1

Definition and Short Time Existence

1.1 Notations and Preliminaries

In this section we introduce some basic notations and facts about Riemannian manifolds and their submanifolds, a good reference is [50].

In all the lectures the convention of summing over the repeated indices will be adopted.

The main objects we will consider are n–dimensional, complete hypersurfaces immersed in $\mathbb{R}^{n+1}$, that is, pairs (M, φ) where M is an n–dimensional, smooth manifold with empty boundary and $\varphi : M \to \mathbb{R}^{n+1}$ is a smooth immersion (the rank of the differential $d\varphi$ is equal to n everywhere on M).

The manifold M gets in a natural way a metric tensor g turning it into a Riemannian manifold (M, g) by pulling back the standard scalar product of $\mathbb{R}^{n+1}$ with the immersion map φ.

Taking local coordinates around $p \in M$, we have local bases of T_pM and T_p^*M, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and 1–forms $\{dx_j\}$.

We will denote the vectors on M by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, the 1–forms by $\omega = \omega_j$, that is, $\omega = \omega_j dx_j$ and a general mixed tensor by $T = T^{i_1 \dots i_k}_{j_1 \dots j_l}$, where the indices refer to the local basis.

Sometimes we will consider tensors along M viewing it as a submanifold of $\mathbb{R}^{n+1}$ via the map φ, in such case we will use the Greek indices to denote the components of the tensors in the canonical basis $\{e_\alpha\}$ of $\mathbb{R}^{n+1}$, for instance, given a vector field X along M, not necessarily tangent, we will have $X = X^\alpha e_\alpha$.

The metric g of M extended to tensors is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{s_1 \dots s_k}_{z_1 \dots z_l} \,,$$

where g_{ij} is the matrix of the coefficients of g in local coordinates and g^{ij} is its inverse matrix. Clearly, the norm of a tensor is then

$$|T| = \sqrt{g(T, T)} \,.$$

The scalar product of $\mathbb{R}^{n+1}$ will be denoted by $\langle \cdot \,|\, \cdot \rangle$. As the metric g is obtained by pulling it back via φ, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (d\varphi^* \langle \cdot \,|\, \cdot \rangle)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left\langle \frac{\partial \varphi}{\partial x_i} \,\middle|\, \frac{\partial \varphi}{\partial x_j} \right\rangle .$$

The canonical measure induced by the metric g is given in a coordinate chart by $\mu = \sqrt{G}\, \mathcal{L}^n$ where $G = \det(g_{ij})$ and $\mathcal{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The induced covariant derivative on (M, g) of a vector field X and of a 1–form ω are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k \,, \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji} \omega_k \,,$$

where the Christoffel symbols Γ^i_{jk} are expressed by the formula,

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \right) .$$

The covariant derivative ∇T of a general tensor $T = T^{i_1 \dots i_k}_{j_1 \dots j_l}$ will be denoted by $\nabla_s T^{i_1 \dots i_k}_{j_1 \dots j_l} = (\nabla T)^{i_1 \dots i_k}_{s j_1 \dots j_l}$ (we recall that such extension of the covariant derivative is uniquely defined on the full tensor algebra by imposing the Leibniz rule and the commutativity with any metric contraction).
$\nabla^m T$ will stand for the m–th iterated covariant derivative of T.

The gradient ∇f of a function and the divergence $\operatorname{div} X$ of a vector field at a point $p \in M$ are defined respectively by

$$g(\nabla f(p), v) = df_p(v) \qquad \forall v \in T_p M$$

and

$$\operatorname{div} X = \operatorname{tr} \nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k \,.$$

The (rough) Laplacian ΔT of a tensor T is given by

$$\Delta T = g^{ij} \nabla_i \nabla_j T \,.$$

If X is a smooth vector field with compact support on M, as $\partial M = \emptyset$ the following *divergence theorem* holds

$$\int_M \operatorname{div} X \, d\mu = 0 \,,$$

which clearly implies, in particular,

$$\int_M \Delta f \, d\mu = 0$$

for every smooth function $f : M \to \mathbb{R}$ with compact support.

