

Monotonicity Formula and Type I Singularities

In all this chapter $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ is the mean curvature flow of an n -dimensional, compact hypersurface in the maximal interval of smooth existence $[0, T)$.

As before we will use the notation $\varphi_t = \varphi(\cdot, t)$ and $\tilde{\mathcal{H}}^n$ will be the n -dimensional Hausdorff measure in \mathbb{R}^{n+1} counting multiplicities.

3.1. The Monotonicity Formula for Mean Curvature Flow

We show the fundamental monotonicity formula for mean curvature flow, discovered by Huisken in [68] and then generalized by Hamilton in [59, 60].

LEMMA 3.1.1. *Let $f : \mathbb{R}^{n+1} \times I \rightarrow \mathbb{R}$ be a smooth function. By a little abuse of notation, we denote by $\int_M f d\mu_t$ the integral $\int_M f(\varphi(p, t), t) d\mu_t(p)$. Then the following formula holds*

$$\frac{d}{dt} \int_M f d\mu_t = \int_M (f_t - H^2 f + H\langle \nabla f | \nu \rangle) d\mu_t.$$

PROOF. Straightforward computation. \square

If $u : \mathbb{R}^{n+1} \times [0, \tau) \rightarrow \mathbb{R}$ is a smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$, by this lemma, we have

$$\begin{aligned} \frac{d}{dt} \int_M u d\mu_t &= \int_M (u_t - H^2 u + H\langle \nabla u | \nu \rangle) d\mu_t \\ &= - \int_M (\Delta^{\mathbb{R}^{n+1}} u + H^2 u - H\langle \nabla u | \nu \rangle) d\mu_t. \end{aligned} \tag{3.1.1}$$

LEMMA 3.1.2. *If $\psi : M \rightarrow \mathbb{R}^{n+1}$ is a smooth isometric immersion of an n -dimensional Riemannian manifold (M, g) , for every smooth function u defined in a neighborhood of $\psi(M)$ we have,*

$$\Delta_g(u(\psi)) = (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla_{\nu\nu}^2 u)(\psi) + H\langle (\nabla u)(\psi) | \nu \rangle,$$

where $(\nabla_{\nu\nu}^2 u)(\psi(p))$ is the second derivative of u in the normal direction $\nu(p) \in \mathbb{R}^{n+1}$ at the point $\psi(p)$.

PROOF. Let $p \in M$ and choose normal coordinates at p . Then,

$$\begin{aligned} \Delta_g(u(\psi)) &= \nabla_{ii}^2(u(\psi)) \\ &= \nabla_i \left(\frac{\partial u}{\partial y_\alpha}(\psi) \frac{\partial \psi^\alpha}{\partial x_i} \right) \\ &= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}(\psi) \frac{\partial \psi^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha}(\psi) \frac{\partial^2 \psi^\alpha}{\partial x_i^2} \\ &= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}(\psi) \frac{\partial \psi^\alpha}{\partial x_i} \frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha}(\psi) h_{ii} \nu^\alpha \\ &= (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla_{\nu\nu}^2 u)(\psi) + H\langle (\nabla u)(\psi) | \nu \rangle, \end{aligned}$$

where we used the Gauss–Weingarten relations (1.1.1). \square

It follows that, substituting $\Delta^{\mathbb{R}^{n+1}} u$ in formula (3.1.1) and using the previous lemma, if the function u is positive we get

$$\begin{aligned} \frac{d}{dt} \int_M u \, d\mu_t &= - \int_M (\Delta_{g(t)}(u(\varphi_t)) + \nabla_{\nu\nu}^2 u + H^2 u - 2H \langle \nabla u | \nu \rangle) \, d\mu_t \\ &= - \int_M (\nabla_{\nu\nu}^2 u + H^2 u - 2H \langle \nabla u | \nu \rangle) \, d\mu_t \\ &= - \int_M \left| H - \frac{\langle \nabla u | \nu \rangle}{u} \right|^2 u \, d\mu_t + \int_M \left(\frac{|\nabla^\perp u|^2}{u} - \nabla_{\nu\nu}^2 u \right) \, d\mu_t, \end{aligned}$$

where $\nabla^\perp u$ denotes the projection on the normal space to M of the gradient of u .

Then, assuming that $u : \mathbb{R}^{n+1} \times [0, \tau) \rightarrow \mathbb{R}$ is a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ for some $\tau > 0$, the following formula easily follows,

$$\begin{aligned} \frac{d}{dt} \left[\sqrt{4\pi(\tau-t)} \int_M u \, d\mu_t \right] &= - \sqrt{4\pi(\tau-t)} \int_M |H - \langle \nabla \log u | \nu \rangle|^2 u \, d\mu_t \\ &\quad - \sqrt{4\pi(\tau-t)} \int_M \left(\nabla_{\nu\nu}^2 u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)} \right) \, d\mu_t \end{aligned} \quad (3.1.2)$$

in the time interval $[0, \min\{\tau, T\})$.

As we can see, the right hand side consists of a nonpositive quantity and a term which is non-positive if $\frac{\nabla_{\nu\nu}^2 u}{u} - \frac{|\nabla^\perp u|^2}{u^2} + \frac{1}{2(\tau-t)} = \nabla_{\nu\nu}^2 \log u + \frac{1}{2(\tau-t)}$ is nonnegative.

Setting $v(x, s) = u(x, \tau-s)$, the function $v : \mathbb{R}^{n+1} \times (0, \tau] \rightarrow \mathbb{R}$ is a positive solution of the standard forward heat equation in all \mathbb{R}^{n+1} and setting $t = \tau-s$ we have $\nabla_{\nu\nu}^2 \log u + \frac{1}{2(\tau-t)} = \nabla_{\nu\nu}^2 \log v + \frac{1}{2s}$.

This last expression is exactly the Li–Yau–Hamilton 2–form $\nabla^2 \log v + g/(2s)$ for positive solutions of the heat equation on a compact manifold (M, g) , evaluated on $\nu \otimes \nu$ (see [59]).

In the paper [59] (see also [95]) Hamilton generalized the Li–Yau differential Harnack inequality in [88] (concerning the nonnegativity of $\Delta \log v + \frac{\dim M}{2s}$) showing that, under the assumptions that the compact manifold (M, g) has parallel Ricci tensor ($\nabla \text{Ric} = 0$) and nonnegative sectional curvatures, the 2–form $\nabla^2 \log v + g/(2s)$ is nonnegative definite (Hamilton’s matrix Li–Yau–Harnack inequality). Even if it is not compact, this result also holds in \mathbb{R}^{n+1} with the canonical flat metric (which clearly satisfies the above hypotheses on the curvature), assuming the boundedness in space of the function v (equivalently of u), at every fixed time, see Appendix D for details. Hence, $\nabla_{\nu\nu}^2 \log u + \frac{1}{2(\tau-t)} = \left(\nabla^2 \log v + g_{\text{can}}^{\mathbb{R}^{n+1}}/(2s) \right) (\nu \otimes \nu) \geq 0$. It follows that, if a smooth solution u of the backward heat equation is bounded in space at every fixed time, the monotonicity formula implies that $\sqrt{4\pi(\tau-t)} \int_M u \, d\mu_t$ is nonincreasing in time.

We resume this discussion in the following theorem by Hamilton [59, 60].

THEOREM 3.1.3 (Huisken’s Monotonicity Formula – Hamilton’s Extension in \mathbb{R}^{n+1}). *Assume that for some $\tau > 0$ we have a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ in $\mathbb{R}^{n+1} \times [0, \tau)$, bounded in space for every fixed $t \in [0, \tau)$, then*

$$\frac{d}{dt} \left[\sqrt{4\pi(\tau-t)} \int_M u \, d\mu_t \right] \leq - \sqrt{4\pi(\tau-t)} \int_M |H - \langle \nabla \log u | \nu \rangle|^2 u \, d\mu_t$$

in the time interval $[0, \min\{\tau, T\})$.

Choosing in particular a backward heat kernel of \mathbb{R}^{n+1} , that is,

$$u(x, t) = \rho_{x_0, \tau}(x, t) = \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}}$$

in formula (3.1.2), we get the standard Huisken’s monotonicity formula, as the Li–Yau–Hamilton expression $\nabla_{\nu\nu}^2 u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)}$ is identically zero in this case.

THEOREM 3.1.4 (Huisken's Monotonicity Formula). *For every $x_0 \in \mathbb{R}^{n+1}$ and $\tau > 0$ we have (see [68])*

$$\frac{d}{dt} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\mu_t = - \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} \left| \mathbf{H} + \frac{\langle x-x_0 | \nu \rangle}{2(\tau-t)} \right|^2 d\mu_t$$

in the time interval $[0, \min\{\tau, T\})$.

Hence, the integral $\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\mu_t$ is nonincreasing during the flow in $[0, \min\{\tau, T\})$.

EXERCISE 3.1.5. Show that for every $x_0 \in \mathbb{R}^{n+1}$, $\tau > 0$ and a smooth function $v : M \times [0, T) \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \frac{d}{dt} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} v d\mu_t &= - \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} \left| \mathbf{H} + \frac{\langle x-x_0 | \nu \rangle}{2(\tau-t)} \right|^2 v d\mu_t \\ &\quad + \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} (v_t - \Delta_{g(t)} v) d\mu_t, \end{aligned}$$

in the time interval $[0, \min\{\tau, T\})$.

In particular if $v : M \times [0, T) \rightarrow \mathbb{R}$ is a smooth solution of $v_t = \Delta_{g(t)} v$, it follows

$$\frac{d}{dt} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} v d\mu_t = - \int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} \left| \mathbf{H} + \frac{\langle x-x_0 | \nu \rangle}{2(\tau-t)} \right|^2 v d\mu_t$$

in $[0, \min\{\tau, T\})$.

3.2. Type I Singularities and the Rescaling Procedure

In the previous chapter we showed that the curvature must blow up at the maximal time T with the following lower bound

$$\max_{p \in M} |A(p, t)| \geq \frac{1}{\sqrt{2(T-t)}}.$$

DEFINITION 3.2.1. Let T be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that we have the upper bound

$$\max_{p \in M} |A(p, t)| \leq \frac{C}{\sqrt{2(T-t)}},$$

we say that the flow is developing at time T a *type I singularity*.

If such a constant does not exist, that is,

$$\limsup_{t \rightarrow T} \max_{p \in M} |A(p, t)| \sqrt{T-t} = +\infty$$

we say that we have a *type II singularity*.

In this chapter we will deal exclusively with type I singularities and the monotonicity formula will be the main tool for the analysis. The next chapter will be devoted to type II singularities.

From now on, we assume that there exists some constant $C_0 > 1$ such that

$$\frac{1}{\sqrt{2(T-t)}} \leq \max_{p \in M} |A(p, t)| \leq \frac{C_0}{\sqrt{2(T-t)}}, \quad (3.2.1)$$

for every $t \in [0, T)$.

Let $p \in M$ and $0 \leq t \leq s < T$, then

$$|\varphi(p, s) - \varphi(p, t)| = \left| \int_t^s \frac{\partial \varphi(p, \xi)}{\partial t} d\xi \right| \leq \int_t^s |\mathbf{H}(p, \xi)| d\xi \leq \int_t^s \frac{C_0 \sqrt{n}}{\sqrt{2(T-\xi)}} d\xi \leq C_0 \sqrt{n(T-t)}$$

which implies that the sequence of functions $\varphi(\cdot, t)$ converges as $t \rightarrow T$ to some function $\varphi_T : M \rightarrow \mathbb{R}^{n+1}$. Moreover, as the constant C_0 is independent of $p \in M$, such convergence is uniform

and the limit function φ_T is continuous. Finally, passing to the limit in the above inequality, we get

$$|\varphi(p, t) - \varphi_T(p)| \leq C_0 \sqrt{n(T-t)}. \quad (3.2.2)$$

In all the chapter we will denote $\varphi_T(p)$ also by \widehat{p} .

DEFINITION 3.2.2. Let \mathcal{S} be the set of points $x \in \mathbb{R}^{n+1}$ such that there exists a sequence of pairs $(p_i, t_i) \in M \times [0, T)$ with $t_i \nearrow T$ and $\varphi(p_i, t_i) \rightarrow x$. We call \mathcal{S} the set of *reachable* points.

We have seen in Proposition 2.2.6 that \mathcal{S} is compact and that $x \in \mathcal{S}$ if and only if, for every $t \in [0, T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center x intersects $\varphi(M, t)$. We show now that $\mathcal{S} = \{\widehat{p} | p \in M\}$.

Clearly $\{\widehat{p} | p \in M\} \subset \mathcal{S}$, suppose that $x \in \mathcal{S}$ and $\varphi(p_i, t_i) \rightarrow x$, then, by inequality (3.2.2) we have $|\varphi(p_i, t_i) - \widehat{p}_i| \leq C_0 \sqrt{n(T-t_i)}$, hence, $\widehat{p}_i \rightarrow x$ as $i \rightarrow \infty$. As the set $\{\widehat{p} | p \in M\}$ is closed it follows that it must contain the point x .

We define now a tool which will be fundamental in the sequel.

DEFINITION 3.2.3. For every $p \in M$, we define the *heat density* function

$$\theta(p, t) = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t$$

and the *limit heat density* function

$$\Theta(p) = \lim_{t \rightarrow T} \theta(p, t).$$

Since M is compact, we can also define the following *maximal heat density* function

$$\sigma(t) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t \quad (3.2.3)$$

and its limit $\Sigma = \lim_{t \rightarrow T} \sigma(t)$.

Clearly, $\theta(p, t) \leq \sigma(t)$ for every $p \in M$ and $t \in [0, T)$ and $\Theta(p) \leq \Sigma$ for every $p \in M$. The function Θ is well defined as the limit exists finite since $\theta(p, t)$ is monotone nonincreasing in t and positive. Moreover, the functions $\theta(\cdot, t)$ are all continuous and monotonically converging to Θ , hence this latter is upper semicontinuous and nonnegative.

