3

Monotonicity Formula and Type I Singularities

In all this lecture $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is the mean curvature flow of an n–dimensional, compact hypersurface in the maximal interval of smooth existence $[0,T)$.

As before we will use the notation $\varphi_t = \varphi(\,\cdot\,,t)$ and $\widetilde{\mathcal{H}}^n$ will be the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ *counting multiplicities*.

3.1 The Monotonicity Formula for Mean Curvature Flow

We show the fundamental monotonicity formula for mean curvature flow, discovered by Huisken in [68] and then generalized by Hamilton in [59, 60].

Lemma 3.1.1. *Let $f : \mathbb{R}^{n+1} \times I \to \mathbb{R}$ be a smooth function. By a little abuse of notation, we denote by $\int_M f\, d\mu_t$ the integral $\int_M f(\varphi(p,t),t)\, d\mu_t(p)$.*
Then the following formula holds

$$\frac{d}{dt}\int_M f\, d\mu_t = \int_M (f_t - \mathrm{H}^2 f + \mathrm{H}\langle \nabla f \,|\, \nu\rangle)\, d\mu_t\,.$$

Proof. Straightforward computation. ☐

If $u : \mathbb{R}^{n+1} \times [0,\tau) \to \mathbb{R}$ is a smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$, by this lemma, we have

$$\begin{aligned}\frac{d}{dt}\int_M u\, d\mu_t &= \int_M (u_t - \mathrm{H}^2 u + \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t \\ &= -\int_M (\Delta^{\mathbb{R}^{n+1}} u + \mathrm{H}^2 u - \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t\,.\end{aligned} \tag{3.1.1}$$

Lemma 3.1.2. *If $\psi : M \to \mathbb{R}^{n+1}$ is a smooth isometric immersion of an n–dimensional Riemannian manifold (M,g), for every smooth function u defined in a neighborhood of $\psi(M)$ we have,*

$$\Delta_g(u(\psi)) = (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle (\nabla u)(\psi) \,|\, \nu\rangle\,,$$

where $(\nabla^2_{\nu\nu} u)(\psi(p))$ is the second derivative of u in the normal direction $\nu(p) \in \mathbb{R}^{n+1}$ at the point $\psi(p)$.

Proof. Let $p \in M$ and choose normal coordinates at p. Then,

$$\begin{aligned}
\Delta_g(u(\psi)) &= \nabla^2_{ii}(u(\psi))\\
&= \nabla_i\left(\frac{\partial u}{\partial y_\alpha}(\psi)\,\frac{\partial \psi^\alpha}{\partial x_i}\right)\\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}(\psi)\,\frac{\partial \psi^\alpha}{\partial x_i}\,\frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha}(\psi)\,\frac{\partial^2 \psi^\alpha}{\partial x_i^2}\\
&= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}(\psi)\,\frac{\partial \psi^\alpha}{\partial x_i}\,\frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha}(\psi)\,h_{ii}\nu^\alpha\\
&= (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle (\nabla u)(\psi)\,|\,\nu\rangle\,,
\end{aligned}$$

where we used the Gauss–Weingarten relations (1.1.1). □

It follows that, substituting $\Delta^{\mathbb{R}^{n+1}} u$ in formula (3.1.1) and using the previous lemma, if the function u is positive we get

$$\begin{aligned}
\frac{d}{dt}\int_M u\,d\mu_t &= -\int_M \left(\Delta_{g(t)}(u(\varphi_t)) + \nabla^2_{\nu\nu}u + \mathrm{H}^2 u - 2\mathrm{H}\langle \nabla u\,|\,\nu\rangle\right) d\mu_t\\
&= -\int_M \left(\nabla^2_{\nu\nu}u + \mathrm{H}^2 u - 2\mathrm{H}\langle \nabla u\,|\,\nu\rangle\right) d\mu_t\\
&= -\int_M \left|\mathrm{H} - \frac{\langle \nabla u\,|\,\nu\rangle}{u}\right|^2 u\,d\mu_t + \int_M \left(\frac{|\nabla^\perp u|^2}{u} - \nabla^2_{\nu\nu}u\right) d\mu_t\,,
\end{aligned}$$

