ERC Workshop on Geometric Analysis on sub-Riemannian and metric spaces

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Palazzo del Castelletto, Aula Dini, via del Castelletto, Pisa

Titles and Abstracts

Zoltan Balogh (Universität Bern)

`Modulus method and radial stretch map in the Heisenberg group`

Abstract. We propose a method by modulus of curve families to identify extremal quasiconformal mappings in the Heisenberg group. This approach allows to study minimizers not only for the maximal distortion but also for a mean distortion functional, where the candidate for the extremal map is not required to have constant distortion. As a counterpart of a classical Euclidean extremal problem, we consider the class of quasiconformal mappings between two spherical annuli in the Heisenberg group. Using logarithmic-type coordinates we can define an analog of the classical Euclidean radial stretch map and discuss its extremal properties both with respect to the maximal and the mean distortion. We prove that our stretch map is a minimizer for a mean distortion functional and it minimizes the maximal distortion within the smaller subclass of sphere-preserving mappings.

Martino Bardi (Università di Padova)

`On some fully nonlinear subelliptic PDEs`

Abstract. We present some joint work with Paola Mannucci on two classes of Hessian equations involving a given family of vector fields such as the generators of a Carnot group. The first class contains Monge-Ampere equations, possibly with terms involving the gradient of the solution. The second type of equations prescribes the minimal eigenvalue of the symmetrized Hessian matrix with respect to the given vector fields. Under general assumptions we prove comparison principles for viscosity solutions and uniqueness for the Dirichlet problem. Existence of solutions is established under stronger conditions. The notion of convexity with respect to vector fields plays an important role for both classes of equations.

Davide Barilari (SISSA, Trieste)

`Invariants and heat kernels in 3D contact sub-Riemannian Geometry`

Abstract. In this talk describe the two functional invariants of a 3D contact sub-Riemannian structure, discussing their role in the classification of left-invariant sub-Riemannian structures on three dimensional Lie groups. We then explicitly compute, in the case of a 3D contact structure, the first two coefficients of the small time asymptotics expansion of the heat kernel on the diagonal, expressing them in terms of the two basic functional invariants.

Jana Bjorn (Linköping University)
`De Giorgi's method: in(s) and out(s)`
Abstract. One of the main break-throughs in the 20th century mathematics was De Giorgi's proof of Hölder continuity for solutions of elliptic PDEs. His method has since then been used to prove interior regularity in various contexts. It is maybe less known, though not entirely surprising that De Giorgi's method also yields sufficient conditions for boundary regularity. In the talk, I will discuss a recent variation of De Giorgi's method which goes in the opposite direction, leading to a necessary condition for boundary regularity of PDEs and variational integrals.

Marc Bourdon (Université de Lille)
L^p-cohomology and conformal dimension
Abstract. We construct examples of hyperbolic group boundaries, homeomorphic to the Sierpinsky carpet, with or without the Combinatorial Loewner property, whose conformal dimensions are arbitrarily close to 1. It answers a question of M. Bonk and of J. Mackay. (Joint work with Bruce Kleiner).

Giovanna Citti (Università di Bologna)
`A Poincaré inequality for Lipschitz intrinsic vector fields in the Heisenberg group`
Abstract. This result is a joint work with M. Manfredini, A. Pinamonti, F. Serra Cassano. We prove a Poincaré inequality for Lipschitz intrinsic vector fields in any Heisenberg group of dimension $n > 1$. If a subriemannian metric is defined in this group, a regular surface implicitly defines a graph ϕ, which is regular with respect to non linear vector fields, defined in terms of ϕ itself. Geometric equations can be written in terms of these nonlinear vector fields. Hence it is necessary to establish a Poincar formula for vector fields with minimal assumptions on the coefficients. This inequality has been already established in case of coefficients Lipchitz continuous in the standard Euclidean sense, but the intrinsic Lipschitz condition is weaker. Hence we will use a different technique based on approximation of the given vector fields with polynomial ones. These approximating vector fields satisfy a representation formula, from which we get the Poincar inequality for the nonlinear vector fields.

