The Riemannian Penrose Inequality via Nonlinear Potential Theory

Carlo Mantegazza

(Joint work with Virginia Agostiniani, Lorenzo Mazzieri and Francesca Oronzio)

Asymptotically flat manifolds

A smooth, connected, complete Riemannian manifold (with or without boundary) (M, g) of dimension $n \geqslant 3$ is called *asymptotically flat* (AF) if there exists a compact subset K such that

- $M \setminus K = M_1 \sqcup \cdots \sqcup M_k$ (*ends* of the manifold M)
- $M_i \cong \mathbb{R}^n \setminus \overline{B}$ via a diffeomorphism (AF chart) Φ such that

$$\Phi_* g = g_{\mathrm{Eucl}} + \sigma_{kl}\, dx^k \otimes dx^l$$

with $\sigma_{kl} = O_2(|x|^{-\tau})$, for some $\tau > \frac{n-2}{2}$ (*order of decay* of the chart Φ)

meaning that there exists a constant $C > 0$ such that

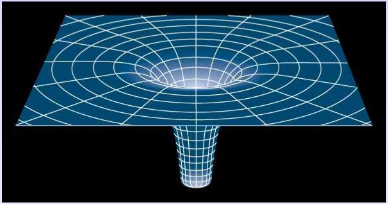
$$|\sigma_{kl}| \leqslant \frac{C}{|x|^{\tau}},$$

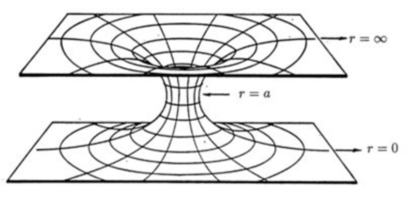
$$\sum_{s=1}^{n} \left| \frac{\partial \sigma_{kl}}{\partial x^s} \right| \leqslant \frac{C}{|x|^{\tau+1}},$$

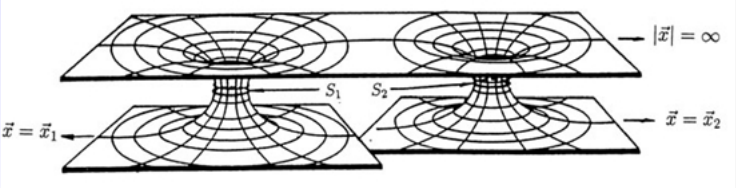
$$\sum_{s,t=1}^{n} \left| \frac{\partial^2 \sigma_{kl}}{\partial x^s \partial x^t} \right| \leqslant \frac{C}{|x|^{\tau+2}}.$$

on $\{|x| \geqslant R\}$, for R large.

Examples







Remarks

The definition of AF manifold consists of two informations.

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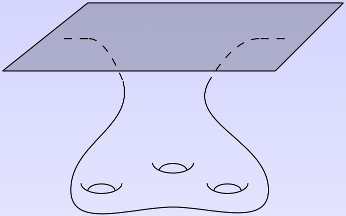
- *One is of topological nature: manifolds can possibly have a quite complicated topology but all concentrated in a bounded subset*
- *One is about the behavior of the metric "in the large"*



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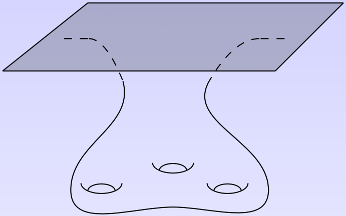

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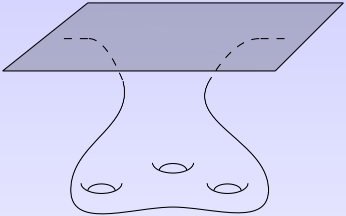

(2) The definition of AF manifold is not uniform in literature.
For smooth metrics, the completeness or the number of ends or the rate of decay of the metric or even assumptions involving the scalar curvature are some of the points where the definitions may differ.

(3) Physically, one expects asymptotically flat 3–manifolds to arise as spacelike hypersurfaces of spacetimes (i.e. 4–dimensional Lorentzian manifolds $(\mathcal{M}, \boldsymbol{g})$ satisfying the Einstein equation, $\mathbf{Ric} - \frac{1}{2}\mathbf{R}\boldsymbol{g} = \frac{8\pi G}{c^4}\mathbf{T}$) modeling *isolated gravitational systems*.