where $\nabla^\perp u$ denotes the projection on the normal space to M of the gradient of u.
Then, assuming that $u : \mathbb{R}^{n+1} \times [0,\tau) \to \mathbb{R}$ is a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ for some $\tau > 0$, the following formula easily follows,

$$\begin{aligned}
\frac{d}{dt}\left[\sqrt{4\pi(\tau-t)}\int_M u\,d\mu_t\right] = &-\sqrt{4\pi(\tau-t)}\int_M |\mathrm{H} - \langle \nabla \log u\,|\,\nu\rangle|^2 u\,d\mu_t \qquad (3.1.2)\\
&-\sqrt{4\pi(\tau-t)}\int_M \left(\nabla^2_{\nu\nu}u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)}\right) d\mu_t
\end{aligned}$$

in the time interval $[0, \min\{\tau, T\})$.
As we can see, the right hand side consists of a nonpositive quantity and a term which is nonpositive if $\frac{\nabla^2_{\nu\nu}u}{u} - \frac{|\nabla^\perp u|^2}{u^2} + \frac{1}{2(\tau-t)} = \nabla^2_{\nu\nu}\log u + \frac{1}{2(\tau-t)}$ is nonnegative.
Setting $v(x,s) = u(x,\tau-s)$, the function $v : \mathbb{R}^{n+1}\times(0,\tau] \to \mathbb{R}$ is a positive solution of the standard *forward* heat equation in all $\mathbb{R}^{n+1}$ and setting $t = \tau - s$ we have $\nabla^2_{\nu\nu}\log u + \frac{1}{2(\tau-t)} = \nabla^2_{\nu\nu}\log v + \frac{1}{2s}$.
This last expression is exactly the Li–Yau–Hamilton 2–form $\nabla^2 \log v + g/(2s)$ for positive solutions of the heat equation on a compact manifold (M,g), evaluated on $\nu\otimes\nu$ (see [59]).
In the paper [59] (see also [95]) Hamilton generalized the Li–Yau differential Harnack inequality in [88] (concerning the nonnegativity of $\Delta \log v + \frac{\dim M}{2s}$) showing that, under the assumptions that the compact manifold (M,g) has parallel Ricci tensor ($\nabla\mathrm{Ric} = 0$) and nonnegative sectional curvatures, the 2–form $\nabla^2 \log v + g/(2s)$ is nonnegative definite (Hamilton's matrix Li–Yau–Harnack inequality). Even if it is not compact, this result also holds in $\mathbb{R}^{n+1}$ with the canonical flat metric (which clearly satisfies the above hypotheses on the curvature), assuming the boundedness in space of the function v (equivalently of u), at every fixed time, see Appendix D for details. Hence, $\nabla^2_{\nu\nu}\log u + \frac{1}{2(\tau-t)} = \left(\nabla^2\log v + g^{\mathbb{R}^{n+1}}_{\mathrm{can}}/(2s)\right)(\nu\otimes\nu) \geq 0$. It follows that, if a smooth solution u of the backward heat equation is bounded in space at every fixed time, the monotonicity formula implies that $\sqrt{4\pi(\tau-t)}\int_M u\,d\mu_t$ is nonincreasing in time.
We resume this discussion in the following theorem by Hamilton [59, 60].

