Some Variations on Ricci Flow

Carlo Mantegazza

The Ricci flow

At the end of ’70s–beginning of ’80s the study of Ricci and Einstein tensors from an analytic point of view gets a strong interest, for instance in the works (static) of Dennis DeTurck. A proposal of investigation of a family of flows, among them the Ricci flow, was done by Jean–Pierre Bourguignon (”Ricci curvature and Einstein metrics”, Lecture Notes in Math 838, 1981). In 1982 Richard Hamilton defines and studies the Ricci flow, that is, the system of partial differential equations

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)}$$

describing the evolution $g(t)$ of the metric of a Riemannian manifold.

R. Hamilton – “Three–manifolds with positive Ricci curvature”, *Journal of Differential Geometry* **17**, 1982, 255–306.

Ricci flow is a sort of geometric heat equations, indeed, the Ricci tensor can be expressed as

$$Ric_g = -\frac{1}{2}\Delta g + \mathrm{LOT}$$

with an appropriate choice of local coordinates.

It can be actually shown that this is a quasilinear, (degenerate) parabolic system of PDE on a manifold. It has a unique smooth solution for small time if the initial manifold is compact. In addition, the solutions satisfy comparison principles and derivative estimates similar to the case of parabolic equations in Euclidean space. Unfortunately, it is well known that the solutions exist in general only in a finite time interval. This means that singularities, for geometric or analytic reasons, develop. The study of such singularities is the key point in the subject of geometric evolutions.

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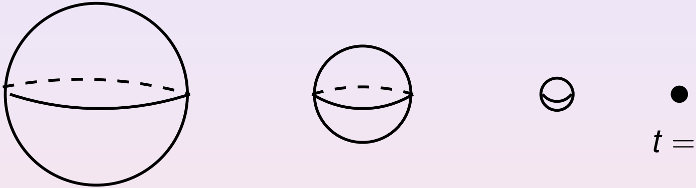

$t = 1/4$

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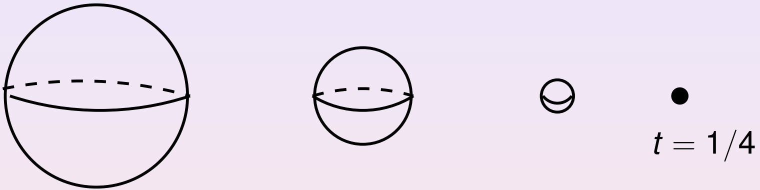
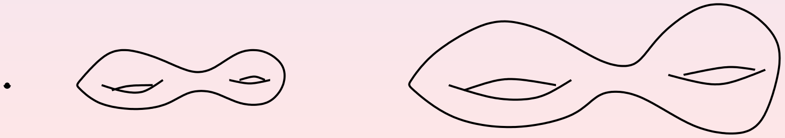


Hyperbolic surface (constant negative curvature):
$g(t) = (1 + 4t)g_0$.



Negative examples: the neckpinch

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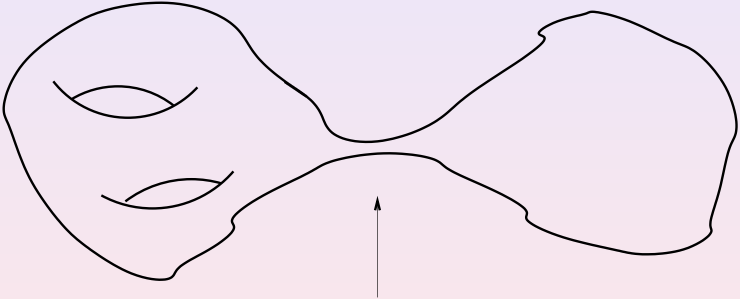