Since φ is locally an embedding in $\mathbb{R}^{n+1}$, at every point $p \in M$ we can define up to a sign a unit normal vector $\nu(p)$. Locally, we can always choose ν in order that it is smooth.
If the hypersurface M is compact and embedded, that is, the map φ is one–to-one, the *inside* of M is easily defined and we will consider ν to be the *inner pointing* unit normal vector at every point of M. In this case the vector field $\nu : M \to \mathbb{R}^{n+1}$ is globally smooth.

The *second fundamental form* $\mathrm{A} = h_{ij}$ of M is the symmetric 2–form defined as follows,

$$h_{ij} = \left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle$$

and the *mean curvature* H is the trace of A, that is $\mathrm{H} = g^{ij} h_{ij}$. Despite its name, H is the *sum* of the eigenvalues of the second fundamental form, not their average mean (some few authors actually define H/n as the mean curvature).

Remark 1.1.1. Notice that since the unit normal ν is defined up to a sign, the same is true for A and H. Instead, the *vector valued second fundamental form* $h_{ij}\nu$, which is a 2–form with values in $\mathbb{R}^{n+1}$, and the *mean curvature vector* $\mathrm{H}\nu$ are uniquely defined.
With our choice of ν as the inner pointing unit normal, the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has a positive definite second fundamental form and positive mean curvature and the same holds for every strictly *convex* hypersurface of $\mathbb{R}^{n+1}$.

We advise the reader that, by a little abuse of terminology, we will always say that a hypersurface is convex when its second fundamental form is nonnegative definite, strictly convex when it is positive definite. If the hypersurface is embedded, convexity in such sense is equivalent to the usual definition that the hypersurface bounds a convex subset of the Euclidean space.

The linear map $W_p : T_pM \to T_pM$ given by $W_p(v) = h^i_j(p)v^j \frac{\partial}{\partial x_i}$ is called the *Weingarten operator* and its eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$ the *principal curvatures* at the point $p \in M$. It is easy to see that $\mathrm{H} = \lambda_1 + \dots + \lambda_n$ and $|\mathrm{A}|^2 = \lambda_1^2 + \dots + \lambda_n^2$.

Exercise 1.1.2. Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $\varphi(x) = (x, f(x))$, we have

$$g_{ij} = \delta_{ij} + f_i f_j\,, \qquad \nu = -\frac{(\nabla f, -1)}{\sqrt{1+|\nabla f|^2}}$$

$$h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1+|\nabla f|^2}}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1+|\nabla f|^2}} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{(\sqrt{1+|\nabla f|^2})^3} = \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$$

where $f_i = \partial_i f$ and $\mathrm{Hess} f$ is the Hessian of the function f.

Exercise 1.1.3. Show that if the hypersurface $M \subset \mathbb{R}^{n+1}$ is locally the *zero set* of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, with $\nabla f \neq 0$ on such level set, we have

$$\mathrm{H} = \frac{\Delta f}{|\nabla f|} - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^3} = \mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right)\,.$$

The following Gauss–Weingarten relations will be fundamental,

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial \varphi}{\partial x_k} + h_{ij}\nu\,, \qquad \frac{\partial \nu}{\partial x_j} = -h_{jl} g^{ls} \frac{\partial \varphi}{\partial x_s}\,. \tag{1.1.1}$$

Actually, they express the fact that $\nabla^M = \nabla^{\mathbb{R}^{n+1}} - \mathrm{A}\nu$. We recall that considering M locally as a regular submanifold of $\mathbb{R}^{n+1}$, we have $\nabla^M_X Y = (\nabla^{\mathbb{R}^{n+1}}_X \widetilde{Y})^M$ where the sign ^{M} denotes the projection on the tangent space to M and $\widetilde{Y}$ is a local extension of the field Y in a local neighborhood $\Omega \subset \mathbb{R}^{n+1}$ of $\varphi(M)$.
Notice that, by these relations, it follows

$$\Delta \varphi = g^{ij}\nabla^2_{ij}\varphi = g^{ij}\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k}\right) = g^{ij}h_{ij}\nu = \mathrm{H}\nu\,. \tag{1.1.2}$$

