2

Evolution of Geometric Quantities

In studying the long term behavior of solutions of parabolic equations and systems, in particular in the analysis of singularities, a basic step is always to obtain a priori estimates. These can be integral or pointwise, the main tool in order to get these latter is the *maximum principle*, in particular in the context of mean curvature flow.

2.1 Maximum Principle

Theorem 2.1.1. *Assume that g_t, for $t \in [0, T)$, is a family of Riemannian metrics on a manifold M, with a possible boundary ∂M, such that the dependence on t is smooth.*
Let $u : M \times [0, T) \to \mathbb{R}$ be a smooth function satisfying

$$\partial_t u \leq \Delta_{g_t} u + g_t\big(\nabla u, X(p, u, \nabla u, t)\big) + b(u)$$

where X and b are respectively a continuous vector field and a locally Lipschitz function in their arguments. Then, suppose that for every $t \in [0, T)$ there exists a value $\delta > 0$ and a compact subset $K \subset M \setminus \partial M$ such that at every time $t' \in (t - \delta, t + \delta) \cap [0, T)$ the maximum of $u(\,\cdot\,, t')$ is attained at least at one point of K (this is clearly true if M is compact without boundary).
Setting $u_{\max}(t) = \max_{p \in M} u(p, t)$ we have that the function $u_{\max}$ is locally Lipschitz, hence differentiable at almost every time $t \in [0, T)$ and at every differentiability time there holds

$$\frac{du_{\max}(t)}{dt} \leq b(u_{\max}(t))\,.$$

As a consequence, if $h : [0, T') \to \mathbb{R}$ is a solution of the ODE

$$\begin{cases} h'(t) = & b(h(t)) \\ h(0) = & u_{\max}(0) \end{cases}$$

for $T' \leq T$, then $u \leq h$ in $M \times [0, T')$.
Moreover, if M is connected and at some time $\tau \in (0, T')$ we have $u_{\max}(\tau) = h(\tau)$ then $u(p, t) = h(t)$ for every $p \in M$ and $t \in [0, \tau]$, that is, $u(\,\cdot\,, t)$ is constant in space for every $t \in [0, \tau]$.

Corollary 2.1.2. *Under the same hypotheses, when M is connected and the function b is nonpositive (in particular if it is identically zero), if the maximum of u is nondecreasing in a time interval I, the function u is constant in $M \times I$.*

The first part of the theorem is a consequence of the following lemma. The last claim, the *strong* maximum principle, is more involved, see the book of Landis [84] for a proof and the extensive discussion in [28, Chapter 12].

Lemma 2.1.3 (Hamilton's Trick [57])**.** *Let $u : M\times(0,T)\to\mathbb{R}$ be a C^1 function such that for every time t, there exists a value $\delta>0$ and a compact subset $K\subset M\setminus\partial M$ such that at every time $t'\in(t-\delta,t+\delta)$ the maximum $u_{\max}(t')=\max_{p\in M}u(p,t')$ is attained at least at one point of K.*
Then, $u_{\max}$ is a locally Lipschitz function in $(0,T)$ and at every differentiability time $t\in(0,T)$ we have

$$\frac{du_{\max}(t)}{dt}=\frac{\partial u(p,t)}{\partial t}$$

where $p\in M\setminus\partial M$ is any interior point where $u(\,\cdot\,,t)$ gets its maximum.

Proof. Fixing $t\in(0,T)$, we have $\delta>0$ and K as in the hypotheses, hence on $K\times(t-\delta,t+\delta)$ the function u is Lipschitz with some Lipschitz constant C. Consider a value $0<\varepsilon<\delta$, then we have

$$u_{\max}(t+\varepsilon)=u(q,t+\varepsilon)\leq u(q,t)+\varepsilon C\leq u_{\max}(t)+\varepsilon C\,,$$

for some $q\in K$, hence,

$$\frac{u_{\max}(t+\varepsilon)-u_{\max}(t)}{\varepsilon}\leq C\,.$$

Analogously,

$$u_{\max}(t)=u(p,t)\leq u(p,t+\varepsilon)+\varepsilon C\leq u_{\max}(t+\varepsilon)+\varepsilon C\,,$$

for some $p\in K$, hence,

$$\frac{u_{\max}(t)-u_{\max}(t+\varepsilon)}{\varepsilon}\leq C\,.$$

With the same argument, considering $-\delta<\varepsilon<0$, we conclude that $u_{\max}$ is a locally Lipschitz function in $(0,T)$, hence differentiable at almost every time.
Suppose that t is one of such times, let p be a point in the nonempty set
$\{p\in M\setminus\partial M\,|\,u(p,t)=u_{\max}(t)\}$.
By Lagrange's theorem, for every $0<\varepsilon<\delta$, $u(p,t+\varepsilon)=u(p,t)+\varepsilon\frac{\partial u(p,\xi)}{\partial t}$ for some ξ, hence

$$u_{\max}(t+\varepsilon)\geq u(p,t+\varepsilon)=u_{\max}(t)+\varepsilon\frac{\partial u(p,\xi)}{\partial t}\,,$$

which implies, as $\varepsilon>0$,

$$\frac{u_{\max}(t+\varepsilon)-u_{\max}(t)}{\varepsilon}\geq\frac{\partial u(p,\xi)}{\partial t}\,.$$

Sending ε to zero, we get $u'_{\max}(t)\geq\frac{\partial u(p,t)}{\partial t}$.
If instead we choose $-\delta<\varepsilon<0$ we get

$$\frac{u_{\max}(t+\varepsilon)-u_{\max}(t)}{\varepsilon}\leq\frac{\partial u(p,\xi)}{\partial t}$$

and when $\varepsilon\to0$, we have $u'_{\max}(t)\leq\frac{\partial u(p,t)}{\partial t}$. Thus, we are done. □

Exercise 2.1.4. Prove that the conclusion of the lemma holds also if the function u is merely locally Lipschitz, provided that all the derivatives involved in the computations there exist.