Federica Dragoni (University of Cardiff)
$\mathcal{X}$-convexity and applications

Abstract. We introduce a new notion of convexity, namely $\mathcal{X}$-convexity, which applies to any given family of vector fields and in particular to the sub-Riemannian case. We then show a PDE-characterization for $\mathcal{X}$-convex functions and we investigate some properties of these functions.

11.40-12.30. **Bruno Franchi** (Università di Bologna)

`Maxwell's equations in Carnot groups`

Abstract. In this talk we define Maxwell's equations in the setting of the intrinsic complex of differential forms in Carnot groups introduced by M. Rumin. It turns out that these equations are higher order equations in the horizontal derivatives. In addition, when looking for a vector potential, we have to deal with a new class of higher order evolution equations that replace usual wave equations of the Euclidean setting and that are no more hyperbolic. We prove equivalence of these equations with the "geometric equations" defined in the intrinsic complex, as well as existence and properties of solutions. Finally, we prove that, in spite of their higher order, these equations can be seen as the limit (in a suitable sense) of usual Maxwell's equations.

Nicola Garofalo (Purdue University)

`Boundedness of the Riesz Transforms for some Subelliptic Operators`

Abstract. In this lecture I will describe some recent joint work with Fabrice Baudoin. Let $\mathbb{M}$ be a smooth connected non-compact manifold endowed with a smooth measure μ and a smooth locally subelliptic diffusion operator $\mathcal{L}$ satisfying $\mathcal{L}1 = 0$, and which is symmetric with respect to μ. We show that if $\mathcal{L}$ satisfies, with a non negative curvature parameter ρ_1, a generalization of the curvature-dimension inequality from Riemannian geometry, then the Riesz transform is bounded in $L^p(\mathbb{M})$ for every $p > 1$, that is

$$\left\| \sqrt{\Gamma((-L)^{-1/2} f))} \right\|_p \leq C_p \, \|f\|_p, \quad f \in C_0^\infty(\mathbb{M})$$

where Γ is the *carré du champ* associated to $\mathcal{L}$. Our results apply in particular to all Sasakian manifolds whose horizontal Tanaka-Webster Ricci curvature is nonnegative, all Carnot groups with step two, and wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is nonnegative.

Roberta Ghezzi (CMAP, École Polytechnique)

`A new class of` $(\mathcal{H}^k, 1)$`-rectifiable subsets of metric spaces`

Abstract. The main motivation of this talk arises from the study of Carnot-Carathéodory spaces, where the class of 1-rectifiable sets does not contain smooth non-horizontal curves; therefore a new definition of rectifiable sets including non-horizontal curves is needed. This is why we introduce in any metric space a new class of curves, called continuously metric differentiable of degree k, which are Hölder but not Lipschitz continuous when $k > 1$. Replacing Lipschitz curves by this kind of curves we define $(\mathcal{H}^k, 1)$-rectifiable sets and show a density result generalizing the corresponding one in Euclidean geometry. This theorem is a consequence of computations of Hausdorff measures along curves, for which we give an integral formula. In particular, we show that both spherical and usual Hausdorff measures along curves coincide with a class of dimensioned lengths and are related to an interpolation complexity, for which estimates have already been obtained in Carnot-Carathéodory spaces.

Nicola Gigli (Université de Nice)

`On the interplay between horizontal and vertical derivation`

Abstract. I will talk about some recent developments on the interplay between the L^2 and W_2 geometries. Results include the possibility of studying the continuity equation in a genuine metric framework, and the identification of the heat flow as gradient flow in spaces with Ricci curvature bounded from below.

Peter Haissinsky (Université de Provence)

`Conformal dimension and characterization of conformal dynamical systems`

Abstract. We will illustrate the role of conformal dimension in the determination of whether a topological dynamical system acting on the two-sphere is conjugate to a conformal one.

