5

Conclusions and Research Directions

From the material of the previous lectures, the analysis of singularity formation and the classification of their asymptotic shape is almost complete for some classes of hypersurfaces. For others it seems difficult and quite far. We collect here some known facts and we discuss some research directions and related problems.

5.1 Curves in the Plane

5.1.1 Embedded Curves

A closed, smooth and embedded curve in the plane evolves, remaining embedded, without developing any singularity till it becomes convex. Then it shrinks smoothly to a point in finite time, becoming asymptotically (exponentially fast) round.

Gage, Hamilton, Grayson [48, 49, 50, 53].

5.1.2 General Curves

A closed, smooth curve immersed in the plane at a type I singularity has a blow up limit which is a finite superposition of lines, $\mathbb{S}^1$ and Abresch–Langer curves, possibly with multiplicities larger than one.

In the case that the blow up limit is compact, the curve shrinks to a point at the singular time.

If the singularity is of type II, the only possible blow up limit arising from the Hamilton's procedure is the grim reaper (possibly with multiplicity larger than one).

Abresch & Langer, Angenent, Altschuler, Grayson, Hamilton, Huisken, Stone [1, 3, 5, 14, 15, 16, 53, 64, 69, 119].

5.2 Hypersurfaces

5.2.1 Entire Graphs

The graph of a locally Lipschitz, entire function $u : \mathbb{R}^n \to \mathbb{R}$ has a smooth, global mean curvature flow, remaining a graph, for every positive time.

Ecker & Huisken [39, 40].

5.2.2 Convex Hypersurfaces

A compact, smooth and convex initial hypersurface becomes immediately *strictly* convex and shrinks smoothly to a point in finite time, becoming asymptotically (exponentially fast) round.

Huisken [67].

5.2.3 Embedded Mean Convex Hypersurfaces

A compact, smooth, embedded, n–dimensional initial hypersurface with $\mathrm{H} \geq 0$ evolves remaining embedded and becoming immediately *strictly* mean convex.
If we have a type I singularity we can produce a blow up limit which is one among $\mathbb{S}^m \times \mathbb{R}^{n-m}$ for $m \in \{1, \dots, n\}$.
If the singularity is of type II, the only possible blow up limits arising by the Hamilton's (modified) procedure are the products of an m–dimensional, strictly convex, translating mean curvature flow with bounded curvature, with a factor $\mathbb{R}^{n-m}$, for $m \in \{2, \dots, n\}$.
Notice that the grim reaper times $\mathbb{R}^{n-1}$ cannot be among the possible blow up limits (White [126]).

Hamilton, Huisken, Sinestrari, Stone, White [64, 69, 75, 76, 119, 126, 127].

5.2.4 Two–Convex Hypersurfaces

The class of two–convex hypersurfaces stays in the middle between the classes of mean convex and convex ones, we recall that $M \subset \mathbb{R}^{n+1}$ is (weakly) two–convex if the sum of the two smallest eigenvalues of the second fundamental form is nonnegative at every point (in particular this implies $\mathrm{H} \geq 0$). In dimension two this class clearly coincides with the class of the mean convex surfaces.
This condition, for a smooth, compact, initial hypersurface, is preserved under the mean curvature flow as we saw in Proposition 2.5.10 and actually after some arbitrarily small positive time, there exists a constant $\alpha > 0$ such that $\lambda_1 + \lambda_2 \geq \alpha\mathrm{H}$ at every point of M and for every positive time (here $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ are the eigenvalues of A).
Being this condition invariant by rescaling, it must also be satisfied by every blow up limit and this implies that these latter at a type I singularity can only be the sphere $\mathbb{S}^n$ or the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$.
At a type II singularity only a strictly convex, translating blow up limit $\widetilde{M}$ is possible, since the product of a factor $\mathbb{R}^{n-m}$ with a strictly convex, m–dimensional, translating hypersurface N of $\mathbb{R}^{m+1}$, does not satisfy the condition $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 \geq \alpha\widetilde{\mathrm{H}}$. If $m < n-1$ this is obvious, if $m = n-1$ it is a consequence of the fact that the infimum of the ratio of the minimum eigenvalue of the second fundamental form of N with its mean curvature $\lambda^N_{\min}/\mathrm{H}^N$ must be zero, otherwise by Theorem 4.1.9 the hypersurface N of $\mathbb{R}^n$ is compact, hence cannot be translating. Thus, the blow up limit $\widetilde{M}$ would have $\widetilde{\lambda}_1 = 0$ (because of the flat factor $\mathbb{R}$) and