Theorem 3.1.3 (Huisken's Monotonicity Formula – Hamilton's Extension in $\mathbb{R}^{n+1}$)**.** *Assume that for some $\tau > 0$ we have a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ in $\mathbb{R}^{n+1} \times [0, \tau)$, bounded in space for every fixed $t \in [0, \tau)$, then*

$$\frac{d}{dt}\Big[\sqrt{4\pi(\tau - t)}\int_M u\, d\mu_t\Big] \leq -\sqrt{4\pi(\tau - t)}\int_M |\mathrm{H} - \langle \nabla \log u \,|\, \nu\rangle|^2 u\, d\mu_t$$

in the time interval $[0, \min\{\tau, T\})$.

Choosing in particular a *backward* heat kernel of $\mathbb{R}^{n+1}$, that is,

$$u(x, t) = \rho_{x_0,\tau}(x, t) = \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{(n+1)/2}}$$

in formula (3.1.2), we get the standard Huisken's monotonicity formula, as the Li–Yau–Hamilton expression $\nabla^2_{\nu\nu}u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)}$ is identically zero in this case.

Theorem 3.1.4 (Huisken's Monotonicity Formula)**.** *For every $x_0 \in \mathbb{R}^{n+1}$ and $\tau > 0$ we have (see [68])*

$$\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\, d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0 \,|\, \nu\rangle}{2(\tau - t)}\right|^2 d\mu_t$$

in the time interval $[0, \min\{\tau, T\})$.
Hence, the integral $\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\, d\mu_t$ is nonincreasing during the flow in $[0, \min\{\tau, T\})$.

Exercise 3.1.5. Show that for every $x_0 \in \mathbb{R}^{n+1}$, $\tau > 0$ and a smooth function $v : M \times [0, T) \to \mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\, v\, d\mu_t = &-\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0 \,|\, \nu\rangle}{2(\tau - t)}\right|^2 v\, d\mu_t\\ &+\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\,(v_t - \Delta_{g(t)}v)\, d\mu_t\,,\end{aligned}$$

in the time interval $[0, \min\{\tau, T\})$.
In particular if $v : M \times [0, T) \to \mathbb{R}$ is a smooth solution of $v_t = \Delta_{g(t)}v$, it follows

$$\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\, v\, d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau - t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0 \,|\, \nu\rangle}{2(\tau - t)}\right|^2 v\, d\mu_t$$

in $[0, \min\{\tau, T\})$.

3.2 Type I Singularities and the Rescaling Procedure

In the previous lecture we showed that the curvature must blow up at the maximal time T with the following lower bound

$$\max_{p\in M} |\mathrm{A}(p, t)| \geq \frac{1}{\sqrt{2(T - t)}}\,.$$

Definition 3.2.1. Let T be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that we have the upper bound

$$\max_{p\in M} |\mathrm{A}(p, t)| \leq \frac{C}{\sqrt{2(T - t)}}\,,$$

we say that the flow is developing at time T a *type I singularity*.
If such a constant does not exist, that is,

$$\limsup_{t\to T}\max_{p\in M}|\mathrm{A}(p,t)|\sqrt{T-t}=+\infty$$

we say that we have a *type II singularity*.

In this lecture we will deal exclusively with type I singularities and the monotonicity formula will be the main tool for the analysis. The next lecture will be devoted to type II singularities.

From now on, we assume that there exists some constant $C_0>1$ such that

$$\frac{1}{\sqrt{2(T-t)}}\leq\max_{p\in M}|\mathrm{A}(p,t)|\leq\frac{C_0}{\sqrt{2(T-t)}}\,, \tag{3.2.1}$$

for every $t\in[0,T)$.

Let $p\in M$ and $0\leq t\leq s<T$, then

$$|\varphi(p,s)-\varphi(p,t)|=\left|\int_t^s\frac{\partial\varphi(p,\xi)}{\partial t}\,d\xi\right|\leq\int_t^s|\mathrm{H}(p,\xi)|\,d\xi\leq\int_t^s\frac{C_0\sqrt{n}}{\sqrt{2(T-\xi)}}\,d\xi\leq C_0\sqrt{n(T-t)}$$

which implies that the sequence of functions $\varphi(\,\cdot\,,t)$ converges as $t\to T$ to some function $\varphi_T:M\to\mathbb{R}^{n+1}$. Moreover, as the constant C_0 is independent of $p\in M$, such convergence is uniform and the limit function φ_T is continuous. Finally, passing to the limit in the above inequality, we get

$$|\varphi(p,t)-\varphi_T(p)|\leq C_0\sqrt{n(T-t)}\,. \tag{3.2.2}$$

In all the lecture we will denote $\varphi_T(p)$ also by $\widehat{p}$.