By straightforward computations, we can see that the Riemann tensor, the Ricci tensor and the scalar curvature can be expressed by means of the second fundamental form as follows,

$$\begin{aligned}
\mathrm{R}_{ijkl} &= g\Big(\nabla^2_{ji}\frac{\partial}{\partial x_k} - \nabla^2_{ij}\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\Big) = h_{ik}h_{jl} - h_{il}h_{jk}\,,\\
\mathrm{Ric}_{ij} &= g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}\,h_{ij} - h_{il}g^{lk}h_{kj}\,,\\
\mathrm{R} &= g^{ij}\mathrm{Ric}_{ij} = g^{ij}g^{kl}\mathrm{R}_{ikjl} = \mathrm{H}^2 - |\mathrm{A}|^2\,.
\end{aligned}$$

Hence, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned}
\nabla_i\nabla_j X^s - \nabla_j\nabla_i X^s &= \mathrm{R}_{ijkl}g^{ks}X^l = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ks}X^l\,,\\
\nabla_i\nabla_j \omega_k - \nabla_j\nabla_i \omega_k &= \mathrm{R}_{ijkl}g^{ls}\omega_s = (h_{ik}h_{jl} - h_{il}h_{jk})\,g^{ls}\omega_s\,.
\end{aligned}$$

The symmetry properties of the covariant derivative of A are given by the following Codazzi equations,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij} \tag{1.1.3}$$

which imply the following *Simons' identity* (see [109]),

$$\Delta h_{ij} = \nabla_i \nabla_j \mathrm{H} + \mathrm{H}\, h_{il} g^{ls} h_{sj} - |\mathrm{A}|^2 h_{ij} \,. \tag{1.1.4}$$

We will write $T * S$, following Hamilton [56], to denote a tensor formed by a sum of terms each one of them obtained by contracting some indices of the pair T, S with the metric g_{ij} and/or its inverse g^{ij}.
A very useful property of such $*$–product is that

$$|T * S| \leq C\, |T|\, |S|$$

where the constant C depends only on the algebraic "structure" of $T * S$.

Sometimes we will need the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$, we will denote it by $\mathcal{H}^n$.

We advise the reader that in all the computations the constants could vary between different formulas and from a line to another.

1.2 First Variation of the Area Functional

Given an immersion $\varphi : M \to \mathbb{R}^{n+1}$ of a smooth hypersurface in $\mathbb{R}^{n+1}$, we consider the Area functional

$$\mathrm{Area}(\varphi) = \int_M d\mu$$

where μ is the canonical measure associated to the metric g induced by the immersion.

In this section we are going to analyze the first variation of such functional which is nothing else than the volume of the hypersurface.

We consider a smooth one parameter family of immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, with $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = \varphi$, such that, outside of a compact set $K \subset M$, we have $\varphi_t(p) = \varphi(p)$ for every $t \in (-\varepsilon, \varepsilon)$.
Defining the field $X = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ along M (as a submanifold of $\mathbb{R}^{n+1}$) we see that X is zero outside K, we call such field the *infinitesimal generator* of the variation φ_t.
We compute

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \varphi_t}{\partial x_i} \,\Big|\, \frac{\partial \varphi_t}{\partial x_j} \right\rangle \Big|_{t=0} \\
&= \left\langle \frac{\partial X}{\partial x_i} \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \left\langle \frac{\partial X}{\partial x_j} \,\Big|\, \frac{\partial \varphi}{\partial x_i} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \frac{\partial}{\partial x_j} \left\langle X \,\Big|\, \frac{\partial \varphi}{\partial x_i} \right\rangle - 2 \left\langle X \,\Big|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle + \frac{\partial}{\partial x_j} \left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_i} \right\rangle - 2\Gamma_{ij}^k \left\langle X^M \,\Big|\, \frac{\partial \varphi}{\partial x_k} \right\rangle - 2h_{ij} \langle X \,|\, \nu \rangle \,,
\end{aligned}$$

where X^M is the tangent component of the field X and we used the Gauss–Weingarten relations (1.1.1) in the last step.
Letting ω be the 1–form defined by $\omega(Y) = g(d\varphi^*(X^M), Y)$, this formula can be rewritten as

$$\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} = \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} - 2\Gamma_{ij}^k \omega_k - 2h_{ij} \langle X \,|\, \nu \rangle = \nabla_i \omega_j + \nabla_j \omega_i - 2h_{ij} \langle X \,|\, \nu \rangle \,.$$