Proof of Theorem 2.1.1 – First Part. By the previous lemma, the function $u_{\max}$ is locally Lipschitz and letting t be a differentiability time of $u_{\max}$, we have, choosing any $p\in M\setminus\partial M$ such that $u(p,t)=u_{\max}(t)$,

$$\begin{aligned}u'_{\max}(t)=\frac{\partial u(p,t)}{\partial t}&\leq\Delta_{g_t}u+g_t\big(\nabla u,X(p,u,\nabla u,t)\big)+b(u(p,t))\\&\leq b(u(p,t))\\&=b(u_{\max}(t))\,.\end{aligned}$$

Let now $h : [0, T') \to \mathbb{R}$ be as in the hypothesis. We define, for $\varepsilon > 0$, the approximating functions $h_\varepsilon : [0, T'') \to \mathbb{R}$ to be the maximal solutions of the family of ODE's

$$\begin{cases} h'_\varepsilon(t) = & b(h_\varepsilon(t)) \\ h_\varepsilon(0) = & u_{\max}(0) + \varepsilon\,. \end{cases}$$

It is easy to see that, as the function b is locally Lipschitz, there holds $\lim_{\varepsilon\to 0} h_\varepsilon = h$ uniformly on $[0, T' - \delta]$ for any $\delta > 0$. Suppose that at some positive time $u_{\max} > h_\varepsilon$ and set $\bar{t} > 0$ to be the positive infimum of such times (at time zero $u_{\max}(0) = h_\varepsilon(0) - \varepsilon$). Then, $u_{\max}(\bar{t}) = h_\varepsilon(\bar{t})$ and, setting $H_\varepsilon = h_\varepsilon - u_{\max}$, at every differentiability point of $u_{\max}$ in the interval $[0, \bar{t})$ we have $H_\varepsilon(0) = \varepsilon > 0$ and

$$H'_\varepsilon(t) \geq b(h_\varepsilon(t)) - b(u_{\max}(t)) \geq -C(h_\varepsilon(t) - u_{\max}(t)) = -CH_\varepsilon(t)$$

where $C > 0$ is a local Lipschitz constant for b.
Then, $(\log H_\varepsilon)'(t) \geq -C$ and integrating, $\log H_\varepsilon|_0^t \geq -Ct$, that is, $H_\varepsilon(t) \geq H_\varepsilon(0)e^{-Ct} = \varepsilon e^{-Ct}$. In particular, if $t \to \bar{t}$, we conclude $H_\varepsilon(\bar{t}) \geq \varepsilon e^{-C\bar{t}} > 0$ which is in contradiction with $H_\varepsilon(\bar{t}) = 0$. Hence, $u_{\max}(t) \leq h_\varepsilon(t)$ for every $t \in [0, T' - \delta)$ and sending ε to zero, $u_{\max}(t) \leq h(t)$ for every $t \in [0, T' - \delta)$. As $\delta > 0$ was arbitrary, we conclude the proof of the first part of the theorem. $\square$

Exercise 2.1.5. When the function $u_{\max}$ is not differentiable at t, actually one can still say something using the upper derivative, that is the $\limsup$ of the incremental ratios, we call this operator d^+. Prove that

$$\frac{d^+ u_{\max}(t)}{dt} = \sup_{\{p\in M \,|\, u(p,t) = u_{\max}(t)\}} \frac{\partial u(p,t)}{\partial t}\,.$$

Roughly speaking, the $\sup$ and the upper derivative operators can be interchanged.
The same holds for the $\inf$ and the lower derivative defined analogously.
What can be said about the left/right derivatives of $u_{\max}$?

Remark 2.1.6. Clearly, there hold analogous results for the minimum of the solution of the opposite partial differential inequality. Moreover, the maximum principle for elliptic equations easily follows as the special case where all the quantities around do not depend on the time variable t.

2.2 Comparison Principle

Theorem 2.2.1 (Comparison Principle for Mean Curvature Flow). *Let $\varphi : M_1 \times [0, T) \to \mathbb{R}^{n+1}$ and $\psi : M_2 \times [0, T) \to \mathbb{R}^{n+1}$ be two hypersurfaces moving by mean curvature, with M_1 compact and the mean curvature of M_2 uniformly bounded in space and locally in time. Then the distance between them is nondecreasing in time.*