The function $\sigma : [0, T) \rightarrow \mathbb{R}$ is also positive and monotone nonincreasing, being the maximum of a family of nonincreasing smooth functions, hence the limit Σ is well defined and finite. Moreover, such family is uniformly locally Lipschitz (look at the right hand side of the monotonicity formula), hence also σ is locally Lipschitz, then by Hamilton's trick 2.1.3, at every differentiability time $t \in [0, T)$ of σ we have the following *maximal monotonicity* formula

$$\sigma'(t) = - \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} \left| \mathbf{H} + \frac{\langle x - x_t | \nu \rangle}{2(T-t)} \right|^2 d\mu_t \quad (3.2.4)$$

where $x_t \in \mathbb{R}^{n+1}$ is any point where the maximum defining $\sigma(t)$ is attained, that is,

$$\sigma(t) = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t.$$

REMARK 3.2.4. Notice that we did not define $\sigma(t)$ as the maximum of $\theta(\cdot, t)$

$$\max_{p \in M} \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t$$

which is *taken among* $p \in M$. Clearly, this latter can be smaller than $\sigma(t)$.

For any point $p \in M$, we rescale now the moving hypersurfaces around $\widehat{p} = \lim_{t \rightarrow T} \varphi(p, t)$, following Huisken [68],

$$\widetilde{\varphi}(q, s) = \frac{\varphi(q, t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}} \quad s = s(t) = -\frac{1}{2} \log(T-t)$$

and we compute the evolution equation for the map $\widetilde{\varphi}(q, s)$ in the time interval $[-\frac{1}{2} \log T, +\infty)$,

$$\begin{aligned} \frac{\partial \widetilde{\varphi}(q, s)}{\partial s} &= \left(\frac{ds}{dt} \right)^{-1} \frac{\partial}{\partial t} \left(\frac{\varphi(q, t) - \widehat{p}}{\sqrt{2(T-t)}} \right) \\ &= \sqrt{2(T-t)} \frac{\partial \varphi(q, t)}{\partial t} + \frac{\varphi(q, t) - \widehat{p}}{\sqrt{2(T-t)}} \\ &= \sqrt{2(T-t)} H(q, t) \nu(q, t) + \widetilde{\varphi}(q, s) \\ &= \widetilde{H}(q, s) \widetilde{\nu}(q, s) + \widetilde{\varphi}(q, s), \end{aligned}$$

where \widetilde{H} is the mean curvature of the rescaled hypersurfaces $\widetilde{\varphi}_s = \widetilde{\varphi}(\cdot, s)$.

As $|\widetilde{A}| = \sqrt{2(T-t)} |A| \leq C_0 < +\infty$, all the hypersurfaces $\widetilde{\varphi}_s$ have equibounded curvatures, moreover,

$$|\widetilde{\varphi}(p, s)| = \left| \frac{\varphi(p, t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}} \right| \leq \frac{C_0 \sqrt{2n(T-t(s))}}{\sqrt{2(T-t(s))}} = C_0 \sqrt{n} \quad (3.2.5)$$

which implies that at every time $s \in [-\frac{1}{2} \log T, +\infty)$ the open ball of radius $C_0 \sqrt{2n}$ centered at the origin of \mathbb{R}^{n+1} intersects the hypersurface $\widetilde{\varphi}(\cdot, s)$. More precisely, the point $\widetilde{\varphi}(p, s)$ belongs to the interior of such ball.

Then, we rescale also the monotonicity formula. In the following $\widetilde{\mu}_s = \frac{\mu_t}{[2(T-t)]^{n/2}}$ will be the canonical measure associated to the rescaled hypersurface $\widetilde{\varphi}_s$ which, by means of equation (2.3.1), satisfies

$$\frac{d}{ds} \widetilde{\mu}_s = (n - \widetilde{H}^2) \widetilde{\mu}_s,$$

as

$$\begin{aligned} \frac{\partial}{\partial s} \widetilde{\mu}_s &= \left(\frac{ds}{dt} \right)^{-1} \frac{\partial}{\partial t} \left(\frac{\mu_t}{[2(T-t)]^{n/2}} \right) \\ &= n \left(\frac{\mu_t}{[2(T-t)]^{n/2}} \right) + \frac{1}{[2(T-t)]^{n/2-1}} \frac{\partial}{\partial t} \mu_t \\ &= n \widetilde{\mu}_s - \frac{1}{[2(T-t)]^{n/2-1}} H^2 \mu_t \\ &= n \widetilde{\mu}_s - \widetilde{H}^2 \widetilde{\mu}_s. \end{aligned}$$

PROPOSITION 3.2.5 (Rescaled Monotonicity Formula). *We have*

$$\frac{d}{ds} \int_M e^{-\frac{|y|^2}{2}} d\widetilde{\mu}_s = - \int_M e^{-\frac{|y|^2}{2}} \left| \widetilde{H} + \langle y | \widetilde{\nu} \rangle \right|^2 d\widetilde{\mu}_s \leq 0, \quad (3.2.6)$$

which integrated becomes

$$\int_M e^{-\frac{|y|^2}{2}} d\widetilde{\mu}_{s_1} - \int_M e^{-\frac{|y|^2}{2}} d\widetilde{\mu}_{s_2} = \int_{s_1}^{s_2} \int_M e^{-\frac{|y|^2}{2}} \left| \widetilde{H} + \langle y | \widetilde{\nu} \rangle \right|^2 d\widetilde{\mu}_s ds.$$

In particular,

$$\int_{-\frac{1}{2} \log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left| \widetilde{H} + \langle y | \widetilde{\nu} \rangle \right|^2 d\widetilde{\mu}_s ds \leq \int_M e^{-\frac{|y|^2}{2}} d\widetilde{\mu}_{-\frac{1}{2} \log T} \leq C < +\infty,$$

for a uniform constant $C = C(\text{Area}(\varphi_0), T)$ independent of $s \in [-\frac{1}{2} \log T, +\infty)$ and $p \in M$.

PROOF. Keeping in mind that $y = \frac{x-\widehat{p}}{\sqrt{2(T-t)}}$ and $s = -\frac{1}{2} \log(T-t)$ we have,

$$\begin{aligned}
\frac{d}{ds} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_s &= \left(\frac{ds}{dt}\right)^{-1} \frac{d}{dt} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_s \\
&= 2(T-t) \frac{d}{dt} \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}} d\mu_t \\
&= -2(T-t) \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}} \left| \mathbf{H} + \frac{\langle x-\widehat{p} | \nu \rangle}{2(T-t)} \right|^2 d\mu_t \\
&= -2(T-t) \int_M e^{-\frac{|y|^2}{2}} \left| \frac{\widetilde{\mathbf{H}}}{\sqrt{2(T-t)}} + \frac{\langle y | \widetilde{\nu} \rangle}{\sqrt{2(T-t)}} \right|^2 d\tilde{\mu}_s \\
&= - \int_M e^{-\frac{|y|^2}{2}} \left| \widetilde{\mathbf{H}} + \langle y | \widetilde{\nu} \rangle \right|^2 d\tilde{\mu}_s.
\end{aligned}$$

The other two statements trivially follow. \square

As a first consequence, we work out an upper estimate on the volume of the rescaled hypersurfaces in the balls of \mathbb{R}^{n+1} .

Fix a radius $R > 0$, if $B_R = B_R(0) \subset \mathbb{R}^{n+1}$, then we have

$$\begin{aligned}
\widetilde{\mathcal{H}}^n(\widetilde{\varphi}(M, s) \cap B_R) &= \int_M \chi_{B_R}(y) d\tilde{\mu}_s \\
&\leq \int_M \chi_{B_R}(y) e^{\frac{R^2-|y|^2}{2}} d\tilde{\mu}_s \\
&\leq e^{R^2/2} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_s \\
&\leq e^{R^2/2} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_{-\frac{1}{2} \log T} \\
&\leq \widehat{C} e^{R^2/2}
\end{aligned} \tag{3.2.7}$$

where the constant \widehat{C} is independent of R and s .

REMARK 3.2.6. As

$$\int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_{-\frac{1}{2} \log T} = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4T}}}{(2T)^{n/2}} d\mu_0 \leq \frac{\text{Area}(\varphi_0)}{(2T)^{n/2}},$$

we can choose the constant \widehat{C} to be independent also of $p \in M$.

Another consequence is the following key technical lemma which is necessary in order to take the limits of integrals of functions on the sequences of rescaled hypersurfaces.

LEMMA 3.2.7 (Stone [118]). *The following estimates hold.*

- (1) *There is a uniform constant $C = C(n, \text{Area}(\varphi_0), T)$ such that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2} \log T, +\infty\right)$,*

$$\int_M e^{-|y|} d\tilde{\mu}_s \leq C.$$

- (2) *For any $\varepsilon > 0$ there is a uniform radius $R = R(\varepsilon, n, \text{Area}(\varphi_0), T)$ such that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2} \log T, +\infty\right)$,*

$$\int_{\widetilde{\varphi}_s(M) \setminus B_R(0)} e^{-|y|^2/2} d\widetilde{\mathcal{H}}^n \leq \varepsilon,$$

that is, the family of measures $e^{-|y|^2/2} d\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}_s(M)$ is tight (see [33]).

PROOF. By the rescaled monotonicity formula (3.2.6) we have that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2} \log T, +\infty\right)$,

$$\int_M e^{-|y|^2/2} d\tilde{\mu}_s \leq \int_M e^{-|y|^2/2} d\tilde{\mu}_{-\frac{1}{2} \log T}.$$

According to Remark 3.2.6, the right hand integral may be estimated by a constant depending only on T and $\text{Area}(\varphi_0)$, not on $p \in M$. Hence, we have the following estimates for all $p \in M$ and for all $s \in \left[-\frac{1}{2} \log T, +\infty\right)$,

$$\int_{\tilde{\varphi}_s(M) \cap B_{n+1}(0)} e^{-|y|} d\tilde{\mathcal{H}}^n \leq C_1 \quad (3.2.8)$$

and

$$\int_{\tilde{\varphi}_s(M) \cap B_{2n+2}(0)} e^{-|y|} d\tilde{\mathcal{H}}^n \leq C_2 \quad (3.2.9)$$

where C_1 and C_2 are constants depending only on n, T and $\text{Area}(\varphi_0)$. Then, we compute for any p and s ,

$$\begin{aligned} \frac{d}{ds} \int_M e^{-|y|} d\tilde{\mu}_s &= \int_M \left\{ n - \tilde{\mathbf{H}}^2 - \frac{1}{|y|} \langle y | \tilde{\mathbf{H}}\tilde{\nu} + y \rangle \right\} e^{-|y|} d\tilde{\mu}_s \\ &\leq \int_M \{ n - \tilde{\mathbf{H}}^2 - |y| + |\tilde{\mathbf{H}}| \} e^{-|y|} d\tilde{\mu}_s \\ &< \int_M \{ n + 1 - |y| \} e^{-|y|} d\tilde{\mu}_s \\ &\leq (n+1) \left\{ \int_{\tilde{\varphi}_s(M) \cap B_{n+1}(0)} e^{-|y|} d\tilde{\mathcal{H}}^n - \int_{\tilde{\varphi}_s(M) \setminus B_{2n+2}(0)} e^{-|y|} d\tilde{\mathcal{H}}^n \right\}. \end{aligned}$$

But then, by inequality (3.2.8) we see that we must have either

$$\frac{d}{ds} \int_M e^{-|x|} d\tilde{\mu}_s < 0,$$

or

$$\int_{\tilde{\varphi}_s(M) \setminus B_{2n+2}(0)} e^{-|y|} d\tilde{\mathcal{H}}^n \leq C_1.$$

Hence, in view of inequality (3.2.9), it follows that either

$$\frac{d}{ds} \int_M e^{-|y|} d\tilde{\mu}_s < 0,$$

or

$$\int_M e^{-|y|} d\tilde{\mu}_s \leq C_1 + C_2,$$

which implies

$$\int_M e^{-|y|} d\tilde{\mu}_s \leq \max \left\{ (C_1 + C_2), \int_M e^{-|y|} d\tilde{\mu}_{-\frac{1}{2} \log T} \right\} = C_3$$

for any p and s .

The proof of part (1) of the lemma follows by noticing that the integral quantity on the right hand side can clearly be estimated by a constant depending on T and $\text{Area}(\varphi_0)$ but not on $p \in M$.

Let now again $p \in M$ and $s \in \left[-\frac{1}{2} \log T, +\infty\right)$ arbitrary. Now subdivide $\tilde{\varphi}_s(M)$ into "annular pieces" $\{\tilde{M}_s^k\}_{k=0}^\infty$ by setting

$$\tilde{M}_s^0 = \tilde{\varphi}_s(M) \cap B_1(0),$$

and for each $k \geq 1$,

$$\tilde{M}_s^k = \{y \in \tilde{\varphi}_s(M) \mid 2^{k-1} \leq |y| < 2^k\}.$$

Then, by part (1) of the lemma $\tilde{\mathcal{H}}^n(\tilde{M}_s^k) \leq C_3 e^{(2^k)}$ for each k , independently of the choice of p and s . Hence in turn, for each k we have

$$\int_{\tilde{M}_s^k} e^{-|y|^2/2} d\tilde{\mathcal{H}}^n \leq C_3 e^{-\frac{1}{2}(2^{k-1})^2} e^{(2^k)} = C_3 e^{(2^k - 2^{2k-3})}$$

again independently of the choice of p and s .

For any $\varepsilon > 0$ we can find a $k_0 = k_0(\varepsilon, n, \text{Area}(\varphi_0), T)$ such that

$$\sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon,$$

then, if $R = R(\varepsilon, n, \text{Area}(\varphi_0), T)$ is simply taken to be equal to 2^{k_0-1} , we have

$$\int_{\tilde{\varphi}_s(M) \setminus B_R(0)} e^{-|y|^2/2} d\tilde{\mathcal{H}}^n = \sum_{k=k_0}^{\infty} \int_{\tilde{M}_s^k} e^{-|y|^2/2} d\tilde{\mathcal{H}}^n \leq \sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon$$

and we are done also with part (2) of the lemma. \square

COROLLARY 3.2.8. *If a sequence of rescaled hypersurfaces $\tilde{\varphi}_{s_i}$ locally smoothly converges (up to reparametrization) to some limit hypersurface \tilde{M}_∞ , we have*

$$\int_{\tilde{M}_\infty} e^{-|y|} d\tilde{\mathcal{H}}^n \leq C$$

and

$$\lim_{i \rightarrow \infty} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_{s_i} = \int_{\tilde{M}_\infty} e^{-\frac{|y|^2}{2}} d\tilde{\mathcal{H}}^n,$$

where the constant C is the same of the previous lemma.

