Evolution by curvature of networks in the plane
The state of the art

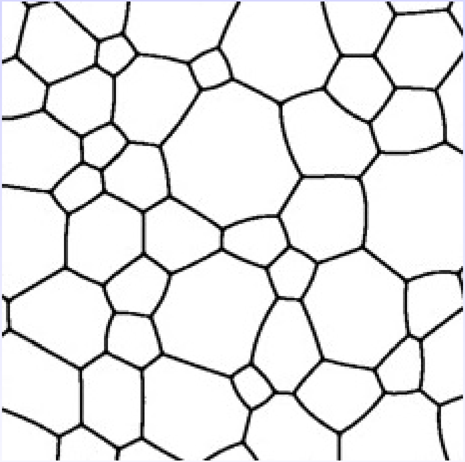
Carlo Mantegazza

Curve Shortening Flow

YouTube – https://www.youtube.com/watch?v=wHfpacPLHIA

Interactive – https://a.carapetis.com/csf

Networks of Curves



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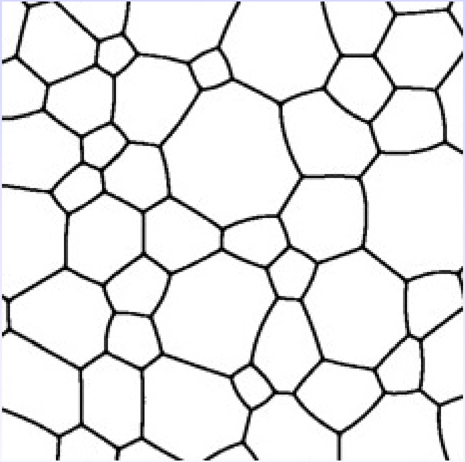

... flowing

Joint project with

- Matteo Novaga & Vincenzo Tortorelli, 2003 – 2005
- Annibale Magni & Matteo Novaga, 2010 – 2014
- Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 – 2016
- Pietro Baldi & Emanuele Haus, 2015 – 2019
- Matteo Novaga, Alessandra Pluda & Marco Pozzetta, 2022 –

After the works of Huisken et alt. about the mean curvature flow of curves and hypersurfaces, weak definitions of mean curvature flow of any merely *closed set* in the Euclidean space appeared.

We were interested in the study of the possibly "least singular" set: a network of curves in the plane. This is clearly a (toy) model for the time evolution of the interfaces of a multiphase planar system where the energy is given only by the total length of such interfaces.

Even if it is still possible to continue to use several of the ideas and techniques of the "parametric" (smooth, classical) approach (differential geometry/PDEs), some extra variational "weak" methods are needed, due to the presence of the multi–points.

We say that a network *moves by curvature* if any of its time–dependent curves $\gamma^i : [0,1] \times (0,T) \to \mathbb{R}^2$ satisfy

$$\gamma_t^i(x,t)^\perp = \underline{k}^i(x,t) = \frac{\langle \gamma_{xx}^i(x,t) \,|\, \nu^i(x,t)\rangle}{|\gamma_x^i(x,t)|^2}\nu^i(x,t) = \left(\frac{\gamma_{xx}^i(x,t)}{|\gamma_x^i(x,t)|^2}\right)^\perp$$

for every $x \in [0,1]$ and $t \in (0,T)$.

The *normal component* of the velocity at every point is given by the curvature vector of the curve (till the endpoints of the curves).

With the right choice of the tangential component of the velocity the problem becomes a non–degenerate system (with several geometric properties) of *quasilinear parabolic partial differential equations*.

This evolution can be seen as the *geometric gradient flow* of the *length functional*, that is, the sum of the lengths of all the curves of the network.

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If no region is collapsing, the geometric changes are only given by pairs of triple junctions colliding (the curve connecting them vanishes - its length goes to zero), producing a 4–point in the network.

Immediately after such a collision of two triple junctions, the network becomes again *regular* (only triple junctions, with curves forming angles of 120 degrees): a new pair of triple junctions “emerges” from every 4–point.