Proof. The distance between the two hypersurfaces $\varphi_t : M_1 \to \mathbb{R}^{n+1}$ and $\psi_t : M_2 \to \mathbb{R}^{n+1}$ at time t is given by $d^\varphi_\psi(t) = \inf_{(p,q)\in M_1\times M_2} |\varphi(p,t) - \psi(q,t)|$. As the mean curvature is uniformly bounded in space and locally in time for both hypersurfaces, this function is locally Lipschitz, hence differentiable almost everywhere, we assume in the following that t is a differentiability point.
This infimum is actually a minimum as M_1 is compact, suppose then that it is positive and let (p_t, q_t) be any pair realizing such minimum.
It is easy to see that, by minimality, the respective tangent hyperplanes $d\varphi_t(T_{p_t}M_1)$ and $d\psi_t(T_{q_t}M_2)$ of the two hypersurfaces, seen as submanifolds of $\mathbb{R}^{n+1}$, have to be parallel. Then we can write locally $\varphi(p,t)$ and $\psi(p,t)$ as graphs of two functions $f(p,t)$ and $h(p,t)$ over one of these tangent spaces for a small interval of time $(t-\varepsilon, t+\varepsilon)$. We can assume that $\langle e_1, \ldots, e_n\rangle \subset \mathbb{R}^{n+1}$ is such tangent space with $\varphi(p_t, t) = (0, f(0,t))$ and $\psi(q_t, t) = (0, h(0,t))$ at time t, moreover $f(0,t) > h(0,t)$. We know, by Exercise 1.3.8 that $f_t = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1+|\nabla f|^2}$ and $h_t = \Delta h - \frac{\mathrm{Hess} h(\nabla h, \nabla h)}{1+|\nabla h|^2}$.
Again, by minimality, the function $x \mapsto f(x,t) - h(x,t)$ has a minimum at $x = 0$, hence, $\Delta f(0,t) -$

$\Delta h(0,t) \geq 0$ and $\nabla f(0,t) = \nabla h(0,t) = 0$, but we saw that for graphs, $\Delta f(0,t) = \mathrm{H}^\varphi(p_t,t)\langle \nu^\varphi(p_t,t)\,|\,e_{n+1}\rangle$ and $\Delta h(0,t) = \mathrm{H}^\psi(q_t,t)\langle \nu^\psi(q_t,t)\,|\,e_{n+1}\rangle$, thus,

$$\langle \mathrm{H}^\varphi(p_t,t)\nu^\varphi(p_t,t) - \mathrm{H}^\psi(q_t,t)\nu^\psi(q_t,t)\,|\,e_{n+1}\rangle = \Delta f(0,t) - \Delta h(0,t) \geq 0\,.$$

Now we have $\frac{\varphi(p_t,t)-\psi(q_t,t)}{|\varphi(p_t,t)-\psi(q_t,t)|} = e_{n+1}$ by construction and, by Lemma 2.1.3, we can conclude that

$$\begin{aligned}
\frac{d}{dt}d^\varphi_\psi(t) &= \frac{\partial}{\partial t}|\varphi(p_t,t) - \psi(q_t,t)| \\
&= \frac{\langle \varphi(p_t,t) - \psi(q_t,t)\,|\,\mathrm{H}^\varphi(p_t,t)\nu^\varphi(p_t,t) - \mathrm{H}^\psi(q_t,t)\nu^\psi(q_t,t)\rangle}{|\varphi(p_t,t) - \psi(q_t,t)|} \\
&= \langle \mathrm{H}^\varphi(p_t,t)\nu^\varphi(p_t,t) - \mathrm{H}^\psi(q_t,t)\nu^\psi(q_t,t)\,|\,e_{n+1}\rangle \\
&\geq 0\,.
\end{aligned}$$

If the minimum is zero, there is nothing to show, obviously the derivative, if it exists, cannot be negative. □

Exercise 2.2.2. Show the following facts for a compact hypersurface moving by mean curvature.

- The diameter of the hypersurface decreases during the flow.
- The circumradius of the hypersurface (the radius of the smallest sphere enclosing the hypersurface) decreases.

Corollary 2.2.3. *Let $\varphi : M_1 \times [0,T) \to \mathbb{R}^{n+1}$ and $\psi : M_2 \times [0,T) \to \mathbb{R}^{n+1}$ be two hypersurfaces moving by mean curvature such that M_1 is compact, M_2 is embedded and $\varphi(M_1,0)$ is strictly "inside" $\psi(M_2,0)$. Then $\varphi(M_1,t)$ remains strictly "inside" $\psi(M_2,t)$ for every time $t \in [0,T)$.*

Proof. This is an easy consequence of the fact that the distance between the two hypersurfaces is nondecreasing, so it cannot get to zero, as it starts positive. Hence, the hypersurface "inside" cannot "touch" the other during the flow. □

Remark 2.2.4. By means of the continuous dependence result in Theorem 1.5.1 one has a slight improvement of the previous corollary, allowing the two hypersurfaces, one "inside" the other, to have common points at the initial time. To prove this fact one can "push" a little inside the initial hypersurface φ_0 along the gradient of the distance function from $\psi(M_2,0)$ in a local small tubular neighborhood (M_1 is compact), then conclude by the above corollary and the continuous dependence of the flow on the initial hypersurface.

By means of the strong maximum principle we can actually show something more, that is, evolving by mean curvature, the distance between two connected hypersurfaces (with at least one compact) with possibly only tangent intersections and such that they "do not cross each other", is always increasing, otherwise they must coincide.
This can be seen by using again the idea of the proof of Theorem 1.5.1, writing the two hypersurfaces as graphs over the initial "external" hypersurface in a small regular tubular neighborhood of this latter and applying the strong maximum principle to the "height" functions representing them. As a preliminary step, one has to consider an "intermediate" hypersurface close enough to the "external" one which stays in its tubular neighborhood for some positive time. We leave the technical details to the reader as an exercise.