Hence, using the formula $\partial_t \det A(t) = \det A(t)\mathrm{Trace}[A^{-1}(t)\partial_t A(t)]$, we get

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\,\Big|_{t=0} &= \frac{\sqrt{\det(g_{ij})}\,g^{ij}\frac{\partial}{\partial t}g_{ij}\big|_{t=0}}{2}\\
&= \frac{\sqrt{\det(g_{ij})}\,g^{ij}\big(\nabla_i\omega_j+\nabla_j\omega_i-2h_{ij}\langle X\,|\,\nu\rangle\big)}{2}\\
&= \sqrt{\det(g_{ij})}\big(\mathrm{div}\,d\varphi^*(X^M)-\mathrm{H}\langle X\,|\,\nu\rangle\big)\,.
\end{aligned}$$

If the Area of the immersion φ is finite, the same holds for all the maps φ_t, as they are compact deformations of φ. Assuming that the compact K is contained in a single coordinate chart, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} &= \frac{\partial}{\partial t}\int_K d\mu_t\,\Big|_{t=0}\\
&= \frac{\partial}{\partial t}\int_K \sqrt{\det(g_{ij})}\,d\mathcal{L}^n\,\Big|_{t=0}\\
&= \int_K \frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\,\Big|_{t=0} d\mathcal{L}^n\\
&= \int_K \big(\mathrm{div}\,d\varphi^*(X^M)-\mathrm{H}\langle X\,|\,\nu\rangle\big)\sqrt{\det(g_{ij})}\,d\mathcal{L}^n\\
&= \int_M \big(\mathrm{div}\,d\varphi^*(X^M)-\mathrm{H}\langle X\,|\,\nu\rangle\big)\,d\mu\\
&= -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu
\end{aligned}$$

where we used the fact that X is zero outside K and in the last step we applied the divergence theorem. Notice that all the integrals are well defined because we are actually integrating only on the compact set K.
In the case that K is contained in several charts, the same conclusion follows from a standard argument using a partition of unity.

Proposition 1.2.1. *The first variation of the* Area *functional depends only on the normal component of the infinitesimal generator* $X = \frac{\partial\varphi_t}{\partial t}\Big|_{t=0}$ *of the variation* φ_t*, precisely*

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\Big|_{t=0} = -\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu\,.$$

Clearly, such dependence is linear.

Given any immersion $\varphi : M \to \mathbb{R}^{n+1}$ and any vector field X with compact support along M, we can always construct a variation with infinitesimal generator X as $\varphi_t(p) = \varphi(p) + tX(p)$. It is easy to see that for $|t|$ small the map φ_t is still a smooth immersion.
Hence, as the hypersurfaces which are critical points of the Area functional must satisfy

$$\int_M \mathrm{H}\langle X\,|\,\nu\rangle\,d\mu = 0$$

for every field X with compact support, they must have $\mathrm{H} = 0$ everywhere, that is, zero mean curvature (and conversely). This is the well known definition of the so called *minimal surfaces*.

As the quantity $-\mathrm{H}\nu$ can be interpreted as the *gradient* of the Area functional (be careful here, the measure μ is varying with the immersion, we are not computing the gradient with respect to some fixed L^2–structure on the space of immersions of M in $\mathbb{R}^{n+1}$), we can consider the motion of a hypersurface by minus this gradient, that is, the *mean curvature flow*. So, one looks for hypersurfaces moving with velocity $\mathrm{H}\nu$ at every point. This means choosing, among all the velocity functions with fixed $L^2(\mu)$–norm equal to $\left(\int_M \mathrm{H}^2\,d\mu\right)^{1/2}$, the one such that the Area of hypersurface decreases most rapidly.

