4

Type II Singularities

We assume now that we are in the type II singularity case, that is,

$$\limsup_{t\to T} \max_{p\in M} |\mathrm{A}(p,t)|\sqrt{T-t} = +\infty$$

for the mean curvature flow of a compact hypersurface $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ in its maximal interval of existence.

A good question is actually whether type II singularities there exist.
An example is given by a closed, symmetric, self–intersecting curve with the shape of a symmetric "eight" figure in the plane, which has zero *rotation number*. Pushing a little the analysis of the previous lectures and keeping into account the symmetries of the curve, if the curve develops a type I singularity, we can produce a nonflat blow up limit which is homothetic and nonflat. Then such a limit must be a circle or one of Abresch–Langer curves. In both cases, the limit would be a compact closed curve and by the smooth convergence, the rotation number would still be zero. Hence, the circle has to be excluded and the contradiction is given by the fact that there are no Abresch–Langer curves with zero rotation number. Hence, type I singularities do not describe all the possible ones.
Another example is given by a cardioid–like curve in the plane with a very small loop, hence high curvature: one can right guess that at some time the loop shrinks while the rest of the curve remains smooth and a cusp develops. Such a singularity is of type II, since if we have a type I singularity we would get an Abresch–Langer curve as a blow up limit and this implies, as these latter are compact, that the entire curve has vanished in a single point (see the analysis in [15] and also [14, 16]).
As we will see in Theorem 4.5.5 that embedded curves do not develop type II singularities, one could reasonably conjecture that also for embedded hypersurfaces (at least in low dimension) all the singularities are of type I. Unfortunately, this is not true even if the dimension is only two, indeed, the following example excludes such a good behavior.

Example (The Degenerate Neckpinch). For a given $\lambda > 0$, let us set

$$\phi_\lambda(x) = \sqrt{(1-x^2)(x^2+\lambda)}, \qquad -1 \leq x \leq 1.$$

For any $n \geq 2$, let M^λ be the n–dimensional hypersurface in $\mathbb{R}^{n+1}$ obtained by rotation of the graph of ϕ_λ in $\mathbb{R}^2$. The hypersurface M^λ looks like a dumbbell, where the parameter λ measures the width of the central part. Then, it is possible to prove the following properties (see [4]):

1. if λ is large enough, the hypersurface M^λ_t eventually becomes convex and shrinks to a point in finite time;

2. if λ is small enough, M^λ_t exhibits a neckpinch singularity as in the case of the *standard neckpinch* (see Section 1.4);

3. there exists at least one intermediate value of $\lambda > 0$ such that M_t^λ shrinks to a point in finite time, has positive mean curvature up to the singular time, but never becomes convex. The maximum of the curvature is attained at the two points where the surface meets the axis of rotation;

4. in this latter case the singularity is of type II, otherwise the blow up at the singular time would give a sphere (for all $p \in M$ we would have $\widehat{p} = O \in \mathbb{R}^{n+1}$ hence, by estimate (3.2.2), any limit hypersurface is bounded). This is impossible as it would imply that the surface would have been convex at some time.

The flowing hypersurface at point (3) is called the *degenerate neckpinch* and was first conjectured by Hamilton for the Ricci flow [63, Section 3]. Intuitively speaking, it is a limiting case of the neckpinch where the cylinder in the middle and the two spheres on the sides shrink at the same time. One can also build the example in an asymmetric way, with only one of the two spheres shrinking simultaneously with the neck, while the other one remains nonsingular.
A sharp analysis of the singular behavior for a class of rotationally symmetric surfaces exhibiting a degenerate neckpinch has been done by Angenent and Velázquez in [19].
Another interesting example of singularity formation (a family of evolving tori, proposed by De Giorgi) was carefully studied by Soner and Souganidis in [114, Proposition 3] (see also the numerical analysis performed by Paolini and Verdi in [104, Section 7.5]).

4.1 Hamilton's Blow Up

In order to deal with the blow up around type II singularities we need a new set of estimates which are actually independent of the type II hypothesis and scaling invariant (see [3] and [108]).