In other words, if two connected hypersurfaces (one compact "inside" the other) touch each other at time zero but they are not the same, immediately they get disjoint, at every positive time.

Even more, in the special case of curves in the plane the number of intersections (or of self–intersections) is nonincreasing in time, see [14, 16].

Applying Corollary 2.2.3 to the case that $\varphi(M_2,0)$ is a sphere of radius R, we have the following estimate for the maximal time of smooth existence.

Corollary 2.2.5. *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface. If $\varphi(M,0) \subset B_R(x_0)$ then the flow is contained in $B_R(x_0)$ at every time and $T \leq R^2/(2n)$.*
Hence, the mean curvature flow of every compact immersed hypersurface develops a singularity in finite time.
In particular, if $T_{\max}$ is the maximal time of smooth existence of the flow, then $T_{\max} \leq \operatorname{diam}^2_{\mathbb{R}^{n+1}}[\varphi(M,0)]/2n$.

Proof. We have already seen that a sphere of radius R shrinks to a point with the rule $R(t) = \sqrt{R^2 - 2nt}$, hence at time $t = R^2/(2n)$ its radius gets to zero. As $\varphi(M,t) \subset B_{\sqrt{R^2-2nt}}(x_0)$, at most at time $t = R^2/(2n)$ the evolving hypersurface φ_t must develop a singularity, since at such time it cannot be an immersion.
The last claim is trivial. □

Another consequence of the maximum principle is the following characterization of the points of $\mathbb{R}^{n+1}$ "reached" by the flow at time T, that is, an estimate on the rate of convergence to a limit hypersurface as $t \to T$ (this will be particularly interesting when T is a singular time). Roughly speaking, if a hypersurface moving by mean curvature is "reaching" a point of the Euclidean space at some time, then it cannot stay "too far" from such point in the past.

Proposition 2.2.6. *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be a mean curvature flow and define $\mathcal{S}$ to be the set of points $x \in \mathbb{R}^{n+1}$ such that there exists a sequence of pairs $(p_i, t_i) \in M \times [0,T)$ with $t_i \nearrow T$ and $\varphi(p_i, t_i) \to x$.*
Then, $\mathcal{S}$ is closed (and bounded if M is compact), moreover $x \in \mathcal{S}$ if and only if for every $t \in [0,T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center x intersects $\varphi(M,t)$.

Proof. One implication is obvious.
Suppose that $x \in \mathcal{S}$ and let $d_t(x) = \min_{p\in M} |\varphi(p,t) - x|$, that is, the Euclidean distance from x to the hypersurface at time t.
The function $d_t : [0,T) \to \mathbb{R}$ is obviously locally Lipschitz and at a differentiability time with $d_t(x) > 0$, by the Hamilton's trick 2.1.3, we have

$$d_t'(x) = \frac{\partial}{\partial t}|\varphi(q,t) - x| = \frac{\mathrm{H}(q,t)\langle \nu(q,t) \,|\, \varphi(q,t) - x\rangle}{|\varphi(q,t) - x|}$$

for any point $q \in M$ such that $d_t(x) = |\varphi(q,t) - x|$.
As the closed ball $\overline{B}_{d_t(x)}(x)$ intersects the hypersurface φ_t only on its boundary and the vector $\frac{\varphi(q,t)-x}{|\varphi(q,t)-x|}$ is parallel to the normal $\nu(q,t)$ by minimality, an easy geometric argument on the principal eigenvalues of the second fundamental form shows that

$$\frac{\mathrm{H}(q,t)\langle \nu(q,t) \,|\, \varphi(q,t) - x\rangle}{|\varphi(q,t) - x|} \geq -n/d_t(x)\,.$$

Hence, we conclude that for almost every time $t \in [0,T)$,

$$d_t'(x) \geq -n/d_t(x)$$

if $d_t(x) \neq 0$.
Integrating this differential inequality on $[t,s]$ we get $d_t^2(x) - d_s^2(x) \leq 2n(s-t)$ and by the hypothesis on x we have $d^2_{t_i}(x) \to 0$, hence

$$d_t^2(x) = \lim_{i\to\infty} d_t^2(x) - d^2_{t_i}(x) \leq \lim_{i\to\infty} 2n(t_i - t) = 2n(T-t)$$

which is the thesis of the proposition.
The closure of $\mathcal{S}$ is obvious, if M is compact $\mathcal{S}$ is clearly also bounded by Corollary 2.2.5. □

A very important fact about hypersurfaces moving by mean curvature is the following.

Proposition 2.2.7. *If the initial hypersurface is compact and embedded, then it remains embedded during the flow.*

Proof. Given the mean curvature flow φ_t, if the hypersurface φ_0 is embedded it remains so for a small positive time, otherwise we will have a sequence of points and times, with $\varphi(p_i, t_i) = \varphi(q_i, t_i)$ and $t_i \to 0$, then, extracting a subsequence (not relabeled) such that $p_i \to p$ and $q_i \to q$, either $p \neq q$ so $\varphi(p, 0) = \varphi(q, 0)$, which is a contradiction, or $p = q$. By the smooth existence of the flow, in particular by the nonsingularity of the differential $d\varphi_t(p)$ there exists a ball $B \subset M$ around p such that for $t \in [0, \varepsilon)$ the map $\varphi_t|_B$ is one–to–one, which is in contradiction with the hypotheses.
This short time embeddedness property is also immediate by revisiting the proof of the short time existence theorem, representing the moving hypersurfaces as graphs on the initial one.