$$\liminf_{\widetilde{M}} \widetilde{\lambda}_2/\widetilde{\mathrm{H}} = \liminf_{N} \lambda^N_{\min}/\mathrm{H}^N = 0\,,$$

which is clearly in contradiction with $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 \geq \alpha\widetilde{\mathrm{H}}$.

The interest in the special class of two–convex hypersurfaces is related to the possibility, fully exploited by Huisken and Sinestrari [77], to perform a surgery procedure in order to continue the flow after the singular time, analogous to the one introduced by Hamilton [66] for the Ricci flow.

Huisken & Sinestrari [77].

5.2.5 General Hypersurfaces

About the evolution of a generic compact, smooth, n–dimensional, initial hypersurface we can only say that if it is initially embedded, it stays embedded, when it develops a type I singularity we can produce a (possibly flat) homothetically shrinking hypersurface as a blow up limit, nonflat in the embedded case. If the singularity is of type II then Hamilton's procedure gives a blow up limit which is an eternal flow with bounded curvature, such that $|\mathrm{A}|$ achieves its absolute maximum at some point in space and time.

For some special classes of hypersurfaces it is possible to reduce the family of possible blow up limits at a singularity. For instance, starshaped hypersurfaces [112] or rotationally symmetric ones [4].

Angenent, Ecker, Huisken, Ilmanen, White [17, 37, 40, 69, 80, 81, 124, 126, 127].

5.3 Mean Curvature Flow with Surgeries

Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth, compact, two–convex hypersurface, with $n \geq 3$. In this section we sketchily describe the surgery procedure by Huisken and Sinestrari at a singular time of its mean curvature flow and some of its geometric consequences. We suggest to consult the survey [111] for more details. All the results of this section come from the paper [77].
We remark that the analogous results in the case of surfaces in the Euclidean space ($n = 2$) is an open problem, see anyway Colding and Kleiner [32].

We already said that the evolving hypersurface remains two–convex and embedded if φ_0 was embedded. Now we discuss a couple of results about the properties of the flow of two–convex hypersurfaces which are essential for the surgery procedure.

Theorem 5.3.1. *Let $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$, be a family of compact, two–convex hypersurfaces evolving by mean curvature. Then, for any $\eta > 0$ there exists a constant C_η such that*

$$|\lambda_1| \leq \eta \mathrm{H} \Longrightarrow |\lambda_i - \lambda_j| \leq c(n)\eta \mathrm{H} + C_\eta, \qquad \forall i, j > 1$$

everywhere on M and for every $t \in [0, T)$, where the constant $c(n) > 0$ depends only on the dimension $n \geq 3$. Here $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of the second fundamental form.

The above inequality is called a *cylindrical estimate* because it shows that, at a point where H is large and λ_1/H is smaller, the second fundamental form is close to the one of a cylinder, since all the eigenvalues are close each other with the exception of λ_1 which is small. Such a property is important because in the surgery procedure one needs to "operate" on regions of the hypersurface which are almost cylinders.

Next, we have the following key inequality for the gradient of the second fundamental form. With respect to the gradient estimates for mean curvature flow already available in the literature, e.g. [28, 40], it should be noticed that this estimate does not depend on the maximum of the curvature in a neighborhood of the point under consideration.

Theorem 5.3.2. *Let $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0, T)$ be a compact, n–dimensional, two–convex hypersurface moving by mean curvature flow, of dimension $n \geq 3$. Then, there exist constants $C = C(n)$ and $D = D(n, \varphi_0)$ such that the flow satisfies the uniform estimate*

$$|\nabla \mathrm{A}|^2 \leq C|\mathrm{A}|^4 + D\,,$$

for every $t \in [0, T)$.

