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J. Differential Equations 192 (2003) 1–46

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Steady supersonic flow past an almost straight wedge with large vertex angle

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Received May 10, 2000; revised December 18, 2002

Abstract

This paper studies the problem on the steady supersonic flow at the constant speed past an almost straight wedge with a piecewise smooth boundary. It is well known that if each vertex angle of the straight wedge is less than an extreme angle determined by the shock polar, the shock wave is attached to the tip of the wedge and constant states on both side of the shock are supersonic. This paper is devoted to generalizing this result. Under the hypotheses that each vertex angle is less than the extreme angle and the total variation of tangent angle along each edge is sufficiently small, a sequence of approximate solutions constructed by a modified Glimm scheme is proved to be convergent to a global weak solution of the steady problem. A sequence of the corresponding approximate leading shock fronts issuing from the tip is shown to be convergent to the leading shock front of the obtained solution. The regularity of the leading shock front is established and the asymptotic behaviour of the obtained solution at infinity is also studied.

Keywords: Supersonic; Wedge; Glimm scheme; Global solution; Asymptotic behaviour

1. Introduction

The problem of steady supersonic flow past a wedge has been studied extensively by many authors (for references, see [2–5,8,11,14–16,21,23,27] and references

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0022-0396/03/$ - see front matter
doi:10.1016/S0022-0396(03)00037-8

therein). In [5,11,14,15,23], the local solution around the vertex has been constructed. The global solution has been constructed in [3–5,8,27] when the wedge has straight edges, or when the curved wedge has small vertex angles and each edge is the perturbation of a straight one. Here by a vertex angle, we mean a lower vertex angle, or an upper vertex angle which is the angle between the velocity of the oncoming flow and the tangent line of the lower edge or upper edge at the vertex, respectively.

In this paper we are concerned with the problem of planar steady supersonic potential flow past a two-dimensional wedge which has a piecewise smooth boundary with each vertex angle less than the extreme angle. For simplicity, we will study here the problem for a half-wedge, that is, we will consider the problem

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0 & \text{in } \Omega, \\ v_x - u_y = 0 & \text{in } \Omega, \\ (u,v)\cdot \vec{n} = 0 & \text{on } \Gamma, \\ (u,v)|_{x<0} = U_\infty, \end{cases} \tag{1.1}$$

under the following assumptions:

(A1) The function $\rho = \rho(\sqrt{u^2+v^2})$ is given by the following Bernoulli relation:

$$\frac{\gamma-1}{\gamma+1}(u^2+v^2) + \frac{2}{\gamma+1}c^2(\rho) = c_*^2, \tag{1.2}$$

where (u,v) is the velocity of flow and ρ is the density; $p = A\rho^{\gamma-1}$ and $(c(\rho))^2 = p'(\rho) = \gamma A\rho^{\gamma-1}$, $A>0$ is a constant and $\gamma>1$ is adiabatic exponent; $c_*>0$, is the constant critical speed given in [2].

(A2) There exists a piecewise C^1 function $b\in C[0,+\infty)$ with $b'_+\in BV([0,+\infty))$, $b'(0+)=0$ and $b(0)=0$, such that

$$\Omega = \{(x,y)|y<b(x), x>0\}, \quad \Gamma = \{(x,y)|y=b(x),\ x>0\},$$

where

$$b'_+(x) = b'(x+) = \lim_{\substack{y\to x\\ y>x}} \frac{b(y)-b(x)}{y-x}$$

and

$$\vec{n} = \vec{n}(x,b(x)) = \frac{(-b'(x+),1)}{\sqrt{(b'(x+))^2+1}}$$

is the outer normal vector to Γ at the continuity points of b' (see Fig. 1).

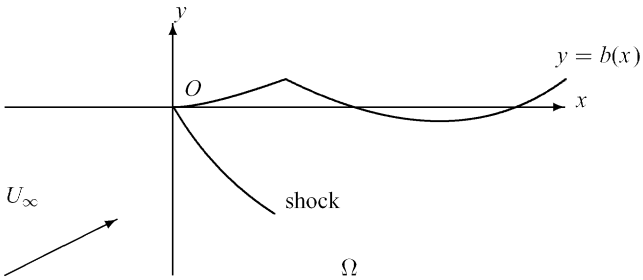

Fig. 1. Supersonic flow past a curved wedge.

(A3) The velocity of the oncoming flow is a constant vector $U_\infty = (u_\infty, v_\infty)$ which satisfies

$$q_\infty = \sqrt{u_\infty^2 + v_\infty^2} > c_*, \tag{1.3}$$

$$u_\infty > 0, \quad v_\infty > 0 \tag{1.4}$$

and

$$0 < \arctan \frac{v_\infty}{u_\infty} < \omega_{\text{ext}}, \tag{1.5}$$

where

$$\omega_{\text{ext}} = \sup\left\{ \left| \arctan \frac{v}{u} - \arctan \frac{v_\infty}{u_\infty} \right|, \ (u, v) \in S(U_\infty), c_*^2 < u^2 + v^2 < q_\infty^2 \right\},$$

and $S(U_\infty)$ is the shock polar associated with U_∞ as given in [12].

A simple case of problem (1.1) is the case that $b(x) \equiv 0$. It has been shown in [8,12] that if $b \equiv 0$ and if assumptions (A1) and (A3) hold, then problem (1.1) admits an entropy solution that consists of the constant state U_∞ and a constant state U_0, with $U_0 = (u_0, 0)$ and

$$u_0 > c_0 > 0 \tag{1.6}$$

in subdomains of Ω separated by a straight shock line issuing from the vertex. In other words, the state ahead of the shock front is U_∞ while the state behind the shock front is U_0 (see Fig. 2), and there holds the entropy condition as follows:

$$\rho_0 > \rho_\infty. \tag{1.7}$$

Moreover by Bernoulli relation, (1.7) has the equivalent form as follows:

$$u_\infty^2 + v_\infty^2 > u_0^2. \tag{1.8}$$

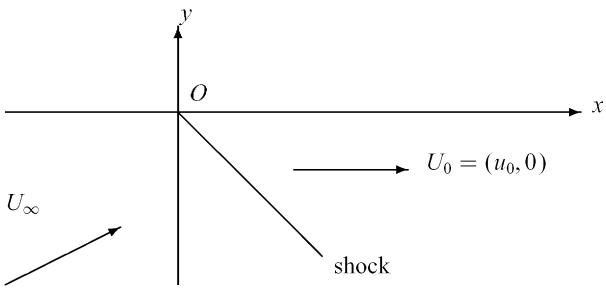

Fig. 2. The case $b \equiv 0$.

Here c_0 and c_∞ are sonic speed given by Bernoulli relation in (A1), corresponding to U_0 and U_∞, respectively.

In this paper, we will generalize the above result, that is, under assumptions (A1)–(A3) and the hypothesis that the total variation of b'_+ is sufficiently small, we will find a global solution U satisfying the following properties:

(s-i) U is a weak solution to problem (1.1), that is, U solves the problem in the following sense as in [12,21]:

$$\int_\Omega \rho u \phi_{1x} + \rho v \phi_{1y} = \int_{-\infty}^{0} \rho_\infty u_\infty \phi_1(0, y)\, dy, \tag{1.9}$$

$$\int_\Omega v \phi_{2x} - u \phi_{2y} = 0 \tag{1.10}$$

for $\forall \phi_1 \in C_c^\infty(R^2), \phi_2 \in C_c^\infty(\Omega)$, where $U = (u, v)$, and $\rho_\infty = \rho(U_\infty)$ is given by the Bernoulli relation;

(s-ii) There is a shock front of U, $y = \chi(x)$, issuing from the vertex point, such that $U|_{y<\chi(x)} = U_\infty$ and such that $U|_{\chi(x)<y<b(x)}$ is close to the state U_0; moreover, $q_\infty > q(U)|_{\chi(x)<y<b(x)}$. Here and throughout the paper the constant state U_0 denotes the state given above.

Meanwhile, we will show that the asymptotic behaviour of the obtained solution at $x = +\infty$ is determined only by the limit, $\lim_{x\to+\infty} b'(x+0)$, and the velocity of the oncoming flow, U_∞ (see Theorems 5.1 and 5.2).

In Ref. [27], we have got a global weak solution when each vertex angle and the total curvature of each edge of the wedge are sufficiently small. In that case, the vertex angle and the total variation of the tangent angle along each edge of the wedge are so small that the shock wave issuing from the tip and the waves produced by the flow moving along each edge are weak (see Lemma 3.3 or [27]) and only the estimates on the interactions between the weak waves and the estimates on weak interactions at the boundary are needed to prove the decreasing of the Glimm

functional. When the vertex angles are less than the extreme angle ω_{ext} but do not satisfy the requirement as given in [27], the shocks issuing from tip of the wedge will be relatively strong and may fail to satisfy the requirement of small strength in Glimm's theorem on waves interactions. Additional estimates are needed to deal with the interactions between these strong shocks and other weak waves. In this paper, we are concerned with the general case that $0<\arctan\frac{v_\infty}{u_\infty}<\omega_{\text{ext}}$, which includes the case of a large vertex angle. Here by a large vertex angle, we mean a vertex angle which is less than ω_{ext} but does not satisfy the requirement of smallness in [27]. We will establish some estimates to deal with the interactions between the strong shock wave issuing from tip and the weak waves produced by the flow moving along the boundary when the total variation of tangent angles along each edge of the wedge is very small. Moreover, to show that the strong shock wave issuing from the tip will not disappear, we will regard the shock front $y=\chi(x)$ as a free boundary and will have to establish the estimates on reflection coefficients, (3.13) and (3.26), which lead to the contraction inequality (4.1) (or equivalently (4.13)) when the total variation of tangent angles along each edge of the wedge is very small. This contraction inequality, which is analogous to the finiteness condition in [24] and the condition of contraction in [22], implies that the strengths of weak waves will diminish after multi-reflections against the leading shock front $y=\chi(x)$ and the fixed boundary, therefore the strong shock wave attached to the tip will be stable and will not disappear here. The Glimm scheme is modified to construct the approximate solutions and to trace the leading shock front $y=\chi(x)$.