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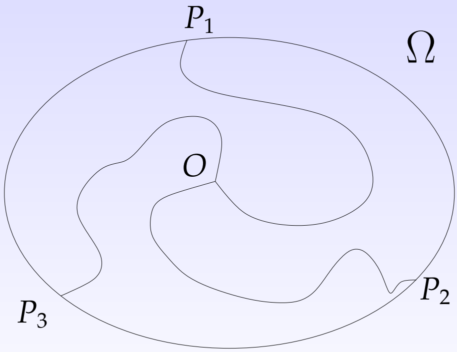
Actually, despite the (apparently) simple problem/behavior/statements, to show in a mathematically satisfactory way these observations, a lot of "technology" from analysis and geometry is needed.

We started dealing with the local problem, that is, the study of the evolution by curvature of the simplest network of three non–intersecting curves with fixed endpoints and a single triple junction with angles of 120 degrees, called a *regular triod*.



Theorem (L. Bronsard, F. Reitich – 1992 & CM, M. Novaga, V. Tortorelli – 2004)

For any initial regular smooth triod there exists a smooth flow by curvature in a positive maximal time interval. Moreover, the evolving triod stays regular.

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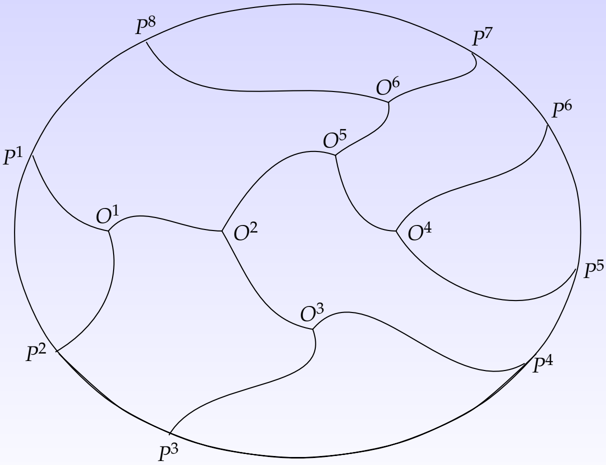
It can be seen as a "local regularity result" for the flow of a general *regular* network.

A *regular* network is given by a finite family of non–intersecting curves such that there are only a finite number of triple junctions with angles of 120 degrees between the concurring curves.



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At the maximal (singular) time T at least one (or both) of the following two conditions holds:

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Even if there is no collapse of curves (or regions), so the topological structure of the network is not going to change, the connection between the "local regularity" (the special case of a triod) to the "global regularity" (general network) is not direct. The main tool is blow–up analysis (after integral estimates) and, in order to get regularity of the flow, one has to exclude that curves with multiplicity larger than one appear in the limit of rescaled networks (which are *shrinkers* – networks self–similarly moving by curvature).

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Hence, to proceed in the analysis, we have to deal with the situation when the length of at least one curve of the network goes to zero as $t \to T$.
There are two cases:

- The curvature stays bounded.
- The curvature is unbounded as $t \to T$.

The analysis in the first case (actually, bounded curvature $\Longrightarrow$ no collapse of regions) consists in understanding the possible limit networks that can arise as $t \to T$ and finding out how to continue the flow (if possible).

It can be shown that, as $t \to T$, such limit network, is unique. Anyway, it can be *non–regular* since multiple points can appear.

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If the collapsing curve is not one of the family containing the fixed boundary points (*boundary curves*), we have the following result.

Lemma

If **M1** *is true, every* interior *vertex of such limit network either is a regular triple junction or it is a* 4*–point where the four concurring curves have opposite unit tangents in pairs and form angles of* 120/60 *degrees among them.*

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We underline that, being this limit network non–regular since it has also 4–junctions, the previous short time existence theorem does not apply.

