Some results and problems about evolutions of geometric structures

CARLO MANTEGAZZA

The Ricci flow

At the end of ’70s–beginning of ’80s the study of Ricci and Einstein tensors from an analytic point of view gets a strong interest, for instance in the works (static) of Dennis DeTurck. A proposal of investigation of a family of flows, among them the Ricci flow, was done by Jean–Pierre Bourguignon (”Ricci curvature and Einstein metrics”, Lecture Notes in Math 838, 1981). In 1982 Richard Hamilton defines and studies the Ricci flow, that is, the system of partial differential equations

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)}$$

describing the evolution $g(t)$ of the metric of a Riemannian manifold.

R. Hamilton – “Three–manifolds with positive Ricci curvature”, *Journal of Differential Geometry* **17**, 1982, 255–306.

The mean curvature flow

Mean curvature flow occurs in the description of the interface evolution in certain physical models, indeed, one can date the genesis of the problem to a paper of Mullins in 1956. The topic received a strong interest in the second part of ’80s and ’90s, after the works of Gerhard Huisken and Michael Gage – Richard Hamilton.

Let $\varphi_0 : M \to \mathbb{R}^n$ be a smooth hypersurface of $\mathbb{R}^n$. Its mean curvature flow is the system of partial differential equations

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases}$$

where $\mathrm{H}(p,t)$ is the mean curvature vector of the hypersurface $M_t = \varphi_t(M)$ at the point $p \in M$, where $\varphi_t = \varphi(\cdot, t)$.

G. Huisken – “Flow by mean curvature of convex surfaces into spheres”, *Journal of Differential Geometry* **20**, 1984, 237–266.

Both are a sort of geometric heat equations, indeed, the Ricci tensor can be expressed as

$$Ric_g = -\frac{1}{2}\Delta g + \mathrm{LOT}$$

with an appropriate choice of local coordinates, and the mean curvature vector is given by

$$\mathrm{H}(p,t) = \Delta\varphi_t(p,t)$$

where Δ is the Laplace–Beltrami operator on M with the metric induced by the immersion φ_t.

It can be actually shown that these are quasilinear, (degenerate) parabolic systems of PDE on manifolds. They possess a unique solution for small times if the initial manifolds are compact. In addition, the solutions satisfy comparison principles and derivative estimates similar to the case of parabolic equations in Euclidean space. Unfortunately, it is well known that the solutions exist in general only in a finite time interval. This means that singularities, for geometric or analytic reasons, develop. The study of such singularities is the key point in the subject of geometric evolutions.

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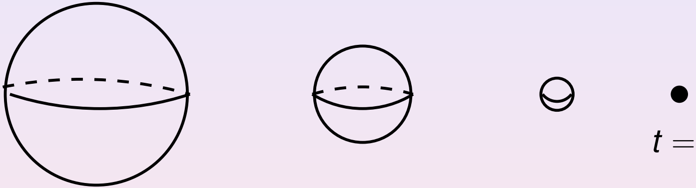

$t = 1/4$

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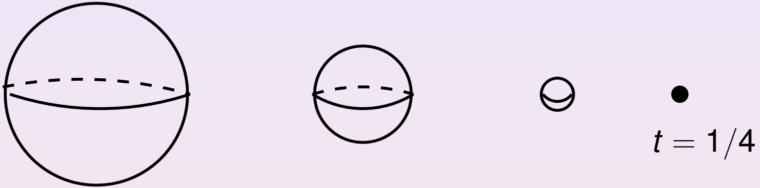
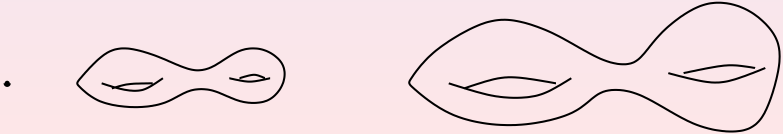


Hyperbolic surface (constant negative curvature):
$g(t) = (1 + 4t)g_0$.



