# LECTURE COURSE 'EINSTEIN MANIFOLDS'

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## Basics: Dates, Duration, Prerequisites

- Introduction, outline of the course, etc. on February 14, 2011.
- Expected duration of lecture course is 24 hours, 2 hours per week for 12 weeks (with short interruptions in between).
- The level of the lectures is for graduate students. Since almost no preliminary knowledge is required, also talented undergraduate students may attend the lecture.
- The lectures will be held in English.
- As for prerequisites, an elementary knowledge of the basics of Riemannian geometry (manifolds, metrics, curvatures and other tensors) is assumed (however, the most important objects are briefly recalled in the first lectures). Moreover, a basic knowledge of the theory of elliptic partial differential equations and of the calculus of variations is certainly an advantage, but not a necessity.

## Abstract

A natural and general question in geometry is: given a manifold $M$, are there any best or canonical Riemannian metrics / structures on $M$? For surfaces (i.e. 2-dimensional manifolds), the best Riemannian metrics are those of constant curvature. In dimension 2, there is only one notion of curvature, namely the Gauss curvature. In higher dimensions, we have three main notions of curvature: i) the Riemannian curvature tensor (or equivalently the sectional curvatures), ii) the Ricci tensor (the trace of the Riemann tensor) and iii) the scalar curvature (the trace of the Ricci tensor). While constant sectional curvatures is a too restrictive condition (satisfied only by $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$) and constant scalar curvature is too weak (for any compact manifold there exists an infinite-dimensional family of such metrics), the condition of constant Ricci curvature seems the most natural and interesting one. Manifolds with constant Ricci curvature are called Einstein manifolds.

In this lecture course, we are interested in a) the existence and b) the uniqueness of Einstein metrics on a given manifolds as well as c) the properties of Einstein manifolds - in particular, we want to investigate functionals which have Einstein manifolds as critical points and we want to understand the moduli space of Einstein manifolds. We are mostly (but not exclusively) interested in dimension 3 and 4 and the course is mainly based on the classical theory found in the book [1].

## More detailed outline for planned course

I plan to cover the following topics during the lecture course.

- First, we will repeat the basics from Riemannian geometry (in particular the different notions of curvature) and prove some elementary properties of Einstein manifolds. [1,2]
- We will *very briefly* study the role of Einstein manifolds in general relativity and the differences in the Riemannian and Lorentzian settings. [1,3]

- Variational structures: We will study general variation formulas for the curvatures on a manifold and for functionals depending on the curvatures, so-called Riemannian functionals. We will see that Einstein metrics are the critical points of the Einstein-Hilbert functional. [1,4] We also study quadratic curvature functionals and their critical points, in particular in dimension three and four. [1,5,6]
- Existence: We will interpret the Einstein condition as a nonlinear partial differential equation for the Riemannian metric. The existence and non-existence of solutions to such equations is studied. [1,2]
- We will briefly study the topic of uniqueness for Einstein metrics on given manifolds. [1]
- We will give a *very brief* overview on the Kähler case. Although the main goal of the course is to work with general Riemannan manifolds, Kähler manifolds form an important subclass where the Einstein condition is better understood. [1]
- We will study compactness theorems for Riemannian manifolds and review the important notions of Gromov-Hausdorff and Cheeger-Gromov convergence in the classical as well as in a weak sense. This is relevant for the remaining lectures.
- Finally, we will study the moduli spaces of Einstein manifolds and try to give an outlook into exciting ongoing research topics.

## Bibliography

[1] A. Besse. *Einstein Manifolds.* Classics in Mathematics. Springer, 1987.
[2] J. Jost. *Riemannian geometry and geometric analysis.* Springer, 2002.
[3] R. Wald. *General Relativity.* University of Chicago Press, 1984.
[4] R. Müller. *Differential Harnack inequalities and the Ricci flow.* EMS, 2006.
[5] M. Anderson. *Canonical metrics on 3-manifolds and 4-manifolds.* AJM, 2006.
[6] M. Gursky and J. Viaclovsky. *A new variational characterization of three-dimensional space forms.* Inventiones, 2001.