Schwarzschild manifold

For $m \geqslant 0$, the *Schwarzschild manifold of mass* m, is the Riemannian manifold

$$\left(M_{\mathrm{Sch}(m)}, g_{\mathrm{Sch}(m)}\right) = \left(\mathbb{R}^n \setminus \overline{B}_{(m/2)^{\frac{1}{n-2}}}(0), \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathrm{Eucl}}\right)$$

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The Schwarzschild manifolds are asymptotically flat with only one end. The Schwarzschild metrics are clearly conformal to the Euclidean metric (via a power of a harmonic function) and have zero scalar curvature.

Moreover, the sphere $\{|x| = (m/2)^{1/n-2}\}$ is a totally geodesic (hence, minimal) hypersurface.

ADM mass

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Let (M,g) be an asymptotically flat manifold with integrable or nonnegative scalar curvature. Let M_i be an end of (M,g) and let Φ be an asymptotically flat chart for M_i.

$$m_{\mathrm{ADM}}(M_i) = \lim_{r\to+\infty} \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\{|x|=r\}} (\partial_l g_{kl} - \partial_k g_{ll}) \frac{x^k}{|x|} d\sigma_{\mathrm{can}}$$

where $\Phi_* g = g_{kl}\, dx^k \otimes dx^l$.

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Bartnik (1986) and Chruściel (1988) proved independently that the ADM mass of each end is well–defined and does not depend on the particular AF coordinate chart chosen.

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- For the Schwarzschild manifolds of mass m, the ADM mass coincides with the parameter m. This condition must be satisfied by any "good" definition of *total mass* of an isolated gravitational system, as in general relativity, Schwarzschild manifolds model the (spatial) exterior of non–moving, spherically symmetric, static stars or black holes of mass m.

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- **A natural requirement for a "mass" is that it is POSITIVE!**

The positive mass theorem

Theorem (Positive Mass Theorem – R. Schoen, S. T. Yau – 1979)

Let (M, g) *be a 3–dimensional Riemannian manifold.*

$$\textit{If} \quad \begin{cases} (M,g) \text{ is complete} \\ (M,g) \text{ is a one–ended asymptotically flat manifold} \\ \text{the scalar curvature } \mathrm{R} \text{ of } (M,g) \text{ is nonnegative, } \mathrm{R} \geqslant 0 \end{cases}$$

then, the positive mass inequality *holds*

$$m_{\mathrm{ADM}} \geqslant 0$$

Moreover, the ADM mass of (M, g) *is zero if and only if* (M, g) *is isometric to* $(\mathbb{R}^3, g_{\mathrm{Eucl}})$.

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Asymptotically flat 3–manifolds with nonnegative scalar curvature arise as spacelike hypersurfaces of spacetimes modeling isolated systems, then the positive mass theorem can be interpreted saying that a nonnegative "local mass density" ($\mathrm{R} \geqslant 0$) implies a nonnegative total mass ($m_{\mathrm{ADM}} \geqslant 0$).

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Several approaches/tools have been used to prove the positive mass theorem:

- Schoen, Yau (1979, minimal surfaces)
- Witten (1981, spin manifolds)
- Lohkamp (1991, nonexistence theorems of positive scalar curvature metrics)
- Huisken, Ilmanen (2001, weak inverse mean curvature flow)
- Li (2018, Ricci flow)
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Virginia Agostiniani, Lorenzo Mazzieri and Francesca Oronzio obtained a "simple" (more elementary) proof of the positive mass inequality via a monotonicity formula holding along the level sets of an appropriate harmonic function related to the minimal positive Green's function with a pole.

