Proposition 3.4.3 (Stone [119]). *If the limit of rescaled hypersurfaces around $\widehat{p}$ is a unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$ or equivalently by Lemma 3.2.15 there holds $\Theta(p) = 1$, then p cannot be a singular point.*

Proof. By Corollary 3.2.16, the point $p \in M$ is a minimum of $\Theta : M \to \mathbb{R}$ which is an upper semicontinuous function. Hence p is actually a continuity point for Θ. We want to show that for every sequence $p_i \to p$ and $t_i \to T$ we have $\theta(p_i, t_i) \to 1 = \Theta(p)$.
Suppose that there exists $\delta > 0$ such that $\theta(p_i, t_i) \to 1 + \delta$. For every $j \in \mathbb{N}$ there exists i_0 such that $t_i \geq t_j$ for every $i > i_0$, hence $\theta(p_i, t_i) \leq \theta(p_i, t_j)$. Sending $i \to \infty$ we then get $1 + \delta \leq \theta(p, t_j)$. This is clearly a contradiction, as sending now $j \to \infty$, we have $\theta(p, t_j) \to \Theta(p) = 1$ (what we did is closely related to Dini's theorem on monotone convergence of continuous functions).
If p is a singular point with $p_i \to p$ and $t_i \to T$ such that for some constant $\delta > 0$ there holds $|\mathrm{A}(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T-t_i)}}$, we consider the families of rescaled hypersurfaces around $\widehat{p}_i$,

$$\widetilde{\varphi}_i(q, s) = \frac{\varphi(q, t) - \widehat{p}_i}{\sqrt{2(T-t)}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)$$

with associated measures $\widetilde{\mu}_{i,s}$ and we set

$$\psi_i(q) = \widetilde{\varphi}_i(q, s_i) = \frac{\varphi(q, t_i) - \widehat{p}_i}{\sqrt{2(T-t_i)}} \qquad s_i = -\frac{1}{2}\log(T-t_i)\,,$$

with associated measures $\widetilde{\mu}_{i,s_i}$.
For every $\varepsilon > 0$, as $\Theta(p_i) \geq 1$ we have definitely

$$\begin{aligned}
\varepsilon \geq \theta(p_i, t_i) - 1 &\geq \theta(p_i, t_i) - \Theta(p_i)\\
&= \int_M \frac{e^{-\frac{|x-\widehat{p}_i|^2}{4(T-t_i)}}}{[4\pi(T-t_i)]^{n/2}}\, d\mu_{t_i} - \Theta(p_i)\\
&= \frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{i,s_i} - \Theta(p_i)\\
&= \frac{1}{(2\pi)^{n/2}} \int_{s_i}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{i,s}\, ds\,.
\end{aligned}$$

Hence, since by the uniform curvature estimates of Proposition 3.2.9, see computation (3.2.10), we have,

$$\left| \frac{d}{ds} \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \right| \leq C$$

where $C = C(\mathrm{Area}(\varphi_0), T)$ is a positive constant independent of s, we get

$$\begin{aligned}
\varepsilon &\geq \frac{1}{(2\pi)^{n/2}} \int_{s_i}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{i,s}\, ds\\
&\geq \frac{1}{(2\pi)^{n/2}} \int_{s_i}^{s_i + \frac{1}{C}\int_M e^{-\frac{|y|^2}{2}} |\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle|^2 d\widetilde{\mu}_{i,s_i}} \Big(\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{i,s_i} - C(s - s_i) \Big)\, ds\\
&= \frac{1}{(2\pi)^{n/2}} \frac{1}{2C} \Big(\int_M e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_{i,s_i} \Big)^2.
\end{aligned}$$

If now we proceed like in Proposition 3.2.10 and we extract from the sequence of hypersurfaces ψ_i a locally smoothly converging subsequence (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$, by Lemma 3.2.7 we have

$$\varepsilon \geq \frac{1}{(2\pi)^{n/2}} \frac{1}{2C} \Big(\int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}} \left|\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 d\mathcal{H}^n \Big)^2,$$

for every $\varepsilon > 0$, hence $\widetilde{M}_\infty$ satisfies $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$.
Finally, by Corollary 3.2.8,

$$\frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n = \lim_{i\to\infty} \frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{i,s_i} = \lim_{i\to\infty} \theta(p_i, t_i) = 1$$

then, by Lemma 3.2.15, the hypersurface $\widetilde{M}_\infty$ has to be a hyperplane. But since all the points $\psi_i(p_i)$ belong to the ball of radius $C_0\sqrt{2n} \subset \mathbb{R}^{n+1}$ and the second fundamental form $\widetilde{\mathrm{A}}_i$ of ψ_i satisfies $|\widetilde{\mathrm{A}}_i(p_i)| \geq \delta > 0$ for every $i \in \mathbb{N}$, by the hypothesis, it follows that the second fundamental form of $\widetilde{M}_\infty$ is not zero at some point in the ball $B_{C_0\sqrt{2n}}(0)$.
Since we have a contradiction, p cannot be a singular point of the flow. □

Remark 3.4.4. This proposition is an immediate consequence of White's Theorem 3.2.19, which actually implies that there must exists $p \in M$ such that $\Theta(p) \geq 1 + \varepsilon(n)$, for some constant $\varepsilon(n)$. Indeed, if $\Theta(p) < 1 + \varepsilon$ for every $p \in M$, then, as the set of reachable points $\mathcal{S}$ is compact, by a covering argument and White's Theorem we can conclude that the curvature is uniformly bounded as $t \to T$, which is a contradiction.