This idea is quite natural and arises often in studying the dynamics of models of physical situations where the energy is given by the "Area" of the interfaces between the phases of a system. Moreover, as the Area functional is the simplest (in terms of derivatives of the parametrization) geometric functional, that is, invariant by isometries of $\mathbb{R}^{n+1}$ and diffeomorphisms of M, the motion by mean curvature is the simplest *variational* geometric flow for immersed hypersurfaces. Other geometric functionals (for instance, depending on the next simpler geometric invariant, the curvature) generally produce a first variation of order higher than two in the derivatives of the parametrization and a relative higher order PDE's system.
One can consider second order flows where the velocity of the motion is related to different functions of the curvature, like the Gauss flow of surfaces, for instance, where the velocity is given by $\mathrm{G}\nu$ (G is the Gauss curvature of M, that is, $\mathrm{G} = \det \mathrm{A}$) or more complicated flows, but these evolutions are usually not variational, they do not arise as "gradients" (in the above sense) of geometric functionals (see Section 1.6).

1.3 The Mean Curvature Flow

Definition 1.3.1. Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an n–dimensional manifold. The mean curvature flow of φ_0 is a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$ for $t \in [0,T)$ such that setting $\varphi(p,t) = \varphi_t(p)$ the map $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is a smooth solution of the following system of PDE's

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) \\ \varphi(p,0) = \varphi_0(p) \end{cases} \tag{1.3.1}$$

where $\mathrm{H}(p,t)$ and $\nu(p,t)$ are respectively the mean curvature and the unit normal of the hypersurface φ_t at the point $p \in M$.

Remark 1.3.2. Notice that even if the unit normal vector is defined up to a sign, the field $\mathrm{H}(p,t)\nu(p,t)$ is independent of such choice.

Using equation (1.1.2), this system can be rewritten in the appealing form

$$\frac{\partial \varphi}{\partial t} = \Delta\varphi$$

but, despite its formal analogy with the heat equation, actually, it is a second order, *quasilinear* and *degenerate*, parabolic system, as the Laplacian is the one associated to the evolving hypersurfaces at time t,

$$\Delta\varphi(p,t) = \Delta_{g(p,t)}\varphi(p,t) = g^{ij}(p,t)\nabla_i^{g(p,t)}\nabla_j^{g(p,t)}\varphi(p,t)$$

and its coefficients as second order partial differential operator depend on the first derivatives of φ. Moreover, this operator is degenerate, as its symbol (the symbol of the linearized operator) admits zero eigenvalues due to the invariance of the Laplacian by diffeomorphisms, see [49] for details.

Like the Area functional, the flow is obviously invariant by rotations and translations, or more generally under any isometry of $\mathbb{R}^{n+1}$. Moreover, if $\varphi(p,t)$ is a mean curvature flow and $\Psi : M \to M$ is a diffeomorphism, then the reparametrization $\widetilde{\varphi}(p,t) = \varphi(\Psi(p),t)$ is still a mean curvature flow. This last property can be reread as "the flow is invariant under reparametrization", suggesting that the important objects in the flow are actually the subsets $M_t = \varphi(M,t)$ of $\mathbb{R}^{n+1}$.

The problem also satisfies the following parabolic invariance under rescaling (consequence of the property $\mathrm{Area}(\lambda\varphi) = \lambda^n \mathrm{Area}(\varphi)$, for any n–dimensional immersion), if $\varphi(p, t)$ is a mean curvature flow of φ_0 and $\lambda > 0$, then $\widetilde{\varphi}(p, t) = \lambda\varphi(p, \lambda^{-2}t)$ is a mean curvature flow of the initial hypersurface $\lambda\varphi_0$.

During the flow the Area of the hypersurfaces (which is the natural energy of the problem) is nonincreasing, indeed, by the same computation for the first variation of such functional in the previous section, we have

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t) = \frac{\partial}{\partial t}\int_M d\mu_t = -\int_M \mathrm{H}^2\, d\mu_t\,.$$

This clearly implies the estimate

$$\int_0^{T_{\max}}\int_M \mathrm{H}^2\, d\mu_t \leq \mathrm{Area}(\varphi_0)$$

in the maximal time interval $[0, T_{\max})$ of smooth existence of the flow.

Exercise 1.3.3. By means of this last inequality, try to get an estimate from above for the maximal time of smooth existence $T_{\max}$ for closed curves in $\mathbb{R}^2$ and compact *surfaces* in $\mathbb{R}^3$.