This argument also implies that the embeddedness holds in an open time interval, then we assume that $T > 0$ is the first time such that the hypersurface φ_t is no more embedded. The set S of pairs (p, q) with $p \neq q$ and $\varphi(p, T) = \varphi(q, T)$ is a nonempty closed set disjoint from the diagonal in $M \times M$, otherwise φ_T fails to be an immersion at some point in M. Then, we can find a smooth open neighborhood Ω of the diagonal with $\overline{\Omega} \cap S = \emptyset$.
We consider the following quantity,

$$C = \inf_{t\in[0,T]} \inf_{(p,q)\in\partial\Omega} |\varphi(p, t) - \varphi(q, t)| \,,$$

then C is positive, as $\overline{\Omega} \cap S = \emptyset$ and $\partial\Omega$ is compact. We claim that the following function

$$L(t) = \min_{(p,q)\in M\times M\setminus\Omega} |\varphi(p, t) - \varphi(q, t)| \,,$$

is bounded from below by $\min\{L(0), C\} > 0$ on $[0, T]$, this is clearly in contradiction with the fact that S is nonempty and contained in $M \times M \setminus \Omega$.
If at some time $L(t) < C$ it follows that $L(t)$ is achieved by some pairs (p, q) not belonging to $\partial\Omega$, then (p, q) are inner points of $M \times M \setminus \Omega$ and a geometric argument analogous to the one of the comparison Theorem 2.2.1 shows that $\frac{dL(t)}{dt} \geq 0$, hence $L(t)$ is nondecreasing in time. This last fact clearly implies the claim. ☐

Remark 2.2.8. Theorem 2.2.1 and Proposition 2.2.7 also hold if the involved hypersurfaces are not compact, with some additional assumptions on the behavior at infinity (for instance, uniform bounds on the curvature), the analysis is anyway more complicated.

2.3 Evolution of Curvature

Now we derive the evolution equations for g, ν, μ, Γ^i_{jk}, A and H. Computing as in Section 1.2, we have

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \varphi}{\partial x_i} \,\middle|\, \frac{\partial \varphi}{\partial x_j} \right\rangle \\
&= \left\langle \frac{\partial (\mathrm{H}\nu)}{\partial x_i} \,\middle|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \left\langle \frac{\partial (\mathrm{H}\nu)}{\partial x_j} \,\middle|\, \frac{\partial \varphi}{\partial x_i} \right\rangle \\
&= \mathrm{H} \left[\left\langle \frac{\partial \nu}{\partial x_i} \,\middle|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \left\langle \frac{\partial \nu}{\partial x_j} \,\middle|\, \frac{\partial \varphi}{\partial x_i} \right\rangle \right] \\
&= -\, 2\mathrm{H} h_{ij} \,,
\end{aligned}$$

where we used the Gauss–Weingarten relations (1.1.1) in the last step.
It follows that the canonical measure μ associated to the hypersurface satisfies

$$\frac{d}{dt}\mu = -\mathrm{H}^2 \mu \,, \tag{2.3.1}$$

as

$$\frac{\partial}{\partial t} \sqrt{\det(g_{ij})} = \frac{\sqrt{\det(g_{ij})}\, g^{ij} \frac{\partial}{\partial t} g_{ij}}{2} = -\mathrm{H}^2 \sqrt{\det(g_{ij})} \,.$$

Differentiating the formula $g_{is}g^{sj} = \delta_i^j$ we get

$$\frac{\partial}{\partial t}g^{ij} = -g^{is}\frac{\partial}{\partial t}g_{sl}g^{lj} = 2\mathrm{H}g^{is}h_{sl}g^{lj} = 2\mathrm{H}h^{ij}\,.$$

The time derivative of the normal ν is determined by

$$\left\langle \frac{\partial \nu}{\partial t} \,\middle|\, \frac{\partial \varphi}{\partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial t \partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial (\mathrm{H}\nu)}{\partial x_i} \right\rangle = -\frac{\partial \mathrm{H}}{\partial x_i}\,,$$

hence,

$$\frac{\partial \nu}{\partial t} = -\frac{\partial \mathrm{H}}{\partial x_l}g^{ls}\frac{\partial \varphi}{\partial x_s}$$

or simply (with a little abuse of notation), identifying the tangent space at every point $p \in M$ with the tangent hyperplane in $\mathbb{R}^{n+1}$ given by $d\varphi_t(T_pM)$,