PROOF. Actually, it is only sufficient to show that the measures $\tilde{\mathcal{H}}^n \llcorner \tilde{\varphi}(M, s_i)$ associated to the hypersurfaces weakly*-converge to the measure $\tilde{\mathcal{H}}^n \llcorner \tilde{M}_\infty$. Indeed, for every $R > 0$ we have,

$$\int_{\tilde{M}_\infty \cap B_R(0)} e^{-|y|} d\tilde{\mathcal{H}}^n \leq \liminf_{i \rightarrow \infty} \int_{\tilde{\varphi}(M, s_i) \cap B_R(0)} e^{-|y|} d\tilde{\mathcal{H}}^n \leq \liminf_{i \rightarrow \infty} \int_M e^{-|y|} d\tilde{\mu}_{s_i} \leq C$$

by the first part of the lemma above. Sending R to $+\infty$, the first inequality follows.

The second statement is an easy consequence of the estimates in the second part of the lemma. \square

Now we want to estimate the covariant derivatives of the rescaled hypersurfaces.

PROPOSITION 3.2.9 (Huisken [68]). *For every $k \in \mathbb{N}$ there exists a constant C_k depending only on k, n, C_0 (the constant in formula (3.2.1)) and the initial hypersurface such that $|\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}} \leq C_k$ for every $p \in M$ and $s \in \left[-\frac{1}{2} \log T, +\infty\right)$.*

PROOF. By Proposition 2.3.5 we have for the original flow,

$$\frac{\partial}{\partial t} |\nabla^k \mathbf{A}|^2 = \Delta |\nabla^k \mathbf{A}|^2 - 2 |\nabla^{k+1} \mathbf{A}|^2 + \sum_{p+q+r=k} \nabla^p \mathbf{A} * \nabla^q \mathbf{A} * \nabla^r \mathbf{A} * \nabla^k \mathbf{A},$$

hence, with a straightforward computation, noticing that $|\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}^2 = |\nabla^k \mathbf{A}|_{g_0}^2 [2(T-t)]^{k+1}$ we get

$$\begin{aligned} \frac{\partial}{\partial s} |\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}^2 &\leq -2(k+1) |\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}^2 + \tilde{\Delta} |\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}^2 - 2 |\tilde{\nabla}^{k+1} \tilde{\mathbf{A}}|_{\tilde{g}}^2 \\ &\quad + C(n, k) \sum_{p+q+r=k} |\tilde{\nabla}^p \tilde{\mathbf{A}}|_{\tilde{g}} |\tilde{\nabla}^q \tilde{\mathbf{A}}|_{\tilde{g}} |\tilde{\nabla}^r \tilde{\mathbf{A}}|_{\tilde{g}} |\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}. \end{aligned}$$

As $|\tilde{\mathbf{A}}|_{\tilde{g}}$ is bounded by the constant C_0 , supposing by induction that for $i = 0, \dots, k-1$ we have uniform bounds on $|\tilde{\nabla}^i \tilde{\mathbf{A}}|_{\tilde{g}}$ with constants $C_i = C_i(n, C_0, \varphi_0)$, we can conclude by means of Peter–Paul inequality

$$\frac{\partial}{\partial s} |\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}^2 \leq \tilde{\Delta} |\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^k \tilde{\mathbf{A}}|_{\tilde{g}}^2 - 2 |\tilde{\nabla}^{k+1} \tilde{\mathbf{A}}|_{\tilde{g}}^2 + D_k$$

for some constants B_k and D_k depending only on n, k, C_0 and the initial hypersurface. Then,

$$\begin{aligned}
\frac{\partial}{\partial s} (|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2) &\leq \tilde{\Delta} |\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 - 2 |\tilde{\nabla}^{k+1} \tilde{\mathbb{A}}|_{\tilde{g}}^2 \\
&\quad + B_k \tilde{\Delta} |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k B_{k-1} |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2 \\
&\quad - 2 B_k |\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + D_k + B_k D_{k-1} \\
&\leq \tilde{\Delta} (|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2) - B_k |\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 \\
&\quad + B_k B_{k-1} |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2 + D_k + B_k D_{k-1} \\
&\leq \tilde{\Delta} (|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2) - B_k |\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 \\
&\quad + B_k B_{k-1} C_{k-1}^2 + D_k + B_k D_{k-1} \\
&\leq \tilde{\Delta} (|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2) \\
&\quad - B_k (|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2) \\
&\quad + B_k B_{k-1} C_{k-1}^2 + D_k + B_k D_{k-1} + B_k^2 C_{k-1}^2
\end{aligned}$$

where we used the inductive hypothesis $|\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}} \leq C_{k-1}$.

By the maximum principle, the function $|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}}^2 + B_k |\tilde{\nabla}^{k-1} \tilde{\mathbb{A}}|_{\tilde{g}}^2$ is then uniformly bounded in space and time by the square of some constant C_k depending on n, k, C_0 and the initial hypersurface, hence $|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}} \leq C_k$. By the inductive hypothesis, the thesis of the proposition follows. \square

We are now ready to study the convergence of the rescaled hypersurfaces as $s \rightarrow +\infty$.

PROPOSITION 3.2.10. *For every point $p \in M$ and every sequence of times $s_i \rightarrow +\infty$ there exists a subsequence (not relabeled) of times such that the hypersurfaces $\tilde{\varphi}_{s_i}$, rescaled around \hat{p} , locally smoothly converge (up to reparametrization) to some nonempty, smooth, complete limit hypersurface \tilde{M}_∞ such that $\tilde{\mathbb{H}} + \langle y | \tilde{\nu} \rangle = 0$ for every $y \in \tilde{M}_\infty$.*

Any limit hypersurface satisfies $\tilde{\mathcal{H}}^n(\tilde{M}_\infty \cap B_R) \leq C_R$ for every ball of radius R in \mathbb{R}^{n+1} and for every $k \in \mathbb{N}$ there are constants C_k such that $|\tilde{\nabla}^k \tilde{\mathbb{A}}|_{\tilde{g}} \leq C_k$.

Moreover, if the initial hypersurface was embedded, \tilde{M}_∞ is embedded.

PROOF. We give a sketch of the proof, following Huisken [68].

By estimate (3.2.7) there is a uniform upper bound on $\mathcal{H}^n(\tilde{\varphi}(M, s) \cap B_R)$ for each R , independent of s . Moreover, by the uniform control on the norm of the second fundamental form of the rescaled hypersurfaces in Proposition 3.2.9, there is a number $r_0 > 0$ such that, for each $s \in \left[-\frac{1}{2} \log T, +\infty\right)$ and each $q \in M$, if $U_{r_0, q}^s$ is the connected component of $\tilde{\varphi}_s^{-1}(B_{r_0}(\tilde{\varphi}_s(q)))$ in M containing q , then $\tilde{\varphi}_s(U_{r_0, q}^s)$ can be written as a graph of a smooth function f over a subset of the ball of radius r_0 in the tangent hyperplane to $\tilde{\varphi}_s(M) \subset \mathbb{R}^{n+1}$ at the point $\tilde{\varphi}_s(q)$.

The estimates of Proposition 3.2.9 then imply that all the derivatives of such function f up to the order $\alpha \in \mathbb{N}$ are bounded by constants C_α independent of s .

Following now the method in [85] we can see that, for each $R > 0$, a subsequence of the hypersurfaces $\tilde{\varphi}(M, s) \cap B_R(0)$ must converge smoothly to a limit hypersurface in $B_R(0)$. Then, the existence of a smooth, complete limit hypersurface \tilde{M}_∞ follows from a diagonal argument, letting $R \rightarrow +\infty$. Recalling the fact that every rescaled hypersurface intersects the ball of radius $C_0 \sqrt{2n}$ centered at the origin of \mathbb{R}^{n+1} , this limit cannot be empty. The estimates on the volume and derivatives of the curvature follow from the analogous properties for the converging sequence.

The fact that \tilde{M}_∞ satisfies $\tilde{\mathbb{H}} + \langle y | \tilde{\nu} \rangle = 0$ is a consequence of the rescaled monotonicity formula

and of the uniform estimates on the curvature and its covariant derivatives for the rescaled hypersurfaces in Proposition 3.2.9. Indeed, by means of equation (2.3.4) we have

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1} \frac{\partial}{\partial t} \left(\sqrt{2(T-t)} H\right) \\ &= \left(2(T-t)\right)^{3/2} (\Delta H + H|A|^2) - \sqrt{2(T-t)} H \\ &= \tilde{\Delta} \tilde{H} + \tilde{H} |\tilde{A}|^2 - \tilde{H}\end{aligned}$$

and since $\tilde{\nu} = \nu$,

$$\frac{\partial \tilde{\nu}}{\partial s} = \left(\frac{ds}{dt}\right)^{-1} \frac{\partial \nu}{\partial t} = -2(T-t) \nabla H = -\tilde{\nabla} \tilde{H}.$$

Hence,

$$\begin{aligned}\left|\frac{\partial}{\partial s} \left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2\right| &= 2 \left| \left(\tilde{\Delta} \tilde{H} + \tilde{H} |\tilde{A}|^2 - \tilde{H} + \langle \tilde{H} \tilde{\nu} + y | \tilde{\nu} \rangle - \langle y | \tilde{\nabla} \tilde{H} \rangle \right) \left(\tilde{H} + \langle y | \tilde{\nu} \rangle \right) \right| \\ &= 2 \left| \tilde{\Delta} \tilde{H} + \tilde{H} |\tilde{A}|^2 + \langle y | \tilde{\nu} \rangle - \langle y | \tilde{\nabla} \tilde{H} \rangle \right| \left| \tilde{H} + \langle y | \tilde{\nu} \rangle \right| \\ &\leq C(|y| + C)(|y| + C) \\ &\leq C(|y|^2 + 1)\end{aligned}$$

for some constant C independent of s .

Then,

$$\begin{aligned}\left|\frac{d}{ds} \int_M e^{-\frac{|y|^2}{2}} \left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2 d\tilde{\mu}_s\right| & \tag{3.2.10} \\ &= \left| \int_M e^{-\frac{|y|^2}{2}} \left[\left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2 \left(n - \tilde{H}^2 - \langle y | \tilde{H} \tilde{\nu} + y \rangle\right) + \frac{\partial}{\partial s} \left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2 \right] d\tilde{\mu}_s \right| \\ &\leq \int_M e^{-\frac{|y|^2}{2}} [C(|y|^2 + 1)(|y|^2 + 1) + C(|y|^2 + 1)] d\tilde{\mu}_s \\ &\leq C \int_M e^{-\frac{|y|^2}{2}} (|y|^4 + 1) d\tilde{\mu}_s\end{aligned}$$

and this last term is bounded uniformly in $s \in \left[-\frac{1}{2} \log T, +\infty\right)$ by a positive constant $C = C(\text{Area}(\varphi_0), T)$ by the estimates in Stone's Lemma 3.2.7.

Supposing that there is a sequence of times $s_i \rightarrow +\infty$ such that

$$\int_M e^{-\frac{|y|^2}{2}} \left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2 d\tilde{\mu}_{s_i} \geq \delta$$

for some $\delta > 0$, then we have that in all the intervals $[s_i, s_i + \delta/(2C))$ such integral is larger than $\delta/2$. This is clearly in contradiction with the fact that

$$\int_{-\frac{1}{2} \log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2 d\tilde{\mu}_s ds < +\infty,$$

stated in Proposition 3.2.5.

If $\tilde{\varphi}_{s_i}$ is a locally smoothly converging subsequence of rescaled hypersurfaces (up to reparametrization), we have then that for every ball B_R

$$\int_{\tilde{\varphi}_{s_i}(M, s_i) \cap B_R} e^{-\frac{|y|^2}{2}} \left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2 d\tilde{\mathcal{H}}^n \leq \int_M e^{-\frac{|y|^2}{2}} \left|\tilde{H} + \langle y | \tilde{\nu} \rangle\right|^2 d\tilde{\mu}_{s_i} \rightarrow 0,$$

hence, the limit hypersurface \tilde{M}_∞ satisfies $\tilde{H} + \langle y | \tilde{\nu} \rangle = 0$ at all its points.

Assume now that the initial hypersurface was embedded, then by Proposition 2.2.7 all the hypersurfaces $\tilde{\varphi}_s$ are embedded and the only possibility for \tilde{M}_∞ not to be embedded is that two or more of its regions "touch" each other at some point $y \in \mathbb{R}^{n+1}$ with a common tangent space. Let $g(t)$ be the metrics induced on the moving hypersurfaces, we consider the following set $\Omega_\varepsilon \subset$

$M \times M \times [0, T)$ given by $\{(p, q, t) \mid d_t(p, q) \leq \varepsilon\sqrt{2(T-t)}\}$, where d_t is the geodesic distance in the Riemannian manifold $(M, g(t))$. Let

$$B_\varepsilon = \inf_{\partial\Omega_\varepsilon} |\varphi(p, t) - \varphi(q, t)| / \sqrt{2(T-t)},$$

we claim that $B_\varepsilon > 0$ for any $\varepsilon > 0$ small enough. Suppose that $B_\varepsilon = 0$ for some $\varepsilon > 0$, this means that there exists a sequence of times $t_i \nearrow T$ and points p_i, q_i with $d_{t_i}(p_i, q_i) = \varepsilon\sqrt{2(T-t_i)}$ and $|\varphi(p_i, t_i) - \varphi(q_i, t_i)| / \sqrt{2(T-t_i)} \rightarrow 0$, hence, $|\tilde{\varphi}_i(p_i) - \tilde{\varphi}_i(q_i)| \rightarrow 0$ and $\tilde{d}_i(p_i, q_i) = \varepsilon$, where we denoted by $\tilde{\varphi}_i$ the rescaled hypersurfaces $\tilde{\varphi}_i(p) = \frac{\varphi(p, t_i) - \varphi(p_i, t_i)}{\sqrt{2(T-t_i)}}$ and $\tilde{d}_i = d_{t_i} / \sqrt{2(T-t_i)}$.