Proposition 4.1.1. *Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface such that $\sup_{p\in M} |\mathrm{A}(p,0)| \leq \Lambda < +\infty$. Then, there exists a time $\tau = \tau(\Lambda) > 0$ and constants $C_m = C_m(\Lambda)$, for every $m \in \mathbb{N}$ such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m/t^m$ for every $p \in M$ and $t \in (0,\tau)$.*

Proof. We prove the claim by induction. By the evolution equation for $|\mathrm{A}|^2$,

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

we get

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,,$$

hence, there exists a time $\tau = \tau(\Lambda) > 0$ and a constant $C_0 = C_0(\Lambda)$ such that $|\mathrm{A}(p,t)|^2 \leq C_0$ for every $p \in M$ and $t \in [0,\tau)$. This is the case $m = 0$.
Recalling equation (2.3.5), setting $f = \sum_{k=0}^m |\nabla^k \mathrm{A}|^2 \lambda_k t^k$ for some positive constants $\lambda_0, \dots, \lambda_m$ and assuming the inductive hypothesis $|\nabla^k \mathrm{A}(p,t)|^2 \leq C_k(\Lambda)/t^k$ for any $k \in \{0, \dots, m-1\}$, $p \in M$

and $t \in (0, \tau)$, we compute

$$\begin{aligned}
\frac{\partial}{\partial t} f &= \frac{\partial}{\partial t} \sum_{k=0}^{m} |\nabla^k \mathrm{A}|^2 \lambda_k t^k \\
&= \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 k \lambda_k t^{k-1} \\
&\quad + \sum_{k=0}^{m} \lambda_k t^k \Big(\Delta |\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \Big) \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} - 2|\nabla^{m+1}\mathrm{A}|^2 \lambda_m t^m \\
&\quad + \sum_{k=0}^{m} \lambda_k t^k C(k) \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} |\nabla^p \mathrm{A}||\nabla^q \mathrm{A}||\nabla^r \mathrm{A}||\nabla^k \mathrm{A}| \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + \sum_{k=0}^{m-1} \lambda_k C(k) \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} C_p C_q C_r C_k \\
&\quad + \lambda_m t^{m/2} C(m) \Big(\sum_{p+q+r=m \,|\, p,q,r<m} C_p C_q C_r \Big) |\nabla^m \mathrm{A}| + \lambda_m t^m C(m) |\mathrm{A}|^2 |\nabla^m \mathrm{A}|^2 \\
&\leq \Delta f + \sum_{k=1}^{m} |\nabla^k \mathrm{A}|^2 (k\lambda_k - 2\lambda_{k-1}) t^{k-1} + C\lambda_m t^m |\nabla^m \mathrm{A}|^2 + D
\end{aligned}$$

where in the last passage we applied Peter–Paul inequality. If we choose now inductively positive constants $\lambda_1, \ldots, \lambda_m$ such that $\lambda_k = 2\lambda_{k-1}/k$ starting with $\lambda_0 = 1$ (easily $\lambda_k = 2^k/k!$), we have

$$\frac{\partial}{\partial t} f \leq \Delta f + C\lambda_m t^m |\nabla^m \mathrm{A}|^2 + D \leq \Delta f + Cf + D\,,$$

for every $p \in M$ and $t \in (0, \tau)$, where the constants C and D depend only on m and Λ, by the inductive hypothesis. Notice that the inequality holds also at $t = 0$ as the function f is smooth on $M \times [0, \tau)$.
This differential inequality, by the maximum principle, then implies that $f_{\max}(t)$ is bounded in the interval $[0, \tau)$ by some constant C depending only on m, Λ and $f_{\max}(0) = |\mathrm{A}|^2_{\max}(0) \leq \Lambda^2$, hence

$$t^m |\nabla^m \mathrm{A}(p,t)|^2 \leq f(t)/\lambda_m \leq C/\lambda_m = C_m$$

in the interval $t \in [0, \tau)$, then we are done as $C_m = C_m(\Lambda)$. □

The following corollary is an easy consequence.