$$\frac{\partial \nu}{\partial t} = -\nabla \mathrm{H}\,.$$

Finally, the derivative of the Christoffel symbols is

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^i_{jk} =& \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}g_{kl}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial t}g_{jl}\right) - \frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
&+ \frac{1}{2}\frac{\partial}{\partial t}g^{il}\left\{\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} - \frac{\partial}{\partial x_l}g_{jk}\right\}\\
=& \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
&+ \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial t}g_{kz}\Gamma^z_{jl} + \frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} + \frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl} + \frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} - \frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl} - \frac{\partial}{\partial t}g_{kz}\Gamma^z_{jl}\right\}\\
&- \frac{1}{2}g^{is}\frac{\partial}{\partial t}g_{sz}g^{zl}\left\{\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} - \frac{\partial}{\partial x_l}g_{jk}\right\}\\
=& \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
&+ g^{il}\frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} - g^{is}\frac{\partial}{\partial t}g_{sz}\Gamma^z_{jk}\\
=& \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\
=& - g^{il}\left\{\nabla_j(\mathrm{H}h_{kl}) + \nabla_k(\mathrm{H}h_{jl}) - \nabla_l(\mathrm{H}h_{jk})\right\}\\
=& - h^i_k\nabla_j\mathrm{H} - h^i_j\nabla_k\mathrm{H} + h_{jk}\nabla^i\mathrm{H} - \mathrm{H}(\nabla_j h^i_k + \nabla_k h^i_j - \nabla^i h_{jk})\,.
\end{aligned}$$

Summarizing, we have

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= -\,2\mathrm{H}h_{ij}\\
\frac{\partial}{\partial t}g^{ij} &= 2\mathrm{H}h^{ij}\\
\frac{\partial}{\partial t}\nu &= -\,\nabla\mathrm{H}\\
\frac{\partial}{\partial t}\Gamma^i_{jk} &= \nabla\mathrm{H} * \mathrm{A} + \mathrm{H} * \nabla\mathrm{A} = \nabla\mathrm{A} * \mathrm{A}\,.
\end{aligned}$$

Proposition 2.3.1. *The second fundamental form satisfies the evolution equation*

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2\mathrm{H}h_{il}g^{ls}h_{sj} + |\mathrm{A}|^2 h_{ij}\,. \tag{2.3.2}$$

It follows

$$\frac{\partial}{\partial t}h_i^j = \Delta h_i^j + |\mathrm{A}|^2 h_i^j\,, \tag{2.3.3}$$

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,. \tag{2.3.4}$$

Proof. Keeping in mind the Gauss–Weingarten relations (1.1.1) and the previous evolution equations, we compute

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}\left\langle \nu \,\middle|\, \frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle \\
&= \left\langle \nu \,\middle|\, \frac{\partial^2(\mathrm{H}\nu)}{\partial x_i\partial x_j}\right\rangle - \left\langle \nabla\mathrm{H} \,\middle|\, \frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle \\
&= \frac{\partial^2\mathrm{H}}{\partial x_i\partial x_j} - \mathrm{H}\left\langle \nu \,\middle|\, \frac{\partial}{\partial x_i}\left(h_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s}\right)\right\rangle \\
&\quad - \left\langle \frac{\partial\mathrm{H}}{\partial x_l}g^{ls}\frac{\partial\varphi}{\partial x_s} \,\middle|\, \Gamma_{ij}^k\frac{\partial\varphi}{\partial x_k} + h_{ij}\nu\right\rangle \\
&= \frac{\partial^2\mathrm{H}}{\partial x_i\partial x_j} - \mathrm{H}h_{jl}g^{ls}\left\langle \nu \,\middle|\, \frac{\partial^2\varphi}{\partial x_i\partial x_s}\right\rangle - \Gamma_{ij}^k\frac{\partial\mathrm{H}}{\partial x_k} \\
&= \nabla_i\nabla_j\mathrm{H} - \mathrm{H}h_{il}g^{ls}h_{sj}\,.
\end{aligned}$$

Then using Simons' identity (1.1.4) we conclude

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2\mathrm{H}h_{il}g^{ls}h_{sj} + |\mathrm{A}|^2 h_{ij}\,.$$

The other equations follow from straightforward computations, as $\frac{\partial}{\partial t}g^{ij} = 2\mathrm{H}h^{ij}$. □

Remark 2.3.2. Since it will be useful in the sequel, we see in detail the evolution equations in the special one–dimensional case of the flow by curvature $\gamma : \mathbb{S}^1 \times [0,T)$ of a closed curve in the plane.
We denote by θ the parameter on $\mathbb{S}^1$ and by $s = s(\theta,t) = \int_0^\theta |\partial_\theta\gamma(\theta,t)|\,d\theta$ the arclength, $\tau = \partial_s\gamma$ is the tangent unit vector and $\nu = \mathrm{R}\tau$ is the unit normal, where $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of an angle of $\pi/2$, finally $k = \langle \partial_s\tau \,|\, \nu\rangle$ is the curvature.
Notice that $\partial_s = |\gamma_\theta|^{-1}\partial_\theta$ and that the evolution equation reads $\partial_t\gamma = k\nu = \partial^2_{ss}\gamma$. Then, we easily get the following commutation rule $\partial_t\partial_s = \partial_s\partial_t + k^2\partial_s$ which implies

$$\begin{aligned}
\partial_t\tau &= \partial_t\partial_s\gamma = \partial_s\partial_t\gamma + k^2\partial_s\gamma = \partial_s(k\nu) + k^2\tau = k_s\nu \\
\partial_t\nu &= \partial_t(\mathrm{R}\tau) = \mathrm{R}\,\partial_t\tau = -k_s\tau \\
\partial_t k &= k_{ss} + k^3\,.
\end{aligned}$$

Now we deal with the covariant derivatives of A.