We wanted anyway to emphasize the fact that the only really needed "ingredient", by means of the line of analysis of Stone, is the uniqueness of the hyperplanes as minimizers of the integral $\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n$ among the hypersurfaces satisfying $\mathrm{H} + \langle y \,|\, \nu\rangle = 0$ with $\int_M e^{-|y|}\, d\widetilde{\mathcal{H}}^n < +\infty$ (Lemma 3.2.15).

Corollary 3.4.5. *At a singular point $p \in M$ a limit $\widetilde{M}_\infty$ of rescaled hypersurfaces is a smooth, nonempty, complete hypersurface with bounded local volume and bounded curvature with all its covariant derivatives, which satisfies $\widetilde{\mathrm{H}} + \langle y \,|\, \widetilde{\nu}\rangle = 0$ and it is not a single unit multiplicity hyperplane through the origin of $\mathbb{R}^{n+1}$.*
Moreover, if the initial hypersurface is embedded $\widetilde{M}_\infty$ is also embedded and nonflat.

Hence, we have the following conclusion.

Theorem 3.4.6. *Let the compact, initial hypersurface be embedded and with $\mathrm{H} \geq 0$. Then, every limit hypersurface obtained by rescaling around a type I singular point, up to a rotation in $\mathbb{R}^{n+1}$, must be either the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.*

Corollary 3.4.7. Every *singular point of a type I singularity of the motion by mean curvature of a compact, embedded initial hypersurface with $\mathrm{H} \geq 0$ is a special singular point.*

Proof. If $p \in M$ is a singular point which is not special, then the limit of rescalings around p is an embedded hypersurface with at least one point with $\mathrm{A} = 0$. This is in contradiction with the previous theorem. □

Corollary 3.4.8. *There always exists at least one special singular point at a type I singularity of the motion by mean curvature of a compact, embedded initial hypersurface with $\mathrm{H} \geq 0$.*

Proof. We know that there always exists at least one singular point (see the beginning of the section), hence the statement follows by Corollary 3.4.7. □

Open Problem 3.4.9. To the author's knowledge, even if we are dealing with the flow of general embedded hypersurfaces (without the assumption $\mathrm{H} \geq 0$), the existence of at least one special singular point is an open problem (the fact that $\Theta(p) > 1$ does not necessarily implies that the point $p \in M$ is a special singular point).
A related stronger statement would be that *every singular point is a special singular point*.

As a consequence, if the flow develops a type I singularity and a blow up limit is a sphere (or a circle for curves), the flow is smooth till the hypersurface shrinks to a point becoming asymptotically spherical.
This also implies that at some time the hypersurface has become convex.

Actually, more in general, the following pair of theorems describe the flow of convex curves and hypersurfaces.

Theorem 3.4.10 (Gage and Hamilton [48, 49, 50])**.** *Under the curvature flow a convex closed curve in $\mathbb{R}^2$ smoothly shrinks to a point in finite time. Rescaling in order to keep the length constant, it converges to a circle in C^∞.*

Theorem 3.4.11 (Huisken [67])**.** *Under the mean curvature flow a compact and convex hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 2$ smoothly shrinks to a point in finite time. Rescaling in order to keep the* Area *constant, it converges to a sphere in C^∞.*

Remark 3.4.12. The theorem for curves is not merely a consequence of the general result. The proof in dimension $n \geq 2$ does not work in the one–dimensional case.

Actually, the C^∞–convergence to a circle or to a sphere is exponential.

At the end of Section 4.1 of the next lecture, we will show a line of proof of Theorem 3.4.11 by Hamilton in [62], different from the original one. Another proof was also given by Andrews in [9], analyzing the behavior of the eigenvalues of the second fundamental form close to the singular time.

Theorem 3.4.10 will follow from the strong fact that a simple closed curve in $\mathbb{R}^2$ cannot develop type II singularities at all.

Remark 3.4.13. The last point missing in all this story, even in the embedded mean convex case when $n \geq 2$, is a full answer to Problem 3.2.11. We concluded that any blow up limit gives the same value of the Huisken's functional, hence its "shape" is fixed: hyperplane, sphere or cylinder. If the limit is a sphere, the limit is unique and there is full convergence, if it is a hyperplane we already had such conclusion by White's Theorem 3.2.19. But, if the limit is a cylinder, its axis could possibly change, depending on the choice of the converging sequence (there are anyway recent results on this problem by Colding–Minicozzi [30, 31] and see also Schulze [106]). Clearly, in the case of curves Problem 3.2.11 is solved affirmatively as there are no "cylinders".

3.5 Embedded Closed Curves in the Plane

The case of an embedded, closed curve γ in $\mathbb{R}^2$ is special, indeed the classification theorem 3.3.1 holds without *a priori* assumptions on the curvature. So there are only two possible limits of rescaled curves without self–intersections, either a line through the origin or the circle $\mathbb{S}^1$. This gives immediately a general positive answer to Problem 3.2.12 and implies as before that every singular point is a special singular point. In this very special case also Problem 3.2.11 is solved affirmatively, the blow up limit curve is always unique.

Arguing as in the previous section, we then have the following conclusion.

Theorem 3.5.1. *Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve, then every curve obtained by limit of rescalings around a type I singular point of its motion by curvature is the circle $\mathbb{S}^1$.*
As a consequence, if a simple closed curve is developing a type I singularity, at some time the curve becomes convex and it shrinks to a point getting asymptotically circular at the singular time.

We mention here that an extensive and deep analysis of the behavior of general curves moving by curvature (even when the ambient is a generic surface different from $\mathbb{R}^2$) is provided by the pair of papers by Angenent [14, 16] (see also the discussion in [17]).