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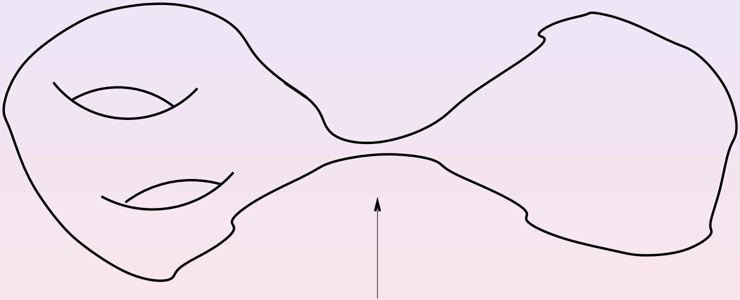

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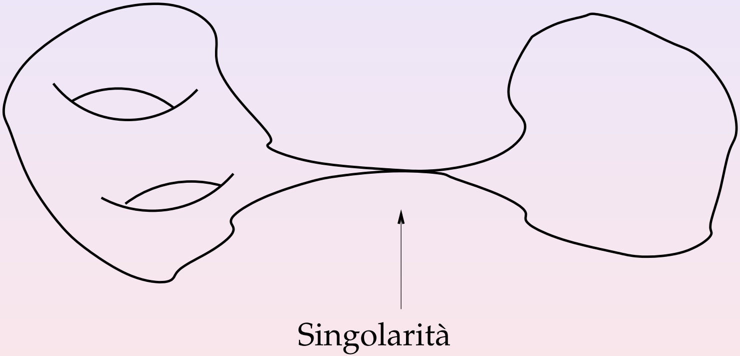

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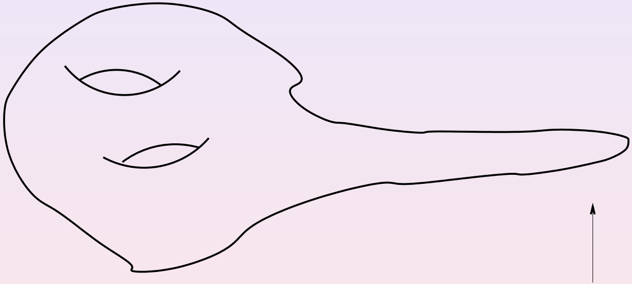

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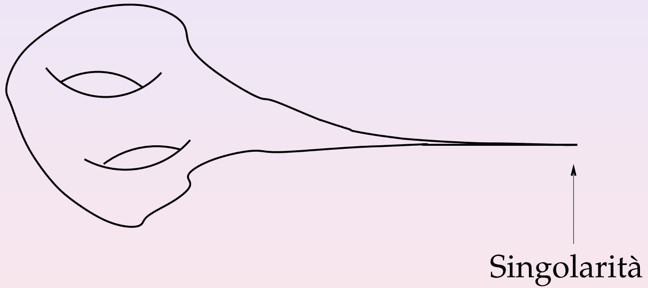

Generalizations of Ricci flow: the renormalization group flow

The *renormalization group* flow arises in modern theoretical physics as a method to investigate the changes of a system viewed at different distance scales. Anyway, it still lacks of a strong mathematical foundation and it is defined by a formal flow of metric on a manifold satisfying the evolution equation

$$\frac{\partial g_{ij}(t)}{\partial \tau} = -\beta_{ij}(g(t))\,,$$

for some functions β_{ij} depending on the metric, the curvature and its derivatives.

In the "perturbative regime" (that is, when $a|Riem(g)| \ll 1$) the functions β_{ij} can be expanded in powers of a,

$$\frac{\partial g_{ij}}{\partial \tau} = -aR_{ij} + o(a)\,,$$

as $a \to 0$.

Hence, the first order truncation (after the substitution $\tau = t/2a$) coincides with the *Ricci flow* $\partial_t g = -2Ric$, as noted by Friedan and Lott (and also Carfora).
It is interesting then to consider also the second order term in the expansion of the beta functions, whose coefficients are quadratic in the curvature and therefore are (possibly) dominating, even when $a|Riem(g)| \to 0$.

The resulting flow is called *two–loop* renormalization group flow

$$\frac{\partial g_{ij}}{\partial \tau} = -aR_{ij} - \frac{a^2}{2} R_{iklm} R_{jstu} g^{ks} g^{lt} g^{mu} \,.$$

Joint work with L. Cremaschi

Theorem (Laura Cremaschi, CM)

Let (M^3, g_0) be a compact, smooth, three–dimensional Riemannian manifold and $a \in \mathbb{R}$. Assume that the sectional curvature K_0 of the initial metric g_0 satisfies

$$1 + 2a\mathrm{K}_0(X, Y) > 0$$

for every point $p \in M^3$ and vectors $X, Y \in T_pM^3$. Then, there exists some $T > 0$ such that the two–loop renormalization group flow has a unique smooth solution $g(t)$ in a maximal time interval $[0, T)$.

