Beyond fractional laplacean: fractional gradient and divergence

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The paper deals with the fractional operators in the class of distributions on $\mathbf{R^n}$:

(1) a fractional vector valued gradient $\nabla^\alpha f$ of a complex order $\alpha \in \mathbf{C} \setminus \{-n-2k-1 : k = 0, 1, \ldots\}$ of a function (or a distribution) f;

(2) a fractional scalar valued divergence $\operatorname{div}^\alpha \mathbf{q}$ of a complex order $\alpha \in \mathbf{C} \setminus \{-n-2k-1 : k = 0, 1, \ldots\}$ of a vector field (or a vector valued distribution) $\mathbf{q}$;

(3) the well–known fractional laplacean $(-\triangle)^\alpha f$ of a complex order $\alpha \in \mathbf{C} \setminus \{n - 2k : k = 0, 1, \ldots\}$ of a function (or a distribution) f.

With f and $\mathbf{q}$ fixed, the maps $\alpha \mapsto \nabla^\alpha f$, $\alpha \mapsto \operatorname{div}^\alpha \mathbf{q}$ and $\alpha \mapsto (-\triangle)^\alpha f$ are analytic maps of the complex variable α with values in the space of distributions. For $\alpha = 1$ the operators ∇^α and $\operatorname{div}^\alpha$ coincide with the usual gradient and divergence. The composition formula

$$\operatorname{div}^\alpha(\nabla^\beta f) = -(-\triangle)^{(\alpha+\beta)/2}$$

holds for any pair of complex numbers α, β whenever the two sides of the equality make sense. The applied methods permit a strengthening of the well–known result of J. Bourgain, H. Brezis & P. Mironescu which says that a suitably renormalized Cagliardo–Sobolevskii fractional norm of order $s \in (0, 1)$ of a function f converges to the integer norm of order 1 if $s \uparrow 1$. The strengthened form says that the renormalized fractional norm is the (non–renormalized) norm of the fractional gradient and that not only the norm, but the fractional gradient itself converges to the standard gradient. The result holds for the same limit to an integer gradient of any order. A new light is shed on the renormalization factor of J. Bourgain, H. Brezis & P. Mironescu: it is related to the residue of Euler's gamma function $\Gamma(z)$ at $z = 0$.