*In the same hypotheses of the previous theorem. Let* $f : \mathscr{L}(TM, TN) \to \mathbb{R}$ *be Carathéodory. If for every open and bounded subset* $\Omega \subseteq M$*, the functional*

$$F(u, \Omega) = \int_\Omega f(du)\, d\mu$$

*is sequentially lower semicontinuous in the weak* topology of* $W^{1,\infty}(\Omega, N)$*, then the function* $f$ *is R–quasiconvex.*

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*is sequentially lower semicontinuous in the weak* topology of $W^{1,\infty}(\Omega, N)$, then the function $f$ is R–quasiconvex.*

This implication is actually easier. One defines standard "roof functions" (as in $\mathbb{R}^n$) on a small ball in the tangent space at a point $x_0$ and "maps" them on $M$ by means of the exponential map $\exp_{x_0}$. Then, following the same proof as in $\mathbb{R}^n$, one gets the quasiconvexity inequality with an error due to the use of $\exp_{x_0}$. Such an error goes to zero, as $\exp_{x_0}$ goes to the identity when the radius of the balls goes to zero. Hence we get the R–quasiconvexity of $f$.

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# R–quasiconvexity, R–polyconvexity and R–rank–one convexity

Let $f : \mathscr{L}(TM, TN) \to \mathbb{R}$ such that for $\mu$–almost every $x_0 \in M$ and every $y_0 \in N$, the "restricted" map $f_{x_0}^{y_0}$ is continuous.

$$f_{x_0}^{y_0} = f|_{\mathscr{L}(T_{x_0}M, T_{y_0}N)} : \mathscr{L}(T_{x_0}M, T_{y_0}N) \to \mathbb{R}$$

## Definition

*The function $f : \mathscr{L}(TM, TN) \to \mathbb{R}$ is* R–convex/polyconvex/rank–one convex*, if for $\mu$–almost every $x_0 \in M$ and every $y_0 \in N$, the "restricted" map is convex/polyconvex/rank–one convex in the "standard way".*

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Then, thanks to the third equivalent definition of R–quasiconvexity, we have the "standard" hierarchy:

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as in the Euclidean setting.

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and if $f = f(\xi)$ is of class $C^2$, we have the *Legendre–Hadamard condition*:

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} \frac{\partial^2 f}{\partial \xi_i^\alpha \partial \xi_j^\beta} \lambda_i \lambda_j \eta^\alpha \eta^\beta \geqslant 0$$

# Work in progress
# The case $p < +\infty$ in the theorem of Acerbi & Fusco

**Theorem (Acerbi–Fusco, 1984)**

*Let $1 \leqslant p < +\infty$ and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$0 \leqslant f(x,s,\xi) \leqslant a(x) + C\big(|s|^p + |\xi|^p\big),$$

*for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^{m\times n}$, where $a : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and locally summable and $C$ is a nonnegative constant.*

*Then, $f$ is quasiconvex in $\xi$ if and only if for every bounded open set $\Omega$ in $\mathbb{R}^n$ the functional $u \mapsto F(u,\Omega)$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega,\mathbb{R}^m)$.*

Thanks for your attention