Remark 1.1. The form of the system of steady irrotational flow is invariant under the same rotation of coordinate systems for both (u,v) and (x,y). Moreover, the Rankine–Hugoniot relation and entropy condition are invariant under the same rotation of coordinate system, thus the corresponding boundary problems are equivalent in the sense of distribution as (1.9) and (1.10). Then for the case that $b'(0)\neq 0$, due to the fact $U_0//b'(0)$, we can choose a suitable coordinate system such that $b'(0)=0$ and $U_0=(u_0,0), u_0>0$ hold in the new coordinate system.

The remaining part of the paper is organized as follows. In Section 2 we study shock polar and epicycloid and distinguish a family of relatively strong shocks, which are small perturbations of the shock $\{U_\infty, U_0\}$, from the relatively weak waves, and we call them strong shocks. These wave curves give the solutions to Riemann problems and the strong shocks will be used to trace the dominant shock front $y=\chi(x)$. In Section 3, we establish by the results above the estimates on the boundary interactions of weak waves and the estimates on the boundary interactions of strong shock waves. Also, we study the interactions between the weak waves and the strong shocks. Sharp estimates (3.13) and (3.26) on the coefficients of reflecting waves are established there. In Section 4 we first approximate the boundary by piecewise line segments and construct the approximate solution in approximate domain. Then we define a modified Glimm functional, which is analogous to that used in [27] (see also [22,24]) and includes the terms needed to take into account the reflections

on the strong shock front issuing from the tip and the reflections on a fixed boundary, and we apply the estimates obtained in Section 3 to prove the desired decreasing of the modified Glimm functional in each approximate domain. Therefore, the approximate solutions can be globally defined and some estimates on the approximate solutions and the approximate strong shock fronts are obtained. The contraction inequality (4.1) (or equivalently (4.13)), which is the consequence of (3.13) and (3.26) when the total variation of tangent angles along each edge of the wedge is very small, plays a crucial role in the proof of the decreasing of the modified Glimm functional. In Section 5 the convergence of the approximate solutions and the convergence of the approximate strong shock fronts are shown, and the limits are proved to be a solution of problem (1.1) and its shock front. The asymptotic behaviour of the obtained solution at $x = +\infty$ is also studied. The main results, Theorems 5.1 and 5.2, are stated there.

2. Riemann problem

2.1. Riemann problem involving only weak waves

First, we recall some basic facts that will be used in the sequel. As usual case we regard the *x-direction* as the time-like direction. Then the system in (1.1) is genuinely nonlinear and strictly hyperbolic in the supersonic subregion D, where $D = \{(u,v)|u > c_*, u^2 + v^2 < q_*^2\}$ and $q_* = \sqrt{\frac{\gamma+1}{\gamma-1}}c_*$. Moreover, we can choose a neighbourhood of $(u_0, 0)$, V_0, with $\overline{V_0} \subset \{(u,v)|u > c_*,\ u^2 + v^2 < u_\infty^2 + v_\infty^2\}$, such that the system possesses two distinct characteristics

$$\lambda_1 = \frac{uv - c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}$$

and

$$\lambda_2 = \frac{uv + c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}$$

in V_0, with $\lambda_1 < 0 < \lambda_2$ in V_0, and two right eigenvectors

$$r_j(u,v) = e_j \begin{pmatrix} -\lambda_j \\ 1 \end{pmatrix}$$

$(j = 1, 2)$ in V_0. Here $U_0 = (u_0, 0)$ is the constant state given in Section 1, and $e_j(u,v)$ $(j = 1, 2)$ are smooth functions in V_0 which satisfy

$$r_j \cdot \nabla \lambda_j = 1. \tag{2.1}$$

Moreover, we have

Lemma 2.1. (i) $\frac{\partial}{\partial u}(\rho u)<0$, $\forall(u,v)\in D$, *therefore* $\Psi: D\mapsto\Psi(D)$ *is a smooth diffeomorphism where* $\Psi(u,v)=(\rho u,v)$; (ii) $e_1(u_0,0)=e_2(u_0,0)>0$, *therefore*

$$e_j(u,v)>0 \quad (j=1,2) \tag{2.2}$$

for any state (u,v) *near* $(u_0,0)$.

Proof. The proof of (i) has been given in [27]. We only have to prove (ii). Indeed, differentiating the Bernoulli relation in assumption (A1) with respect to u and v, we can get

$$(c^2)_u|_{(u,v)=(u_0,0)}=-(\gamma-1)u_0,$$

$$(c^2)_v|_{(u,v)=(u_0,0)}=0,$$

then

$$\lambda_{ju}|_{U=U_0}=\pi_j\left\{\frac{(\gamma-1)u_0}{2c_0\sqrt{u_0^2-c_0^2}}+\frac{(\gamma+1)u_0c_0}{2\sqrt{(u_0^2-c_0^2)^3}}\right\}$$

and

$$\lambda_{jv}|_{U=U_0}=\frac{u_0}{u_0^2-c_0^2}>0.$$

Here $j=1,2$ and $\pi_1=1$ while $\pi_2=-1$. Thus it follows that

$$\nabla_U\lambda_1\cdot(-\lambda_1,1)|_{U=U_0}=\nabla_U\lambda_2\cdot(-\lambda_2,1)|_{U=U_0}>0.$$

This yields the result (ii). □

Thus, by Lemma 2.1 and by shrinking V_0, we can assume in the sequel that $e_j(u,v)>0$ $(j=1,2)$ for any $(u,v)\in V_0$.

In the rest of this section, we consider the problem

$$\begin{cases} W(U)_x+H(U)_y=0, \\ U|_{x=a}=\begin{cases} U_r & y>b, \\ U_l & y<b, \end{cases} \end{cases} \tag{2.3}$$

where $W(U)=\binom{\rho u}{v}$ and $H(U)=\binom{\rho v}{-u}$. Here for any b, the state U_r defined in $\{y>b\}$ is regarded as the *right* state, and the state U_l defined in $\{y<b\}$ is regarded as the *left* state.

Let us recall some basic facts on the wave curves related to problem (2.3). It has been shown in [8,12] that for any constant state $(\underline{u},\underline{v})$ lying in the supersonic region,

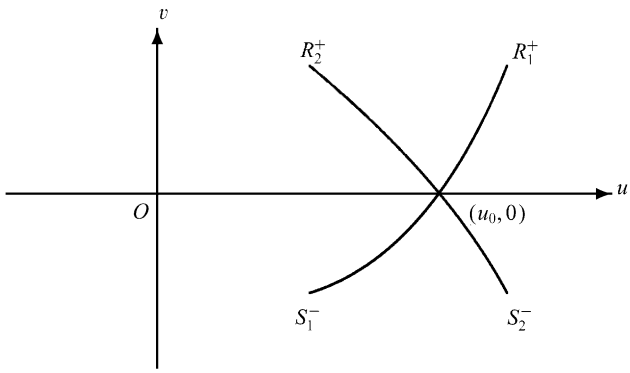

Fig. 3. Wave curve for the case: $\underline{u} = u_0, \underline{v} = 0$.

the states which can be connected with the state $(\underline{u}, \underline{v})$ by a simple wave form a curve called epicycloid, while the states which can be connected with the state $(\underline{u}, \underline{v})$ by a shock form a curve called shock polar. And we denote by $R(\underline{u}, \underline{v})$ the epicycloid and denote by $S(\underline{u}, \underline{v})$ the shock polar. Let $R_j(\underline{u}, \underline{v})$ and $S_j(\underline{u}, \underline{v})$ be the part of the epicycloid and the shock polar in the supersonic region corresponding to the λ_j-characteristic field, respectively. Denote

$$R_2^+(\underline{u}, \underline{v}) = \{(u, v) \in R_2(\underline{u}, \underline{v}) | q \leqslant \underline{q}\},$$

$$S_2^-(\underline{u}, \underline{v}) = \{(u, v) \in S_2(\underline{u}, \underline{v}) | q \geqslant \underline{q}\},$$

$$R_1^+(\underline{u}, \underline{v}) = \{(u, v) \in R_1(\underline{u}, \underline{v}) | q \geqslant \underline{q}\},$$

$$S_1^-(\underline{u}, \underline{v}) = \{(u, v) \in S_1(\underline{u}, \underline{v}) | q \leqslant \underline{q}\}$$

and

$$T_j(\underline{u}, \underline{v}) = R_j^+(\underline{u}, \underline{v}) \cup S_j^-(\underline{u}, \underline{v}), \quad j = 1, 2.$$

Here $q = \sqrt{u^2 + v^2}$ and $\underline{q} = \sqrt{\underline{u}^2 + \underline{v}^2}$ (see Fig. 3).