Reasoning like in the first part of this proof, by the uniform bound on the second fundamental form of the rescaled hypersurfaces, if U^i is the connected component of $\tilde{\varphi}_i^{-1}(B_{r_0}(\tilde{\varphi}_i(p_i)))$ containing p_i , then $\tilde{\varphi}_i(U^i)$ can be written as a graph of a smooth function f_i over a subset of the tangent hyperplane to $\tilde{\varphi}_i(M) \subset \mathbb{R}^{n+1}$ at the point $\tilde{\varphi}_i(p_i)$. As $\tilde{d}_i(p_i, q_i) = \varepsilon$, when $\varepsilon > 0$ is small enough (depending on r_0 and C_0) the Lipschitz constants of these functions f_i are uniformly bounded by a constant depending on r_0 and C_0 , moreover, for every $i \in \mathbb{N}$ the point q_i stays in U^i and $\tilde{\varphi}_i(q_i)$ belongs to the graph of f_i .

It is then easy to see that there exists a uniform positive bound from below on $|\tilde{\varphi}_i(p_i) - \tilde{\varphi}_i(q_i)|$, hence the constant B_ε cannot be zero for such $\varepsilon > 0$.

Supposing that \tilde{M}_∞ has a self-intersection, we can parametrize it locally with a map $\tilde{\varphi}_\infty : U \rightarrow \mathbb{R}^{n+1}$ such that a sequence of reparametrizations of the rescaled hypersurfaces $\tilde{\varphi}_i$ converges smoothly to $\tilde{\varphi}_\infty$ and $\tilde{\varphi}_\infty(p) = \tilde{\varphi}_\infty(q)$ for a couple of points $p, q \in U$.

Choosing $\varepsilon > 0$ smaller than the intrinsic distance between p and q in \tilde{M}_∞ and such that $B_\varepsilon > 0$, we consider the function

$$L(p, q, t) = |\varphi(p, t) - \varphi(q, t)| / \sqrt{2(T-t)}$$

on $\mathbb{C}\Omega_\varepsilon \subset M \times M \times [0, T)$. If the minimum of L at time t is lower than $B_\varepsilon > 0$ then it cannot be attained on the boundary of Ω_ε and arguing as in the proof of Proposition 2.2.7, such minimum is nondecreasing. Hence, there is a positive lower bound on

$$\inf_{\mathbb{C}\Omega_\varepsilon} |\varphi(p, t) - \varphi(q, t)| / \sqrt{2(T-t)}.$$

Now we are done, since if we consider two sequences $p_i \rightarrow p$ and $q_i \rightarrow q$ we have definitely $\tilde{d}_i(p_i, q_i) > \varepsilon$ and $|\tilde{\varphi}_i(p_i) - \tilde{\varphi}_i(q_i)| \rightarrow 0$, hence $d_{t_i}(p_i, q_i) > \varepsilon\sqrt{2(T-t_i)}$ which implies that $(p_i, q_i, t_i) \in \mathbb{C}\Omega_\varepsilon$ and $|\varphi(p_i, t_i) - \varphi(q_i, t_i)| / \sqrt{2(T-t_i)} \rightarrow 0$, in contradiction with the previous conclusion. \square

OPEN PROBLEM 3.2.11. The limit hypersurface \tilde{M}_∞ is unique? That is, independent of the sequence $s_i \rightarrow +\infty$?

This problem is the parabolic analogue to the long-standing problem of uniqueness of the tangent cone in minimal surface theory.

If the limit hypersurface is compact the answer is positive, as shown recently by Schulze [105].

We have seen in Proposition 1.4.1 that any of these limit hypersurfaces \tilde{M}_∞ satisfying $\tilde{H} + \langle y | \tilde{\nu} \rangle = 0$, that we call *homothetic*, generates a homothetically shrinking mean curvature flow given by $M_t = \tilde{M}_\infty \sqrt{1-2t}$, vanishing at $T = 1/2$.

As we said few explicit examples are available, hyperplanes through the origin, the sphere $\mathbb{S}^n(\sqrt{n})$, the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and the Angenent's torus we mentioned in Section 1.4, described in [17].

OPEN PROBLEM 3.2.12. Classify all the complete hypersurfaces (compact or not) satisfying $H + \langle y | \nu \rangle = 0$, or at least the ones arising as blow up limits of the flow of a compact and embedded hypersurface.

This problem is difficult, an equivalent formulation is to find the critical points of the Huisken's functional

$$\int_M e^{-\frac{|y|^2}{2}} d\mathcal{H}^n.$$

As we will see in the next sections, the classification is possible for curves or under the extra hypothesis $H \geq 0$ when $n \geq 2$.

REMARK 3.2.13. In the case of a homothetically shrinking hypersurface around a point $x_0 \in \mathbb{R}^{n+1}$ and vanishing at time T , the derivative in the monotonicity formula with the backward heat kernel $\rho_{x_0, T}$ is zero, that is, the integral

$$\int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t$$

is constant in time. Conversely, it follows from Proposition 1.4.1 and the uniqueness of the flow, that if such derivative is zero at some time the hypersurface is homothetically shrinking around x_0 as at such time it must satisfy $H + \frac{\langle x-x_0 | \nu \rangle}{2(T-t)} = 0$ for all its points.

Finally, notice that if the flow φ_t is homothetically shrinking around x_0 for $t \in [0, T)$, the relative rescaled hypersurfaces $\tilde{\varphi}_s = \frac{\varphi_t - x_0}{\sqrt{2(T-t)}}$ are not moving at all (as a subset of \mathbb{R}^{n+1}) and conversely.

REMARK 3.2.14. By means of a slight modification of the function σ defined in formula (3.2.3), it is possible to show that a compact *breather* for mean curvature flow is actually homothetically shrinking. A breather is a hypersurface moving by mean curvature such that $M_t = \lambda L(M_{t'})$ for a couple of times $t > t'$, a constant $\lambda > 0$ and an isometry L of \mathbb{R}^{n+1} . Here and below we use the notation $M_s = \varphi(M, s)$.

Notice that *steady* or *expanding* compact breathers, that is when $\lambda = 1$ or $\lambda > 1$, do not exist by comparison with evolving spheres.

We can assume that $t' = 0$ and $t > 0$, for a compact hypersurface M moving by mean curvature in $[0, T)$ and $\tau > 0$ we consider the function

$$\tilde{\sigma}(M, \tau) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4\tau}}}{[4\pi\tau]^{n/2}} d\tilde{\mathcal{H}}^n,$$

then it is easy to see that $\tilde{\sigma}(L(M), \tau) = \tilde{\sigma}(M, \tau)$ and for every $\lambda > 0$

$$\tilde{\sigma}(\lambda M, \lambda^2 \tau) = \tilde{\sigma}(M, \tau). \quad (3.2.11)$$

Setting $\tau(t) = C - t$ for some $C > 0$ and integrating the analogue of equation (3.2.4) we have

$$\tilde{\sigma}(M_0, \tau(0)) - \tilde{\sigma}(M_t, \tau(t)) = \int_0^t \int_{M_s} \frac{e^{-\frac{|x-x_{\tau(s)}|^2}{4\tau(s)}}}{[4\pi\tau(s)]^{n/2}} \left| H + \frac{\langle x - x_{\tau(s)} | \nu \rangle}{2\tau(s)} \right|^2 d\tilde{\mathcal{H}}^n ds.$$

By the rescaling property of $\tilde{\sigma}$ in formula (3.2.11) and the hypothesis, we get

$$\tilde{\sigma}(M_0, C) \geq \tilde{\sigma}(M_t, C - t) = \tilde{\sigma}(\lambda M_0, C - t) = \tilde{\sigma}(M_0, (C - t)/\lambda^2).$$

If now we choose $C = \frac{t}{1-\lambda^2} > t$ as $\lambda < 1$, we have $(C - t)/\lambda^2 = C$, it follows that

$$\tilde{\sigma}(M_0, \tau(0)) = \tilde{\sigma}(M_0, C) = \tilde{\sigma}(M_t, C - t) = \tilde{\sigma}(M_t, \tau(t)),$$

hence, by the formula above for almost every $s \in (0, t)$ we have $H + \frac{\langle x - y_s | \nu \rangle}{2(C-s)}$ for some $y_s \in \mathbb{R}^{n+1}$. This clearly implies that we are dealing with a homothetically shrinking hypersurface.

We now fix a point $p \in M$ and consider a sequence of rescaled hypersurfaces $\tilde{\varphi}_{s_i}$, locally smoothly converging (up to reparametrization) to some limit hypersurface \tilde{M}_∞ which satisfies $\tilde{H} + \langle y | \tilde{\nu} \rangle = 0$ for every $y \in \tilde{M}_\infty$.

We want to relate the limit heat density $\Theta(p)$ in Definition 3.2.3 with \widetilde{M}_∞ ,

$$\begin{aligned}\Theta(p) &= \lim_{t \rightarrow T} \theta(p, t) \\ &= \lim_{i \rightarrow \infty} \int_M \frac{e^{-\frac{|x - \widehat{p}|^2}{4(T-t(s_i))}}}{[4\pi(T-t(s_i))]^{n/2}} d\mu_{t(s_i)} \\ &= \lim_{i \rightarrow \infty} \int_M \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{n/2}} d\widetilde{\mu}_{s_i} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}} d\widetilde{\mathcal{H}}^n,\end{aligned}$$

where in the last passage we applied Corollary 3.2.8.

In particular, if \widetilde{M}_∞ is a unit multiplicity hyperplane through the origin of \mathbb{R}^{n+1} then $\Theta(p) = \frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}} d\widetilde{\mathcal{H}}^n = 1$.

REMARK 3.2.15. If we choose a time $\tau > 0$ which is strictly less than the maximal time T of existence of the flow and we perform the rescaling procedure around the *nonsingular* point $\widehat{p} = \lim_{t \rightarrow \tau} \varphi(p, t) = \varphi(p, \tau)$, being the hypersurface regular around p at time τ , every limit of rescaled hypersurfaces must be flat, actually a union of hyperplanes through the origin. If moreover at $\varphi(p, \tau)$ the hypersurface has no self-intersections, such limit is a *single* hyperplane through the origin and

$$\lim_{t \rightarrow \tau} \int_M \frac{e^{-\frac{|x - \varphi(p, \tau)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\mu_t = 1.$$

This clearly holds for every $p \in M$ if the initial hypersurface is embedded.

REMARK 3.2.16. By the previous remark, if $\tau \in (0, T)$ and $x_0 = \varphi_\tau(p)$ we have

$$\lim_{t \rightarrow \tau} \int_M \frac{e^{-\frac{|x - x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\mu_t = 1$$

and

$$\int_M \frac{e^{-\frac{|x - x_0|^2}{4\tau}}}{[4\pi\tau]^{n/2}} d\mu_0 \geq 1$$

by the monotonicity formula, for every $p \in M$.

Then,

$$\text{Area}(\varphi_0) \geq \int_M e^{-\frac{|x - x_0|^2}{4\tau}} d\mu_0 \geq [4\pi\tau]^{n/2}$$

and $\tau \leq [\text{Area}]^{2/n}/(4\pi)$. As this holds for every $\tau < T$, we get the estimate $T \leq [\text{Area}]^{2/n}/(4\pi)$ on the maximal time T of existence of the flow (which is independent of the type I singularity hypothesis).

LEMMA 3.2.17 (White [126]). *Among all the smooth, complete, hypersurfaces M in \mathbb{R}^{n+1} satisfying $H + \langle y | \nu \rangle = 0$ and $\int_M e^{-|y|^2} d\widetilde{\mathcal{H}}^n < +\infty$, the hyperplanes with unit multiplicity through the origin are the only minimizers of the functional*

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} d\widetilde{\mathcal{H}}^n.$$

Hence, for all such hypersurfaces the value of this integral is at least 1.

PROOF. Suppose that there exists a smooth hypersurface $M = M_0$ such that

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} d\widetilde{\mathcal{H}}^n \leq 1$$

and satisfies $H + \langle y | \nu \rangle = 0$, then the flow $M_t = M\sqrt{1-2t}$ is a smooth mean curvature flow in the time interval $(-\infty, 1/2)$.

Choosing a point $y_0 \in \mathbb{R}^{n+1}$ and a time $\tau \leq 1/2$ we consider the limit

$$\lim_{t \rightarrow -\infty} \int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\tilde{\mathcal{H}}^n,$$

where all the integrals are well defined since $\int_M e^{-|y|} d\tilde{\mathcal{H}}^n < +\infty$.

Changing variables, we have

$$\lim_{t \rightarrow -\infty} \int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\tilde{\mathcal{H}}^n(y) = \lim_{t \rightarrow -\infty} \int_M \frac{e^{-\frac{|x\sqrt{1-2t}-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)/(1-2t)]^{n/2}} d\tilde{\mathcal{H}}^n(x).$$

As $t \rightarrow -\infty$, the sequence of functions inside the integral pointwise converges to the function $e^{-\frac{|x|^2}{2}}/(2\pi)^{n/2}$ and they are definitely uniformly bounded from above, outside some large fixed ball $B_R(0) \subset \mathbb{R}^{n+1}$, by the function $e^{-|x|}$. Since this last function is integrable on M by the hypothesis, using the dominated convergence theorem we get

$$\lim_{t \rightarrow -\infty} \int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\tilde{\mathcal{H}}^n = \frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|x|^2}{2}} d\tilde{\mathcal{H}}^n \leq 1.$$

By the monotonicity formula this implies that

$$\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\tilde{\mathcal{H}}^n \leq 1$$

for every $y_0 \in \mathbb{R}^{n+1}$ and $t < \tau \in (-\infty, 1/2)$.