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For any initial network of non–intersecting curves there exists a (possibly non–unique) Brakke flow by curvature *in a positive maximal time interval such that for every positive time the evolving network is smooth and regular.*

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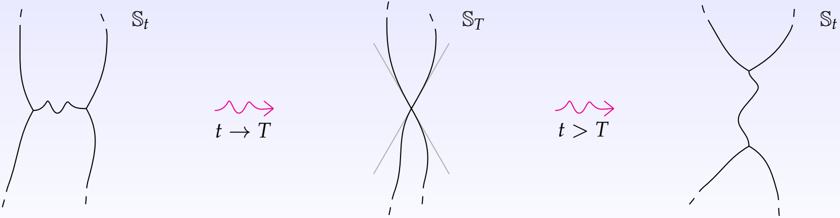
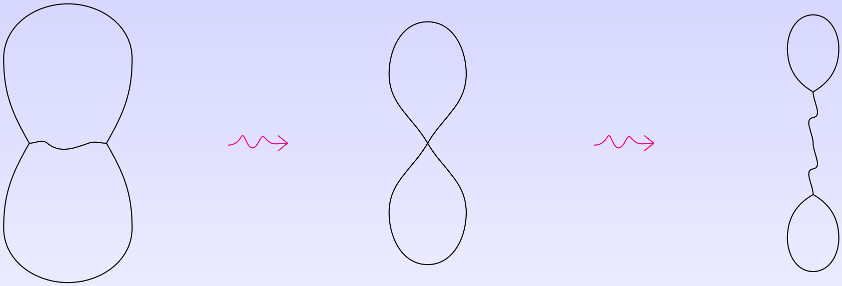


The local description of a “standard” transition.



A "standard" transition for a Θ–shaped network (double cell).

The second situation, when the curvature is unbounded and some curves are vanishing, can be faced again with blow–up methods, but in general, even if **M1** is true, there can be several possible limits of rescaled networks, making the classification quite difficult. Then, the (local) structure (topology) of the evolving network plays an important role in the analysis.

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If **M1** *holds and the network is a* tree *(no loops), the curvature is uniformly bounded during the flow, hence the only "singularities" (after which we can restart the flow as we described before) are given by the collapse of a curve with only two triple junctions going to coincide.*

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bounded curvature $\Longleftrightarrow$ no collapse of regions

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If **M1** *holds and the network is general, as* $t \to T$*, there exists a* unique *limit non–regular network, with multiple points or even with triple junctions not satisfying* 120 *degrees condition, to which the previous "restarting theorem" can be applied to continue the flow.*

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The number of singular times is finite. If no boundary curves collapse, the flow is definitely smooth and the evolving network converges (asymptotically) to a Steiner (minimal) configuration connecting the fixed endpoints.

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Then, as the classification of self–shrinking networks with at most two triple junctions is complete, some special cases of flows of networks with "few" triple junctions can be fully analyzed.

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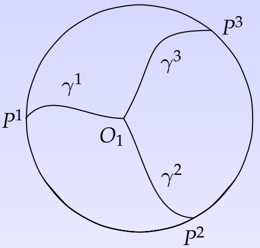

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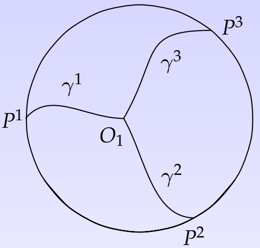
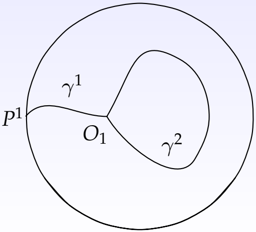
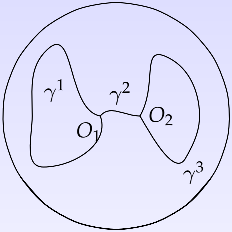