Corollary 4.1.2. *Let $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow of a compact hypersurface such that $\sup_{p \in M} |\mathrm{A}(p, 0)| \leq \Lambda < +\infty$. Then, there exists a value $\tau = \tau(\Lambda) > 0$ and constants C_m for every $m \in \mathbb{N}$, depending only on Λ such that $|\nabla^m \mathrm{A}(p,t)|^2 \leq C_m$ for every $p \in M$ and $t \in (\tau/2, \tau)$.*
For instance, one can choose $\tau = 1/(4\Lambda^2)$.

Proof. Only the last claim needs an explanation, it follows by integrating the differential inequality

$$\frac{\partial}{\partial t} |\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,.$$

□

Remark 4.1.3. These estimates provide another proof of Proposition 2.4.8, moreover they can replace the estimates of Proposition 3.2.9 in the proof of Proposition 3.2.10.

We describe now Hamilton's procedure to get a blow up flow at a type II singularity of the mean curvature flow of a compact hypersurface at time $T > 0$.

Let us choose a sequence of times $t_k \in [0, T - 1/k]$ and points $p_k \in M$ such that

$$|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) = \max_{\substack{t\in[0,T-1/k] \\ p\in M}} |\mathrm{A}(p, t)|^2(T - 1/k - t)\,. \tag{4.1.1}$$

This maximum goes to $+\infty$ as $k \to \infty$, indeed, if it is bounded by some constant C on a subsequence $k_i \to \infty$, for every $t \in [0, T)$ definitely we have $t \in [0, T - 1/k_i]$ and

$$|\mathrm{A}(p, t)|^2(T - t) = \lim_{i\to\infty} |\mathrm{A}(p, t)|^2(T - 1/k_i - t) \leq C$$

for every $p \in M$. This is in contradiction with the type II condition

$$\limsup_{t\to T} \max_{p\in M} |\mathrm{A}(p, t)|\sqrt{T - t} = +\infty\,.$$

This fact also forces the sequence t_k to converge to T as $k \to \infty$. If t_{k_i} is a subsequence not converging to T, we would have that the sequence $|\mathrm{A}(p_{k_i}, t_{k_i})|^2$ is bounded, hence also

$$\max_{\substack{t\in[0,T-1/k_i] \\ p\in M}} |\mathrm{A}(p, t)|^2(T - 1/k_i - t)$$

would be bounded.

Thus, we can choose an increasing (not relabeled) subsequence t_k converging to T, such that $|\mathrm{A}(p_k, t_k)|$ goes monotonically to $+\infty$ and

$$|\mathrm{A}(p_k, t_k)|^2 t_k \to +\infty\,, \qquad |\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) \to +\infty\,,$$

Moreover, we can also assume that $p_k \to p$ for some $p \in M$.

We rescale now *the flow* as follows: let $\varphi_k : M \times I_k \to \mathbb{R}^{n+1}$, where

$$I_k = [-|\mathrm{A}(p_k, t_k)|^2 t_k, |\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k)]\,,$$

be the evolution given by

$$\varphi_k(p, s) = |\mathrm{A}(p_k, t_k)|[\varphi(p, s/|\mathrm{A}(p_k, t_k)|^2 + t_k) - \varphi(p_k, t_k)]$$

and we set $M^k_s = \varphi_k(M, s)$ and A_k the second fundamental form of the flowing hypersurfaces φ_k.

It is easy to check that this is a parabolic rescaling hence, every φ_k is still a mean curvature flow, moreover the following properties hold

- $\varphi_k(p_k, 0) = 0 \in \mathbb{R}^{n+1}$ and $|\mathrm{A}_k(p_k, 0)| = 1$,
- for every $\varepsilon > 0$ and $\omega > 0$ there exists $\overline{k} \in \mathbb{N}$ such that

$$\max_{p\in M} |\mathrm{A}_k(p, s)| \leq 1 + \varepsilon \tag{4.1.2}$$

  for every $k \geq \overline{k}$ and $s \in [-|\mathrm{A}(p_k, t_k)|^2 t_k, \omega]$.