Proposition 1.3.4 (Geometric Invariance under Tangential Perturbations)**.** *If a smooth family of immersions $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ satisfies the system of PDE's*

$$\begin{cases} \frac{\partial}{\partial t}\varphi(p, t) = \mathrm{H}(p, t)\nu(p, t) + X(p, t) \\ \varphi(p, 0) = \varphi_0(p) \end{cases}$$

where X is a time dependent smooth vector field along M such that $X(p, t)$ belongs to $d\varphi_t(T_pM)$ for every $p \in M$ and every time $t \in [0, T)$, then, locally around any point in space and time, there exists a family of reparametrizations (smoothly time dependent) of the maps φ_t which satisfies system (1.3.1).
If the hypersurface M is compact, one can actually find uniquely a family of global *reparametrizations of the maps φ_t as above for every $t \geq 0$, leaving the initial immersion φ_0 unmodified and satisfying system* (1.3.1).
Conversely, if a smooth family of moving hypersurfaces $\varphi : M\times[0, T) \to \mathbb{R}^{n+1}$ can be globally reparametrized for $t \geq 0$ in order that it moves by mean curvature, then the map φ has to satisfy the system above for some (uniquely determined) time dependent vector field X with $X(p, t) \in d\varphi_t(T_pM)$, for every $p \in M$ and every time $t \in [0, T)$.

Proof. First we assume that M is compact, we will produce a smooth global parametrization of the evolving sets in order to check Definition 1.3.1.
By the tangency hypothesis, the time dependent vector field on M given by

$$Y(q, t) = -d\varphi_t^*(q)(X(q, t))$$

is globally well defined and smooth.
Let $\Psi : M \times [0, T) \to M$ be a smooth family of diffeomorphisms of M with $\Psi(p, 0) = p$ for every $p \in M$ and

$$\frac{\partial}{\partial t}\Psi(p, t) = Y(\Psi(p, t), t)\,, \tag{1.3.2}$$

for every time $t \in [0, T)$.
This family exists, is unique and smooth, by the existence and uniqueness theorem for ODE's on the compact manifold M.

Considering the reparametrizations $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$, one has

$$\begin{aligned}\frac{\partial\widetilde{\varphi}}{\partial t}(p,t) &= \frac{\partial\varphi}{\partial t}(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))\Big(\frac{\partial\Psi}{\partial t}(p,t)\Big)\\ &= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t) + X(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))(Y(\Psi(p,t),t))\\ &= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t) + X(\Psi(p,t),t)\\ &\quad - d\varphi_t(\Psi(p,t))(d\varphi_t^*(\Psi(p,t))(X(\Psi(p,t),t)))\\ &= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t)\\ &= \widetilde{\mathrm{H}}(p,t)\widetilde{\nu}(p,t)\,.\end{aligned}$$

Hence, $\widetilde{\varphi}$ satisfies system (1.3.1) and $\widetilde{\varphi}_0 = \varphi_0$.
Conversely, this computation also shows that if $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$ satisfies system (1.3.1), the family of diffeomorphisms $\Psi : M \times [0,T) \to M$ must solve equation (1.3.2), hence, it is unique if we assume $\Psi(\,\cdot\,,0) = \mathrm{Id}_M$ in order that the map φ_0 is unmodified.
In the noncompact case, we have to work locally in space and time, solving the above system of ODE's in some positive interval of time in an open subset $\Omega \subset M$ with compact closure, then obtaining a solution of system (1.3.1) in a possibly smaller open subset of Ω and some interval of time.