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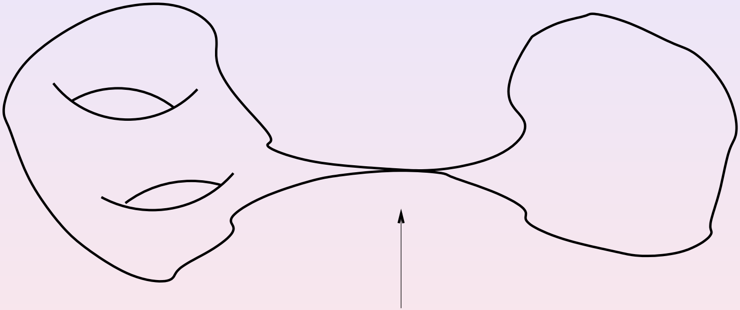

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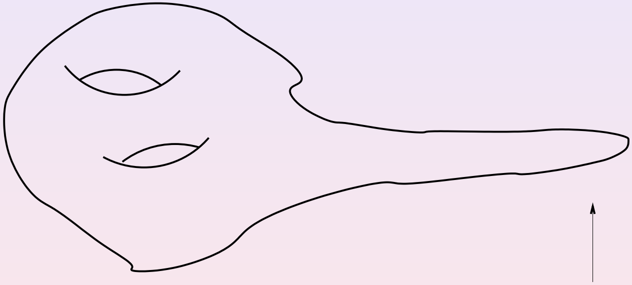

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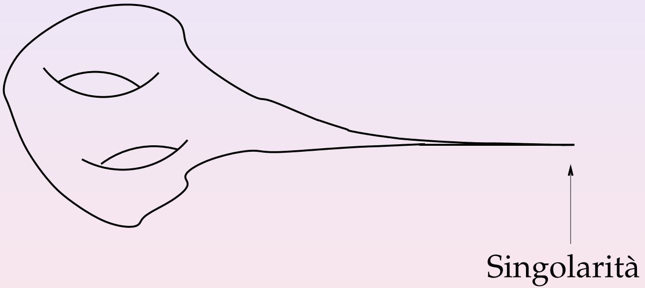

Generalizations of Ricci flow: the renormalization group flow

The *renormalization group* flow arises in modern theoretical physics as a method to investigate the changes of a system viewed at different distance scales. Anyway, it still lacks of a strong mathematical foundation and it is defined by a formal flow of metric on a manifold satisfying the evolution equation

$$\frac{\partial g_{ij}(\tau)}{\partial \tau} = -\beta_{ij}(g(\tau)) \, ,$$

for some functions β_{ij} depending on the metric, the curvature and its derivatives.

In the "perturbative regime" (that is, when $a|Riem(g)| \ll 1$) the functions β_{ij} can be expanded in powers of a,

$$\frac{\partial g_{ij}}{\partial \tau} = -aR_{ij} + o(a) \, ,$$

as $a \to 0$.

Hence, the first order truncation (after the substitution $\tau = t/2a$) coincides with the *Ricci flow* $\partial_t g = -2Ric$, as noted by Friedan and Lott (and also Carfora).
It is interesting then to consider also the second order term in the expansion of such *beta functions*, whose coefficients are quadratic in the curvature and therefore are (possibly) dominating, even when $a|Riem(g)| \to 0$.

The resulting flow is called *two–loop* renormalization group flow

$$\frac{\partial g_{ij}}{\partial t} = -aR_{ij} - \frac{a^2}{2} R_{iklm} R_{jstu} g^{ks} g^{lt} g^{mu} .$$

Joint work with L. Cremaschi

Theorem (Laura Cremaschi, CM)

Let (M^3, g_0) *be a compact, smooth, three–dimensional Riemannian manifold and* $a \in \mathbb{R}$*. Assume that the sectional curvature* K_0 *of the initial metric* g_0 *satisfies*

$$1 + 2a\mathrm{K}_0(X, Y) > 0$$

for every point $p \in M^3$ *and vectors* $X, Y \in T_pM^3$*. Then, there exists some* $T > 0$ *such that the two–loop renormalization group flow has a unique smooth solution* $g(t)$ *in a maximal time interval* $[0, T)$*.*

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for every point $p \in M^3$ and vectors $X, Y \in T_pM^3$. Then, there exists some $T > 0$ such that the two–loop renormalization group flow has a unique smooth solution $g(t)$ in a maximal time interval $[0, T)$.