Let U_r and U_l be two constant states near the constant state $U_0 = (u_0, 0)$. Then according to [8,12], the $T_j(\underline{u}, \underline{v})(j = 1, 2)$ can give the physically admissible solution to problem (2.3). We call the waves given by T_j the elementary waves, or *j-wave* in the sequel. It has been shown in [27] that the following holds.

Lemma 2.2. *There exists a $\delta_1 > 0$ such that the following hold for all points $(\underline{u}, \underline{v})$ belonging to the neighbourhood of $(u_0, 0)$, $O_{\delta_1}(U_0)$, with $O_{\delta_1}(U_0) \subset V_0$:*

$$R_j^+(\underline{u}, \underline{v}) \cap O_{\delta_1}(U_0) = \{(u, v) \in R_j(\underline{u}, \underline{v}) | \lambda_j(u, v) \geqslant \lambda_j(\underline{u}, \underline{v}).\} \cap O_{\delta_1}(U_0),$$

$$S_j^-(\underline{u},\underline{v})\cap O_{\delta_1}(U_0)=\{(u,v)\in S_j(\underline{u},\underline{v})|\lambda_j(u,v)\leqslant\lambda_j(\underline{u},\underline{v})\}\cap O_{\delta_1}(U_0)$$

$(j=1,2)$, *where the equality for* R_j^+ *(or* S_j^-*) holds if and only if* $u=\underline{u}$, $v=\underline{v}$.

Then following Lax [13], by this lemma we can parameterize the curve $T_j(U_l)$ for any state U_l near the state U_0. As in [13] (see also [27]), let $T_j(u_l,v_l)$ be parameterized by $\varepsilon_j\mapsto\Phi_j(\varepsilon_j,U_l)$ in a neighbourhood of U_0, $O_{\delta_2}(U_0)$, with $\Phi\in C^2$ and

$$\Phi_j|_{\varepsilon_j=0}=U_l,$$

$$\left.\frac{\partial\Phi_j}{\partial\varepsilon_j}\right|_{\varepsilon_j=0}=r_j(U_l).$$

Moreover, $\varepsilon_j>0$ along $R_j^+(U_l)\cap O_{\delta_2}(U_0)$ while $\varepsilon_j<0$ along $S_j^-(U_l)\cap O_{\delta_2}(U_0)$ $(j=1,2)$. Here, $\delta_2>0$ is a positive constant independent of ε and U, and $\delta_2\leqslant\delta_1$ with $O_{\delta_2}(U_0)\subset V_0$.

Denote

$$\Phi(\varepsilon_2,\varepsilon_1,U_l)=\Phi_2(\varepsilon_2,\Phi_1(\varepsilon_1,U_l)),\tag{2.4}$$

then we have

Lemma 2.3. *There is a* $\delta_2'\in(0,\delta_2)$ *such that for any pair of states* $U_r,U_l\in O_{\delta_2'}(U_0)$, *problem* (2.3) *admits a unique admissible solution consisting of two elementary waves. In addition, it owns the representation*: $U_r=\Phi(\beta,\alpha,U_l)$ *with*

$$\Phi|_{\alpha=\beta=0}=U_l,$$

$$\left.\frac{\partial\Phi}{\partial\alpha}\right|_{\alpha=\beta=0}=r_1(U_l)$$

and

$$\left.\frac{\partial\Phi}{\partial\beta}\right|_{\alpha=\beta=0}=r_2(U_l).$$

For simplicity if U_r, $U_l\in O_{\delta_2'}(U_0)$, we shall use the notation $\{U_l,U_r\}=(\alpha,\beta)$ to denote that $U_r=\Phi(\beta,\alpha,U_l)$ throughout the paper, and call the parameters α and β the magnitude of weak 1-wave and the magnitude of weak 2-wave, respectively. It is obvious that $\alpha>0$ along R_1^+ and $\beta>0$ along R_2^+ while $\alpha<0$ along S_1^- and $\beta<0$ along S_2^-.

2.2. Riemann problem involving a strong 1-shock

In this subsection we consider the Riemann problem (2.3) in the case that $U_l = U_\infty$ and U_r is a constant state near U_0.

For any $U \in S_1^-(U_\infty)$, we also use $\{U_\infty, U\} = (\sigma, 0)$ to denote the shock that connects U_∞ and U with the speed σ (see Fig. 4). Furthermore, if $U \in O_{\delta_2}(U_0) \cap S_1^-(U_\infty)$ we call shock $\{U_\infty, U\}$ a strong 1-shock throughout the paper.

First, we have the following properties of unperturbed strong 1-shock $\{U_\infty, U_0\}$.

Lemma 2.4. *Let* $\{U_\infty, U_0\} = (\sigma_0, 0)$, *then*
(1) $\sigma_0 < 0$ *and* $u_\infty > u_0 > c_*$, $u_\infty > c_\infty$;
(2) $\lambda_1(U_0) < \sigma_0 < \lambda_1(U_\infty)$.

Proof. First we will prove the first statement (1). To do this, we write the Rankine–Hugoniot relation as

$$\rho_\infty(u_\infty \sigma_0 - v_\infty) = \rho_0 u_0 \sigma_0, \tag{2.5}$$

$$v_\infty \sigma_0 + u_\infty = u_0. \tag{2.6}$$

Assume, to reach a contradiction, that $\sigma_0 \geqslant 0$. Then by entropy condition (1.7) and (2.5), it follows that

$$u_\infty \sigma_0 - v_\infty > u_0 \sigma_0;$$

therefore $u_\infty > u_0$, which yields the contradiction to (2.6). Then it follows that $\sigma_0 < 0$; therefore, from (2.6) and (1.6) we have

$$u_\infty > u_0 > c_*.$$

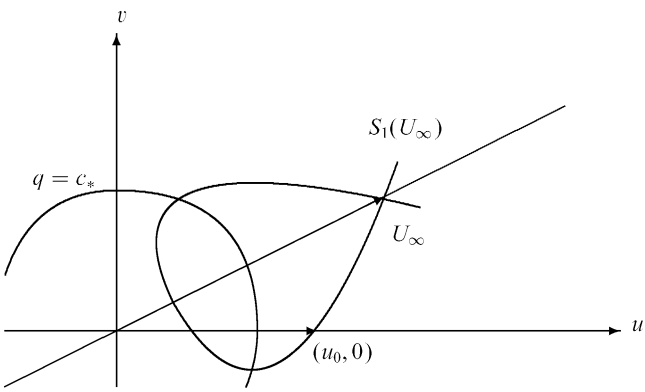

Fig. 4. Shock polar.

In addition, from Bernoulli relation it follows that

$$u_\infty > c_* > c_\infty .$$

Thus, the first statement (1) is proved.

To prove second statement (2), let

$$\beta_0 = \arctan \sigma_0,$$

$$\alpha_0 = \arctan \frac{c_0}{\sqrt{u_0^2 - c_0^2}},$$

$$\alpha_\infty = \arctan \frac{c_\infty}{\sqrt{u_\infty^2 + v_\infty^2 - c_\infty^2}},$$

$$\omega_\infty = \arctan \frac{v_\infty}{u_\infty}.$$

Then $\beta_0 \in (-\pi/2, 0)$ and $\alpha_\infty, \alpha_0, \omega_\infty \in (0, \pi/2)$.

As in [12], let $\sigma_0 = \tan \beta_0$ in (2.5) and (2.6), then we have

$$\sin^2 \beta_0 = \frac{1 - (q_\infty^2/u_0^2)}{1 - (\rho_0^2/\rho_\infty^2)}.$$

Let

$$m(q) = \frac{1 - (q^2/u_0^2)}{1 - (\rho_0^2/\rho^2)},$$

where ρ is a function of $q^2 = u^2 + v^2$ given by Bernoulli equation. Then $\sin^2 \beta_0 = m(q_\infty)$.

By differentiating the Bernoulli relation we have

$$\rho_q = -(q\rho/c^2)$$

and

$$(c^2)_q = -(\gamma - 1)q.$$

Therefore

$$\frac{dm}{dq} = \frac{2qI(q)}{u_0^2 \rho^2 c^2 \{1 - (\rho_0^2/\rho^2)\}^2},$$

where $I(q) = (\rho_0^2 - \rho^2)c^2 + (u_0^2 - q^2)\rho_0^2$.

In addition, from the Bernoulli relation it follows that $0<\rho<\rho_0$ for any $q\in\{q>u_0\}$. This yields

$$\left.\frac{dI}{dq}\right|_{q>u_0}=(\gamma+1)q(\rho^2-\rho_0^2)<0.$$

Thus $I(q)<I(u_0)=0$ for any $q>u_0$, which implies

$$\left.\frac{dm}{dq}\right|_{q>u_0}<0.$$

Then

$$\sin^2\beta_0=m(q_\infty)<\lim_{q\to u_0}m(q).$$

Applying the L'Hospital rule to the limit in this inequality, we have

$$\sin^2\beta_0<\lim_{q\to u_0}m(q)=\frac{c_0^2}{u_0^2}=\sin^2\alpha_0,$$

which yields that $\sigma_0>\lambda_1(U_0)$.