Lemma 2.3.3. *The following formula for the interchange of time and covariant derivative of a tensor T holds*

$$\frac{\partial}{\partial t}\nabla T = \nabla\frac{\partial}{\partial t}T + T * \mathrm{A} * \nabla\mathrm{A}\,.$$

Proof. We suppose that $T = T_{i_1\dots i_k}$ is a covariant tensor, the general case is analogous, as it will be clear by the following computation,

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_j T_{i_1\dots i_k} &= \frac{\partial}{\partial t}\Big(\frac{\partial T_{i_1\dots i_k}}{\partial x_j} - \sum_{s=1}^{k}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\Big)\\
&= \frac{\partial}{\partial x_j}\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}\Gamma^l_{ji_s}\frac{\partial T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}}{\partial t}\\
&\quad - \sum_{s=1}^{k}\frac{\partial}{\partial t}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\\
&= \nabla_j\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}(\mathrm{A}*\nabla\mathrm{A})^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\,,
\end{aligned}$$

which is the formula we wanted. □

Lemma 2.3.4. *We have, for $k \geq 0$, denoting by ∇^k the k–th iterated covariant derivative,*

$$\frac{\partial}{\partial t}\nabla^k h_{ij} = \Delta\nabla^k h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}\,.$$

Proof. We work by induction on $k \in \mathbb{N}$. The case $k = 0$ is given by equation (2.3.2), we then suppose that the formula holds for $k-1$. We have, by the previous lemma,

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^k h_{ij} &= \nabla\frac{\partial}{\partial t}\nabla^{k-1}h_{ij} + \nabla^{k-1}\mathrm{A}*\nabla\mathrm{A}*\mathrm{A}\\
&= \nabla\Big(\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k-1\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}\Big)\\
&\quad + \nabla^{k-1}\mathrm{A}*\nabla\mathrm{A}*\mathrm{A}\\
&= \nabla\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}\,.
\end{aligned}$$

Interchanging now the Laplacian and the covariant derivative and recalling that $\mathrm{Riem} = \mathrm{A}*\mathrm{A}$, we have the conclusion, as all the extra terms we get are of the form $\mathrm{A}*\mathrm{A}*\nabla^k\mathrm{A}$ and $\mathrm{A}*\nabla\mathrm{A}*\nabla^{k-1}\mathrm{A}$. □

Proposition 2.3.5. *The following formula holds,*

$$\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 = \Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}*\nabla^k\mathrm{A}\,. \tag{2.3.5}$$

Proof. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 &= 2g\Big(\nabla^k\mathrm{A}, \frac{\partial}{\partial t}\nabla^k\mathrm{A}\Big) + \nabla^k\mathrm{A}*\nabla^k\mathrm{A}*\mathrm{A}*\mathrm{A}\\
&= 2g\Big(\nabla^k\mathrm{A}, \Delta\nabla^k\mathrm{A} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}\Big)\\
&\quad + \nabla^k\mathrm{A}*\nabla^k\mathrm{A}*\mathrm{A}*\mathrm{A}\\
&= 2g\left(\nabla^k\mathrm{A}, \Delta\nabla^k\mathrm{A}\right) + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}*\nabla^k\mathrm{A}\\
&= \Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}*\nabla^k\mathrm{A}\,.
\end{aligned}$$

□

2.4 Consequences of Evolution Equations

Let us see some consequences of the application of the maximum principle to the evolution equations for the curvature.
Suppose that we have a mean curvature flow of a compact hypersurface M in the time interval $[0,T)$, we have seen that

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,.$$

First we deal with the so called *mean convex* hypersurfaces that play a major role in the subject. A hypersurface is mean convex if $\mathrm{H} \geq 0$ everywhere. We will see in the next proposition that this property is preserved by the mean curvature flow. Mean convexity is a significant generalization of convexity, for instance, it is enough general to allow the neckpinch behavior described in Section 1.4, in particular, mean convex hypersurfaces do not necessarily shrink to a point at the singular time.

Proposition 2.4.1. *Assume that the initial, compact hypersurface satisfies $\mathrm{H} \geq 0$. Then, under the mean curvature flow, the minimum of H is increasing, hence H is positive for every positive time.*

Proof. Arguing by contradiction, suppose that in an interval $(t_0,t_1) \subset \mathbb{R}^+$ we have $\mathrm{H}_{\min}(t) < 0$ and $\mathrm{H}_{\min}(t_0) = 0$ ($\mathrm{H}_{\min}$ is obviously continuous in time and $\mathrm{H}_{\min}(0) \geq 0$).
Let $|\mathrm{A}|^2 \leq C$ in such interval, then

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2$$

implies

$$\frac{\partial \mathrm{H}_{\min}}{\partial t} \geq C\mathrm{H}_{\min}$$

for almost every $t \in (t_0,t_1)$.
Integrating this differential inequality in $[s,t] \subset (t_0,t_1)$ we get $\mathrm{H}_{\min}(t) \geq e^{C(t-s)}\mathrm{H}_{\min}(s)$, then sending $s \to t_0^+$ we conclude $\mathrm{H}_{\min}(t) \geq 0$ for every $t \in (t_0,t_1)$ which is a contradiction.