The Spoon – A. Pluda



2 triple junctions – CM, M. Novaga, A. Pluda

The Eyeglasses



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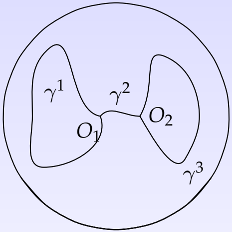
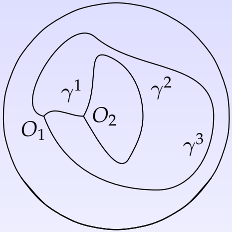
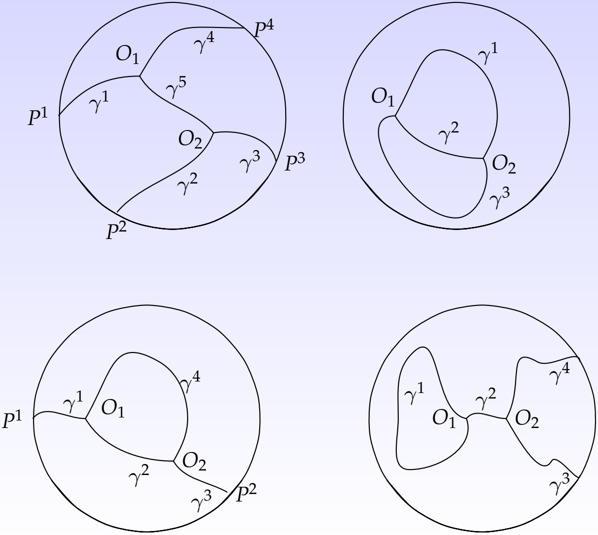
The Eyeglasses and... the Broken Eyeglasses





2 triple junctions – CM, M. Novaga, A. Pluda

The “Steiner”, Theta, Lens and Island



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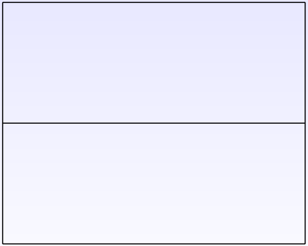
Generic (stable) singularities and generic flows (generic uniqueness, stability, etc.)

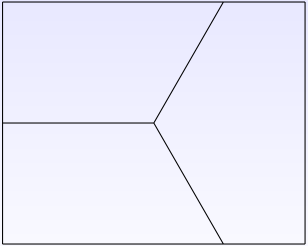
A regular shrinkers gallery (*J. Hättenschweiler – Tom Ilmanen*)

All the following examples are obtained by numeric analysis.
Only the shrinkers with at most one bounded region were completely classified, by Chen and Guo, who proved rigorously also their existence.
All of them have at least one axis of symmetry, we do not know of examples without any symmetries at all.
A natural conjecture is that the number of regular shrinkers is finite (up to rotation).