Finally we will prove that $\sigma_0<\lambda_1(U_\infty)$. By rotation we choose a new coordinate systems $Ox'y'$ with the direction of U_∞ as the direction of new x-axis, $\vec{O}x'$. Then in the new coordinate system $Ox'y'$, we have $U_\infty=(q_\infty,0)$ and $U_0=(u_0',v_0')$ with $u_0'>0$ and $v_0'<0$. In addition, there hold $(u_0')^2+(v_0')^2=u_0^2$ and $u_\infty^2+v_\infty^2=q_\infty^2$. From Remark 1.1 we know that $S_1^-(U_\infty)$ is also invariant under the rotation of the coordinate systems. Then in the new coordinate systems $Ox'y'$, we can get the following in the same way as above by studying $S_1^-(U_\infty)$:

$$\sin^2(\omega_\infty-\beta_0)>\sin^2\alpha_\infty,$$

and we can prove that the shock speed is equal to $\tan(\beta_0-\omega_\infty)$ with

$$\tan(\beta_0-\omega_\infty)<0.$$

Therefore these yield that

$$0<\alpha_\infty<\omega_\infty-\beta_0<\frac{\pi}{2}.$$

Hence $-\pi/2<\beta_0<\omega_\infty-\alpha_\infty<\pi/2$, and this implies that $\sigma_0<\lambda_1(U_\infty)$. The proof is complete. □

Lemma 2.4 implies that the shock $\{U_\infty,U_0\}$ satisfies Lax shock condition if we regard x-direction as time direction. Next, we will prove that it is also a Majda stable 1-shock. To do this, we need the following estimates.

Lemma 2.5.

$$0<\frac{|\lambda_1(U_0)-\sigma_0|}{|\lambda_2(U_0)-\sigma_0|}<1. \tag{2.7}$$

Proof. Since

$$\lambda_1(U_0)=-\frac{c_0}{\sqrt{u_0^2-c_0^2}}<0$$

and

$$\lambda_2(U_0)=\frac{c_0}{\sqrt{u_0^2-c_0^2}}>0,$$

the result follows from Lemma 2.4. □

Lemma 2.6. *Let*

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=[\nabla_U W(U_0)]^{-1}(W(U_0)-W(U_\infty)). \tag{2.8}$$

Then $a_1<0$, $a_2<0$.

Proof. Differentiating the Bernoulli relation with respect to u and v, respectively, and taking $u=u_0, v=0$, we have

$$\rho_u|_{U=U_0}=-\frac{\rho_0 u_0}{c_0^2},$$

$$\rho_v|_{U=U_0}=0.$$

Therefore

$$\nabla_U W(U_0)=\begin{pmatrix} \rho_0(1-\frac{u_0^2}{c_0^2}) & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.9}$$

Moreover, from the Rankine–Hugoniot relation (2.5), we can get

$$(W(U_0)-W(U_\infty))=\begin{pmatrix} -\frac{\rho_\infty v_\infty}{\sigma_0} \\ -v_\infty \end{pmatrix}. \tag{2.10}$$

Then by the supersonic inequality (1.6), Assumption (A3) and Lemma 2.4, we can deduce the result from (2.9) and (2.10). □

Lemma 2.7. *Let* $\vec{t} = \binom{a_1}{a_2}$ *be the vector given by* (2.8). *Then*

$$\det(r_2(U_0), \vec{t}) \neq 0 \tag{2.11}$$

and

$$\frac{|\det(r_1(U_0), \vec{t})|}{|\det(r_2(U_0), \vec{t})|} < 1. \tag{2.12}$$

Proof. By direct calculation and by Lemma 2.6 we have

$$\det(r_2(U_0), \vec{t}) = e_2(U_0)\left(\frac{-c_0 a_2}{\sqrt{u_0^2 - c_0^2}} - a_1\right) > 0 \tag{2.13}$$

and

$$\det(r_1(U_0), \vec{t}) = e_1(U_0)\left(\frac{c_0 a_2}{\sqrt{u_0^2 - c_0^2}} - a_1\right). \tag{2.14}$$

Then by Lemma 2.6 we can deduce the result from (2.13) and (2.14). □

According to Majda [19,20] and Schochet [24], we say 1-shock $\{U_\infty, \underline{U}\} = (\sigma, 0)$ is a Majda stable 1-shock if it satisfies Lax entropy condition and satisfies the following conditions:

1. σ is not an eigenvalue of either of $(\nabla_U W(U))^{-1} \nabla_U H(U)|_{U=\underline{U} \text{ or } U=U_\infty}$;
2. $\det(\underline{r}_2(\underline{U}), W(\underline{U}) - W(U_\infty)) \neq 0$. Here $\underline{r}_2(U) = \nabla_U W(U) r_2(U)$.

Then Lemma 2.4 and (2.11) imply that the unperturbed strong shock $\{U_\infty, U_0\}$ is a Majda stable shock. Therefore as in [7,24] we can parameterize this shock polar near the state U_0 as follows.

Proposition 2.1. *There exists a* $\delta_3' > 0$, *with* $\delta_3' < \delta_2$, *such that the shock polar* $S_1^-(U_\infty) \cap O_{\delta_3'}(U_0)$ *can be parameterized by the shock speed as* $\sigma \mapsto G(\sigma)$ *with* $G \in C^2$ *near* σ_0 *and* $G(\sigma_0) = U_0$. *Moreover* $\{U_\infty, G(\sigma)\}$ *is a Majda stable* 1-*shock with* $\det(\nabla_U W(G(\sigma))) \neq 0$.

Proof. It suffices to find the solution, $U = G(\sigma)$, to the following:

$$\sigma(W(U) - W(U_\infty)) = H(U) - H(U_\infty). \tag{2.15}$$

Since

$$\nabla_U\{\sigma(W(U)-W(U_\infty))-(H(U)-H(U_\infty))\}|_{\sigma=\sigma_0,U=U_0}$$
$$=\sigma_0\nabla_U W(U_0)-\nabla_U H(U_0),$$

and since Lemma 2.4 and (2.9) imply

$$\det(\sigma_0\nabla_U W(U_0)-\nabla_U H(U_0))\neq 0,$$

by the implicit function theorem we can find a unique C^2-function $U=G(\sigma)$ solving (2.15) near $\sigma=\sigma_0$ and $U=U_0$. Moreover, from Lemma 2.4, (2.9) and (2.11) in Lemma 2.7, it follows that $\{U_\infty, G(\sigma)\}$ is a Majda stable shock for any σ close to σ_0. This completes the proof. □

Lemma 2.8. *Let* $U_m=G(\sigma)\in O_{\delta_3'}(U_0)$, *then*

$$([\nabla_U W(U_m)]^{-1}\nabla_U H(U_m)-\sigma I)G_\sigma(\sigma)$$
$$=[\nabla_U W(U_m)]^{-1}(W(U_m)-W(U_\infty)). \tag{2.16}$$

Moreover, let $G_\sigma(\sigma_0)=\binom{b_1}{b_2}$, *then*

$$b_1=\frac{-\sigma_0 a_1+(\lambda_1(U_0))^2 a_2}{\sigma_0^2-(\lambda_1(U_0))^2}>0, \tag{2.17}$$

$$b_2=\frac{a_1-\sigma_0 a_2}{\sigma_0^2-(\lambda_1(U_0))^2}>0. \tag{2.18}$$

Proof. We can get (2.16) by differentiating the following Rankine–Hugoniot Relation with respect to σ:

$$\sigma(W(G(\sigma))-W(U_\infty))=H(G(\sigma))-H(U_\infty).$$

Let $\sigma=\sigma_0$, therefore $U_m=U_0$ in (2.16). As in the proof of Lemma 2.6, we have

$$\nabla_U H(U_0)=\begin{pmatrix}0 & \rho_0\\ -1 & 0\end{pmatrix}.$$

Then by (2.9), Lemmas 2.4 and 2.6, we can get (2.17) and (2.18). The proof is complete. □

To conclude the above discussions we give the solution to the Riemann problem involving a strong 1-shock.

Proposition 2.2. *Let $U_l = U_\infty$ in* (2.3). *There exists a δ_3, with $\delta_3 \in (0, \delta_2)$, such that if $U_r \in O_{\delta_3}(U_0)$ then problem* (2.3) *admits a unique admissible solution consisting of two waves, of which one is a weak* 2*-wave with magnitude β, and the other is a Majda stable strong* 1*-shock with shock speed σ, which is a small perturbation of the shock $\{U_\infty, U_0\}$. In addition, it owns the representation*: $U_r = \Phi(\beta, 0, G(\sigma))$.

Proof. It suffices to solve the following equation for any U_r near the state U_0,

$$\Phi(\beta, 0, G(\sigma)) = U_r. \tag{2.19}$$

Since

$$\left.\frac{\partial \Phi(\beta, 0, G(\sigma))}{\partial(\beta, \sigma)}\right|_{\beta=0, \sigma=\sigma_0} = (r_2(U_0), G_\sigma(\sigma_0)),$$

by Lemmas 2.1 and 2.8 we have

$$\det\left\{\frac{\partial \Phi(\beta, 0, G(\sigma))}{\partial(\beta, \sigma)}\right\}\Bigg|_{\beta=0, \sigma=\sigma_0} = -e_2(U_0)(b_1 + \lambda_2(U_0)b_2) < 0.$$

Therefore, by implicit function theorem we can get the desired result. □

As in Section 2.1 we will also use the notation $\{U_\infty, U_r\} = (\sigma, \beta)$ to denote that $U_r = \Phi(\beta, 0, G(\sigma))$ or to denote the solution to problem (2.3) with $U_l = U_\infty$ throughout the paper.