Choosing now $y_0 \in M$ and $\tau = 0$, repeating the argument in Remark 3.2.15 (in this noncompact case it can be carried on by means of the hypothesis $\int_M e^{-|y|} d\tilde{\mathcal{H}}^n < +\infty$) we have

$$\lim_{t \rightarrow 0^-} \int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}} d\tilde{\mathcal{H}}^n = 1,$$

hence, we conclude that the function

$$\int_{M_t} \frac{e^{-\frac{|y-y_0|^2}{-4t}}}{[-4\pi t]^{n/2}} d\tilde{\mathcal{H}}^n$$

is constant equal to 1 for every $t \in (-\infty, 0)$. Even if the evolving hypersurfaces M_t are not compact, by the hypothesis $\int_M e^{-|y|} d\tilde{\mathcal{H}}^n < +\infty$ it is straightforward to check (writing every integral as an integral on M fixed) that the monotonicity formula still holds. Hence, we must have that the right hand side of such formula is identically zero and $H(y, t) + \frac{\langle y-y_0 | \nu(y) \rangle}{-2t} = 0$ for every $t < 0$ and $y \in M_t$. Multiplying by $-2t$ and sending t to zero, as $M_t \rightarrow M$, we conclude that $\langle y-y_0 | \nu(y) \rangle = 0$ for every $y, y_0 \in M$. This condition easily implies that M is a hyperplane through the origin of \mathbb{R}^{n+1} . \square

REMARK 3.2.18. The smoothness hypothesis can be weakened in this lemma, provided that the set M satisfies some definition of mean curvature to give sense to the condition $H + \langle y | \nu \rangle = 0$ and coherent with the monotonicity formula (for instance, one can allow integral varifolds with bounded first variation, see [21, 80]).

It is not known by the author whether the hypothesis $\int_M e^{-|y|} d\tilde{\mathcal{H}}^n < +\infty$ can be removed. Anyway, it is satisfied by every limit hypersurface obtained as blow up limit, by Corollary 3.2.8.

The following corollary is the consequence of Lemma 3.2.17 and the previous discussion about the relation between Θ and the limits of sequences of rescaled hypersurfaces.

COROLLARY 3.2.19. *The function $\Theta : M \rightarrow \mathbb{R}$ satisfies $\Theta \geq 1$ on all M . Moreover, if $\Theta(p) = 1$, every converging sequence of rescaled hypersurfaces $\tilde{\varphi}_{s_i}$ around \hat{p} converges to a unit multiplicity hyperplane through the origin of \mathbb{R}^{n+1} . It follows that $\Sigma \geq 1$ (recall Definition 3.2.3).*

REMARK 3.2.20. The fact that $\Theta \geq 1$ on all M can also be proved directly using the argument in Remark 3.2.16. Since for every $\tau < T$ we have

$$\lim_{t \rightarrow \tau} \int_M \frac{e^{-\frac{|x - \varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\mu_t = 1$$

we get

$$\int_M \frac{e^{-\frac{|x - \varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\mu_t \geq 1$$

for every $t < \tau$. Keeping now $t < T$ fixed and sending $\tau \rightarrow T$ we have $\varphi_\tau(p) \rightarrow \hat{p}$ and

$$\theta(p, t) = \int_M \frac{e^{-\frac{|x - \hat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t = \lim_{\tau \rightarrow T} \int_M \frac{e^{-\frac{|x - \varphi_\tau(p)|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}} d\mu_t \geq 1.$$

This clearly implies that $\Theta(p) = \lim_{t \rightarrow T} \theta(p, t) \geq 1$.

REMARK 3.2.21. Rescaling around some \hat{p} , by the discussion after Definition 3.2.2, means rescaling around some *reachable* point. Actually, we could rescale the evolving hypersurfaces around *any* point $x_0 \in \mathbb{R}^{n+1}$ but if $x_0 \notin \mathcal{S}$, as the distance from $\varphi(M, t)$ and x_0 is definitely positive, the limit hypersurface is empty. This would imply that

$$\int_M \frac{e^{-\frac{|x - x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t \rightarrow 0$$

as $t \rightarrow T$.

By Corollary 3.2.19, if instead we consider $x_0 \in \mathcal{S}$, that is, $x_0 = \hat{p}$ for some $p \in M$, there holds $\Theta(p) \geq 1$. Hence, there is a dichotomy between the points of \mathbb{R}^{n+1} , according to the value of this extended limit heat density function which can be either zero or at least one.

Moreover, by looking carefully at the first part of the proof of Lemma 3.2.17 we can see that this fact is independent of the type I hypothesis, it is indeed only a consequence of the upper semicontinuity of $\theta(p, t)$.

Actually, one can say more by the following result of White [126] (see also [36, Theorems 5.6, 5.7] and [118], moreover compare with [21, Theorem 6.11]), which also gives a partial answer to Problem 3.2.11.

THEOREM 3.2.22 (White [126, Theorem 3.5]). *There exist constants $\varepsilon = \varepsilon(n) > 0$ and $C = C(\varphi_0)$ such that if $\Theta(p) < 1 + \varepsilon$, then $|A| \leq C$ in a ball of \mathbb{R}^{n+1} around \hat{p} uniformly in time $t \in [0, T)$.*

If the limit of a subsequence of rescaled hypersurfaces is a hyperplane through the origin, then $\Theta(p) = 1$ and by this theorem there is a ball around \hat{p} where the curvature is bounded. Then in such a ball, the *unscaled* hypersurfaces φ_t (possibly after a reparametrization) converge locally uniformly in C^0 to some φ_T with uniformly bounded curvature, this implies that the convergence is actually in C^∞ by the interior estimates of Ecker and Huisken in [39]. Hence, it follows easily that the tangent hyperplane to φ_T at the point \hat{p} coincides with the limit of *any* sequence of rescaled hypersurfaces, that is, there is full convergence and the limit hypersurface is unique, solving affirmatively Problem 3.2.11 in this very special case.

REMARK 3.2.23. The strength of White's result is that it does not assume any condition on the sign of H and on the blow up rate of the curvature. The theorem also holds without the type I hypothesis.

Another consequence is that there is a “gap” between the value 1 realized by the hyperplanes through the origin of \mathbb{R}^{n+1} in the functional

$$\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mathcal{H}}^n$$

and any other smooth, complete hypersurface M in \mathbb{R}^{n+1} satisfying $H + \langle y | \nu \rangle = 0$ and $\int_M e^{-|y|} d\tilde{\mathcal{H}}^n < +\infty$.

3.3. Analysis of Singularities

DEFINITION 3.3.1. We say that $p \in M$ is a *singular point* if there exists a sequence of points $p_i \rightarrow p$ in M and times $t_i \rightarrow T$ such that for some constant $\delta > 0$ there holds

$$|A(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T - t_i)}}, \quad (3.3.1)$$

for every $i \in \mathbb{N}$.

We say that $p \in M$ is a *special singular point* if there exists a sequence of times $t_i \rightarrow T$ such that for some constant $\delta > 0$ there holds

$$|A(p, t_i)| \geq \frac{\delta}{\sqrt{2(T - t_i)}},$$

for every $i \in \mathbb{N}$.

The lower estimate on the blow up rate of the curvature

$$\max_{p \in M} |A(p, t)| \geq \frac{1}{\sqrt{2(T - t)}}.$$

and the compactness of M clearly imply that there always exists at least one singular point, but not that any *special* singular point is present.

REMARK 3.3.2. Though we will concentrate on the above definition of singular points at a type I singularity, it could possibly happen that while around some point $|A|$ is blowing up at least like $\frac{1}{\sqrt{2(T-t)}}$, at the same time in another region the hypersurface is getting singular with the curvature going to infinity at a lower rate. So one can say that $p \in M$ is a *mild singular point* if there exists a sequence of points $p_i \rightarrow p$ in M and times $t_i \rightarrow T$ such that simply

$$|A(p_i, t_i)| \rightarrow +\infty,$$

but for no constant $\delta > 0$ there exist a sequence of points $p_i \rightarrow p$ and times $t_i \rightarrow T$ such that inequality 3.3.1 holds, for every $i \in \mathbb{N}$.

It is easy to see that, performing the rescaling procedure of the previous section around a mild singular point $p \in M$, any limit hypersurface must be flat, hence, the union of a finite number of hyperspaces. Then, in the case of the flow of an embedded hypersurface, actually it has to be a single hyperspace, which implies $\Theta(p) = 1$. Thus, White’s Theorem 3.2.22 tell us that the curvature must be uniformly bounded in a neighborhood $U \subset \mathbb{R}^{n+1}$ of the point \hat{p} , for every time $t \in [0, T)$. This is in contradiction with the fact that p is a mild singular point, indeed, $\varphi(p_i, t_i) \rightarrow \hat{p}$, as $i \rightarrow \infty$, since, by inequality 3.2.2, we have

$$|\varphi(p_i, t_i) - \hat{p}| \leq |\varphi(p_i, t_i) - \hat{p}_i| + |\hat{p}_i - \hat{p}| \leq C_0 \sqrt{n(T - t_i)} + |\varphi_T(p_i) - \varphi_T(p)|,$$

for every $i \in \mathbb{N}$ and the map $\varphi_T : M \rightarrow \mathbb{R}^{n+1}$ is continuous, so the point $\varphi(p_i, t_i)$ is definitely in the open set U , but $A(p_i, t_i) \rightarrow +\infty$.

A different line to exclude such mild singular points in the case of a flow of embedded hypersurfaces with nonnegative mean curvature was followed by Stone in [118, Section 4].

Notice that this fact is a sort of “localization” of the bound from below (2.4.1) on the blow up rate of the curvature. It is unknown to the author if the presence of points with such “bad” behavior at a type I singularity can also be excluded when the evolving hypersurfaces are not necessarily embedded, at least in the case of nonnegative mean curvature.

Suppose that $p \in M$ is a special singular point, then after rescaling the hypersurface as before around \hat{p} we have for $s_i = -\frac{1}{2} \log(T - t_i)$,

$$|\tilde{A}(p, s_i)| = \sqrt{2(T - t_i)} |A(p, t_i)| \geq \delta > 0$$

which implies, by estimate (3.2.5), that taking a subsequence of $s_i \rightarrow +\infty$, any limit hypersurface obtained by Proposition 3.2.10 cannot be flat, as $\tilde{A} \neq 0$ at some point in the ball $B_{C_0\sqrt{2n}}$.

If $p \in M$ is not a special singular point clearly, for every sequence $s_i \rightarrow +\infty$,

$$|\tilde{A}(p, s_i)| = \sqrt{2(T - t_i)} |A(p, t_i)| \rightarrow 0,$$

that is, any limit hypersurface satisfies $\tilde{A} = 0$ at some point in the ball $B_{C_0\sqrt{2n}}$.

OPEN PROBLEM 3.3.3. Is it true that any limit hypersurface associated to a nonspecial singular point is a union of hyperplanes through the origin?

This conclusion would follow if any nonflat hypersurface M satisfying $H + \langle y | \nu \rangle = 0$ and $\int_M e^{-|y|} d\tilde{\mathcal{H}}^n < +\infty$ cannot have a point where the second fundamental form is zero.

By means of a small variation of an argument by Stone, we have a good description when the limit hypersurface is a single hyperplane.

PROPOSITION 3.3.4 (Stone [118]). *If the limit of rescaled hypersurfaces around \hat{p} is a unit multiplicity hyperplane through the origin of \mathbb{R}^{n+1} or equivalently by Lemma 3.2.17 there holds $\Theta(p) = 1$, then p cannot be a singular point.*

PROOF. By Corollary 3.2.19, the point $p \in M$ is a minimum of $\Theta : M \rightarrow \mathbb{R}$ which is an upper semicontinuous function. Hence p is actually a continuity point for Θ . We want to show that for every sequence $p_i \rightarrow p$ and $t_i \rightarrow T$ we have $\theta(p_i, t_i) \rightarrow 1 = \Theta(p)$.

Suppose that there exists $\delta > 0$ such that $\theta(p_i, t_i) \rightarrow 1 + \delta$. For every $j \in \mathbb{N}$ there exists i_0 such that $t_i \geq t_j$ for every $i > i_0$, hence $\theta(p_i, t_i) \leq \theta(p_i, t_j)$. Sending $i \rightarrow \infty$ we then get $1 + \delta \leq \theta(p, t_j)$. This is clearly a contradiction, as sending now $j \rightarrow \infty$, we have $\theta(p, t_j) \rightarrow \Theta(p) = 1$ (what we did is closely related to Dini's theorem on monotone convergence of continuous functions).

If p is a singular point with $p_i \rightarrow p$ and $t_i \rightarrow T$ such that for some constant $\delta > 0$ there holds $|A(p_i, t_i)| \geq \frac{\delta}{\sqrt{2(T - t_i)}}$, we consider the families of rescaled hypersurfaces around \hat{p}_i ,

$$\tilde{\varphi}_i(q, s) = \frac{\varphi(q, t) - \hat{p}_i}{\sqrt{2(T - t)}} \quad s = s(t) = -\frac{1}{2} \log(T - t)$$

with associated measures $\tilde{\mu}_{i,s}$ and we set

$$\psi_i(q) = \tilde{\varphi}_i(q, s_i) = \frac{\varphi(q, t_i) - \hat{p}_i}{\sqrt{2(T - t_i)}} \quad s_i = -\frac{1}{2} \log(T - t_i),$$

with associated measures $\tilde{\mu}_{i,s_i}$.