Idea of the proof

- By the work of H. Bray, D. Kazaras, M. Khuri and D. Stern, it is sufficient to show that the positive mass inequality is true in the class of Riemannian manifolds satisfying the assumptions of the positive mass theorem and also the following properties:

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- By the work of H. Bray, D. Kazaras, M. Khuri and D. Stern, it is sufficient to show that the positive mass inequality is true in the class of Riemannian manifolds satisfying the assumptions of the positive mass theorem and also the following properties:
  - M is diffeomorphic to $\mathbb{R}^3$ (*topological simplification*)
  - There exists a distinguished AF coordinate chart Φ such that

$$\Phi_* g = \left(1 + \frac{m_{\mathrm{ADM}}}{2|x|}\right)^4 g_{\mathbb{R}^3}$$

  outside of a compact set (*simplification of the metric at infinity*)

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Then, there holds

$$0 < s < t < +\infty : 1 - 1/s,\ 1 - 1/t \text{ are regular values of } u$$
$$\Downarrow$$
$$F(s) \leqslant F(t)$$

Indeed, after some (heavy, but straightforward) computations

$$F'(t) = 4\pi + \int_{\{u=1-\frac{1}{t}\}} \Big[-\frac{\mathrm{R}^{\Sigma_t}}{2} + \underbrace{\frac{|\nabla^{\Sigma_t}|\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3}{4}\Big(\frac{2\,|\nabla u|}{1-u} - \mathrm{H}\Big)^2}_{Q \geqslant 0} \Big]\, d\sigma$$

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$$\left.\begin{array}{l} M \cong \mathbb{R}^3 \\ u \text{ is harmonic} \end{array}\right\} \Longrightarrow \begin{array}{c}\text{every regular level set}\\ \text{of } u \text{ is connected}\end{array} \xrightarrow{\text{Gauss–Bonnet theorem}} 4\pi - \int_{\{u=1-\frac{1}{t}\}} (R^{\Sigma_t}/2)\, d\sigma \geqslant 0$$

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Hence, $F'(t) \geqslant 0$ a.e. $\Longrightarrow F$ is monotone nondecreasing
(F is absolutely continuous)

- Consequently,

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  - The limit at the right hand side is a consequence of the asymptotic behavior of the minimal positive Green's function $\mathcal{G}_o$ at the pole o
  - The limit at the left hand side can be computed by means of the special AF coordinate chart

$$\Phi_* g = \Big(1 + \frac{m_{\mathrm{ADM}}}{2|x|}\Big)^4 g_{\mathbb{R}^3}$$

In such a chart, the expansion of the function u near infinity is given by

$$u = 1 - \frac{1}{|x|} + \frac{1}{2|x|^2}\big(m_{\mathrm{ADM}} + \phi(x/|x|)\big) + O_2\big(|x|^{-2-\alpha}\big)$$

for every $\alpha \in (0,1)$, where $\Delta^{\mathbb{S}^2}\phi = -2\phi$

The Riemannian Penrose inequality

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Theorem (Riemannian Penrose Inequality G. Huisken–T. Ilmanen & H. Bray – 2001)

Let (M, g) be a 3–dimensional, complete AF manifold with a smooth, compact, connected boundary and one single end. Assume that

- *the metric g has nonnegative scalar curvature* $\mathrm{R} \geqslant 0$
- ∂M *is the unique minimal surface in* (M, g),

then, the ADM mass satisfies the inequality

$$m_{\mathrm{ADM}} \geqslant \sqrt{\frac{|\partial M|}{16\pi}}$$

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The proof of Huisken–Ilmanen is based on the (weak) inverse mean curvature flow, the one of Bray on a conformal flow of metrics. One would like to try to follow the same "level sets line" as in the proof of the positive mass theorem (with a different, suitable harmonic function).

The Riemannian Penrose inequality

Unfortunately, the natural choice of a solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases}$$

with an analogously associated monotone function F, only leads to the following "capacitary" Riemannian Penrose inequality

$$m_{\mathrm{ADM}} \geqslant \frac{\mathrm{Cap}_2(\partial M)}{8\pi}$$

where the *p–capacity* of ∂M is defined as

$$\mathrm{Cap}_p(\partial M) = \inf\left\{ \int_M |\nabla v|^p \, d\mu : v \in \mathscr{C}_c^\infty(M),\ v = 1 \text{ on } \partial M \right\}$$

for $p \in [1, +\infty)$.