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Subsequently generalized to any dimensions by Gimre, Guenther and Isenberg.

Open problems

It is unknown if the condition

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Investigate higher order truncations of the flow (derivatives of the Riemann tensor also appear $\Longrightarrow$ higher order flows).

The Ricci–Bourguignon flow

$$\frac{\partial g}{\partial t} = -2\big(Ric - \rho R g\big)$$

- Einstein flow: $\rho = 1/2$
- Traceless Ricci flow: $\rho = 1/n$
- Schouten flow: $\rho = 1/2(n-1)$
- Ricci flow: $\rho = 0$

It can be seen as an interpolation between the Ricci flow and the Yamabe flow

$$\frac{\partial g(t)}{\partial t} = -2Rg$$

Joint work with G. Catino, L. Cremaschi, Z. Djadli, L. Mazzieri

General Results:

- short time existence and uniqueness for any metric on M compact, if $\rho < 1/2(n-1)$
- blow–up of the curvature at a singularity
- preservation of positive scalar curvature
- preservation of positive Riemann operator
- easier classification of solitons (see later) when $\rho \neq 0$ (easier than for Ricci flow), in particular when $n = 3$

When $n = 3$ also:

- preservation of positive Ricci tensor
- preservation of positive sectional curvature
- Hamilton–Ivey estimate

Open problems

- Schouten case $\rho = 1/2(n-1)$ very interesting but critical for the short time existence
- missing the analogue of a monotonicity quantity, like Perelman's one for Ricci flow
- missing an injectivity radius estimate at the scale of the curvature
- no possibility to get a blow–up limit at a singularity

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Conjecture (Laura Cremaschi, CM)

If a compact 3–dimensional Riemannian manifold has positive Ricci tensor, the (normalized) RB flow deforms it in a sphere (asymptotically).

Key estimates – Uniform in time

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Roundness estimate: there exist constants C and D such that

$$\left| Ric - \frac{1}{3} Rg \right| \leq CR^{1-\delta} + D$$

for some $\delta > 0$.

Gradient estimate: for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\frac{|\nabla R|^2}{R} \leq \varepsilon R^2 + C(\varepsilon) \, .$$

Ricci solitons

In several cases the asymptotic profile of a singularity of the Ricci flow is given by a so called Ricci soliton. They are Riemannian manifolds (M, g) such that there exists a smooth function $f : M \to \mathbb{R}$ and a constant $\lambda \in \mathbb{R}$ satisfying

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They describe selfsimilar solutions of the Ricci flow and their study and classification is necessary to “continue” the flow after a singularity, performing a surgery, in order to get geometric conclusions.

Joint work with G. Catino and L. Mazzieri (et alt.)

We obtained several classifications results for Ricci solitons, mainly shrinking and steady, in low dimensions ($n = 2, 3$) or in general dimension with positive Ricci tensor, under various hypotheses on some (derived) curvature tensor, for instance,

- null Weyl tensor, that is, locally conformally flatness of the manifold
- null Cotton tensor (with M. Rimoldi and S. Mongodi)
- null Bach tensor (with H.-D. Cao and Q. Chen)

These results are actually symmetry result, showing that actually the solitons share rotational symmetry. This then leads to their full classification.

Moreover, we investigate more deeply the LCF condition also for “ancient” solutions of the Ricci flow, that are important in the cases where it is not possible to conclude that the asymptotic profile of a singularity is a Ricci soliton. Our result is that actually under such hypothesis the two classes coincide, then, for instance in low dimension they can be classified.

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We underline that actually one of the major open problems for Ricci flow is to classify the ancient solutions in dimension three.

Einstein–type manifolds

A natural generalization of the concept of Ricci solitons (already appeared in other fields, some related to physics) is the family of the so called *Einstein–type* manifold (a Ricci soliton is already a generalization of an Einstein manifold). They are Riemannian manifolds (M, g) such that there exists smooth functions $f, \mu : M \to \mathbb{R}$ and a constant $\alpha \in \mathbb{R}$ satisfying

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As a special case, choosing $\alpha = 0$ and $\mu = \rho R + \lambda$, for constants $\rho, \lambda \in \mathbb{R}$, one gets the solitons for the Ricci–Bourguignon flow

$$Ric - \rho R g + \nabla^2 f = \lambda g$$

describing the selfsimilar solutions of such flow.