No regions

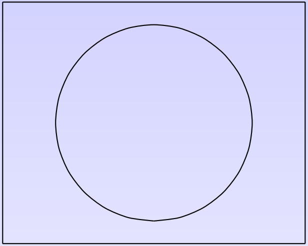


Line

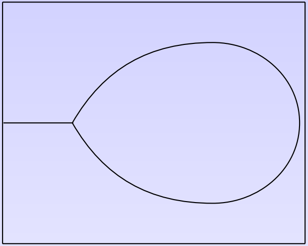


Triod

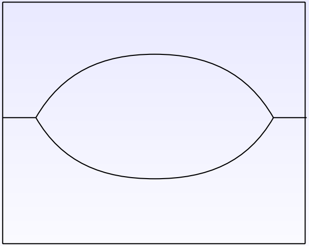
1 region



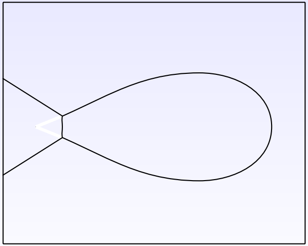
Circle



Spoon

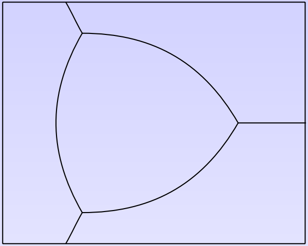


Lens

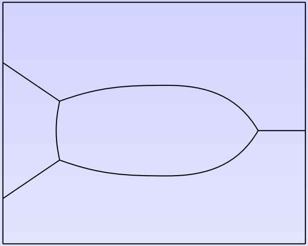


Fish

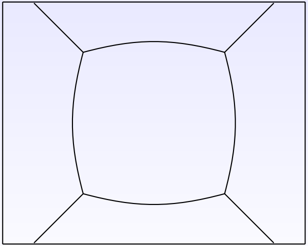
1 region



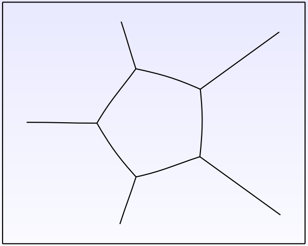
3–ray star



Rocket

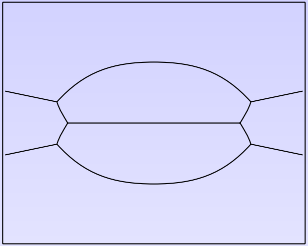


4–ray star

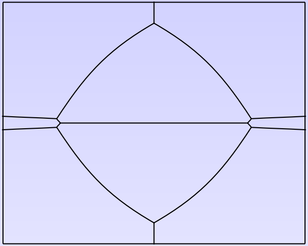


5–ray star

2 regions

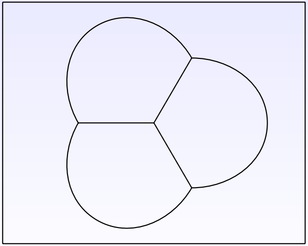


Cisgeminate eye

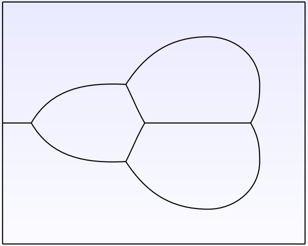


Cisgeminate 4–ray star

3 regions

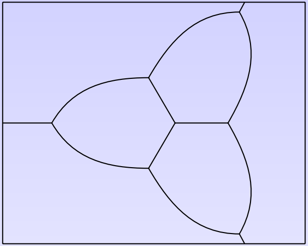


Mercedes–Benz

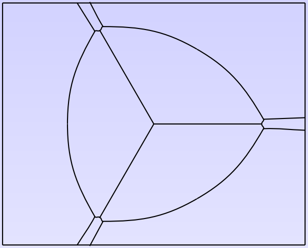


1–ray Mercedes–Benz

3 regions

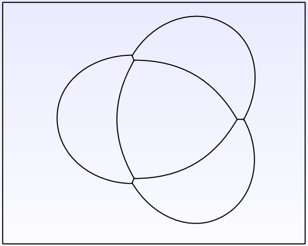


3–ray Mercedes–Benz

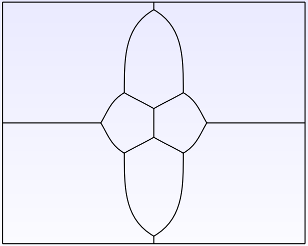


Cisgeminate 3–ray star

4 regions

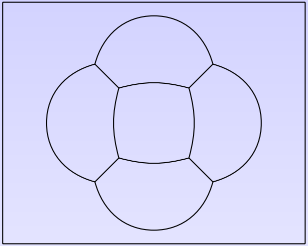


3–leaf clover

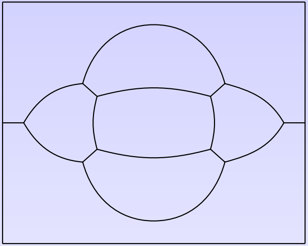


2–ray 2–floc

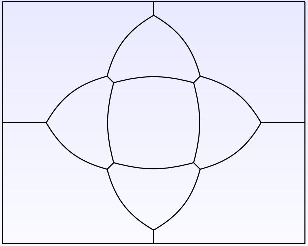
5 regions



4–leaf clover

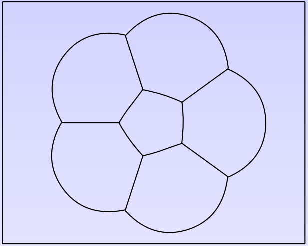


2–ray 4–leaf clover

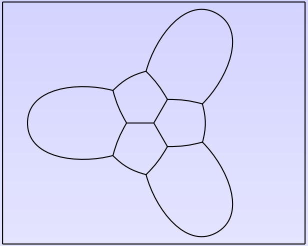