Once the estimate for $|\nabla \mathrm{A}|$ is obtained, it is easy to obtain similar estimates for all the higher order derivatives, as well as the time derivatives.

We describe now the construction of the mean curvature flow with surgeries, defined in [77], for two–convex hypersurfaces of dimension $n \geq 3$. The aim of performing surgeries is to define a continuation of the flow past the first singular time and all the subsequent ones until the hypersurface vanishes. Another possibility to do this would be to consider weak solutions of the mean curvature flow by the level sets approach, however, weak solutions have generally low regularity past the singular time and it is difficult to analyze the topological changes passing through a singularity. The flow with surgeries is based on a different strategy: if at a singular time T the whole hypersurface vanishes, then we do nothing and consider the flow terminated at time T. Otherwise, we consider the flow at some time t_0 slightly smaller than T and we cut from the hypersurface $\varphi_{t_0} : M \to \mathbb{R}^{n+1}$ the regions with large curvature replacing them with less curved ones. Such an operation is called a *surgery*. Possibly, in doing that we could disconnect the hypersurface into several components, in this case we restart the flow independently for each component (or sometimes we simply forget some of them when their topology is trivial) until a new singular time is approached. This procedure is repeated till the vanishing of all these independent components.

The rigorous definition of such a procedure involves a precise knowledge of the geometric properties of the regions that are removed and of the ones that are added as replacement. To this purpose, one introduces the notion of *neck*. The precise definition is given in [77]; roughly speaking, a neck is a portion of a hypersurface which is close, up to a homothety and a rigid motion, to a standard cylinder $[a, b] \times \mathbb{S}^{n-1}$. The surgeries which we consider consist of removing a neck and of replacing it with two regions diffeomorphic to disks which fill smoothly the two holes left at the two ends of the neck. We have then two situations: if the remaining hypersurface is still connected this topological operation was the inverse of "adding an handle" to the hypersurface (a connected sum with $\mathbb{S}^{n-1} \times \mathbb{S}^1$), if instead the removal of the neck disconnected the hypersurface in two pieces, what we actually performed was the inverse of the connected sum of the two components.
If after a finite number of surgeries we know the topology of all the remaining components, keeping track of all the operations we performed, we can reconstruct backward the topology of the initial hypersurface.

This program can be actually carried out and we have the following result.

Theorem 5.3.3. *Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a compact, immersed, n–dimensional, two–convex, initial hypersurface, with $n \geq 3$. Then there is a mean curvature flow with surgeries such that, after a finite number of surgeries, all the remaining connected components are diffeomorphic either to $\mathbb{S}^n$ or to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

As we said, this theorem implies that the initial manifold can be recovered (up to diffeomorphisms) adding a finite number of handles to the connected sum of a finite family of components all diffeomorphic to $\mathbb{S}^n$ or to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ (all these topological operations are commutative and associative). Recalling that a connected sum with $\mathbb{S}^n$ leaves the topology unchanged, we obtain the following classification of two–convex hypersurfaces.

Corollary 5.3.4. *Any smooth, compact, n–dimensional, two–convex, immersed hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$ is diffeomorphic either to $\mathbb{S}^n$ or to a finite connected sum of $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Topological results on k–convex hypersurfaces were already known in the literature (see e.g. [128]). However, these results were based on Morse theory and only ensured homotopic equivalence.

Since the only simply connected hypersurface in the family above is the sphere, another consequence (with some extra arguments) of this surgery procedure based on mean curvature flow is the following Schoenflies type theorem for simply connected, two–convex hypersurfaces.

Corollary 5.3.5. *Any smooth, compact, simply connected, n–dimensional, two–convex, embedded hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$ is diffeomorphic to $\mathbb{S}^n$ and bounds a region whose closure is diffeomorphic to the $(n + 1)$–dimensional, standard closed ball.*

The proof of Theorem 5.3.3 is quite long and technical, we only describe the main points and ideas.