We generalized several results for Ricci solitons to this more general case (with M. Rimoldi et alt.) and we realize that for the RB–solitons when $\rho \neq 0$ the analysis and classification is easier than for the Ricci flow ($\rho = 0$). In particular for the Schouten flow, which makes its study of special interest.

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Moreover, these techniques, leading to symmetry (rotational) results for manifolds, were recently used by V. Agostiniani e L. Mazzieri to get symmetry results for overdetermined problems for semilinear elliptic PDEs in exterior domains of $\mathbb{R}^n$, transforming the PDE problems to geometric ones, by conformal deformations of the canonical metric of $\mathbb{R}^n$.

Mean Curvature Flow, Ricci Flow and Variations

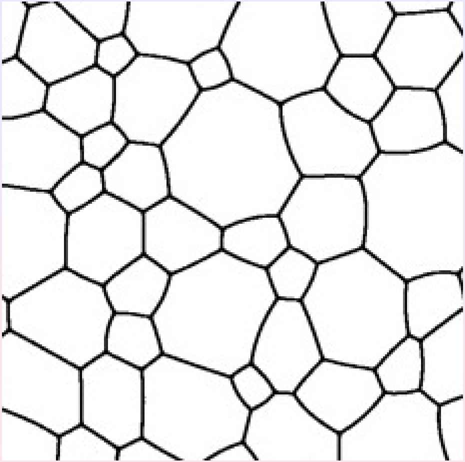
Ricci Solitons and other Einstein–Type Manifolds

Flows of Singular Structures

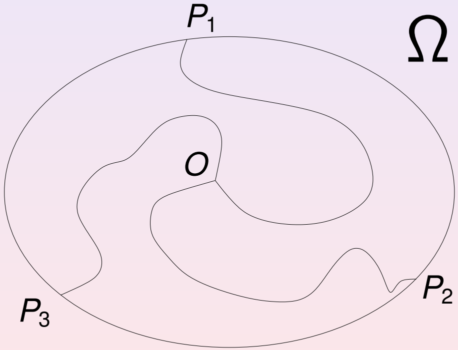
Connecting the Two Flows

Motion of networks of curves by curvature
Joint work with A. Magni, M. Novaga and V. Tortorelli

After the works of Huisken et alt. about the mean curvature flow of hypersurfaces, weak definitions of mean curvature flow of even any *closed set* in the plane appeared. The techniques to study such weak evolutions are no more the ones of differential geometry but more variational and the results obviously weaker. We were interested to continue to use the ideas of the “parametric” approach even if the evolving set was singular, then we chose to study the possibly “least singular” set: a network of curves in the plane, connected by triple junctions forming 120 degrees among the curves.



We concentrated on the local problem, that is, the study of the evolution by curvature of the simplest network of three curves with fixed endpoints and a single triple junction, called a *triod*.



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Recently T. Ilmanen, A. Neves and F. Schulze were able to deal with a full network and also consider the continuation of the flow after a singularity (collapse of one or more curves).

Open problem

Generalizing the techniques and in particular the estimates to the motion of 2–dimensional interfaces in $\mathbb{R}^3$.

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- A short time existence/uniqueness for special initial interfaces is already present in literature.
- Basic computations and estimates with A. Magni and M. Novaga – Work in progress.

A weak flow tangent to Ricci flow
Joint work with N. Gigli

Let (M, g) be a compact Riemannian manifold, $\mathscr{P}(M)$ the space of Borel probability measures on M and let $\mathrm{K}_t : \mathscr{P}(M) \to \mathscr{P}(M)$ be the heat semigroup. Given a couple of points $p, q \in M$ and a smooth curve $s \mapsto \gamma(s)$ connecting them, for every $t > 0$ we have a curve $s \mapsto \gamma_t(s)$ in $\mathscr{P}(M)$ defined by

$$\gamma_t(s) = \mathrm{K}_t(\delta_\gamma(s))\,.$$

Such curves turns out to be absolutely continuous with respect to the Wasserstein distance W_2 on $\mathscr{P}(M)$ so their lengths are well defined. Taking the infimum of such lengths on all smooth curves connecting the points p and q in M, we can define a new distance d_t on the manifold M.

