Appendix D

Hamilton's Matrix Li–Yau–Harnack Inequality in $\mathbb{R}^n$

Theorem D.1.1. *Let $u : \mathbb{R}^n \times (0,T] \to \mathbb{R}$ be a smooth positive solution of the heat equation such that for every $t \in (0,T]$ the function $u(\,\cdot\,,t)$ is bounded by some constant $C(t) > 0$ (possibly unbounded as $t \to 0$).*
Then, the Hamilton's quadratic

$$H_{ij} = \nabla^2_{ij}u - \frac{\nabla_i u \nabla_j u}{u} + \frac{u}{2t}\delta_{ij}$$

is nonnegative definite for every $x \in \mathbb{R}^n$ and $t > 0$.

Proof. It is well known that being a solution of the heat equation, the boundedness of u in space implies that also $|\nabla u|$ and $|\nabla^2 u|$ are bounded in space by some constant depending only on t, that we still call $C(t)$.
We suppose for the moment that these constants $C(t)$ are uniformly bounded from above by $C < +\infty$ and that $u > \theta > 0$ for some $\theta \in \mathbb{R}$.
We set, for some positive constants ε and A,

$$H^\varepsilon_{ij} = H_{ij} + \frac{\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}\delta_{ij}\,.$$

With this choice we see that

$$H^\varepsilon_{ij}(x,t) \geq -\left|\nabla^2_{ij}u - \frac{\nabla_i u\nabla_j u}{u}\right|\delta_{ij} + \frac{\theta}{2t}\delta_{ij} \geq -(C + C^2/\theta)\delta_{ij} + \frac{\theta}{2t}\delta_{ij}\,, \tag{D.1}$$

and since the last term goes to $+\infty$ when $t \to 0$, for t small H^ε_{ij} is uniformly positive definite.
After a straightforward computation (see [60] or [96]) we get

$$\partial_t H_{ij} \geq \Delta H_{ij} + \frac{2}{u}H^2_{ij} - \frac{2}{t}H_{ij}\,,$$

hence, as $\frac{e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}$ solves the heat equation in $\mathbb{R}^n \times [0,A)$,

$$\begin{aligned}(\partial_t - \Delta)H^\varepsilon_{ij} &\geq \frac{2}{u}H^2_{ij} - \frac{2}{t}H_{ij}\\ &= \frac{2}{u}(H^\varepsilon_{ij})^2 - \frac{2}{t}H^\varepsilon_{ij} - \frac{4\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{u(A-t)^{n/2}}H^\varepsilon_{ij} + \frac{2\varepsilon^2 e^{\frac{|x|^2}{2(A-t)}}}{u(A-t)^n}\delta_{ij} + \frac{2\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{t(A-t)^{n/2}}\delta_{ij}\,.\end{aligned}$$

For every $t \in (0, A)$ the form H^ε_{ij} gets its smallest eigenvalue at some point $x_0 \in \mathbb{R}^n$ as the term H^ε_{ij} is bounded and the term $\frac{\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}\delta_{ij}$ goes to $+\infty$ as $|x| \to +\infty$. If at some $t_0 < A/2$ the smallest eigenvalue of H^ε_{ij} at $x_0 \in \mathbb{R}^n$ is zero and $\{v^i\}$ is a relative unit eigenvector, at such point x_0 we have

$$(\partial_t - \Delta)(H^\varepsilon_{ij}v^iv^j) \geq \frac{2\varepsilon^2 e^{\frac{|x|^2}{2(A-t)}}}{u(A-t)^n} + \frac{2\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{t(A-t)^{n/2}} > 0\,.$$

Thus, considering the *first point* in space–time (x_0, t_0), with $t_0 < A/2$ such that H^ε_{ij} is not positive definite (such a first point must exist by the estimate (D.1)) and considering a relative unit eigenvector $\{v^i\}$, we have that the function $h(x,t) = H^\varepsilon_{ij}(x,t)v^iv^j$ is positive for every (x,t) with $t < t_0$, hence $(\partial_t - \Delta)h(x_0,t_0) \leq 0$, which is in contradiction with the previous estimate.

Hence, for every $\varepsilon > 0$, the matrix $H^\varepsilon_{ij}(x,t)$ is positive definite for every $t \in (0, A/2)$ and $x \in \mathbb{R}^n$.

Since $H^\varepsilon_{ij}(x,t) = H_{ij}(x,t) + \frac{\varepsilon e^{\frac{|x|^2}{4(A-t)}}}{(A-t)^{n/2}}\delta_{ij}$, sending ε to zero and A to $+\infty$, we conclude that for every $x \in \mathbb{R}^n$ and $t > 0$, the Harnack quadratic H_{ij} is nonnegative definite.

Let now u be general (not uniformly bounded from below by some $\theta > 0$), given $\varepsilon > 0$ we consider the positive solution of the heat equation $w(x,s) = \varepsilon + u(x, s+\varepsilon) : \mathbb{R}^n \times [0, T-\varepsilon) \to \mathbb{R}$. As w, $|\nabla w|$ and $|\nabla^2 w|$ are uniformly bounded respectively by $\sup_{\mathbb{R}^n\times[\varepsilon,T]} u$, $\sup_{\mathbb{R}^n\times[\varepsilon,T]} |\nabla u|$ and $\sup_{\mathbb{R}^n\times[\varepsilon,T]} |\nabla^2 u|$, by what we proved we conclude

$$0 \leq \nabla^2_{ij}w - \frac{\nabla_i w\nabla_j w}{w} + \frac{w}{2s}\delta_{ij} = \nabla^2_{ij}u - \frac{\nabla_i u\nabla_j u}{u+\varepsilon} + \frac{u+\varepsilon}{2(t-\varepsilon)}\delta_{ij}$$

for every $x \in \mathbb{R}^n$ and $t \in (\varepsilon, T]$, where we substituted $s + \varepsilon = t$.
As u is positive at every point, sending ε to zero we have the thesis.

$\square$

For more details on this topic consult the book [27].