Piotr Hajłasz (University of Pittsburgh)

`On the lack of density of Lipschitz mappings in Sobolev spaces with Heisenberg target`

Abstract. I am going to talk about my recent joint paper with N. De Jarnette, A. Lukyanenko, and J. Tyson. We study the question: When are Lipschitz mappings dense in the Sobolev space $W^{1,p}(M, \mathbb{H}^n)$? Here M denotes a compact Riemannian manifold with or without boundary, while $\mathbb{H}^n$ denotes the nth Heisenberg group equipped with a sub-Riemannian metric. We show that Lipschitz maps are dense in $W^{1,p}(M, \mathbb{H}^n)$ for all $1 \leq p < \infty$ if $\dim M = n$, but that Lipschitz maps are not dense in $W^{1,p}(M, \mathbb{H}^n)$ if $\dim M \geq n+1$ and $n \leq p < n+1$. The proofs rely on the construction of smooth horizontal embeddings of the sphere $\mathbb{S}^n$ into $\mathbb{H}^n$. The nondensity assertion is strictly related to the fact that the n-th Lipschitz homotopy group of $\mathbb{H}^n$ is nontrivial. We initiate a study of Lipschitz homotopy groups for sub-Riemannian spaces.

11.40-12.30. **Ilkka Holopainen** (University of Helsinki)

`Nonsolvability of the Dirichlet Problem at Infinity for p-Laplacian on Cartan-Hadamard Manifolds`

Abstract. In 1979 Greene and Wu conjectured that a Cartan-Hadamard manifold M admits non-constant bounded harmonic functions if the sectional curvatures of M have an upper bound

$$K_M(P) \leq \frac{-C}{r^2(x)}$$

outside a compact set for some constant $C > 0$, where $r = d(\cdot, o)$ is the distance function to a fixed point $o \in M$ and P is any 2-dimensional subspace of T_xM. A Cartan-Hadamard manifold M can be compactified by adding a sphere at infinity (or a boundary at infinity), denoted by $M(\infty)$, so that the resulting space $\bar{M} = M \cup M(\infty)$ equipped with the cone topology will be homeomorphic to a closed Euclidean ball. The conjecture of Greene and Wu is still open for dimensions $n \geq 3$. It can be approached by studying the so-called Dirichlet problem at infinity (or the asymptotic Dirichlet problem). Thus one asks whether every continuous function on $M(\infty)$ has a (unique) harmonic extension to M. In general, the answer is no since the simplest Cartan-Hadamard manifold Rn admits no positive harmonic functions other than constants. On the other hand, some kind of curvature lower bounds are needed even in the case of strictly negative sectional curvatures by counterexamples due to Ancona (1994) and Borbély (1998). The Dirichlet problem at infinity has been extensively studied during the last 30 years under various curvature assumptions. In the talk I will survey studies on the Dirichlet problem at infini ty for p-harmonic functions on Cartan Hadamard manifolds. I will also describe the counterexample by Borbély and show that after a modification it applies to the case of p-harmonic functions as well.

Amos Nathan Koeller (Tübingen University)

`Possibilities Stemming from the Euclidean Setting`

Abstract. We observe possible directions of research inspired by work in a Euclidean setting. Firstly we consider various forms of a Reifenberg-type multi-resolutional linear approximation analysis, not dissimilar to ideas already used in the traveling salesman problem and in other applications of Jones' β-numbers. Secondly we consider a possible form of dimension determination for scattered, or dust sets, given by a comparison principle.

Urs Lang (ETH Zurich)

`Asymptotic Plateau problems in spaces of higher asymptotic rank`

Abstract. (Joint work with Bruce Kleiner.) We study n-dimensional (quasi-) minimizing varieties (locally integral currents) in nonpositively curved metric spaces of rank n in an asymptotic sense. The varieties considered have polynomial volume growth of order n. We prove several results regarding the existence, stability, persistence under deformations of the metric, and the asymptotic geometry of such (quasi-) minimizers. Some of these are parallel to known results on quasi-geodesics or higher-dimensional quasi-minimizers in hyperbolic spaces.