As we said in the previous section, the only possible blow up limits of the flow of a two–convex hypersurface are the sphere $\mathbb{S}^n$, the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ and the n–dimensional, strictly convex, translating hypersurfaces.
When the limit is a sphere, this means that at some time the hypersurface became convex, thus, no surgery is necessary in this case.
If the limit is a cylinder, then we already have the right geometric structure to perform a surgery
In the case of a translating hypersurface, corresponding to a type II singularity, we have a paraboloid–like hypersurface, but the cylindrical estimates above tell us that a strip of this paraboloid far from the vertex, where the first eigenvalue of the second fundamental form is smaller compared

to the others, is actually very close to a strip of a cylinder. Hence, in this case we can choose to perform the surgery not at the point where the curvature takes its maximum, but in a region nearby where the curvature is still quite large. After the surgery, the region containing the vertex of the paraboloid will be thrown away, since it is diffeomorphic to a disc (alternatively, one can "close" such a region in order to have a convex hypersurface that will shrink to a point).

An important point in the surgery procedure is being sure that the curvature actually decreases. To achieve this, one has to choose the necks to be removed in such a way that the "radius" in the central part is much smaller than at the ends. In this way, it is possible to "close" smoothly the holes that are formed in the hypersurface, with two convex *caps* with small curvature.
The above estimates in Theorems 5.3.1 and 5.3.3 (hence the two–convexity, which also restricts the family of the possible blow up limits) play a fundamental role to prove the existence of necks suitable for the surgery.

To conclude, there are other two essential technical points to be showed in order to make this procedure effective. First, the estimates, with the same constants, must "survive" every surgery. Second, at every step the volume of the hypersurface have to decrease of a fixed positive amount, this clearly implies that the flow necessarily terminates after a finite number of steps, as the volume is decreasing in the time intervals where the flow is smooth.

5.4 Some Problems and Research Directions

We mention some problems that are recently receiving attention by the community.

- Noncollapsing results in order to exclude multiplicities in the blow up limits and to reduce the possible singularity profiles (discussion at the end of Section 4.4).
  See White [126], Ecker [38] and the recent paper by Sheng and Wang [107] where a direct argument by Andrews [10] is also quoted.

- Surgery in dimension two: the extension of the work by Huisken and Sinestrari to the case of mean convex surfaces in $\mathbb{R}^3$ (see Colding and Kleiner [32]) and surgery without the assumption of two–convexity.

- Generic singularities of the flow, that is, showing that the only singularity profiles of the flow for a generic initial hypersurface are spheres or cylinders (a long–standing conjecture of Huisken).
  See Colding and Minicozzi [29].

Finally, we list here some references to research directions related to the mean curvature flow, present in literature. We do not pretend to be exhaustive, we simply want to suggest some starting points for the interested reader.

5.4.1 Motion of Noncompact Hypersurfaces

Some very nice results about existence for short time and regularity in the large were obtained by Ecker and Huisken [39, 40], in particular, for graphs of functions and by Chou and Zhu [25] for unbounded curves in the plane.

5.4.2 Motion of Hypersurfaces with Boundary

One can consider the mean curvature flow of a hypersurface such that its boundary is fixed or it is forced to have a prescribed angle with another hypersurface, see Huisken [68] or Stahl [116, 117], for instance.

5.4.3 Higher Codimension

See the survey of White [125] and, for instance, the global results of Wang [121], the works of Altschuler and Grayson [3, 5] about the evolution of curves in space and the recent paper by Andrews and Baker [11].

5.4.4 Evolutions by Different Functions of the Curvature

Instead of taking the mean curvature as the normal speed of the hypersurface, one can consider different functions of the curvature, in particular, any expression in the symmetric functions of the eigenvalues of the second fundamental form, see [8], for instance.

Other possibilities are adding "forcing terms" to a function of the curvature, driving the flow of a hypersurface, or considering evolutions in an ambient different from the Euclidean space, like a Riemannian manifold.

5.4.5 Weak Solutions

The literature on weak formulations of mean curvature flow is quite huge, we simply list some of the main papers that established such weak approaches, see [2, 7, 20, 23, 44, 80].