Subsequently generalized to any dimensions by Gimre, Guenther and Isenberg.

Open problems

It is unknown if the condition

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To investigate higher order truncations of the RG flow (derivatives of the Riemann tensor also appear $\Longrightarrow$ higher order flows).

The Ricci–Bourguignon flow

$$\frac{\partial g}{\partial t} = -2\big(Ric - \rho R g\big)$$

- Einstein flow: $\rho = 1/2$
- Traceless Ricci flow: $\rho = 1/n$
- Schouten flow: $\rho = 1/2(n-1)$
- Ricci flow: $\rho = 0$

It can be seen as an interpolation between the Ricci flow and the Yamabe flow

$$\frac{\partial g(t)}{\partial t} = -2Rg$$

Joint work with G. Catino, L. Cremaschi, Z. Djadli, L. Mazzieri

General Results:

- short time existence and uniqueness for any metric on M compact, if $\rho < 1/2(n-1)$
- blow–up of the curvature at a singularity
- preservation of positive scalar curvature
- preservation of positive Riemann operator
- easier classification of solitons (see later) when $\rho \neq 0$ (easier than for Ricci flow), in particular when $n = 3$

When $n = 3$ also:

- preservation of positive Ricci tensor
- preservation of positive sectional curvature
- Hamilton–Ivey estimate

Open problems

- Schouten case $\rho = 1/2(n-1)$ very interesting but ”critical” for the short time existence of the flow
- missing the analogue of a monotonicity formula, like Perelman’s one for Ricci flow
- missing an injectivity radius estimate at the scale of the curvature

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Key estimates – Uniform in time

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Roundness estimate: there exist constants C and D such that

$$\left| Ric - \frac{1}{3} R g \right| \leq C R^{1-\delta} + D$$

for some $\delta > 0$.

Gradient estimate: for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\frac{|\nabla R|^2}{R} \leq \varepsilon R^2 + C(\varepsilon) \, .$$

Ricci solitons

In several cases the asymptotic profile of a singularity of the Ricci flow is given by a so called Ricci soliton. They are Riemannian manifolds (M, g) such that there exists a smooth function $f : M \to \mathbb{R}$ and a constant $\lambda \in \mathbb{R}$ satisfying

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They describe selfsimilar solutions of the Ricci flow and their study and classification is necessary to “continue” the flow after a singularity, performing a surgery, in order to get geometric conclusions.

Joint work with G. Catino and L. Mazzieri (et alt.)

We obtained several classifications results for Ricci solitons, mainly shrinking and steady, in low dimensions ($n = 2, 3$) or in general dimension with positive Ricci tensor, under various hypotheses on some (derived) curvature tensor, for instance,

- null Weyl tensor, that is, locally conformally flatness of the manifold
- null Cotton tensor (with M. Rimoldi and S. Mongodi)
- null Bach tensor (with H.-D. Cao and Q. Chen)

These results are actually symmetry result, showing that actually the solitons share rotational symmetry. This then leads to their full classification.

Moreover, we investigate more deeply the LCF condition also for “ancient” solutions of the Ricci flow, that are important in the cases where it is not possible to conclude that the asymptotic profile of a singularity is a Ricci soliton. Our result is that actually under such hypothesis the two classes coincide, then, for instance in low dimension they can be classified.

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We underline that actually one of the major open problems for Ricci flow is to classify the ancient solutions in dimension three.

Einstein–type manifolds

A natural generalization of the concept of Ricci solitons (already appeared in other fields, some related to physics) is the family of the so called *Einstein–type* manifold (a Ricci soliton is already a generalization of an Einstein manifold). They are Riemannian manifolds (M, g) such that there exists smooth functions $f, \mu : M \to \mathbb{R}$ and a constant $\alpha \in \mathbb{R}$ satisfying

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As a special case, choosing $\alpha = 0$ and $\mu = \rho R + \lambda$, for constants $\rho, \lambda \in \mathbb{R}$, one gets the solitons for the Ricci–Bourguignon flow

$$Ric - \rho Rg + \nabla^2 f = \lambda g$$

describing the selfsimilar solutions of such flow.