For every $\varepsilon > 0$, as $\Theta(p_i) \geq 1$ we have definitely

$$\begin{aligned} \varepsilon &\geq \theta(p_i, t_i) - 1 \geq \theta(p_i, t_i) - \Theta(p_i) \\ &= \int_M \frac{e^{-\frac{|x - \hat{p}_i|^2}{4(T - t_i)}}}{[4\pi(T - t_i)]^{n/2}} d\mu_{t_i} - \Theta(p_i) \\ &= \frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_{i,s_i} - \Theta(p_i) \\ &= \frac{1}{(2\pi)^{n/2}} \int_{s_i}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} |\tilde{H} + \langle y | \tilde{\nu} \rangle|^2 d\tilde{\mu}_{i,s} ds. \end{aligned}$$

Hence, since by the uniform curvature estimates of Proposition 3.2.9, see computation (3.2.10), we have,

$$\left| \frac{d}{ds} \int_M e^{-\frac{|y|^2}{2}} |\tilde{H} + \langle y | \tilde{\nu} \rangle|^2 d\tilde{\mu}_s \right| \leq C$$

where $C = C(\text{Area}(\varphi_0), T)$ is a positive constant independent of s , we get

$$\begin{aligned} \varepsilon &\geq \frac{1}{(2\pi)^{n/2}} \int_{s_i}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left| \tilde{H} + \langle y | \tilde{\nu} \rangle \right|^2 d\tilde{\mu}_{i,s} ds \\ &\geq \frac{1}{(2\pi)^{n/2}} \int_{s_i}^{s_i + \frac{1}{C}} \int_M e^{-\frac{|y|^2}{2}} \left| \tilde{H} + \langle y | \tilde{\nu} \rangle \right|^2 d\tilde{\mu}_{i,s_i} \left(\int_M e^{-\frac{|y|^2}{2}} \left| \tilde{H} + \langle y | \tilde{\nu} \rangle \right|^2 d\tilde{\mu}_{i,s_i} - C(s - s_i) \right) ds \\ &= \frac{1}{(2\pi)^{n/2}} \frac{1}{2C} \left(\int_M e^{-\frac{|y|^2}{2}} \left| \tilde{H} + \langle y | \tilde{\nu} \rangle \right|^2 d\tilde{\mu}_{i,s_i} \right)^2. \end{aligned}$$

If now we proceed like in Proposition 3.2.10 and we extract from the sequence of hypersurfaces ψ_i a locally smoothly converging subsequence (up to reparametrization) to some limit hypersurface \tilde{M}_∞ , by Lemma 3.2.7 we have

$$\varepsilon \geq \frac{1}{(2\pi)^{n/2}} \frac{1}{2C} \left(\int_{\tilde{M}_\infty} e^{-\frac{|y|^2}{2}} \left| \tilde{H} + \langle y | \tilde{\nu} \rangle \right|^2 d\tilde{\mathcal{H}}^n \right)^2,$$

for every $\varepsilon > 0$, hence \tilde{M}_∞ satisfies $\tilde{H} + \langle y | \tilde{\nu} \rangle = 0$.

Finally, by Corollary 3.2.8,

$$\frac{1}{(2\pi)^{n/2}} \int_{\tilde{M}_\infty} e^{-\frac{|y|^2}{2}} d\tilde{\mathcal{H}}^n = \lim_{i \rightarrow \infty} \frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mu}_{i,s_i} = \lim_{i \rightarrow \infty} \theta(p_i, t_i) = 1$$

then, by Lemma 3.2.17, the hypersurface \tilde{M}_∞ has to be a hyperplane. But since all the points $\psi_i(p_i)$ belong to the ball of radius $C_0\sqrt{2n} \subset \mathbb{R}^{n+1}$ and the second fundamental form \tilde{A}_i of ψ_i satisfies $|\tilde{A}_i(p_i)| \geq \delta > 0$ for every $i \in \mathbb{N}$, by the hypothesis, it follows that the second fundamental form of \tilde{M}_∞ is not zero at some point in the ball $B_{C_0\sqrt{2n}}(0)$.

Since we have a contradiction, p cannot be a singular point of the flow. \square

COROLLARY 3.3.5. *There holds $\Sigma > 1$ (see Definition 3.2.3), hence, there must exist a point $p \in M$ such that $\Theta(p) > 1$.*

PROOF. By Corollary 3.2.19, we have $\Sigma \geq 1$. If $\Sigma = 1$, it would follow that $\Theta(p) = 1$ for every $p \in M$, hence, there are no singular points by the previous proposition. This is in contradiction with the fact that there exists always at least one singular point $p \in M$. \square

REMARK 3.3.6. This proposition and its corollary are an immediate consequence of White's Theorem 3.2.22, which actually implies that $\Sigma \geq 1 + \varepsilon(n)$, for some constant $\varepsilon(n)$. Indeed, if $\Sigma < 1 + \varepsilon$, also $\Theta(p) < 1 + \varepsilon$ for every $p \in M$, then, as the set of reachable points S is compact, by a covering argument and White's Theorem we can conclude that the curvature is uniformly bounded as $t \rightarrow T$, which is a contradiction.

We wanted anyway to emphasize the fact that the only really needed "ingredient", by means of the line of analysis of Stone, is the uniqueness of the hyperplanes as minimizers of the integral $\frac{1}{(2\pi)^{n/2}} \int_M e^{-\frac{|y|^2}{2}} d\tilde{\mathcal{H}}^n$ among the hypersurfaces satisfying $\tilde{H} + \langle y | \tilde{\nu} \rangle = 0$ with $\int_M e^{-|y|} d\tilde{\mathcal{H}}^n < +\infty$ (Lemma 3.2.17).

OPEN PROBLEM 3.3.7. To the author's knowledge, even if we are dealing with the flow of embedded hypersurfaces, the existence of at least one special singular point is an open problem (the fact that $\Theta(p) > 1$ does not necessarily implies that the point $p \in M$ is a special singular point).

A related stronger statement would be that *every singular point is a special singular point*.

This problem and Problem 3.3.3, in the embedded situation are quite connected. Indeed, if the initial hypersurface is embedded any limit hypersurface \tilde{M}_∞ is also embedded, so the union of two or more hyperplanes cannot arise.

This means that if Problem 3.3.3 has a positive answer, for every nonspecial singular point $p \in M$ any limit hypersurfaces can be only a single unit multiplicity hyperplane through the origin, hence, by Proposition 3.3.4 such point is actually not singular.

Then, if there are no special singular points, it follows that there are no singular points at all which is a contradiction and also Problem 3.3.7 has a positive answer.

Repeating this argument in the general nonembedded case, unfortunately one could obtain as limit hypersurface, a *union* of hyperplanes or, even more disturbing, hyperplanes with integer multiplicities higher than one.

Putting together Propositions 3.2.10 and 3.3.4, we have the following conclusion, as the only flat blow up limits of a mean curvature flow of an embedded hypersurface are the hyperplanes through the origin of \mathbb{R}^{n+1} .

COROLLARY 3.3.8. *At a singular point $p \in M$ a limit \widetilde{M}_∞ of rescaled hypersurfaces is a smooth, nonempty, complete hypersurface with bounded local volume and bounded curvature with all its covariant derivatives, which satisfies $\widetilde{H} + \langle y | \widetilde{\nu} \rangle = 0$ and it is not a single unit multiplicity hyperplane through the origin of \mathbb{R}^{n+1} .*

Moreover, if the initial hypersurface is embedded \widetilde{M}_∞ is also embedded and nonflat.

REMARK 3.3.9. We can produce a nonflat homothetic blow up limit also by rescaling the evolving hypersurfaces around some *moving* points as follows. For every $t \in [0, T)$, we let x_t be any maximum point realizing $\sigma(t)$ in formula (3.2.3), that is,

$$\sigma(t) = \max_{x_0 \in \mathbb{R}^{n+1}} \int_M \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t = \int_M \frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}} d\mu_t$$

and we consider the rescaled hypersurfaces with associated measures $\widetilde{\mu}_s$ around the points x_t ,

$$\widetilde{\varphi}(q, s) = \frac{\varphi(q, t(s)) - x_{t(s)}}{\sqrt{2(T-t(s))}} \quad s = s(t) = -\frac{1}{2} \log(T-t).$$

Rescaling also the *maximal* monotonicity formula (3.2.4) we have

$$\frac{d}{ds} \int_M e^{-\frac{|y|^2}{2}} d\widetilde{\mu}_s = - \int_M e^{-\frac{|y|^2}{2}} \left| \widetilde{H} + \langle y | \widetilde{\nu} \rangle \right|^2 d\widetilde{\mu}_s \leq 0.$$

It follows that, integrating this formula as before, we get

$$\sigma(0) - \Sigma = \frac{1}{(2\pi)^{n/2}} \int_{-\frac{1}{2} \log T}^{+\infty} \int_M e^{-\frac{|y|^2}{2}} \left| \widetilde{H} + \langle y | \widetilde{\nu} \rangle \right|^2 d\widetilde{\mu}_s ds < +\infty,$$

and with the same argument we can produce a homothetic limit hypersurface \widetilde{M}_∞ such that

$$\frac{1}{(2\pi)^{n/2}} \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}} d\widetilde{\mathcal{H}}^n = \Sigma.$$

Since we saw that $\Sigma > 1$, the limit hypersurface \widetilde{M}_∞ cannot be a single unit multiplicity hyperplane through the origin.

3.4. Hypersurfaces with Nonnegative Mean Curvature

For most of this section we assume that all the hypersurfaces are embedded. At the end we will discuss the general immersed case.

If the compact initial hypersurface is embedded and has $H \geq 0$ (or at some positive time the evolving hypersurface becomes like that) then the analysis of the previous section can be pushed forward since we can restrict the class of the possible limits of rescaled hypersurfaces to the ones satisfying these two conditions.

In this case *every* singular point is a special singular point and it is actually possible to classify all the embedded limit hypersurfaces in \mathbb{R}^{n+1} such that $H + \langle x | \nu \rangle = 0$ and $H \geq 0$, see [68, 69] and [119].

It will follow that Problem 3.3.3 and consequently Problem 3.3.7 have a positive answer.

Finally, as we mentioned in Remark 3.3.2, there cannot happen that the curvature in some region of the hypersurface goes to infinity with a rate lower than $\frac{\delta}{\sqrt{2(T-t)}}$, for every constant $\delta > 0$.

We recall that if $H \geq 0$ for the initial hypersurface, after a positive time $t_0 > 0$, there exists a constant $\alpha > 0$ such that $\alpha|A| \leq H \leq n|A|$ everywhere on M for every time $t \geq t_0$, by Corollary 2.4.3.

Hence, we can assume in the sequel that for $t \in [0, T)$ we have

$$\frac{\alpha}{\sqrt{2(T-t)}} \leq \max_{p \in M} H(p, t) \leq \frac{C}{\sqrt{2(T-t)}}.$$

PROPOSITION 3.4.1 (Huisken [68, 69], Abresch and Langer [1] in the one-dimensional case). *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, complete, embedded, mean convex hypersurface in \mathbb{R}^{n+1} such that $H + \langle x | \nu \rangle = 0$ at every $x \in M$ and there exists a constant C such that $|A| + |\nabla A| \leq C$ and $\mathcal{H}^n(M \cap B_R) \leq Ce^R$, for every ball of radius $R > 0$ in \mathbb{R}^{n+1} .*

Then, up to a rotation of \mathbb{R}^{n+1} , M must be one of only $n + 1$ possible hypersurfaces, namely, either a hyperplane through the origin or the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.

In the special one-dimensional case the only smooth, complete, embedded curves in \mathbb{R}^2 satisfying the structural equation $k + \langle x | \nu \rangle = 0$ are the lines through the origin and the unit circle (notice that in this case neither the positivity of the curvature nor the two estimates above are assumed).

PROOF. Let us suppose that M is connected. If the theorem is true in this case, it is easy to see that it is not possible to have a nonconnected embedded hypersurface satisfying the hypotheses. Indeed, any connected component has to belong to the list of the statement and every two hypersurfaces in such list either coincide or have some intersections.

We first deal separately with the case $n = 1$.

Fixing a reference point on a curve γ we have an arclength parameter s which gives a unit tangent vector field $\tau = \gamma_s$ and a unit normal vector field ν , which is the counterclockwise rotation of $\pi/2$ in \mathbb{R}^2 of the vector τ , then the curvature is given by $k = \langle \partial_s \tau | \nu \rangle = -\langle \tau | \partial_s \nu \rangle$.

The relation $k = -\langle \gamma | \nu \rangle$ implies the ODE for the curvature $k_s = k \langle \gamma | \tau \rangle$. Suppose that at some point $k = 0$, then also $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODE's we conclude that k is identically zero and we are dealing with a line L which, as $\langle x | \nu \rangle = 0$ for every $x \in L$, must contain the origin of \mathbb{R}^2 .

So we suppose that k is always nonzero and possibly reversing the orientation of the curve, we assume that $k > 0$ at every point, that is, the curve is strictly convex.

Computing the derivative of $|\gamma|^2$,

$$\partial_s |\gamma|^2 = 2 \langle \gamma | \tau \rangle = 2k_s/k = 2\partial_s \log k$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C > 0$, it follows that k is bounded from below by $C > 0$.

We consider now a new coordinate $\theta = \arccos \langle e_1 | \nu \rangle$, this can be done for the whole curve as we know that this latter is convex (obviously, as for the arclength parameter s it is only locally continuous, θ "jumps" after a complete round).

Differentiating with respect to the arclength parameter we have $\partial_s \theta = k$ and

$$k_\theta = k_s/k = \langle \gamma | \tau \rangle \quad k_{\theta\theta} = \frac{\partial_s k_\theta}{k} = \frac{1 + k \langle \gamma | \nu \rangle}{k} = \frac{1}{k} - k. \quad (3.4.1)$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\partial_\theta [k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity $k_\theta^2 + k^2 - \log k^2$ is equal to some constant E along all the curve. Notice that such quantity E cannot be less than 1, moreover, if $E = 1$ we have that k must be constant and equal to 1 along the curve, which consequently must be the unit circle centered at the origin of \mathbb{R}^2 .

When $E > 1$, it follows that k is uniformly bounded from above, hence recalling that $k = Ce^{|\gamma|^2/2}$, the image of the curve is contained in a ball of \mathbb{R}^2 and by the embeddedness and completeness hypotheses, the curve must be closed, simple and strictly convex, as $k > 0$ at every point.

We now suppose that γ is not the unit circle and we look at the critical points of the curvature k . Since $k_{\theta\theta} = \frac{1}{k} - k$, there holds that $k_{\theta\theta} \neq 0$ when $k_\theta = 0$, otherwise this second order ODE for k would imply $k_\theta = 0$ everywhere, hence $k = 1$ identically and we would be in the case of the unit circle. Thus, the critical points of the curvature are not degenerate, hence, by the compactness of the curve they are isolated and finite. Moreover, by looking at the equation for the curvature (3.4.1) we can see easily that $k_{\min} < 1$ and $k_{\max} > 1$.