Indeed, (the first point is immediate), by the choice of the minimizing pairs (p_k, t_k) we get

$$\begin{aligned}
|\mathrm{A}_k(p, s)|^2 &= |\mathrm{A}(p_k, t_k)|^{-2}|\mathrm{A}(p, s/|\mathrm{A}(p_k, t_k)|^2 + t_k)|^2 \\
&\leq |\mathrm{A}(p_k, t_k)|^{-2}|\mathrm{A}(p_k, t_k)|^2 \frac{T - 1/k - t_k}{T - 1/k - t_k - s/|\mathrm{A}(p_k, t_k)|^2} \\
&= \frac{|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k)}{|\mathrm{A}(p_k, t_k)|^2(T - 1/k - t_k) - s}\,,
\end{aligned}$$

if $s/|A(p_k, t_t)|^2 + t_k \in [0, T - 1/k]$, that is, if $s \in I_k$. Then, assuming $s \leq \omega$ and k large enough, the claim follows as we know that $|A(p_k, t_k)|^2(T - 1/k - t_k) \to +\infty$.

This discussion implies that if we are able to take a (subsequential) limit of these *flows*, locally smoothly converging in every compact time interval, we would get a mean curvature flow such that the norm of the second fundamental form is uniformly bounded by one and the time interval of existence is the whole $\mathbb{R}$ as $\lim_{k\to\infty} I_k = (-\infty, +\infty)$.
This is ensured by the next proposition.

Proposition 4.1.4. *The family of flows φ_k converges (up to a subsequence) in the C^∞_{loc} topology to a nonempty, smooth evolution by mean curvature of complete hypersurfaces M^∞_s in the time interval $(-\infty, +\infty)$. Such a flow is called* eternal*, as a consequence it cannot contain compact hypersurfaces.*
Moreover, the second fundamental form and all its covariant derivatives are uniformly bounded and $|A_\infty|$ takes its absolute maximum, which is 1, at time $s = 0$ at the origin of $\mathbb{R}^{n+1}$, hence the limit flow is nonflat. Finally, if the original initial hypersurface was embedded this limit flow consists of embedded hypersurfaces.

Proof. By the previous discussion, in every bounded interval of time $[s_1, s_2]$ the evolutions φ_k have definitely uniformly bounded curvature, precisely $|A_k| \leq (1 + \varepsilon)$, then for $\varepsilon << 1$ by Corollary 4.1.2 in every interval $[s_1 + 1/16, s_1 + 1/8]$ we have uniform estimates $|\nabla^m A_k| \leq C_m$ with C_m independent of s_1, for every $m \in \mathbb{N}$.
By means of the monotonicity formula we can have a uniform estimate on $\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R)$ as follows (we recall that $\widetilde{\mathcal{H}}^n$ is the n–dimensional Hausdorff measure counting multiplicities): we set μ^k_s to be the measure associated to the hypersurface φ_k at time s and μ_0 the measure associated to the initial hypersurface φ_0, then

$$\begin{aligned}
\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) &= \int_M \chi_{B_R}(y)\, d\mu^k_s(y) \\
&\leq \int_M \chi_{B_R}(y) e^{\frac{R^2 - |y|^2}{4}}\, d\mu^k_s(y) \\
&\leq e^{R^2/4} \int_M e^{-\frac{|y|^2}{4}}\, d\mu^k_s(y) \\
&= (4\pi)^{n/2} e^{R^2/4} \int_M \frac{e^{-\frac{|y|^2}{4(s+1-s)}}}{[4\pi(s + 1 - s)]^{n/2}}\, d\mu^k_s(y) \\
&\leq C(R) \int_M \frac{e^{-\frac{|y|^2}{4(s+1+|A(p_k,t_k)|^2 t_k)}}}{[4\pi(s + 1 + |A(p_k, t_k)|^2 t_k)]^{n/2}}\, d\mu^k_{-|A(p_k,t_k)|^2 t_k}(y) \\
&= C(R) \int_M \frac{|A(p_k, t_k)|^n e^{-\frac{|x-\varphi(p_k,t_k)|^2 |A(p_k,t_k)|^2}{4(s+1+|A(p_k,t_k)|^2 t_k)}}}{[4\pi(s + 1 + |A(p_k, t_k)|^2 t_k)]^{n/2}}\, d\mu_0(x) \\
&\leq C(R) \int_M \frac{|A(p_k, t_k)|^n}{[4\pi(s + 1 + |A(p_k, t_k)|^2 t_k)]^{n/2}}\, d\mu_0(x) \\
&\leq C(R) \text{Area}(\varphi_0) \frac{|A(p_k, t_k)|^n}{[4\pi(s + 1 + |A(p_k, t_k)|^2 t_k)]^{n/2}}\,,
\end{aligned}$$