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- The distance d_t comes from a smooth Riemannian metric tensor g_t, for every $t > 0$.
- The dependence on $t \in \mathbb{R}$ is smooth.
- As $t \to 0$ the metrics g_t converge to the original metric tensor g of the manifold M.

Theorem (CM, N. Gigli)

For almost every vector $v \in TM$ there holds

$$\frac{d}{dt} g_t(v, v)\Big|_{t=0} = -2Ric_g(v, v),$$

where g is the original metric on M.

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One can then recover the Ricci flow of a smooth (compact) manifold with successive deformations of the initial metrics by this flow in short intervals of times, then sending to zero the time steps.

Moreover, this result opens the possibility (work in progress) to define the Ricci flow for special classes of nonsmooth spaces, where this flow is defined, that is, metric spaces allowing a well behaved heat kernel.

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A “*mantra*” of geometric flows...

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Ricci flow in dimension n behaves like mean curvature flow in dimension $n-1$

Before Perelman...

Ricci flow
Non variational
Maximum principle

$\xrightarrow{\textit{techniques}}$

$\xleftarrow[\ ]{\textit{examples conjectures}}$

Mean curvature flow
Variational
Maximum principle
Monotonicity formula

After Perelman...

Ricci flow
Almost variational
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Monotonicity of $\mathcal{W}$–functional

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Joint work with R. Müller

After the discovery by Perelman of the monotonicity of the so called $\mathcal{W}$–functional, analogue of Huisken's monotonicity formula for the mean curvature flow, variational methods could also be applied to the study of Ricci flow. We applied then to the Ricci flow the line of analysis of singularity followed in the MCF using Huisken's monotonicity formula.

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Theorem (CM, R. Müller)

Let g_t be a family of metrics on the manifold M evolving by the Ricci flow till the Type–I singular time T. Then, for every $p \in M$ there exists a sequence of times $t_i \nearrow T$ and constants $\lambda_i \to +\infty$ such that the pointed rescaled manifolds $(M, \lambda_i g_{t_i}, p)$ smoothly converge to a Ricci shrinking soliton.

Joint work with R. Müller

In low dimension, this variational method allows to the deal also with the Type–II singularities of mean curvature flow. Analogously (and substained by the estimates), it should also work for the Ricci flow in dimensions two and three (and – hopefully – in the critical case of dimension four).

The major obstacle is the lack of an "environment" where the manifolds "live", playing the role of the ambient Euclidean space for mean curvature flow.

Coupling the two flows
Joint work with A. Magni and E. Tsatis

Following some suggestions from theoretical physics, we investigated the situation of an hypersurface moving by mean curvature flow inside a Riemannian manifold evolving by Ricci flow (or the backward Ricci flow or more general flows). Our main interest was to find monotonicity quantities during this flow and selfsimilar solutions (solitons) for such structure.

Coupling the two flows Joint work with A. Magni and E. Tsatis

Unfortunately, a general natural monotonicity quantity was not available, with the exception of the low dimensional motion of a curve in a surface.

Theorem

If $(M, g(t))$ is a family of compact surfaces moving by backward Ricci flow and γ is a curve moving by its curvature inside $(M, g(t))$, we have

$$\frac{d}{dt}\left(\sqrt{T-t}\int_\gamma R\,ds\right) \leq -\sqrt{T-t}\int_\gamma \left|\mathrm{k} - \nabla^\perp \log R\right|^2 R\,ds$$

and the inequality becomes an equality if and only if M is an expanding Ricci soliton with $R > 0$ and $\mathrm{k} = \nabla^\perp \log R$.

Coupling the two flows
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The only other "good" case is when the ambient is a Ricci soliton.

Theorem

If $(M^{n+1}, g(t))$ is a shrinking or steady Ricci soliton in the time interval $(-\infty, T)$ and $f : M^{n+1} \to \mathbb{R}$ is its "potential function", then, the Huisken's integral

$$\int_N \frac{e^{-f}}{(T-t)^{n/2}}\, d\mu_t$$

of a compact hypersurface N moving by mean curvature inside $(M, g(t))$, is monotone nonincreasing for every $t \in (-\infty, T)$.

Thanks for your attention