Suppose now that $k(0) = k_{\max}$ and $k(\bar{\theta})$ is the first subsequent critical value for k , for some $\bar{\theta} > 0$. Then the curvature is strictly decreasing in the interval $[0, \bar{\theta}]$ and again by the second order ODE, the function k (hence also the curve, by integration) is symmetric with respect to $\theta = 0$ and $\theta = \bar{\theta}$. This clearly implies that $k(\bar{\theta})$ must be the minimum k_{\min} of the curvature, as every critical point is not degenerate.

By the four vertex theorem [94, 102], on every closed curve there are at least four critical points of k , as a consequence our curve is composed of at least four pieces like the one described above. Hence, since the curve is closed and embedded the curvature $k(\theta)$ must be a periodic function with period $T > 0$ not larger than π (since 2π is an obvious multiple of the period) and $\bar{\theta} = T/2$. More precisely, the period T must be $2\pi/n$ for some $n \geq 2$.

By a straightforward computation, starting by differentiating the equation $k_{\theta\theta} = \frac{1}{k} - k$, one gets $(k^2)_{\theta\theta\theta} + 4(k^2)_{\theta} = 4k_{\theta}/k$, then we compute

$$\begin{aligned} 4 \int_0^{T/2} \sin 2\theta \frac{k_{\theta}}{k} d\theta &= \int_0^{T/2} \sin 2\theta [(k^2)_{\theta\theta\theta} + 4(k^2)_{\theta}] d\theta \\ &= \sin 2\theta (k^2)_{\theta\theta} \Big|_0^{T/2} - 2 \int_0^{T/2} \cos 2\theta (k^2)_{\theta\theta} d\theta + 4 \int_0^{T/2} \sin 2\theta (k^2)_{\theta} d\theta \\ &= 2 \sin T [k(T/2)k_{\theta\theta}(T/2) + k_{\theta}^2(T/2)] - 2 \cos 2\theta (k^2)_{\theta} \Big|_0^{T/2} \\ &\quad - 4 \int_0^{T/2} \sin 2\theta (k^2)_{\theta} d\theta + 4 \int_0^{T/2} \sin 2\theta (k^2)_{\theta} d\theta \\ &= 2 \sin T [k(T/2)k_{\theta\theta}(T/2) + k_{\theta}^2(T/2)] \\ &\quad - 4 \cos T k(T/2)k_{\theta}(T/2) + 4k(0)k_{\theta}(0). \end{aligned}$$

Now, since $k_{\theta}(0) = k_{\theta}(T/2) = 0$ and $k(T/2) = k_{\min}$, using the equation for the curvature $k_{\theta\theta} = 1/k - k$ we get

$$4 \int_0^{T/2} \sin 2\theta \frac{k_{\theta}}{k} d\theta = 2 \sin T (1 - k_{\min}^2),$$

and this last term is nonnegative as $k_{\min} < 1$ and $0 < T \leq \pi$.

Looking at the left hand integral we see instead that the factor $\sin 2\theta$ is always nonnegative, since $T \leq \pi$ and k_{θ} is always nonpositive in the interval $[0, T/2]$, as we assumed that we were moving from the maximum k_{\max} at $\theta = 0$ to the minimum k_{\min} at $\theta = T/2$ without crossing any other critical point of k . This gives a contradiction so γ must be the unit circle.

We suppose now that $n \geq 2$.

By covariant differentiation of the equation $H + \langle x | \nu \rangle = 0$ in an orthonormal frame $\{e_1, \dots, e_n\}$ on M we get

$$\begin{aligned} \nabla_i H &= \langle x | e_k \rangle h_{ik} \\ \nabla_i \nabla_j H &= h_{ij} + \langle x | \nu \rangle h_{ik} h_{jk} + \langle x | e_k \rangle \nabla_i h_{jk} = h_{ij} - H h_{ik} h_{jk} + \langle x | e_k \rangle \nabla_k h_{ij} \end{aligned} \quad (3.4.2)$$

where we used Gauss–Weingarten and Codazzi equations (1.1.1), (1.1.3).

Contracting now with g^{ij} and h^{ij} respectively we have

$$\begin{aligned} \Delta H &= H - H|A|^2 + \langle x | e_k \rangle \nabla_k H = H(1 - |A|^2) + \langle x | \nabla H \rangle \\ h^{ij} \nabla_i \nabla_j H &= |A|^2 - \text{tr}(A^3)H + \langle x | e_k \rangle \nabla_k |A|^2/2 \end{aligned} \quad (3.4.3)$$

which implies, by Simons' identity (1.1.4),

$$\Delta |A|^2 = 2|A|^2(1 - |A|^2) + 2|\nabla A|^2 + \langle x | \nabla |A|^2 \rangle.$$

From equation (3.4.3) and the strong maximum principle for elliptic equations we see that, since M satisfies $H \geq 0$ by assumption and $\Delta H \leq H + \langle x | \nabla H \rangle$, we must either have that $H \equiv 0$ or $H > 0$ on all M .

The case $H \equiv 0$ can be easily handled: as M is complete and x is a tangent vector field on M by the equation $\langle x | \nu \rangle = 0$, for every point x_0 of M there is a unique solution of the ODE $\gamma'(s) = x(\gamma(s)) = \gamma(s)$ passing through x_0 and contained in M for every $s \in \mathbb{R}$, but such solution is simply the line in \mathbb{R}^{n+1} passing through x_0 and the origin. Thus, M has to be a cone and being

smooth the only possibility is a hyperplane through the origin of \mathbb{R}^{n+1} .

Therefore we may assume that the mean curvature satisfies the strict inequality $H > 0$ everywhere (so dividing by H and $|A|$ is allowed).

Now let $R > 0$ and define η to be the inward unit conormal to $M \cap B_R(0)$ along $\partial(M \cap B_R(0))$, which is a smooth boundary for almost every $R > 0$ (by Sard's theorem). Then, supposing that R belongs to the set $\Omega \subset \mathbb{R}^+$ of the *regular values* of the function $|\cdot|$ restricted to $M \subset \mathbb{R}^{n+1}$, from equation (3.4.3) and the divergence theorem we compute

$$\begin{aligned} \varepsilon_R &= - \int_{\partial(M \cap B_R(0))} |A| \langle \nabla H | \eta \rangle e^{-R^2/2} d\mathcal{H}^{n-1}(x) \\ &= \int_{M \cap B_R(0)} |A| \Delta H e^{-|x|^2/2} + \langle \nabla(|A| e^{-|x|^2/2}) | \nabla H \rangle d\mathcal{H}^n(x) \\ &= \int_{M \cap B_R(0)} \left(|A| H (1 - |A|^2) + |A| \langle x | \nabla H \rangle \right) e^{-|x|^2/2} d\mathcal{H}^n(x) \\ &\quad + \int_{M \cap B_R(0)} \left(\frac{1}{2|A|} \langle \nabla |A|^2 | \nabla H \rangle - |A| \langle x | \nabla H \rangle \right) e^{-|x|^2/2} d\mathcal{H}^n(x) \\ &= \int_{M \cap B_R(0)} \left(|A| H (1 - |A|^2) + \frac{1}{2|A|} \langle \nabla |A|^2 | \nabla H \rangle \right) e^{-|x|^2/2} d\mathcal{H}^n(x) \end{aligned}$$

and similarly,

$$\begin{aligned} \delta_R &= - \int_{\partial(M \cap B_R(0))} \frac{H}{|A|} \langle \nabla |A|^2 | \eta \rangle e^{-R^2/2} d\mathcal{H}^{n-1}(x) \\ &= \int_{M \cap B_R(0)} \frac{H}{|A|} \Delta |A|^2 e^{-|x|^2/2} + \left\langle \nabla \left(\frac{H}{|A|} e^{-|x|^2/2} \right) | \nabla |A|^2 \right\rangle d\mathcal{H}^n(x) \\ &= \int_{M \cap B_R(0)} \left(2|A| H (1 - |A|^2) + \frac{2H|\nabla A|^2}{|A|} + \frac{H}{|A|} \langle x | \nabla |A|^2 \rangle \right) e^{-|x|^2/2} d\mathcal{H}^n(x) \\ &\quad + \int_{M \cap B_R(0)} \left(\frac{\langle \nabla H | \nabla |A|^2 \rangle}{|A|} - \frac{H|\nabla |A|^2|^2}{2|A|^3} - \frac{H}{|A|} \langle x | \nabla |A|^2 \rangle \right) e^{-|x|^2/2} d\mathcal{H}^n(x) \\ &= \int_{M \cap B_R(0)} \left(2|A| H (1 - |A|^2) + \frac{2H|\nabla A|^2}{|A|} + \frac{\langle \nabla H | \nabla |A|^2 \rangle}{|A|} - \frac{H|\nabla |A|^2|^2}{2|A|^3} \right) e^{-|x|^2/2} d\mathcal{H}^n(x). \end{aligned}$$

Hence,

$$\begin{aligned} \sigma_R = 2\delta_R - 4\varepsilon_R &= \int_{M \cap B_R(0)} \left(\frac{4H|\nabla A|^2}{|A|} - \frac{H|\nabla |A|^2|^2}{|A|^3} \right) e^{-|x|^2/2} d\mathcal{H}^n(x) \\ &= \int_{M \cap B_R(0)} \left(4|A|^2 |\nabla A|^2 - |\nabla |A|^2|^2 \right) \frac{H}{|A|^3} e^{-|x|^2/2} d\mathcal{H}^n(x). \end{aligned}$$

As we have $4|A|^2 |\nabla A|^2 \geq |\nabla |A|^2|^2$, this quantity σ_R is nonnegative and nondecreasing in R .

If now we show that $\liminf_{R \rightarrow +\infty} \sigma_R = 0$ we can conclude that at every point of M

$$4|A|^2 |\nabla A|^2 = |\nabla |A|^2|^2. \quad (3.4.4)$$

Getting back to the definitions of ε_R and δ_R , we have

$$\begin{aligned} |\sigma_R| &= \left| -2 \int_{\partial(M \cap B_R(0))} \frac{H}{|A|} \langle \nabla |A|^2 | \eta \rangle e^{-R^2/2} d\mathcal{H}^{n-1} + 4 \int_{\partial(M \cap B_R(0))} |A| \langle \nabla H | \eta \rangle e^{-R^2/2} d\mathcal{H}^{n-1} \right| \\ &\leq 4e^{-R^2/2} \int_{\partial(M \cap B_R(0))} \frac{H}{|A|} |\nabla |A|^2| + |A| |\nabla H| d\mathcal{H}^{n-1} \\ &\leq 8e^{-R^2/2} \int_{\partial(M \cap B_R(0))} H |\nabla A| + |A| |\nabla H| d\mathcal{H}^{n-1} \\ &\leq C e^{-R^2/2} \mathcal{H}^{n-1}(\partial(M \cap B_R(0))), \end{aligned}$$

by the estimates on A and ∇A in the hypotheses.

Now, suppose that for every R belonging to the set $\Omega \subset \mathbb{R}^+$ (which is of full measure) and R larger than some $R_0 > 0$ we have

$$\mathcal{H}^{n-1}(\partial(M \cap B_R(0))) \geq \delta R e^{R^2/4}$$

for some constant $\delta > 0$. Setting x^M to be the projection of the vector x on the tangent space to M , as the function $R \mapsto \mathcal{H}^n(M \cap B_R(0))$ is monotone and continuous from the left and actually continuous at every value $R \in \Omega$, we can differentiate it almost everywhere in \mathbb{R}^+ and we have (by the coarea formula, see [42] or [108]),

$$\begin{aligned} \mathcal{H}^n(M \cap B_R(0)) - \mathcal{H}^n(M \cap B_r(0)) &\geq \int_r^R \frac{d}{d\xi} \mathcal{H}^n(M \cap B_\xi(0)) d\xi \\ &\geq \int_r^R \int_{\partial(M \cap B_\xi(0))} |\nabla^M |x||^{-1} d\mathcal{H}^{n-1}(x) d\xi \\ &= \int_r^R \int_{\partial(M \cap B_\xi(0))} |x|/|x^M| d\mathcal{H}^{n-1}(x) d\xi \\ &\geq \int_r^R \int_{\partial(M \cap B_\xi(0))} d\mathcal{H}^{n-1}(x) d\xi, \end{aligned}$$

where the derivative in the integral is taken only at the points where it exists and $\nabla^M |x|$ denotes the projection of the \mathbb{R}^{n+1} -gradient of the function $|x|$ on the tangent space to M .

Hence, if $R > r > R_0$ we get

$$\begin{aligned} \mathcal{H}^n(M \cap B_R(0)) - \mathcal{H}^n(M \cap B_r(0)) &\geq \int_r^R \int_{\partial(M \cap B_\xi(0))} d\mathcal{H}^{n-1} d\xi \\ &\geq \delta \int_r^R \xi e^{\xi^2/4} d\xi \\ &= 2\delta(e^{R^2/4} - e^{r^2/4}), \end{aligned}$$

then if R goes to $+\infty$, the quantity $\mathcal{H}^n(M \cap B_R(0))e^{-R}$ diverges, in contradiction with the hypotheses of the proposition. Hence, the \liminf of the quantity $e^{-R^2/4}\mathcal{H}^{n-1}(\partial(M \cap B_R(0)))$ as $R \rightarrow +\infty$ in the set Ω has to be zero. It follows that the same holds for $|\sigma_R|$ and equation (3.4.4) is proved.

Making explicit such equation, by the equality condition in the Cauchy-Schwartz inequality it immediately follows that at every point there exist constants c_k such that

$$\nabla_k h_{ij} = c_k h_{ij}$$

for every i, j . Contracting this equation with the metric g^{ij} and with h^{ij} we get $\nabla_k H = c_k H$ and $\nabla_k |A|^2 = 2c_k |A|^2$, hence $\nabla_k \log H = c_k$ and $\nabla_k \log |A|^2 = 2c_k$.

This implies that locally $|A| = \alpha H$ for some constant $\alpha > 0$ and by connectedness this relation has to hold globally on M .