3. Estimates on the interactions and reflections

3.1. Estimates on the weak interactions and reflections

In this subsection we shall establish the sharp estimates on the interactions and reflections of weak waves. First, by the standard results (see [9,13] or [17]) we have the interaction estimates of weak waves in the interior as follows:

Lemma 3.1. *Suppose that U_l, U_m, U_r are three states close to U_0 with $\{U_l, U_r\} = (\alpha_1, \alpha_2)$, $\{U_l, U_m\} = (\beta_1, \beta_2)$ and $\{U_m, U_R\} = (\gamma_1, \gamma_2)$, then*

$$\alpha_j = \beta_j + \gamma_j + O(1)\Delta'(\beta, \gamma) \tag{3.1}$$

$(j = 1, 2)$. *Here $\Delta'(\alpha, \beta) = \sum |\alpha_i||\beta_j|$, where the sum is over all pairs for which the i-wave from α and j-wave from β are approaching*; $O(1)$ *depends only on the system and U_0.*

By direct computation we have

Lemma 3.2. *Suppose that $f \in C^2(R^2)$; then*

$$f(x,y) - f(x,0) - f(0,y) + f(0,0) = \left(\int_0^1 \int_0^1 f_{xy}(tx, sy)\, dt\, ds\right) xy \tag{3.2}$$

for any $x, y \in R^1$.

Let $C_k(a_k, b_k)(k = 1,2,3)$ be points in R^2 with $a_{k+1} > a_k > 0 (k = 1,2)$ and denote

$$\omega_1 = \arctan \frac{b_2 - b_1}{a_2 - a_1},$$
$$\omega_2 = \arctan \frac{b_3 - b_2}{a_3 - a_2},$$
$$\omega = \omega_2 - \omega_1,$$

$$\Omega_k = \left\{ (x,y) | a_k \leqslant x \leqslant a_{k+1}, y < \frac{b_{k+1} - b_k}{a_{k+1} - a_k}(x - a_k) + b_k \right\},$$
$$\Gamma'_k = \left\{ (x,y) | a_k < x < a_{k+1}, y = \frac{b_{k+1} - b_k}{a_{k+1} - a_k}(x - a_k) + b_k \right\},$$

and denote $\vec{n}_k$ the outer normal vector to Γ'_k, i.e.

$$\vec{n}_k = \frac{(-b_{k+1} + b_k, a_{k+1} - a_k)}{\sqrt{(-b_{k+1} + b_k)^2 + (a_{k+1} - a_k)^2}}$$

(see Fig. 5).

Set

$$\Delta(a,b) = \begin{cases} 0 & \text{if } a \geqslant 0 \text{ and } b \geqslant 0, \\ |a|\,|b| & \text{otherwise} \end{cases}$$

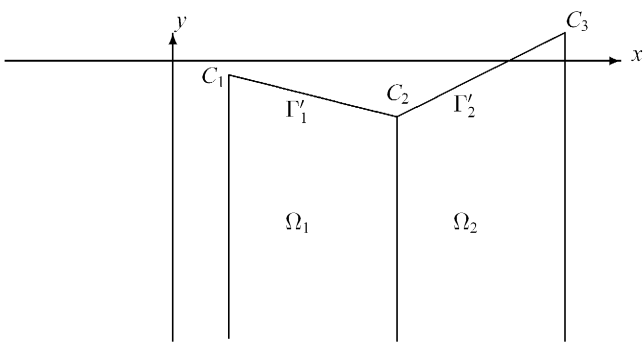

Fig. 5. Initial boundary value problem.

and, without confusion, in the sequel denote $O(1)$ the quantity of which the bound depends only the system and the states U_0 and U_∞. Then consider the following mixed problem:

$$\begin{cases} W(U)_x + H(U)_y = 0 & \text{in } \Omega_2, \\ U|_{x=a_2} = \underline{U}, & \\ U \cdot \vec{n}_2 = 0 & \text{on } \Gamma_2'. \end{cases} \tag{3.3}$$

Lemma 3.3. *There exist $\delta_i > 0$ $(i = 4, 5)$ and $\delta_4' > 0$ such that if $|U_r - U_0| < \delta_4$, $|\omega_1| < \delta_5$ and $|\omega_2| < \delta_5$ with $U_r \cdot \vec{n}_1 = 0$, then there exists a unique $\beta \in (-\delta_4', \delta_4')$ and a constant state U_2, with $\{U_r, U_2\} = (\beta, 0)$, such that the mixed problem* (3.3) *in Ω_2 with the initial data $\underline{U} = U_r$ admits an admissible solution U consisting of a weak* 1*-wave of which the magnitude is β and satisfying that $U = U_2$ in a neighbourhood of Γ_2'. Moreover, there holds*

$$\beta = K_0'\omega + O(1)|\omega|^2 \tag{3.4}$$

with $K_0' > 0$, where the bounds of K_0' and $O(1)$ depend only on the system and the state U_0.

Proof. As in [27], it suffices to find the function $\beta = \beta(\omega + \omega_1, U_r)$ which solves the following equation:

$$\Phi(0, \beta, U_r) \cdot (-\sin(\omega + \omega_1), \cos(\omega + \omega_1)) = 0. \tag{3.5}$$

Since $\Phi(0, 0, U_0) \cdot (0, 1) = 0$ and since Lemma 2.1 implies

$$\frac{\partial}{\partial \beta}(\Phi(0, \beta, U_r) \cdot (-\sin(\omega + \omega_1), \cos(\omega + \omega_1))) = r_1(U_0) \cdot (0, 1) > 0 \tag{3.6}$$

for $\beta = \omega = \omega_1 = 0$ and $U_r = U_0$, by the implicit function theorem we can get a C^2-function of $(\omega + \omega_1, U_r)$, $\beta = \beta(\omega + \omega_1, U_r)$, which solves (3.5) uniquely in some neighbourhood of $\beta = \omega = \omega_1 = 0$ and $U_r = U_0$.

Moreover, since $U_r \cdot \vec{n}_1 = 0$ implies $\beta(\omega_1, U_r) = 0$, we can have (3.4) and the inequality, $K' > 0$, by Taylor formula and (3.6). Therefore the proof is complete. □

Lemma 3.3 deals only with the case that the paralleling flow moves past a straight corner or a straight wedge with a small turning angle. To take into account the reflection of weak waves at boundary, we need the following:

Lemma 3.4. *The following equation*

$$\Phi(0, \varepsilon, U_l) \cdot \vec{n}_2 = \Phi(\gamma_2, \gamma_1, U_l) \cdot \vec{n}_2 \tag{3.7}$$

admits a unique C^2-solution of $(\gamma_2, \gamma_1, \omega_2, U_l)$, $\varepsilon = \varepsilon(\gamma_1, \gamma_2, \omega_2, U_l)$, *in a neighbourhood of* $\varepsilon = \gamma_1 = \gamma_2 = \omega_2 = 0$ *and* $U_l = U_0$. *Moreover, there holds*

$$\varepsilon = \gamma_1 + K_1'\gamma_2 + O(1)(|\gamma_1|\,|\gamma_2| + |\gamma_2|^2) \tag{3.8}$$

with

$$0 \leqslant K_1' = K_1'(\omega_2, U_l) = 1 + O(1)(|\omega_2| + |U_l - U_0|), \tag{3.9}$$

where the bounds of $O(1)$ *depend only on the system and* U_0.

Proof. Since $\vec{n}_2 = (-\sin\omega_2, \cos\omega_2)$, it suffices to find the solution ε to

$$\Phi(0, \varepsilon, U_l) \cdot (-\sin\omega_2, \cos\omega_2) = \Phi(\gamma_2, \gamma_1, U_l) \cdot (-\sin\omega_2, \cos\omega_2). \tag{3.10}$$

Since

$$\frac{\partial}{\partial\varepsilon}(\Phi(0, \varepsilon, U_l) \cdot (-\sin\omega_2, \cos\omega_2))|_{\varepsilon=0, U_l=U_0, \omega_2=0} = e_1(U_0)(-\lambda_1, 1) \cdot (0, 1) > 0,$$

we can find a C^2-function of $(\gamma_2, \gamma_1, \omega_2, U_l)$, $\varepsilon = \varepsilon(\gamma_1, \gamma_2, \omega_2, U_l)$, which solves Eq. (3.10) uniquely in some neighbourhood of $\varepsilon = \gamma_1 = \gamma_2 = \omega_2 = 0$ and $U_l = U_0$.