4–petal flower

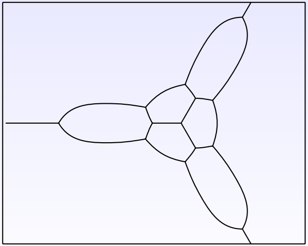
6 regions



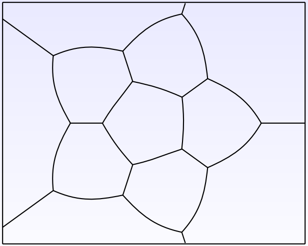
5–leaf clover



3–floc

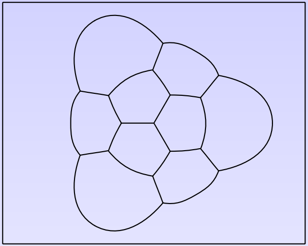


3–ray three–floc

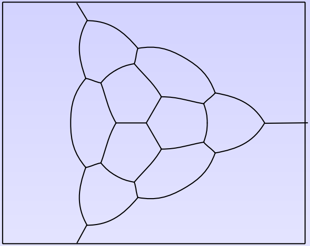


5–petal flower

9 regions

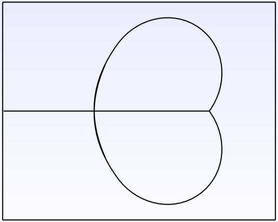


9–floc

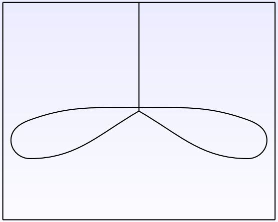


3–ray 9–floc

Non–embedded regular shrinkers

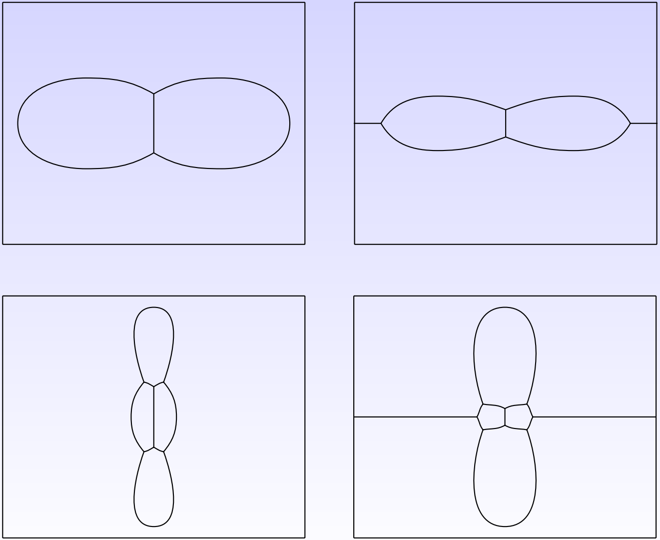


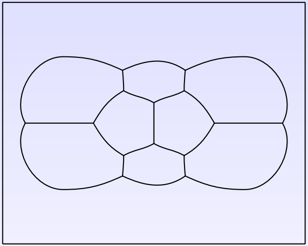
Antispoon



Bowtie

Impossible regular shrinkers





By numerical evidences, there are no regular shrinkers with these topological shapes. The only one whose *non–existence* was rigorously proved is the first one, the Θ–shaped (double cell) shrinker.

The future... clusters of bubbles

Study of the motion by mean curvature of 2–dimensional interfaces in $\mathbb{R}^3$ (a *double–bubble*, for instance)

The future... clusters of bubbles

Study of the motion by mean curvature of 2–dimensional interfaces in $\mathbb{R}^3$ (a *double–bubble*, for instance)

- Short time existence/uniqueness for special initial interfaces by Depner–Garcke–Kohsaka
- Short time existence/estimates for special initial interfaces by Schulze–White

Thanks for your attention