Suppose now that at a point $|\nabla H| \neq 0$, then $\nabla_k h_{ij} = c_k h_{ij} = \frac{\nabla_k H}{H} h_{ij}$ which is a symmetric 3-tensor by the Codazzi equations (1.1.3), hence $\nabla_k H h_{ij} = \nabla_j H h_{ik}$. Computing then in normal coordinates with an orthonormal basis $\{e_1, \dots, e_n\}$ such that $e_1 = \nabla H/|\nabla H|$, we have

$$0 = |\nabla_k H h_{ij} - \nabla_j H h_{ik}|^2 = 2|\nabla H|^2 \left(|A|^2 - \sum_{i=1}^n h_{1i}^2 \right).$$

Hence, $|A|^2 = \sum_{i=1}^n h_{1i}^2$, that is,

$$h_{11}^2 + 2 \sum_{i=2}^n h_{1i}^2 + \sum_{i,j=2}^n h_{ij}^2 = |A|^2 = h_{11}^2 + \sum_{i=2}^n h_{1i}^2,$$

so $h_{ij} = 0$ unless $i = j = 1$, which means that A has rank one.

Thus, we have two possible (not mutually excluding) situations at every point of M , either A has rank one or $\nabla H = 0$.

If the kernel of A is empty everywhere, A must have rank at least two as we assumed $n \geq 2$, then we have $\nabla H = 0$ which implies $\nabla A = 0$ and $h_{ij} = Hh_{ik}h_{kj}$, by equation (3.4.2). This means that all the eigenvalues of A are 0 or $1/H$. As the kernel is empty $A = Hg/n$, precisely $H = \sqrt{n}$ and $A = g/\sqrt{n}$. Then, the complete hypersurface M has to be the sphere $\mathbb{S}^n(\sqrt{n})$, indeed we compute

$$\Delta^M |x|^2 = 2n + 2\langle x | \Delta^M x \rangle = 2n + 2H\langle x | \nu \rangle = 2n - 2H^2 = 0,$$

by means of the structural equation $H + \langle x | \nu \rangle = 0$, hence $|x|^2$ is a harmonic function on M and looking at the point of M of minimum distance from the origin, by the strong maximum principle for elliptic equations, it must be constant on M .

We assume now that the kernel of A is not empty at some point $p \in M$ and let $v_1(p), \dots, v_{n-m}(p) \in T_p M \subset \mathbb{R}^{n+1}$ be a family of unit orthonormal tangent vectors spanning such $(n-m)$ -dimensional kernel, that is $h_{ij}(p)v_k^j(p) = 0$. Then, the geodesic $\gamma(s)$ from p in M (which is complete) with initial velocity $v_k(p)$ satisfies

$$\nabla_s(h_{ij}\gamma_s^j) = H^{-1}\langle \nabla H | \gamma_s \rangle h_{ij}\gamma_s^j$$

hence, by Gronwall's lemma there holds $h_{ij}(\gamma(s))\gamma_s^j(s) = 0$ for every $s \in \mathbb{R}$.

Being γ a geodesic in M , the normal to the curve in \mathbb{R}^{n+1} is also the normal to M , then letting κ be the curvature of γ in \mathbb{R}^{n+1} , we have

$$\kappa = \left\langle \nu \mid \frac{d}{ds}\gamma_s \right\rangle = h_{ij}\gamma_s^i\gamma_s^j = 0,$$

thus γ is a straight line in \mathbb{R}^{n+1} .

Hence, all the $(n-m)$ -dimensional affine subspace $p + S(p) \subset \mathbb{R}^{n+1}$ is contained in M , where we set $S(p) = \langle v_1(p), \dots, v_{n-m}(p) \rangle \subset \mathbb{R}^{n+1}$.

Let now $\sigma(s)$ be a geodesic from p to another point q parametrized by arclength and extend by parallel transport the vectors v_k along σ , then

$$\nabla_s(h_{ij}v_k^j) = H^{-1}\langle \nabla H | \sigma_s \rangle h_{ij}v_k^j$$

and again by Gronwall's lemma it follows that $h_{ij}v_k^j(s) = 0$ for every $s \in \mathbb{R}$, in particular $v_k(q)$ is contained in the kernel of A at $q \in M$. This argument clearly shows that the kernel $S(p)$ of A has constant dimension $n-m$ with $0 < m < n$ (as A is never zero) at every point $p \in M$ and all the affine $(n-m)$ -dimensional subspaces $p + S(p) \subset \mathbb{R}^{n+1}$ are contained in M .

Moreover, as $h_{ij}v_k^j = 0$ along the geodesic σ , denoting by $\nabla^{\mathbb{R}^{n+1}}$ the covariant derivative of \mathbb{R}^{n+1} we have

$$\nabla_s^{\mathbb{R}^{n+1}} v_k = \nabla_s v_k + h_{ij}v_k^j \sigma_s^i \nu = 0,$$

so the extended vectors v_k are constant in \mathbb{R}^{n+1} , which means that the parallel extension is independent of the geodesic σ , that the subspaces $S(p)$ are all a common $(n-m)$ -dimensional vector subspace of \mathbb{R}^{n+1} that we denote by S and that $M = M + S \subset \mathbb{R}^{n+1}$.

Since the orthogonal projection map $\pi : M \rightarrow S$ is then a submersion, for every vector $y \in S$ we have that $N = M \cap (y + S^\perp)$ is a smooth, complete m -dimensional submanifold of \mathbb{R}^{n+1} and, as $M = M + S$, it is easy to see that $M = N \times S$, which implies that $L = S^\perp \cap M$ is a smooth, complete m -dimensional submanifold of $S^\perp = \mathbb{R}^{m+1}$ with $M = L \times S$.

Moreover, as S is in the tangent space to every point of L , the normal ν to M at a point of L stays in S^\perp so it must coincide with the normal ν^L to L in S^\perp , then a simple computation shows that the mean curvature of M at the points of L is equal to the mean curvature H^L of L as a hypersurface of $S^\perp = \mathbb{R}^{m+1}$. This shows that L is a hypersurface in \mathbb{R}^{m+1} satisfying $H^L + \langle z | \nu^L \rangle = 0$ for every $z \in \mathbb{R}^{m+1}$. Finally, as by construction the second fundamental form of L has empty kernel, by the previous discussion we have $L = \mathbb{S}^m(\sqrt{m})$ and $M = \mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ and we are done. \square

REMARK 3.4.2. Notice that it follows that all the possible blow up limits are convex. It is a very important fact, proved by Huisken and Sinestrari [74] (see also White [125]), that the same conclusion also holds at a type II singularity of the flow of a mean convex hypersurface (Theorem 4.2.1 and Proposition 4.2.3 in the next chapter).

REMARK 3.4.3. Actually, Abresch and Langer in [1] (and also Epstein and Weinstein in [41]) classified *all* the closed curves in \mathbb{R}^2 satisfying the structural equation $k + \langle \gamma | \nu \rangle = 0$ (also the curves with self-intersections), see Appendix E for more details. We underline that, even if the techniques are elementary, the proof of such classification result is definitely nontrivial. The result in the embedded case in these papers is a consequence of the general classification theorem. To the author's knowledge, the "shortcut" presented in the proof above is due to Chou and Zhu [27, Proposition 2.3].

We mention that recently Colding and Minicozzi in [30] proved this classification result assuming only a polynomial volume growth, without any bound on the second fundamental form A .

In dimension $n \geq 2$, without the assumption $H > 0$ the conclusion is not true, an example is the Angenent's torus in [17]. The following higher dimensional analogue of Abresch and Langer result is an open question.

OPEN PROBLEM 3.4.4. When $n \geq 2$, is any smooth embedding of \mathbb{S}^n in \mathbb{R}^{n+1} such that $H + \langle x | \nu \rangle = 0$ isometric to a sphere?

This was recently shown by Brendle [22] in the 2-dimensional case.

As by Corollary 3.3.8 the unit multiplicity hyperplanes are excluded as blow up limits at a singular point, we have the following conclusion.

THEOREM 3.4.5. *Let the compact, initial hypersurface be embedded and with $H \geq 0$. Then, every limit hypersurface obtained by rescaling around a type I singular point, up to a rotation in \mathbb{R}^{n+1} , must be either the sphere $\mathbb{S}^n(\sqrt{n})$ or one of the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$.*

We discuss now what are the possible values of the limit heat density function, following Stone [118]. As $\Theta(p)$ is the value of the Huisken's functional on any limit of rescaled hypersurfaces and since these latter are "finite", we have that the possible values of $\Theta(p)$ are 1 in the case of a hyperplane and

$$\Theta^{n,m} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}} e^{-\frac{|x|^2}{2}} d\mathcal{H}^n$$

for $m \in \{1, \dots, n\}$.

A straightforward computation gives for $m > 0$

$$\Theta^{n,m} = \left(\frac{m}{2\pi e}\right)^{m/2} \omega_m$$

where ω_m denotes the volume of the unit m -sphere.

Notice that $\Theta^{n,m}$ does not depend on n so we can simply write $\Theta^m = \Theta^{n,m}$.

LEMMA 3.4.6 (Stone [118]). *The values of Θ^m are all distinct and larger than 1 for $m > 0$. Indeed the numbers $\{\Theta^m \mid m = 1, 2, \dots\}$ form a strictly decreasing sequence in $m \in \mathbb{N}$, with $\Theta^m \searrow \sqrt{2}$ as $m \rightarrow \infty$.*

By all this discussion we conclude that the "shape" of the limit hypersurfaces arising from a blow up at a type I singularity of mean curvature flow of a compact, embedded, mean convex hypersurface, is classified by the value of the limit heat density function at the blow up point.

Proposition 3.4.1 solves Problem 3.2.12 in the case of curves and in the subclass of the embedded, limit hypersurfaces with $H \geq 0$, when $n \geq 2$. A consequence is a positive answer to Problem 3.3.3, indeed, if the limit of rescalings around a nonspecial singular point is an embedded hypersurface with at least one point with $A = 0$, the only possibility is then a single hyperplane with unit multiplicity by the classification result.

Moreover, combining such conclusion with Proposition 3.3.4, also Problem 3.3.7 has a full answer in this subclass, in its stronger form (see the discussion immediately after the problem).

PROPOSITION 3.4.7. Every *singular point of a type I singularity of the motion by mean curvature of a compact, embedded initial hypersurface with $H \geq 0$ is a special singular point.*

As a consequence, if the flow develops a type I singularity and a blow up limit is a sphere (or a circle for curves), the flow is smooth till the hypersurface shrinks to a point becoming asymptotically spherical.

This also implies that at some time the hypersurface has become convex.

Actually, more in general, the following pair of theorems describe the flow of convex curves and hypersurfaces.

THEOREM 3.4.8 (Gage and Hamilton [47, 48, 49]). *Under the curvature flow a convex closed curve in \mathbb{R}^2 smoothly shrinks to a point in finite time. Rescaling in order to keep the length constant, it converges to a circle in C^∞ .*

THEOREM 3.4.9 (Huisken [66]). *Under the mean curvature flow a compact and convex hypersurface in \mathbb{R}^{n+1} with $n \geq 2$ smoothly shrinks to a point in finite time. Rescaling in order to keep the Area constant, it converges to a sphere in C^∞ .*

REMARK 3.4.10. The theorem for curves is not merely a consequence of the general result. The proof in dimension $n \geq 2$ does not work in the one-dimensional case.

Actually, the C^∞ -convergence to a circle or to a sphere is exponential.

At the end of Section 4.1 of the next chapter, we will show a line of proof of Theorem 3.4.9 by Hamilton in [61], different from the original one. Another proof was also given by Andrews in [9], analyzing the behavior of the eigenvalues of the second fundamental form close to the singular time.

Theorem 3.4.8 will follow from the strong fact that a simple closed curve in \mathbb{R}^2 cannot develop type II singularities at all.

The last point missing in all this story, even in the embedded mean convex case when $n \geq 2$, is a full answer to Problem 3.2.11. We concluded that any blow up limit gives the same value of the Huisken's functional, hence its "shape" is fixed: hyperplane, sphere or cylinder. If the limit is a sphere, the limit is unique and there is full convergence, if it is a hyperplane we already had such conclusion by White's Theorem 3.2.22. But, if the limit is a cylinder, its axis could possibly change, depending on the choice of the converging sequence. Clearly, in the case of curves Problem 3.2.11 is solved affirmatively as there are no "cylinders".

If the initial hypersurface is only immersed, the conclusion of the classification theorem 3.4.1 still holds allowing the possibility that M is the *union* of a finite family of hypersurfaces among the hyperplanes through the origin, the spheres $\mathbb{S}^n(\sqrt{n})$ and the cylinders $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ (with possible different axes).

In the one-dimensional case one has to enlarge the possible curves in the conclusion of such theorem to include also all the family of the so called Abresch-Langer curves, described in [1] (see Appendix E).

Anyway, the possible existence of blow up limits which are hyperplanes with multiplicity larger than one prevents the application of the argument leading to Proposition 3.4.7. Actually, to the author's knowledge, in the immersed-only case there is not a general procedure to exhibit a limit of rescaled hypersurfaces which is homothetic and nonflat at a type I singularity.

3.5. Embedded Closed Curves in the Plane

The case of an embedded, closed curve γ in \mathbb{R}^2 is special, indeed the classification theorem 3.4.1 holds without *a priori* assumptions on the curvature. So there are only two possible limits of rescaled curves without self-intersections, either a line through the origin or the circle \mathbb{S}^1 . This gives immediately a general positive answer to Problems 3.2.12 and 3.3.3 and implies as before that every singular point is a special singular point.

As we already said, in this very special case also Problem 3.2.11 is solved affirmatively, the limit is always unique.

Arguing as in the previous section, we then have the following conclusion.

THEOREM 3.5.1. *Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve, then every curve obtained by limit of rescalings around a type I singular point of its motion by curvature is the circle S^1 . As a consequence, if a simple closed curve is developing a type I singularity, at some time the curve becomes convex and it shrinks to a point getting asymptotically circular at the singular time.*

We mention here that an extensive and deep analysis of the behavior of general curves moving by curvature (even when the ambient is a generic surface different from \mathbb{R}^2) is provided by the pair of papers by Angenent [14, 16] (see also the discussion in [17]).