Moreover,

$$\varepsilon(\gamma_1, 0, \omega_2, U_l) = \gamma_1.$$

Let

$$I_1 = \varepsilon(\gamma_1, \gamma_2, \omega_2, U_l) - \varepsilon(\gamma_1, 0, \omega_2, U_l) - \varepsilon(0, \gamma_2, \omega_2, U_l) + \varepsilon(0, 0, \omega_2, U_l),$$

$$I_2 = \varepsilon(\gamma_1, 0, \omega_2, U_l),$$

$$I_3 = \varepsilon(0, \gamma_2, \omega_2, U_l),$$

$$I_4 = \varepsilon(0, 0, \omega_2, U_l),$$

then $\varepsilon = I_1 + I_2 + I_3 - I_4$.

By Lemma 3.2,

$$I_1 = O(1)|\gamma_1|\,|\gamma_2|;$$

and the uniqueness of the solution ε implies that $I_2 = \gamma_1$ and $I_4 = 0$. Moreover, from the Taylor formula, it follows that

$$I_3 = K_1'\gamma_2 + O(1)|\gamma_2|^2,$$

where

$$K_1' = \left.\frac{\partial \varepsilon}{\partial \gamma_2}\right|_{\gamma_1=\gamma_2=0}.$$

Then combining the estimates for I_1, I_2, I_3 and I_4, we can get estimate (3.8). Therefore to finish the proof, it suffices to obtain the estimates on K_1'.

$\frac{\partial}{\partial \gamma_2}$ (3.10) and let $\gamma_2 = \gamma_1 = 0$, then we have

$$\left.\frac{\partial \varepsilon}{\partial \gamma_2}\right|_{\gamma_1=\gamma_2=0} r_1(U_l)\cdot \vec{n}_2 = r_2(U_l)\cdot \vec{n}_2,$$

therefore

$$\left.\frac{\partial \varepsilon}{\partial \gamma_2}\right|_{\gamma_1=\gamma_2=0} = \frac{r_2(U_l)\cdot \vec{n}_2}{r_1(U_l)\cdot \vec{n}_2}. \tag{3.11}$$

Let $\omega_2 = 0$ and $U_l = U_0$ in (3.11), then $K_1'(0, U_0) = 1$. This yields the estimates on K_1'. The proof is complete. □

From the results above, we can deduce the following:

Proposition 3.1. *There exist $\delta_i > 0$ $(i = 6, 7)$ and $\delta_6' > 0$, with $\delta_6 \in (0, \delta_2)$, such that if $U_l, U_m, U_r \in O_{\delta_6}(U_0)$ and $\omega_1, \omega_2 \in (-\delta_7, \delta_7)$ with $\{U_l, U_m\} = (0, \alpha)$, $\{U_m, U_r\} = (\gamma, 0)$ and $U_r \cdot \vec{n}_1 = 0$, then there exists a unique $\varepsilon \in (-\delta_6', \delta_6')$ and a constant state U_2, with $\{U_l, U_2\} = (\varepsilon, 0)$, such that the mixed problem* (3.3) *in Ω_2 with the initial data $U|_{x=a_2} = U_l$ admits an admissible solution U consisting of a weak* 1*-wave of which the magnitude is ε and satisfying that $U = U_2$ in a neighbourhood of Γ_2' (see Fig.* 6*). Moreover, there holds*

$$\varepsilon = \gamma + K_1\alpha + K_0\omega + O(1)\{|\alpha|\,|\gamma| + |\alpha|\,|\omega| + \Delta(\gamma, \omega) + |\alpha|^2 + |\omega|^2\} \tag{3.12}$$

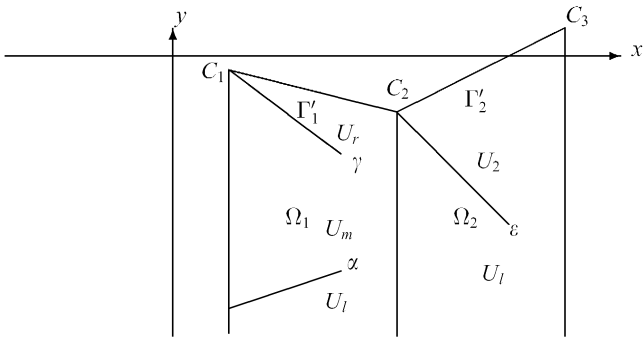

Fig. 6. Boundary interaction involving only weak waves.

with $K_0 > 0$ *and*

$$0 \leqslant K_1 = K_1(\omega_2, U_l) = 1 + O(1)(|\omega_2| + |U_l - U_0|). \tag{3.13}$$

Here the bounds of K_0, K_1 *and* $O(1)$ *depend only on the system and* U_0.

Proof. As in [27], it suffices to find the solution $(\varepsilon, \beta) = (\varepsilon, \beta)(\alpha, \gamma, \omega + \omega_1, U_l)$ to the following equations:

$$\begin{aligned} &\Phi(0, \varepsilon, U_l) \cdot (-\sin(\omega + \omega_1), \cos(\omega + \omega_1)) \\ &\quad = \Phi(0, \beta, \Phi(0, \gamma, \Phi(\alpha, 0, U_l))) \cdot (-\sin(\omega + \omega_1), \cos(\omega + \omega_1)) = 0. \end{aligned} \tag{3.14}$$

Since $U_r \cdot \vec{n}_1 = 0$, by Lemma 3.3 we can find a unique C^2-function of $(\omega + \omega_1, U_r)$, $\beta = \beta(\omega + \omega_1, U_r)$, which solves the following equation:

$$\Phi(0, \beta, U_r) \cdot \vec{n}_2 = 0 \tag{3.15}$$

in some neighbourhood of $\beta = \omega_1 = \omega = \gamma = 0$ and $U_r = U_0$. Moreover, estimate (3.4) holds with $K' > 0$. Therefore, we have to solve (3.14) by finding the solution ε to the following for any β', α, γ, ω and ω_1, U_l:

$$\begin{aligned} &\Phi(0, \varepsilon, U_l) \cdot (-\sin(\omega + \omega_1), \cos(\omega + \omega_1)) \\ &\quad = \Phi(0, \beta', \Phi(0, \gamma, \Phi(\alpha, 0, U_l))) \cdot (-\sin(\omega + \omega_1), \cos(\omega + \omega_1)). \end{aligned} \tag{3.16}$$

Indeed by Lemma 3.1 we have the following equality:

$$\Phi(0, \beta', \Phi(0, \gamma, \Phi(\alpha, 0, U_l))) = \Phi(\gamma_2', \gamma_1', U_l) \tag{3.17}$$

in some neighbourhood of $\gamma_k' = \alpha = \beta' = \gamma = 0$ $(k = 1, 2)$ and $U_l = U_0$ with

$$\gamma_1' = \gamma + \beta' + O(1)\{|\alpha|\,|\gamma| + \Delta(\gamma, \beta') + |\beta'|\,|\alpha|\},$$

$$\gamma_2' = \alpha + O(1)\{|\alpha|\,|\gamma| + \Delta(\gamma, \beta') + |\beta'|\,|\alpha|\}.$$

Then Eq. (3.16) is reduced to the following equation:

$$\Phi(0, \varepsilon, U_l) \cdot (-\sin\omega_2, \cos\omega_2) = \Phi(\gamma_2', \gamma_1', U_l) \cdot (-\sin\omega_2, \cos\omega_2). \tag{3.18}$$

Applying Lemma 3.4 to (3.18), we can find a C^2-function of $(\gamma_2', \gamma', \omega_2, U_l)$, $\varepsilon = \varepsilon(\gamma_1', \gamma_2', \omega_2, U_l)$ solving Eq. (3.18) uniquely in some neighbourhood of $\omega_2 = \gamma_1' = \gamma_2' = \varepsilon = 0$ and $U_l = U_0$. Moreover, estimates (3.8) and (3.9) hold near $\gamma_1' = \gamma_2' = 0$.

Let $\beta' = \beta$. Then by the solutions to (3.15) and (3.18) and by the Eq. (3.17), we can find an ε that solves (3.14) in some neighbourhood of $\varepsilon = \beta = \omega_1 = \omega = \gamma = 0$ and $U_l = U_m = U_r = U_0$, and we can get the desired estimates (3.12) and (3.13). The

uniqueness of the solution ε follows from the fact that

$$\frac{\partial}{\partial \varepsilon}\Phi(0,\varepsilon,U_l)\cdot(-\sin(\omega+\omega_1),\cos(\omega+\omega_1))\neq 0 \tag{3.19}$$

for $\varepsilon=\omega=\omega_1=0$ and $U_l=U_0$.

Moreover, noticing that $\frac{d}{ds}\Phi_i(s,\underline{U})|_{s=0}=r_i(U_0)\neq\vec{0}$ $(i=1,2)$ for $\underline{U}=U_0$, which implies $|\Phi_i(s,U)-U|\geqslant\frac{1}{2}|r_i(U_o)|\,|s|$ for U lying in some neighbourhood of U_0, we can choose the suitable neighbourhood, $O_{\delta_6}(U_0)$. Thus the proof is complete. □

3.2. Estimate on the boundary perturbation of the strong shock

Let $\vec{n}_k$ $(k=1,2)$ and ω,ω_1,ω_2 be given as in Section 3.1, and let $O_{\delta_2}(U_0)$ be the neighbourhood of U_0 given in Section 2.1.