hence, if s stays in a bounded interval $[s_1, s_2] \subset \mathbb{R}$, we have

$$\limsup_{k\to\infty} \widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) \leq C(R) \frac{\text{Area}(\varphi_0)}{[4\pi T]^{n/2}} = C(R, \varphi_0)\,.$$

This implies that

$$\widetilde{\mathcal{H}}^n(\varphi_k(M, s) \cap B_R) \leq C(R, \varphi_0, s_1, s_2)$$

uniformly in $s \in [s_1, s_2]$ and where the constant C is independent of $k \in \mathbb{N}$.
Then we use the same argument of Proposition 3.2.10, but applied to *flows*, that is, we consider

the time–tracks of the flows φ_k as hypersurfaces $\widetilde{\varphi}_k : M \times I_k \to \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$ defined by $\widetilde{\varphi}_k(p, s) = (\varphi_k(p, s), s)$ and we reparametrize them locally as graphs of smooth functions.
Reasoning like in the proof of Proposition 2.4.9, the estimates on the space covariant derivatives of A_k imply uniform locally estimates on space and also time derivatives (using the evolution equation) of the representing functions, so up to a subsequence we can get locally a limit smooth mean curvature flow. By a diagonal argument, we have the existence of a limit flow (follow the proof of Proposition 3.2.10).
The claimed properties of such limit flow are immediate by the above discussion and by the fact that any compact hypersurface cannot give rise to an eternal flow by Corollary 2.2.5. The only point requiring a justification is the embeddedness, if the initial hypersurface is embedded.
In this case, by Proposition 2.2.7, all the hypersurfaces in the flows φ_k are embedded at every time, then the only possibility for M^∞_s not to be embedded is if two or more of its regions "touch" each other at some point $y \in \mathbb{R}^{n+1}$ with a common tangent hyperplane.
We define the monotone nondecreasing function $G(t) = \max_{\substack{s\in[0,t]\\ p\in M}} |\mathrm{A}(p, s)|$ and we choose a smooth, monotone nondecreasing function $K : [0, T) \to \mathbb{R}^+$ such that $G(t) \leq K(t) \leq 2G(t)$ for every $t \in [0, T)$.
Then, we consider the following open set $\Omega_\varepsilon \subset M \times M \times [0, T)$ given by $\{(p, q, t) \,|\, d_{g(t)}(p, q) \leq \varepsilon/K(t)\}$, where $d_{g(t)}$ is the geodesic distance in the Riemannian manifold $(M, g(t))$. Let

$$B_\varepsilon = \inf_{\partial\Omega_\varepsilon} |\varphi(p, t) - \varphi(q, t)|K(t)$$

and suppose that $B_\varepsilon = 0$ for some $\varepsilon > 0$. This means that there exists a sequence of times $t_i \nearrow T$ and points p_i, q_i with $d_{g(t_i)}(p_i, q_i) = \varepsilon/K(t_i)$ and $|\varphi(p_i, t_i) - \varphi(q_i, t_i)|K(t_i) \to 0$, that is, $|\widetilde{\varphi}_i(p_i) - \widetilde{\varphi}_i(q_i)| \to 0$ and $d_{\widetilde{g}(s_i)}(p_i, q_i) = \varepsilon$, where $\widetilde{\varphi}_i$ is the rescaling of the hypersurface φ_{t_i} around the point $\varphi(p_i, t_i)$ by the dilation factor $K(t_i) \geq G(t_i)$.
As the curvatures A_i of these rescaled hypersurfaces $\widetilde{\varphi}_i$ satisfy