Jan Malý (Charles University in Prague)

`Luzin's condition N and Sobolev mappings`

Abstract. We study conditions under which the area formula holds for Sobolev transformation of variables. The most recent results are obtained jointly with Pekka Koskela and Thomas Zürcher.

Pierre Martinetti (Dipartimento di Matematica & CMTP, Università di Roma Tor Vergata)

`Gauge fluctuation in Noncommutative Geometry and Carnot-Carathéodory distance`

Abstract. In noncommutative geometry, starting with an algebra A and an operator D which generalizes to the noncommutative framework the Dirac (or Atiyah) operator of a spin manifold, Connes defines a distance on the space of states of A. We call it the "spectral distance". In case A is chosen as the tensor product of the algebra of smooth function on a compact Riemannian manifold M with the algebra of n-square matrices, the space of state of A is a $U(n)$ trivial bundle on M. Through the so called process of "fluctuation of the metric", one equips this bundle with a connection C. This amounts to turn the initial Dirac operator D into a covariant Dirac operator $D + C$. It was expected that the spectral distance calculated with $D + C$ were equal to the horizontal distance defined by C. We will show that the link between the two distances is more subtle, depending on the holonomy of the connection.

Michele Miranda (Università di Ferrara)

`Two characterizations of BV functions on Carnot groups via the heat semigroup`

Abstract. In this talk I shall present a recent paper, joint work with M. Bramanti and D. Pallara, accepted for publication on International Mathematics Research Notices. The aim of the paper is to provide two different characterizations of sets with finite perimeter and functions of bounded variation in Carnot groups, analogous to those which hold in Euclidean spaces, in terms of the short-time behavior of the heat semigroup. The first characterization follows the original definition given by De Giorgi of functions with bounded variation; if we denote by $(T_t)_{t\geq 0}$ the heat semigroup associated to the sub-Laplacian, then we investigate the behavior as $t \to 0$ of the map

$$f(t) = \int |T_t u|.$$

In the original paper of De Giorgi it was proved that f is a monotone map and that

$$\lim_{t \to 0} f(t) < +\infty$$

if and only if u has bounded variation; in this case the limit coincides with the total variation of u. In the Carnot group setting, we are not able to prove the existence of the limit, but we show that $f(t)$ is bounded if and only if u has finite total variation, with total variation quantitatively controlled by the small time behavior of $f(t)$.
The second characterization was suggested to us by a paper of Ledoux and holds under the hypothesis that the reduced boundary of a set of finite perimeter is rectifiable, a result that presently is known in step 2 Carnot groups. In this case we investigate, when reducing the study on Borel sets E, the function

$$g(t) = \frac{1}{\sqrt{t}} \int_{E^c} T_t \chi_E.$$

It is shown that

$$\lim_{t \to 0} g(t)$$

exists and it is finite if and only if E has finite perimeter and this limit is given by a weighted, possibly anisotropic, perimeter.

Roberto Monti (Università di Padova)

`New results on subriemannian geodesics`

Abstract. We shall discuss some recent ideas and results on the problem of regularity of subriemannian length-minimizing curves. This is a joint research project with E. Le Donne, G.P. Leonardi, and D. Vittone.

Daniele Morbidelli (Università di Bologna)

`Involutive families of vector fields, their orbits and the Poincaré inequality`

Abstract. Given a family of nonsmooth vector fields, under a suitable quantitative involutivity assumption, we discuss some regularity property of their orbits and the related Poincaré inequality.