Assume now that the reparametrized map $\widetilde{\varphi}(p,t) = \varphi(\Psi(p,t),t)$ is a mean curvature flow. Differentiating, we get

$$\begin{aligned}\frac{\partial\widetilde{\varphi}}{\partial t}(p,t) &= \frac{\partial\varphi}{\partial t}(\Psi(p,t),t) + d\varphi_t(\Psi(p,t))\Big(\frac{\partial\Psi}{\partial t}(p,t)\Big)\\ &= \widetilde{\mathrm{H}}(p,t)\widetilde{\nu}(p,t)\\ &= \mathrm{H}(\Psi(p,t),t)\nu(\Psi(p,t),t)\end{aligned}$$

that is,

$$\frac{\partial\varphi}{\partial t}(q,t) = \mathrm{H}(q,t)\nu(q,t) - d\varphi_t(q)\Big(\frac{\partial\Psi}{\partial t}(\Psi_t^{-1}(q),t)\Big)\,,$$

for every $q \in M$ and $t \in [0,T)$. Then, the last statement of the proposition follows by setting $X(q,t) = -d\varphi_t(q)\Big(\frac{\partial\Psi}{\partial t}(\Psi_t^{-1}(q),t)\Big)$. □

Corollary 1.3.5. *If a smooth family of hypersurfaces $\varphi_t = \varphi(\,\cdot\,,t)$ satisfies $\langle\partial_t\varphi\,|\,\nu\rangle = \mathrm{H}$, then it can be everywhere locally reparametrized to a mean curvature flow. If M is compact, this can be done uniquely by global reparametrizations, without modifying φ_0.*

Remark 1.3.6. A short way to state the previous proposition and corollary is to say that the tangential component of the velocity of the points of the hypersurface, does not affect the global "shape" during the motion.
This is particularly meaningful in the case that system (1.3.1) has a unique solution, for instance when M is compact, as we will see in Theorem 1.5.1 in the next section.

By this invariance property one is led to speak of mean curvature flow of hypersurfaces considering them as subsets of $\mathbb{R}^{n+1}$ and forgetting their parametrizations. This is clear in the case of embedded hypersurfaces, where the identification of $(M, g(t))$ with the images of the embeddings $\varphi_t(M)$ is immediate, but it also works for nonembedded hypersurfaces as every immersion is locally an embedding.
We give then a more geometric, alternative definition of the mean curvature flow. In the sequel it will be clear by the context which one we are using.

Definition 1.3.7. We still say that a family of smooth immersions $\varphi_t : M \to \mathbb{R}^{n+1}$, for $t \in [0,T)$, is a mean curvature flow if locally at every point, in space and time, there exists a family of reparametrizations which satisfies system (1.3.1).

Proposition 1.3.4 expresses the substantial equivalence between this definition (Eulerian point of view) and Definition 1.3.1 (Lagrangian point of view).

Exercise 1.3.8 (Motion of Graphs)**.** Show that if the smooth hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$, moving by mean curvature, are locally graphs on some open subset Ω of the hyperplane $\langle e_1, \dots, e_n\rangle \subset \mathbb{R}^{n+1}$, that is, we have a smooth function $f : \Omega \times [0, T) \to \mathbb{R}$, such that

$$\varphi(p, t) = (x_1(p), \dots, x_n(p), f(x_1(p), \dots, x_n(p), t)) ,$$

there holds

$$\partial_t f = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1 + |\nabla f|^2} = \sqrt{1 + |\nabla f|^2}\,\mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) .$$

On the other hand, if we have a function f satisfying the above parabolic equation then its graph is a hypersurface moving by mean curvature (according to Definition 1.3.7).

Exercise 1.3.9 (Motion of Level Sets)**.** Assume that for every time $t \in [0, T)$ the image $\varphi_t(M)$ of the smooth, embedded hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$, moving by mean curvature, is the *zero set* of $f_t = f(\,\cdot\,, t)$, where $f : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}$ is a smooth function and zero is a regular value of f_t for every $t \in [0, T)$. Then at all the points $x \in \mathbb{R}^{n+1}$ and times $t \in [0, T)$ such that $f(x, t) = 0$ there holds

$$\partial_t f = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{|\nabla f|^2} = |\nabla f|\,\mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right) .$$

Conversely, if we have a smooth function f satisfying the above parabolic equation, *every regular* level set of $f(\,\cdot\,, t)$ is a hypersurface moving by mean curvature (according to Definition 1.3.7).