$$|\mathrm{A}_i(p)| = |\mathrm{A}(p, t_i)|/K(t_i) \leq |\mathrm{A}(p, t_i)|/G(t_i) \leq 1\,,$$

reasoning like in the proof of Proposition 3.2.10, we have a contradiction if $\varepsilon > 0$ is small enough.
Now, fixing $\varepsilon > 0$ such that the above constant B_ε is positive and looking at the function

$$L(p, q, t) = |\varphi(p, t) - \varphi(q, t)|K(t)$$

on $\complement\Omega_\varepsilon \subset M \times M \times [0, T)$, we have that if the minimum of L at any time t (which is positive as the hypersurfaces are embedded) is lower than B_ε, then such minimum is not taken on the boundary of the set but in its interior, say at a pair (p, q). Then, we compute at the point (p, q, t)

$$\begin{aligned}\frac{\partial L(p, q, t)}{\partial t} &= K(t)\frac{\partial}{\partial t}|\varphi(p, t) - \varphi(q, t)| + |\varphi(p, t) - \varphi(q, t)|K'(t)\\ &\geq K(t)\frac{\partial}{\partial t}|\varphi(p, t) - \varphi(q, t)|\end{aligned}$$

and a geometric argument analogous to the one in the proof of Proposition 2.2.7 shows that this last partial derivative is nonnegative (when it exists, almost everywhere). Then, by means of the maximum principle (Hamilton's trick, Lemma 2.1.3) we conclude that when the minimum of L at time t is lower than B_ε it is nondecreasing.
Hence, there is a positive lower bound C_ε on

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p, t) - \varphi(q, t)|K(t)\,,$$

consequently,

$$\inf_{\complement\Omega_\varepsilon} |\varphi(p, t) - \varphi(q, t)|G(t) \geq C_\varepsilon/2 > 0\,.$$

Now notice that, for the pairs (p_k, t_k) coming from formula (4.1.1) we have $|\mathrm{A}(p_k, t_k)| = G(t_k)$, otherwise there would exist a time $t < t_k$ with $\max_{p\in M} |\mathrm{A}(p, t)| > |\mathrm{A}(p_k, t_k)|$ which is in contradiction with the maximum in the right hand side of equation (4.1.1).

As $|A(p_k, t_k)| \geq G(t)$ for every $t \leq t_k$, fixed $\omega, \delta > 0$, by inequality (4.1.2) we have definitely $\max_{p \in M} |A_k(p, s)| \leq (1 + \delta)$ for every $s \leq \omega$, hence

$$\begin{aligned} G(s/|A(p_k, t_k)|^2 + t_k) &= \max_{\substack{r \leq s \\ p \in M}} |A(p, r/|A(p_k, t_k)|^2 + t_k)| \\ &\leq \max_{\substack{r \leq \omega \\ p \in M}} |A_k(p, r)||A(p_k, t_k)| \\ &\leq (1 + \delta)|A(p_k, t_k)| \,. \end{aligned} \tag{4.1.3}$$

If $s \in [0, \omega]$ and $d_{g_k(s)}(p, q) > \varepsilon$, definitely

$$\begin{aligned} d_{g(s/|A(p_k,t_k)|^2+t_k)}(p, q) &= d_{g_k(s)}(p, q)/|A(p_k, t_k)| \\ &= d_{g_k(s)}(p, q)/G(t_k) \\ &\geq \frac{\varepsilon}{G(s/|A(p_k, t_k)|^2 + t_k)} \\ &\geq \frac{\varepsilon}{K(s/|A(p_k, t_k)|^2 + t_k)} \end{aligned}$$