Herve Pajot (Université de Grenoble)

`Nonnegative Ricci curvature and Poincaré inequalities`

Abstract. A famous result in geometric analysis states that any Riemannian manifold with nonnegative Ricci curvature supports a Poincar inequality. In this talk, we will discuss (possible) extensions of this result to various settings, for instance (discrete) graphs and (continuous) geodesic spaces. In these situations,

the Ricci curvature condition will be expressed in terms of optimal transportation, the Bakry-Emery inequality for elliptic operators, or the Brunn-Minkowski inequality.

Pierre Pansu (Université Paris-Sud)

`$L^p$-cohomology and pinching`

Abstract. We prove that Riemannian manifolds quasiisometric to complex hyperbolic plane cannot have sectional curvature pinched between -1 and a for $a < -14$. The proof uses the multiplicative structure on L^p-cohomology and considerations on differential forms on the Heisenberg group.

Eero Saksman (University of Helsinki)

`Burkholder functionals, quasiconformal maps and singular integrals`

Abstract. We consider norms of singular integral operators, quasiconvexity of variational integrals, a question of Morrey, and sharp regularity estimates of quasiconformal maps. All these different themes will be linked via Burkholder functionals. The talk is based on joint work with K. Astala (Helsinki), S. Geiss (Innsbruck), T. Iwaniec (Syracuse), S. Montgomery-Smith (Missouri) and I. Prause (Helsinki).

Svetlana Selivanova (Sobolev Institute of Mathematics)

`Local geometry of nonregular quasimetric "Carnot-Caratheodory spaces".`

Abstract. Carnot-Carathéodory spaces are a wide generalization of sub-Riemannian manifolds which model nonholonomic processes and naturally arise in many applications. We consider the case of arbitrary weighted filtration of the tangent bundle (it generalizes the sub-Riemannian framework of a bracket-generating distribution) and study the local geometry of such spaces in a neighborhood of nonregular points (where dimensions of the subbundles generating the filtration may vary from point to point). In particular, we prove analogs of such classical results of sub-Riemannian geometry as Local approximation theorem and Tangent cone theorem. Quasimetrics are needed since, in the considered situation, the intrinsic Carnot-Carathéodory metric might not exist. The motivation of this research stems from nonlinear control theory and subelliptic equations.

Francesco Serra Cassano (Università di Trento)

`Intrinsic $C^1$ and Lipschitz graphs in the Heisenberg group and continuous solutions of Burgers equation`

Abstract. We are going to discuss on a characterization of different (weak) continuous solutions of Burgers' equation and functions which induce intrinsic C^1 and Lipschitz graphs in the first Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$, endowed with its standard Sub-Riemanniam metric structure, named also of Carnot-Carathodory . We will also extend the characterization to higher Heisenberg groups $\mathbb{H}^n = \mathbb{R}^{2n+1}$.

Nages Shanmugalingam (University of Cincinnati)
`Regularity of sets of quasiminimal boundary surfaces in metric setting`
Abstract. The talk will focus on some regularity properties of sets with quasiminimal boundary surfaces in the setting of metric measure spaces with doubling measure supporting a 1-Poincare inequality, and an application to strong A_∞ weights. The talk is based on joint work with Juha Kinnunen, Riikka Korte, and Andrew Lorent.

Jeremy Tyson (University of Illinois)
`The effect of projections on dimension in the Heisenberg group`
Abstract. The semidirect product decomposition of the Heisenberg group into a horizontal subgroup and a complementary vertical subgroup induces a pair of projection-type mappings. We study the effect of these mappings on the sub-Riemannian Hausdorff dimensions of sets. We establish almost sure dimension estimates phrased in terms of the isotropic Grassmannian of m-dimensional horizontal subgroups of the Heisenberg group. Our results generalize classical almost sure dimension theorems of Marstrand, Kaufman and Mattila from the Euclidean setting. This talk is based on joint work with Zoltán Balogh, Estibalitz Durand Cartagena, Katrin Fässler and Pertti Mattila.