Definition 3.2.2. Let $\mathcal{S}$ be the set of points $x\in\mathbb{R}^{n+1}$ such that there exists a sequence of pairs $(p_i,t_i)\in M\times[0,T)$ with $t_i\nearrow T$ and $\varphi(p_i,t_i)\to x$.
We call $\mathcal{S}$ the set of *reachable* points.

We have seen in Proposition 2.2.6 that $\mathcal{S}$ is compact and that $x\in\mathcal{S}$ if and only if, for every $t\in[0,T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center x intersects $\varphi(M,t)$. We show now that $\mathcal{S}=\{\widehat{p}\,|\,p\in M\}$.
Clearly $\{\widehat{p}\,|\,p\in M\}\subset\mathcal{S}$, suppose that $x\in\mathcal{S}$ and $\varphi(p_i,t_i)\to x$, then, by inequality (3.2.2) we have $|\varphi(p_i,t_i)-\widehat{p}_i|\leq C_0\sqrt{n(T-t_i)}$, hence, $\widehat{p}_i\to x$ as $i\to\infty$. As the set $\{\widehat{p}\,|\,p\in M\}$ is closed it follows that it must contain the point x.

We define now a tool which will be fundamental in the sequel.

Definition 3.2.3. For every $p\in M$, we define the *heat density* function

$$\theta(p,t)=\int_M\frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$$

and the *limit heat density* function

$$\Theta(p)=\lim_{t\to T}\theta(p,t)\,.$$

Since M is compact, we can also define the following *maximal heat density* function

$$\sigma(t)=\max_{x_0\in\mathbb{R}^{n+1}}\int_M\frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t \tag{3.2.3}$$

and its limit $\Sigma=\lim_{t\to T}\sigma(t)$.

Clearly, $\theta(p,t) \leq \sigma(t)$ for every $p \in M$ and $t \in [0,T)$ and $\Theta(p) \leq \Sigma$ for every $p \in M$. The function Θ is well defined as the limit exists finite since $\theta(p,t)$ is monotone nonincreasing in t and positive. Moreover, the functions $\theta(\,\cdot\,,t)$ are all continuous and monotonically converging to Θ, hence this latter is upper semicontinuous and nonnegative.

The function $\sigma : [0,T) \to \mathbb{R}$ is also positive and monotone nonincreasing, being the maximum of a family of nonincreasing smooth functions, hence the limit Σ is well defined and finite. Moreover, such family is uniformly locally Lipschitz (look at the right hand side of the monotonicity formula), hence also σ is locally Lipschitz, then by Hamilton's trick 2.1.3, at every differentiability time $t \in [0,T)$ of σ we have the following *maximal* monotonicity formula

$$\sigma'(t) = -\int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \left| \mathrm{H} + \frac{\langle x - x_t \,|\, \nu \rangle}{2(T-t)} \right|^2 d\mu_t \tag{3.2.4}$$

where $x_t \in \mathbb{R}^{n+1}$ is any point where the maximum defining $\sigma(t)$ is attained, that is,

$$\sigma(t) = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \, d\mu_t \,.$$

Remark 3.2.4. Notice that we did not define $\sigma(t)$ as the maximum of $\theta(\,\cdot\,,t)$

$$\max_{p\in M} \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \, d\mu_t$$

which is *taken among* $p \in M$. Clearly, this latter can be smaller than $\sigma(t)$.