Since then $\mathrm{H} \geq 0$ we get

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2 \geq \Delta\mathrm{H} + \mathrm{H}^3/n\,.$$

With the notation of Theorem 2.1.1, we let $u = -\mathrm{H}$, $X = 0$ and $b(x) = x^3/n$, then, if $\mathrm{H}_{\min}(0) = 0$ the ODE solution $h(t)$ is always zero, so if at some positive time $\mathrm{H}_{\min}(\tau) = 0$, we have that $\mathrm{H}(\,\cdot\,,\tau)$ is constant equal to zero on M, but there are no compact hypersurfaces with zero mean curvature. Hence, $\mathrm{H}_{\min}$ is always increasing during the flow and H is positive on all M at every positive time. □

Actually, this proposition can be slightly improved as follows.

Proposition 2.4.2. *If the initial, compact hypersurface satisfies $|\mathrm{A}| \leq \alpha\mathrm{H}$ for some constant α, then $|\mathrm{A}| \leq \alpha\mathrm{H}$ for every positive time.*

Proof. We know that $\mathrm{H} > 0$ for every positive time, hence also $|\mathrm{A}| > 0$ for every positive time which implies that it is smooth as $|\mathrm{A}|^2$.

Let $[0, T)$ be the interval of smooth existence of the flow. Computing the evolution equation of the function $f = |\mathrm{A}| - \alpha\mathrm{H}$, we get

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{2|\mathrm{A}|}(\Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4) - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)\\ &= \Delta|\mathrm{A}| + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla\mathrm{A}|^2) + |\mathrm{A}|^3 - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)\\ &= \Delta f + |\mathrm{A}|^2 f + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla\mathrm{A}|^2)\\ &\leq \Delta f + |\mathrm{A}|^2|f|\,,\end{aligned}$$

as the term $|\nabla|\mathrm{A}||^2 - |\nabla\mathrm{A}|^2$ is nonpositive.
Hence, choosing any $T' < T$, if C is the maximum of $|\mathrm{A}|^2$ on $M \times [0, T']$, we have $\partial_t f \leq \Delta f + C|f|$ on $M \times [0, T']$. By the maximum principle 2.1.1, as $f_{\max}(0) \leq 0$, we conclude $f \leq 0$ on $M \times [0, T']$. By the arbitrariness of $T' < T$, the thesis follows. □

Corollary 2.4.3. *If $H > 0$ for the initial, compact, n–dimensional hypersurface, then there exists $\alpha_0 > 0$ such that $\alpha_0|\mathrm{A}|^2 \leq \mathrm{H}^2 \leq n|\mathrm{A}|^2$ everywhere on M for every time.*
If the initial hypersurface has positive scalar curvature, then the same holds for every positive time.

Proof. The first claim is immediate by the compactness of M and the previous proposition (the second inequality is algebraic).
Recalling that the scalar curvature is equal to $\mathrm{H}^2 - |\mathrm{A}|^2$, positive scalar curvature implies that $\mathrm{H} > 0$ (H cannot change sign on M and there is always a point where it is positive, as M is compact) and $\mathrm{H}^2/|\mathrm{A}|^2 > 1$, the second part of this corollary is also a consequence of Proposition 2.4.2. □

Corollary 2.4.4. *Assume that the initial, compact hypersurface has $\mathrm{H} \geq 0$, then, if A is not bounded as $t \to T$ then H is also not bounded.*

Proof. Immediate consequence of Proposition 2.4.1 and the estimate of the previous corollary. □

Now we consider the evolution equation of $|\mathrm{A}|^2$ which implies

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,.$$

Notice that $|\mathrm{A}|^2_{\max}$ is always positive, otherwise at some time t we would have $\mathrm{A} = 0$ identically on M, which would imply that M is a hyperplane in $\mathbb{R}^{n+1}$ in contradiction with the compactness hypothesis of M. Hence, we can divide both members by $|\mathrm{A}|^2_{\max}$ obtaining the following differential inequality for the locally Lipschitz function $1/|\mathrm{A}|^2_{\max}$, holding at almost every time $t \in [0, T)$,

$$-\frac{d}{dt}\frac{1}{|\mathrm{A}|^2_{\max}} \leq 2\,.$$

Integrating in time in any interval $[t, s] \subset [0, T)$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} - \frac{1}{|\mathrm{A}(\,\cdot\,, s)|^2_{\max}} \leq 2(s - t)\,.$$

Suppose now that A is not bounded in $[0, T)$, that is, there exists a sequence of times $s_i \nearrow T$ such that $|\mathrm{A}(\,\cdot\,, s_i)|^2_{\max} \to +\infty$. Substituting these times s_i in the previous inequality and sending $i \to \infty$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} \leq 2(T - t)\,.$$

Exercise 2.4.5. Show that the only compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature are the spheres. What can be said about a compact hypersurface in $\mathbb{R}^{n+1}$ with constant $|\mathrm{A}|$?

In other words, we proved the following.

Proposition 2.4.6. *If the second fundamental form* A *during the mean curvature flow of a compact hypersurface is not bounded as* $t \to T < +\infty$*, then it must satisfy the following lower bound for its blow up rate*

$$\max_{p \in M} |\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T-t)}} \tag{2.4.1}$$

for every $t \in [0, T)$.
Hence,

$$\lim_{t \to T} \max_{p \in M} |\mathrm{A}(p,t)| = +\infty \, .$$

Exercise 2.4.7. Assume that the initial, compact hypersurface has $\mathrm{H} > 0$, then the maximal time of smooth existence of the flow can be estimated as $T_{\max} \leq \frac{n}{2\mathrm{H}^2_{\min}(0)}$.