Besicovitch's magic method and problems of minimal resistance

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We consider piecewise smooth functions $u : \bar{\Omega} \to \mathbb{R}$ defined on the closure of a bounded domain $\Omega \subset \mathbb{R}^2$ satisfying the conditions $u(x) < 0$ for $x \in \Omega$ and $u(x) = 0$ for $x \in \partial\Omega$ (in other words, the graph of u forms a "dimple" on the plane).

We also consider a flow of particles that fall on the graph of u vertically down and reflect from it in the perfectly elastic manner. It is assumed that u satisfies the so-called "single impact condition" (SIC): each particle reflected at a non-singular point of the graph, further moves freely above the graph until it leaves the dimple. This condition can be stated analytically as follows: for any regular point $x \in \Omega$ and any $t > 0$ such that $x - t\nabla u(x) \in \bar{\Omega}$,

$$\frac{u(x - t\nabla u(x)) - u(x)}{t} \le \frac{1}{2}(1 - |\nabla u(x)|^2). \tag{1}$$

The force of resistance of the dimple to the flow (more precisely, the vertical projection of this force) equals $2\rho|\Omega|\, R(u; \Omega)$, where ρ is the flow density, $|\Omega|$ is the area of Ω, and

$$R(u; \Omega) = \frac{1}{|\Omega|} \int_\Omega \frac{dx}{1 + |\nabla u(x)|^2}. \tag{2}$$

This formula is true provided that the SIC (1) is fulfilled.

The problem is: minimize the value of "specific resistance" $R(u; \Omega)$. It has two versions which are eventually equivalent:

(a) $\inf_{u,\Omega} R(u; \Omega)$ и

(b) $\inf_u R(u; \Omega)$ for a given Ω.

Obviously, $\sup R(u; \Omega) = 1$ and $\inf R(u; \Omega) \ge 1/2$. The main question is to find if $\inf R(u; \Omega) > 1/2$ or $\inf R(u; \Omega) = 1/2$. I will prove that the latter is true. This result is somewhat counterintuitive: one needs to provide a sequence of functions with the slope of the graph being "almost"45^0 in the most part of the region Ω. That is, most part of reflected particles move "almost" horizontally and do not meet obstacles on the way.

A part of the construction is borrowed from Besicovitch's solution of the Kakeya problem: what is the minimum area of a plane region in which a unit line segment can be rotated continuously through 360^0.