hence, $(p, q, s/|A(p_k, t_k)|^2 + t_k) \in \complement\Omega_\varepsilon$.
If instead $s \leq 0$, we define $L(s) = \sup_{M_s^\infty} |A_\infty| \leq 1$ and we see that if $L(s) = 0$ for some $s \leq 0$ then M_s^∞ is a hyperplane and the limit flow is flat till $s = 0$ (by uniqueness of the flow as A_∞ is bounded, see Remark 1.5.4), which is impossible as $|A_\infty(0, 0)| = 1$, hence $L(s) > 0$. Then, for every $s \leq 0$ we must have definitely

$$G(s/|A(p_k, t_k)|^2 + t_k)/|A(p_k, t_k)| \geq L(s)/2$$

and if $d_{g_k(s)}(p, q) > 2\varepsilon/L(s)$,

$$\begin{aligned} d_{g(s/|A(p_k,t_k)|^2+t_k)}(p, q) &= d_{g_k(s)}(p, q)/|A(p_k, t_k)| \\ &> \frac{2\varepsilon}{|A(p_k, t_k)|L(s)} \\ &\geq \frac{\varepsilon}{G(s/|A(p_k, t_k)|^2 + t_k)} \\ &\geq \frac{\varepsilon}{K(s/|A(p_k, t_k)|^2 + t_k)} \end{aligned}$$

hence, also in this case $(p, q, s/|A(p_k, t_k)|^2 + t_k) \in \complement\Omega_\varepsilon$.
Then in both cases, if $d_{g_k(s)}(p, q) > \min\{\varepsilon, 2\varepsilon/L(s)\} = \varepsilon > 0$ (notice that $\varepsilon < 2\varepsilon/L(s)$ as $L(s) \leq 1$),

$$\Big|\varphi(p, s/|A(p_k, t_k)|^2 + t_k) - \varphi(q, s/|A(p_k, t_k)|^2 + t_k)\Big| G(s/|A(p_k, t_k)|^2 + t_k) \geq C_\varepsilon/2 > 0$$

and by inequality (4.1.3) it follows that definitely

$$\begin{aligned} |\varphi_k(p, s) - \varphi_k(q, s)| &= |\varphi(p, s/|A(p_k, t_k)|^2 + t_k) - \varphi(q, s/|A(p_k, t_k)|^2 + t_k)| \, |A(p_k, t_k)| \\ &\geq \frac{C_\varepsilon |A(p_k, t_k)|}{2G(s/|A(p_k, t_k)|^2 + t_k)} \\ &\geq \frac{C_\varepsilon}{2(1 + \delta)} \,. \end{aligned}$$

As ω and δ were arbitrary and the convergence is smooth, this conclusion passes to all the limit hypersurfaces M_s^∞, for every $s \in \mathbb{R}$. That is, if a couple of points of M_s^∞ has intrinsic distance larger than $\varepsilon > 0$, their extrinsic distance is bounded from below by some uniform positive constant. If $\varepsilon > 0$ is then chosen small enough such that any hypersurface with $|A| \leq 1$ (like every M_s^∞) is an embedding when it is restricted to any intrinsic ball of radius smaller than ε, we are done. The hypersurfaces M_s^∞ cannot have self–intersections for every $s \in \mathbb{R}$, hence they are all embedded. □

Exercise 4.1.5. This blow up procedure can be applied also at a type I singularity. There are some differences and the sequence t_k must be chosen in order that $t_k \to T$, since it is not a consequence of the construction.
The limit mean curvature flow that one obtains is no more eternal but only *ancient*, that is, defined on some interval $(-\infty, \Omega)$ with $\Omega > 0$, and $|A_\infty| \leq 1$ holds only on $(\infty, 0]$.
It is an open problem if this limit flow is actually homothetically shrinking, in general.

Remark 4.1.6. Differently by the case of a type I singularity of the flow, we did not and we are not going to define any concept of singular point here. Moreover, it is quite conceivable and actually possible that while we are dealing with a type II singularity via the above Hamilton's procedure, in some other zone of the hypersurface the curvature is locally blowing up at the rate of a type I singularity or even mild singular points could be present, see Remark 3.4.2. These latter, anyway, in the case of an embedded evolving hypersurface, can be excluded by the same argument of Remark 3.4.2, based on White's Theorem 3.2.19 (which holds in general without any bound on the blow up rate of the curvature). In this situation too, it is unknown to the author if the presence of such points can be excluded also for general hypersurfaces or at least in the case of nonnegative mean curvature.