We generalized several results for Ricci solitons to this more general case (with M. Rimoldi et alt.) and we realize that for the RB solitons when $\rho \neq 0$ the analysis and classification is easier than for the Ricci flow ($\rho = 0$). In particular for the Schouten flow, which makes its study of special interest.

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Moreover, these techniques, leading to symmetry (rotational) results for manifolds, were recently used by V. Agostiniani e L. Mazzieri to get symmetry results for overdetermined problems for semilinear elliptic PDEs in exterior domains of $\mathbb{R}^n$, transforming the PDE problems to geometric ones, by conformal deformations of the canonical metric of $\mathbb{R}^n$.

A weak flow tangent to Ricci flow
Joint work with N. Gigli

Let (M, g) be a compact Riemannian manifold, $\mathscr{P}(M)$ the space of Borel probability measures on M and let $\mathrm{K}_t : \mathscr{P}(M) \to \mathscr{P}(M)$ be the heat semigroup. Given a couple of points $p, q \in M$ and a smooth curve $s \mapsto \gamma(s)$ connecting them, for every $t \geq 0$ we have a curve $s \mapsto \gamma_t(s)$ in $\mathscr{P}(M)$ defined by

$$\gamma_t(s) = \mathrm{K}_t(\delta_{\gamma}(s)) .$$

Such curves turns out to be absolutely continuous with respect to the Wasserstein distance W_2 on $\mathscr{P}(M)$ so their lengths are well defined. Taking the infimum of such lengths on all smooth curves connecting the points p and q in M, we can define a new "distance" d_t on the manifold M.

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- The distance d_t comes from a smooth Riemannian metric tensor g_t, for every $t > 0$.
- The dependence on $t \in \mathbb{R}^+$ is smooth.
- As $t \to 0$ the metrics g_t converge to the original metric tensor g of the manifold M.

Theorem (CM, N. Gigli)

For almost every vector $v \in TM$ there holds

$$\frac{d}{dt} g_t(v, v)\Big|_{t=0} = -2Ric_g(v, v),$$

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Moreover, this result opens the possibility (work in progress!!!) to define the Ricci tensor and the Ricci flow for special classes of nonsmooth spaces, where this flow can be defined, that is, metric spaces allowing a well behaved heat kernel.

Open problems

- Defining ”weak” Ricci tensor and Ricci flow for some nonsmooth metric measure spaces
- Understanding ”easy” singular spaces, like a flat cone
- Removing the technical ”almost every vector” conclusion in the theorem

Thanks for your attention

We denote by $(0, +\infty) \times M \times M \ni (t, x, y) \mapsto \rho(t, x, y) \in \mathbb{R}^+$ the heat kernel on M.

Theorem

Let $t > 0$, $x \in M$ and $v \in T_x M$. Then, there exists a unique C^∞ function $\varphi_{t,x,v} : M \to \mathbb{R}$ such that $\int_M \varphi_{t,x,v}(y)\, d\mathrm{Vol}(y) = 0$ and

$$g\big(\nabla_x \rho(t, x, y), v\big) = -\nabla_y \big(\nabla_y \varphi_{t,x,v}(y) \rho(t, x, y)\big). \qquad (1)$$

Such $\varphi_{t,x,v}$ smoothly depends on the data t, x, v.
Moreover, if $v \neq 0$, then $\nabla \varphi_{t,x,v}$ is not identically zero.

Definition

Let $t > 0$, $x \in M$ and $v, w \in T_x M$. Then, $g_t(v, w)$ is defined as

$$g_t(v, w) := \int_M g\big(\nabla_y \varphi_{t,x,v}(y), \nabla_y \varphi_{t,x,w}(y)\big) \rho(t, x, y)\, d\mathrm{Vol}(y).$$