Bozhidar Velichkov (Scuola Normale Superiore, Pisa)
`Shape optimization problems on metric measure spaces`
Abstract. We consider shape optimization problems of the form

$$\min\{\lambda_1(\Omega) : \Omega \subset X, |O| = m\} \tag{1}$$

where X is a metric measure space of finite measure and λ_1 is a generalization, through a variational formulation, of the first eigenvalue of the Dirichlet Laplacian. We work in a purely abstract setting, defining the Sobolev space over a general measure space (X, μ), as a linear subspace of $L^2(\mu)$ which obeys certain properties. A particular nonlinear operator $D : H \to L^2(\mu)$ has the role of the modulus of the weak gradient in H and the first eigenvalue of the Dirichlet Laplacian is defined as

$$\lambda_1(O) = \inf \Big\{ \int |Du|^2 d\mu,\ u \in H,\ \|u\|_{L^2} = 1,\ u = 0 \text{ a.e. on } \Omega^c \}.$$

We adapt the classical γ-convergence techniques to this general abstract setting to prove an existence result for the problem (1). We apply the existence result to the case of a metric measure space (X, d, μ). In fact, the Sobolev space $H^{1,2}(X)$, as defined by Cheeger, satisfies the properties of H, under some classical hypothesis on X (doubling, supporting weak Poincaré inequality) and the assumption that the inclusion $H^{1,2} \hookrightarrow L^2$ is compact. The particular case of a Carnot-Carathéodory space is also discussed.

Davide Vittone (Università di Padova)
`Lipschitz hypersurfaces and perimeter in Carnot-Carathéodory spaces`
Abstract. The notion of "intrinsic Lipschitz" hypersurface has been recently introduced in Carnot groups by B. Franchi, R. Serapioni and F. Serra Cassano. We will discuss an equivalent definition which can be stated in the more general framework of Carnot-Carathodory spaces and prove that intrinsic Lipschitz domains have locally finite perimeter. Properties of the perimeter measure (e.g., an area-type formula, Ahlfors regularity) will be discussed. If time permits, we will also show some application to trace problems for functions with Bounded Variation in Carnot-Carathodory spaces.

Dachun Yang (School of Mathematical Sciences, Beijing Normal University)
`Sobolev Spaces on Metric Measure Spaces`
Abstract. It is well known that the Sobolev spaces play an important role in analysis on metric measure spaces. In this talk, we present some characterizations of Hajłasz-Sobolev spaces on metric measure spaces via grand Littlewood-Paley functions and also the pointwise characterization of Besov spaces, Triebel-Lizorkin spaces, Newton-Besov and Newton-Triebel-Lizorkin spaces on metric measure spaces.

Xiao Zhong (University of Jyväskylä)
`Quantitative isoperimetric inequalities`
Abstract. I will talk about Bonnesen's inequality for John domains. This is joint work with Kai Rajala.

Yuan Zhou (Beijing University of Aeronautics ans Astronatics)
`Geometry and Analysis of Dirichlet forms`
Abstract. Let $\mathcal{E}$ be a regular, strongly local Dirichlet form on $L^2(X, m)$ and d the associated intrinsic distance. Assume that the topology induced by d coincides with the original topology on X, and that X is compact, satisfies a doubling property and supports a weak $(1, 2)$-Poincare inequality. We first discuss the (non-)coincidence of the intrinsic length structure and the gradient structure. Then when the Ricci curvature of X is bounded from below in the sense of Lott-Sturm-Villani, the following are equivalent:
(i) the heat flow of $\mathcal{E}$ gives the unique gradient flow of the entropy $\mathcal{U}_\infty$,
(ii) $\mathcal{E}$ satisfies the Newtonian property,
(iii) the intrinsic length structure coincides with the gradient structure. Moreover, for the standard (resistance) Dirichlet form on the Sierpinski gasket equipped with the Kusuoka measure, we identify the intrinsic length structure with the

measurable Riemannian and the gradient structures. We also apply the above results to the (coarse) Ricci curvatures and symptotics of the gradient of the heat kernel.