Exercise 1.3.10 (Distance Functions)**.** Compute the evolution equation satisfied by the signed distance function $d_{M_t} : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}$ at the points $x \in M_t = \varphi_t(M)$, if the compact and embedded, smooth hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$ move by mean curvature.
The signed distance function is the function which coincides with the distance in the region "outside" a hypersurface and with minus the distance in the "inside" region (show that it is smooth in a tubular neighborhood of the hypersurface).

Exercise 1.3.11 (Brakke's Definition [21])**.** Show that a smooth family of compact and embedded hypersurfaces $\varphi_t : M \to \mathbb{R}^{n+1}$ satisfies

$$\frac{d}{dt}\int_{\varphi_t(M)} f\, d\mathcal{H}^n \leq \int_{\varphi_t(M)} \left(\mathrm{H}\langle \nabla f \,|\, \nu\rangle - \mathrm{H}^2 f\right)\, d\mathcal{H}^n ,$$

for every *positive* function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, if and only if the hypersurfaces are moving by mean curvature flow (according to Definition 1.3.7).
In the formula $\mathcal{H}^n$ is the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$.

1.4 Examples

Spheres and cylinders are the easiest and actually some of the few nontrivial explicitly computable examples of mean curvature flows (minimal surfaces are trivial examples as they are not moving at all, satisfying $\mathrm{H} = 0$).

Let us consider a sphere of radius R which, by the translation invariance of the flow, we can assume to be centered at the origin of $\mathbb{R}^{n+1}$. A right guess is that at every time the hypersurface remains a sphere and the mean curvature flow simply changes its radius $R(t)$, this is actually true by the uniqueness theorem in the next section. As the mean curvature is everywhere equal to n/R and since we chose the inner pointing unit normal, the evolution equation for the radius of the sphere is simply $R'(t) = -n/R(t)$ with $R(0) = R$. Indeed, if we set $M = \mathbb{S}^n$ and $\varphi(p,t) = R(t)\varphi_0(p)$, being φ_0 the standard immersion of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we have

$$R'(t)\varphi_0(p) = \frac{\partial}{\partial t}\varphi(p,t) = \mathrm{H}(p,t)\nu(p,t) = -n\varphi_0(p)/R(t)\,,$$

which is an ODE that can be easily integrated to get $R(t) = \sqrt{R^2 - 2nt}$.
At time $T_{\max} = R^2/(2n)$ the sphere shrinks to a point so the flow becomes singular, this is the maximal time of existence. We can then write the evolution of the radius also as $R(t) = \sqrt{2n(T_{\max} - t)}$.
During the flow the norm of the second fundamental form evolves as

$$|\mathrm{A}(t)| = \sqrt{n}/R(t) = \frac{1}{\sqrt{2(T_{\max} - t)}}\,.$$

Other examples are given by the cylinders $\mathbb{S}^m(R) \times \mathbb{R}^{n-m}$. In general, we can see that if $\varphi : M \times [0,T) \to \mathbb{R}^{m+1}$ is a mean curvature flow of an m–dimensional hypersurface M of $\mathbb{R}^{m+1}$, then the map $\widetilde{\varphi} : (M \times \mathbb{R}^{n-m}) \times [0,T) \to \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} = \mathbb{R}^{n+1}$, defined by $\widetilde{\varphi}(p,s,t) = (\varphi(p,t), s)$, is a mean curvature flow of the immersion of the product manifold $M \times \mathbb{R}^{n-m}$ in $\mathbb{R}^{n+1}$.
Then, by the above discussion, these cylinders evolve homothetically as $\mathbb{S}^m(R(t)) \times \mathbb{R}^{n-m}$, with $R(t) = \sqrt{R^2 - 2mt}$ and collapse to the subspace $\{0\} \times \mathbb{R}^{n-m}$ at time $T_{\max} = R^2/(2m)$. Again, the norm of the second fundamental form satisfies $|\mathrm{A}(t)| = \frac{1}{\sqrt{2(T_{\max}-t)}}$.
As these cylinder are noncompact, it must be remarked here that it is needed here a uniqueness theorem also for noncompact hypersurfaces to conclude that their evolution is actually only the one described above. In this case it actually holds, see Remark 1.5.4.

Spheres and cylinders are special examples of *homothetically shrinking* flows, that is, hypersurfaces that simply move by contraction during the evolution by mean curvature.