Proposition 3.2. *There exist $\delta_8>0$ and $\delta_8'>0$, with $G(O_{\delta_8}(\sigma_0))\subset O_{\delta_2}(U_0)$ and $\delta_8<|\sigma_0|$, such that if $\omega_1,\omega_2\in(-\delta_8',\delta_8')$, then for each $k=1,2$, the following equation*

$$G(\sigma)\cdot\vec{n}_k=0 \tag{3.20}$$

admits a unique solution $\sigma_k\in O_{\delta_8}(\sigma_0)$. Moreover, $\{U_\infty,G(\sigma_k)\}$ is a Majda stable 1-shock, and there hold

$$\sigma_2=\sigma_1+K_0''\omega+O(1)|\omega|^2 \tag{3.21}$$

and

$$\sigma_j=\sigma_0+K_{0j}'''\omega_j+O(1)|\omega_j|^2,\quad j=1,2, \tag{3.22}$$

with $K_0''>0$ and $K_{0j}'''>0$ $(j=1,2)$. Here the bound of $O(1)$ and K_0'', K_{0j}''' $(j=1,2)$ depend only on the system; σ_0 is the speed of the unperturbed shock in the case $b(x)\equiv 0$.

Proof. It suffices to find the solution $\sigma=\sigma(h)$ to the following equation:

$$G(\sigma)\cdot(-\sin h,\cos h)=0. \tag{3.23}$$

Since Lemma 2.8 implies

$$\frac{\partial}{\partial\sigma}(G(\sigma)\cdot(-\sin h,\cos h))\bigg|_{\sigma=\sigma_0,h=0}=b_k>0,$$

we can find a unique C^2 function of h, $\sigma=\sigma(h)$, with $\sigma(0)=\sigma_0$, which solves the (3.23) in some neighbourhood of $\sigma=\sigma_0,h=0$.

Then $\sigma(\omega_k) = \sigma_k$ $(k = 1, 2)$, and by the Taylor formula we have the desired estimates (3.21) and (3.22). This completes the proof. $\square$

3.3. Estimates on the interaction between the strong shock and weak waves

Consider Riemann problem (2.3) given in Section 2 with $U_l = U_\infty$ and $U_r \in O_{\delta_2}(U_0)$. Here $O_{\delta_2}(U_0)$ is the neighbourhood of U_0 given in Section 2.1.

Proposition 3.3. *There exist $\delta_9 > 0$, $\delta_{10} > 0$ and $\delta_9' > 0$, $\delta_9'' > 0$, with $\delta_9 \in (0, \delta_2)$ and $\sigma_0 + \max(\delta_9'', \delta_{10}) < 0$, such that if $U_m, U_r \in O_{\delta_9}(U_0)$ and $\sigma \in (\sigma_0 - \delta_{10}, \sigma_0 + \delta_{10})$ with $\{G(\sigma), U_m\} = (0, \alpha)$ and $\{U_m, U_r\} = (\beta_1, \beta_2)$, then there exists a unique $(\sigma', \gamma) \in (\sigma_0 - \delta_9'', \sigma_0 + \delta_9'') \times (-\delta_9', \delta_9')$ such that Riemann problem* (2.3) *with $U_l = U_\infty$ admits an admissible solution consisting of a Majda stable strong* 1*-shock with the speed σ', and a weak* 2*-wave of which the magnitude is γ, i.e. $\{U_\infty, U_r\} = (\sigma', \gamma)$. Moreover, there hold*

$$\gamma = K_2\beta_1 + \beta_2 + \alpha + O(1)\{|\beta_1|\,|\beta_2| + |\beta_1|^2 + |\alpha|\,|\beta_1| + \Delta(\alpha, \beta_2)\} \tag{3.24}$$

and

$$\sigma' = \sigma + K_3\beta_1 + O(1)\{|\beta_1|\,|\beta_2| + |\beta_1|^2 + |\alpha|\,|\beta_1| + \Delta(\alpha, \beta_2)\} \tag{3.25}$$

with $K_2 = K_2(\sigma) \in C^1(\sigma_0 - \delta_{10}, \sigma_0 + \delta_{10})$ and

$$\sup_{|\sigma - \sigma_0| < \delta_{10}} |K_2(\sigma)| < 1, \tag{3.26}$$

where the bounds of K_3 and $O(1)$ depend only on the system and U_0.

Proof. As in the proof of Proposition 3.1, it suffices to find the solution $(\sigma', \gamma) = (\sigma'(\beta_2, \beta_1, \alpha), \gamma(\beta_2, \beta_1, \alpha))$ to the following equation:

$$\Phi(\beta_2, \beta_1, \Phi(\alpha, 0, G(\sigma)) = \Phi(\gamma, 0, G(\sigma')). \tag{3.27}$$

Indeed by Lemma 3.1, there exist β_1' and β_2' such that

$$\Phi(\beta_2, \beta_1, \Phi(\alpha, 0, G(\sigma))) = \Phi(\beta_2', \beta_1', G(\sigma)) \tag{3.28}$$

with $\beta_j' = \beta_j + \delta_{2j}\alpha + O(1)\{|\alpha|\,|\beta_1| + \Delta(\alpha, \beta_2)\}$ where δ_{2j} is the Kronecker symbol. Thus Eq. (3.27) can be reduced to the following equation:

$$\Phi(\beta_2', \beta_1', G(\sigma)) = \Phi(\gamma, 0, G(\sigma')). \tag{3.29}$$

Since Lemma 2.8 implies

$$\det\left(\frac{\partial\Phi(\gamma,0,G(\sigma'))}{\partial(\gamma,\sigma')}\right)\bigg|_{\gamma=0,\sigma'=\sigma_0} = \det(r_2(U_0), G_\sigma(\sigma_0))$$
$$= -e_2(U_0)\{b_2\lambda_2(U_0)+b_1\}<0, \tag{3.30}$$

by the implicit function theorem we can find the C^2-functions of $(\beta_2', \beta_1', \sigma)$, $\gamma = \gamma(\beta_2', \beta_1', \sigma)$ and $\sigma' = \sigma'(\beta_2', \beta_1', \sigma)$, which solve Eq. (3.29) uniquely in some neighbourhood of $\beta_1' = \beta_2' = \gamma = 0$ and $\sigma' = \sigma = \sigma_0$.

Therefore by the solutions to (3.29) and (3.28) we can get the solution (σ', γ) to (3.27) in a neighbourhood of $\beta_1 = \beta_2 = \alpha = \gamma = 0$ and $\sigma' = \sigma = \sigma_0$. Also the uniqueness of the solution (σ', γ) follows from (3.30).

Let

$$I_1 = \gamma(\beta_2', \beta_1', \sigma) - \gamma(\beta_2', 0, \sigma) - \gamma(0, \beta_1', \sigma) + \gamma(0, 0, \sigma)$$

and let

$$I_2 = \gamma(\beta_2', 0, \sigma),$$

$$I_3 = \gamma(0, \beta_1', \sigma),$$

$$I_4 = \gamma(0, 0, \sigma).$$

Then $\gamma = I_1 + I_2 + I_3 - I_4$. In the same way as in the proof of Lemma 3.4, we obtain

$$I_1 = O(1)|\beta_1'|\,|\beta_2'|,$$

$$I_3 = \gamma(0,0,\sigma) + \frac{\partial\gamma}{\partial\beta_1'}\bigg|_{\beta_1'=\beta_2'=0}\beta_1' + O(1)|\beta_1'|^2.$$

In addition, from the uniqueness it follows that $I_4 = \gamma(0,0,\sigma) = 0$ and

$$I_2 = \beta_2', \quad \sigma'(\beta_2', 0, \sigma) = \sigma.$$

Then combining the estimates for I_1, I_2, I_3 and I_4, we can obtain estimate (3.24).

To prove inequality (3.26) for $K_2 = \frac{\partial\gamma}{\partial\beta_1'}|_{\beta_1'=\beta_2'=0}$, we differentiate the following equation:

$$\Phi(\beta_2', \beta_1', G(\sigma)) = \Phi(\gamma, 0, G(\sigma')),$$

with respect to β_1' and take $\beta_1' = \beta_2' = 0$ and $\sigma = \sigma_0$, then

$$r_1(U_0) = r_2(U_0)\left.\frac{\partial\gamma}{\partial\beta_1'}\right|_{\beta_1'=\beta_2'=0} + G_{\sigma'}(\sigma_0)\left.\frac{\partial\sigma'}{\partial\beta_1'}\right|_{\beta_1'=\beta_2'=0}. \tag{3.31}$$

Multiplying (3.31) by the matrix $([\nabla_U W(U_0)]^{-1}\nabla_U H(U_0) - \sigma_0 I)$, then by Lemma 2.8 we can deduce that

$$(\lambda_1(U_0) - \sigma_0)r_1(U_0) = (\lambda_2(U_0) - \sigma_0)K_2(\sigma_0)r_2(U_0) + \left.\frac{\partial\sigma'}{\partial\beta_1}\right|_{\beta_1'=\beta_2'=0}\vec{t}, \tag{3.32}$$

where $\vec{t} = [\nabla_U W(U_0)]^{-1}(W(U_0) - W(U_\infty))$ is given by Lemmas 2.6 and 2.7. Hence by Lemmas 2.5, 2.7 and (3.32), we have

$$|K_2(\sigma_0)| = \left|\frac{(\lambda_1(U_0) - \sigma)\det(r_1(U_0), \vec{t})}{(\lambda_2(U_0) - \sigma_0)\det(r_2(U_0), \vec{t})}\right| < 1,$$

which yields inequality (3.26).