The analysis of singularities in the type II case is then reduced to classify these eternal flows with bounded curvature (and its covariant derivatives) with the extra property that the norm of the second fundamental form takes its maximum, equal to one, at some point in space and time.

Examples of this class are the *translating* mean curvature flows (with bounded second fundamental form and $|A|$ achieving its maximum), that is, hypersurfaces $M \subset \mathbb{R}^{n+1}$ such that during the motion do not change their shape but simply move in a fixed direction with constant velocity. We have seen in Proposition 1.4.2 that this condition is equivalent to the existence of a vector $v \in \mathbb{R}^{n+1}$ such that $H(p) = \langle v \,|\, \nu(p) \rangle$ at every point $p \in M$. Clearly, by comparison with spheres, these hypersurfaces cannot be compact.

Open Problem 4.1.7. Classify all the eternal mean curvature flows of complete, connected, hypersurfaces in $\mathbb{R}^{n+1}$ such that A and its covariant derivatives are uniformly bounded and $|A|$ takes its maximum at some point in space–time. The same problem assuming embeddedness or supposing that the flow comes from Hamilton's blow up procedure.

Another problem is the analogous classification for *ancient* complete flows with bounded curvature at every fixed time (see the discussion in [125, page 536]). For closed convex curves, this problem has been solved by Daskalopoulos, Hamilton and Sesum [33]. The higher dimensional case was recently studied by Brendle, Huisken and Sinestrari.
Finally, the same questions can be asked also for the *immortal* flows, that is, defined on $[0, +\infty)$.

In view of the results of the next section, we also state the following.

Open Problem 4.1.8. All the eternal mean curvature flows of complete hypersurfaces in $\mathbb{R}^{n+1}$ coming from Hamilton's blow up procedure are translating flows? At least if they are embedded?

These problems are difficult in general, but like in the type I singularity case, if the evolving hypersurfaces are mean convex ($H \geq 0$) or if we are dealing with curves in the plane, they have a positive answer. This will be the subject of the next sections.

We underline that the "bad blow up rate" is an obstacle to the use of Huisken's monotonicity formula in the context of type II singularities.

We conclude this section by giving Hamilton's line of proof of Theorem 3.4.11, which is different from the original one.

Proof of Theorem 3.4.11. Let T be the maximal time of smooth existence of the mean curvature flow of an n–dimensional convex hypersurface. By the results of Section 2.5, in particular Proposition 2.5.8, we have that after any positive time $H > 0$ and there exists a positive constant α, independent of time, such that $A \geq \alpha H g$ as forms.
If at time T we have a type II singularity, we get an unbounded, eternal convex blow up limit flow with $H \geq 0$, using Hamilton's procedure. By the strong maximum principle, actually $H > 0$

for every time (otherwise $\mathrm{H} \equiv 0$ everywhere, but this and the convexity would imply that the limit flow is simply a fixed hyperplane) and the condition $\mathrm{A} \geq \alpha \mathrm{H} g$ passes to the limit. Then, by the following theorem of Hamilton [62], all the hypersurfaces of the limit flow are compact, in contradiction with the unboundedness, hence type II singularities cannot develop.

Theorem 4.1.9. *Let M be a smooth, complete, strictly convex, n–dimensional hypersurface in the Euclidean space, with $n \geq 2$. Suppose that for some $\alpha > 0$ its second fundamental form is α–pinched in the sense that $\mathrm{A} \geq \alpha \mathrm{H} g$, where g is the induced metric and H its mean curvature. Then M is compact.*

Dealing with type I singularities, any blow up limit is embedded, strictly convex and compact, again by this theorem. Hence, by Theorem 3.4.6 it can be only the sphere $\mathbb{S}^n(\sqrt{n})$.
This implies that the full sequence of rescaled hypersurfaces converges in C^∞ to such sphere.
Finally, as the blow up limit is unique and compact, the original hypersurface shrinks to a point in finite time. □