The Riemannian Penrose inequality

Inspired by some formal computations by Geroch in 1973 (and, more recently, of Colding and Minicozzi), actually related to the inverse mean curvature flow, that inspired also the proof of Huisken–Ilmanen, we looked at the level sets of the (weak) solution of the following problem

$$\begin{cases} \Delta_p u = 0 & \text{in } M \\ u = 0 & \text{on } \partial M \\ u \to 1 & \text{at } \infty \end{cases}$$

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Then, computing formally (rigorously, if the solution u has no critical points at all), it turns out that, if $p \in (1,3)$, the function $F_p : [t_p, +\infty) \to \mathbb{R}$, given by

$$F_p(t) = 4\pi t - \frac{t^{\frac{2}{p-1}}}{c_p} \int_{\{u=\alpha_p(t)\}} |\nabla u|\,\mathrm{H}\,d\sigma + \frac{t^{\frac{5-p}{p-1}}}{c_p^2} \int_{\{u=\alpha_p(t)\}} |\nabla u|^2\,d\sigma$$

is monotone nondecreasing, where $\alpha_p(t) = 1 - (t_p/t)^{\frac{3-p}{p-1}}$ and $c_p^{p-1} = \dfrac{\mathrm{Cap}_p(\partial M)}{4\pi}$.

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- Hence, as before, for every $p \in (1,3)$, there holds

$$\lim_{t \to +\infty} F_p(t) \geqslant F_p(t_p)$$

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  - The limit in the left hand side is a consequence of the fact that, by the work of Benatti, Fogagnolo and Mazzieri, the p–harmonic function u has the following asymptotic expansion (in a suitable AF coordinate chart)

$$u(x) = 1 - \frac{p-1}{3-p}\,\frac{c_p}{|x|^{\frac{3-p}{p-1}}} + o_2\big(|x|^{-\frac{3-p}{p-1}}\big)$$

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- The Riemannian Penrose inequality follows by sending $p \to 1$, as

$$\liminf_{p\to1^+} \mathrm{Cap}_p(\partial M) \; \geqslant \; |\partial M|$$

thus, the last term converges to $\sqrt{4\pi|\partial M|}$.

The Riemannian Penrose inequality

- As we said the “monotonicity part” was formal, rigorous only if the critical points of the function u are not present, since the Sard theorem is necessary in the proof and the p–harmonic functions are in general only of class $C^{1,\alpha}$, thus possibly not smooth, so it cannot be applied. Moreover, the set of critical points a priori could even have positive Lebesgue measure (OPEN PROBLEM!), affecting the regularity of “too many” surfaces–level sets of the function u and their evolution (too many “jumps”), since we would like to apply to them the Gauss–Bonnet theorem, as in the previous situation dealing with harmonic functions (case $p = 2$). Finally, another issue (related to these ones) is the possible loss of connectedness of such level sets.

The Riemannian Penrose inequality

- All this can be dealt with by "locally" approximating the p–harmonic function u_p with the solutions $u_p^{\varepsilon,T}$ of the following "perturbed" problem (see the works of Di Benedetto)

$$\begin{cases} \operatorname{div}\left(|\nabla u|_\varepsilon^{p-2}\nabla u\right) = 0 & \text{in } M_T = \{0 \leqslant u_p \leqslant T\}\,, \\ u = 0 & \text{on } \partial M\,, \\ u = T & \text{on } \{u_p = T\}\,, \end{cases}$$

where $|\nabla u|_\varepsilon = \sqrt{|\nabla u|^2 + \varepsilon^2}$.

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where $|\nabla u|_\varepsilon = \sqrt{|\nabla u|^2 + \varepsilon^2}$.

Since they are smooth, the Sard theorem can now be applied, then, arguing as before, there exists a similar function F_p^ε, pointwise converging ("often" enough) to F_p as $\varepsilon \to 0$, which satisfies an "approximate" monotonicity, with an "error" going to zero with ε,

$$F_p^\varepsilon(t) - F_p^\varepsilon(s) \geqslant -\varepsilon Q(t, s, \varepsilon) \qquad \text{with } Q \text{ locally bounded.}$$

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Hence, sending $\varepsilon \to 0$, we rigorously get the monotonicity of the original function F_p, in the general situation. The Riemannian Penrose inequality then follows by the previous arguments.

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Thanks for your attention