The estimate on σ' can be obtained in the same way as above. To do this, let $\sigma' = D_1 + D_2 + D_3 - D_4$ where

$$D_1 = \sigma'(\beta_2', \beta_1', \sigma) - \sigma'(\beta_2', 0, \sigma) - \sigma'(0, \beta_1', \sigma) + \sigma'(0, 0, \sigma),$$

$$D_2 = \sigma'(\beta_2', 0, \sigma),$$

$$D_3 = \sigma'(0, \beta_1', \sigma),$$

$$D_4 = \sigma'(0, 0, \sigma).$$

Then $D_1 = O(1)|\beta_2'|\,|\beta_1'|$. In addition, from the uniqueness of (σ', γ) it follows that

$$D_2 = D_4 = \sigma,$$

and by the Taylor formula we have

$$D_3 = \sigma'(0, 0, \sigma) + K_3\beta_1' + O(1)|\beta_1'|^2.$$

Therefore, estimate (3.25) on σ' follows from the above estimates on D_1, D_2, D_3, and D_4.

Moreover, noticing that $\frac{d}{ds}\Phi_i(s, \underline{U})|_{s=0} = r_i(U_0) \neq \vec{0}$ $(i = 1, 2)$ for $\underline{U} = U_0$, we can choose the suitable neighbourhood $O_{\delta_9}(U_0)$ by the continuity argument. Thus the proof is complete. $\square$

4. Approximate solution

In this section we shall use a modified Glimm scheme as in [27] to obtain the approximate solution in the approximate domain $\Omega_{\Delta x}$ which will be defined. Meanwhile we will establish some estimates on the approximate solutions.

4.1. Some notations

Based on the results in the above sections, we can choose positive constants $\delta(1)$, $\delta(2)$ and $\delta'(2)$, with $\delta(1)<\min(\delta_2', \delta_3, \delta_4, \delta_6, \delta_9)$, $\delta'(2)<\min(\delta_8, \delta_{10}, |\sigma_0|/8)$ and

$$\delta(2)<\min\left\{\delta_5, \delta_7, \delta_8', \min_{V_0}\left(\frac{1}{8}|\arctan\lambda_{1,2}|\right), \arctan\frac{v_\infty}{u_\infty},\ \omega_{ext}-\arctan\frac{v_\infty}{u_\infty}\right\},$$

such that $B=\{U\in D|\,|U-U_0|<\delta(1)\}$, $B_1=\{\omega|\,|\omega|<\delta(2)\}$ and $B_2=\{\sigma|\,|\sigma-\sigma_0|<\delta'(2)\}$ with $G(B_2)\subset B$, in which there hold Lemma 3.1 and the following:

$$\sup_{B\times B_1\times B_2}|K_1|\ \sup_{B\times B_1\times B_2}|K_2|<1. \tag{4.1}$$

Here $K_1=K_1(\omega, U)$ and $K_2=K_2(\sigma)$ are coefficients given in Propositions 3.1 and 3.3; δ_i and δ_i' $(1\leqslant i\leqslant 10)$ are the constants given in the lemmas and propositions in Sections 2 and 3.

To define the difference scheme, first suppose that

$$\sup_{x\geqslant 0}|b'(x+)|<\tfrac{1}{8}\min\left\{\min_{V_0}|\lambda_{1,2}|, \min_{\sigma\in B_2}|\sigma|\right\}. \tag{4.2}$$

For any $\Delta x>0$, let $y_k=b(k\Delta x)$ and let $A_k=(k\Delta x, y_k), 0\leqslant k<\infty$. Then

$$\sup_{k>0}\left\{\frac{|y_k-y_{k-1}|}{\Delta x}\right\}<\frac{1}{4}\min\left\{\min_{V_0}|\lambda_{1,2}|, \min_{\sigma\in B_2}|\sigma|\right\}.$$

Denote

$$\omega(A_k)=\arctan\frac{y_{k+1}-y_k}{\Delta x}-\arctan\frac{y_k-y_{k-1}}{\Delta x},\quad k\geqslant 1,$$

$$\omega(A_0)=\arctan\frac{y_1-y_0}{\Delta x},$$

$$\Gamma_k=\{(x,y)|k\Delta x<x<(k+1)\Delta x,\ y=b(x,k,\Delta x)\},$$

where

$$b(x,k,\Delta x)=y_k+\frac{y_{k+1}-y_k}{\Delta x}(x-k\Delta x),$$

and denote $\vec{n}_k$ the outer normal vector to Γ_k, that is,

$$\vec{n}_k = \frac{(y_k - y_{k+1}, \Delta x)}{\sqrt{(y_{k+1} - y_k)^2 + (\Delta x)^2}}.$$

Define

$$\Omega_{\Delta x,k} = \{(x,y) | k\Delta x \leqslant x < (k+1)\Delta x, y < b(x,k,\Delta x)\}$$

and define the approximate domain as follows:

$$\Omega_{\Delta x} = \bigcup_{k \geqslant 0} \Omega_{\Delta x,k}$$

(see Fig. 7).

Let $\Delta y > 0$ satisfy

$$\frac{\Delta y - m\Delta x}{\Delta x} = 4(|\sigma_0| + \sup\{|\lambda_{1,2}(z)|, z \in B\}),$$

where $m = \sup_{k>0}\{\frac{|y_k - y_{k-1}|}{\Delta x}\}$. Then choose a set of mesh points

$$\{P_{k,n} | P_{k,n} = (k\Delta x, a_{k,n}), k \geqslant 0, -\infty < n < +\infty\}$$

in R^2 where

$$a_{k,n} = (2n + 1 + \theta_k)\Delta y + y_k$$

and θ_k is randomly and independently chosen in $(-1, 1)$. We connect the mesh point $P_{k,n}$ by two line segments to the two mesh points, $P_{k-1,n-1}$ and $P_{k-1,n}$ if $\theta_k \leqslant 0$, or connect the mesh point $P_{k,n}$ by two line segments to the two mesh points $P_{k-1,n}$ and $P_{k-1,n+1}$ if $\theta_k > 0$. Then for any integers $k \geqslant 1$ and $n \in Z$, an interaction diamond $\Lambda_{k,n}$ with the center at $(k\Delta x, 2n\Delta y + y_k)$ is defined to be the domain bounded by four lines

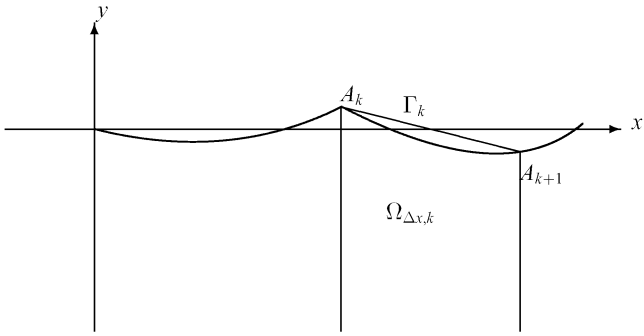

Fig. 7. Approximate domain.

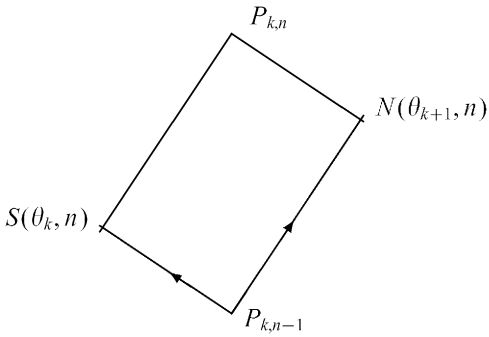

Fig. 8. Interaction diamond $\Lambda_{k,n}$ and orientation of the segments.

segments with vertices $N(\theta_{k+1},n)$, $P_{k,n-1}$, $S(\theta_k,n)$ and $P_{k,n}$ (see Fig. 8), where

$$N(\theta_{k+1},n)=\begin{cases} P_{k+1,n} & \text{if } \theta_{k+1}\leqslant 0,\\ P_{k+1,n-1} & \text{if } \theta_{k+1}>0\end{cases}$$

and

$$S(\theta_k,n)=\begin{cases} P_{k-1,n-1}, & \text{if } \theta_k\leqslant 0,\\ P_{k-1,n}, & \text{if } \theta_k>0.\end{cases}$$

We call the domain bounded by segments $P_{0,n-1}N(\theta_1,n)$, $N(\theta_1,n)P_{0,n}$ and $P_{0,n-1}P_{0,n}$ a half diamond $\Lambda_{0,n}$.

We define a class of space-like and orientable curves in the strip $\{(j-1)\Delta x\leqslant x\leqslant(j+1)\Delta x\}$ for any integer $j>0$ as in [1,25].

Definition 4.1. A j-mesh curve J is defined to be an unbounded piecewise linear curve lying in the strip domain $\{(j-1)\Delta x\leqslant x\leqslant(j+1)\Delta x\}$ and satisfying the following properties:

1. J consists of line segments of the form $P_{k,n-1}N(\theta_{k+1},n)$, $P_{k,n-1}S(\theta_k,n)$ (see Fig. 8);
2. the y-coordinates along J range from $-\infty$ to $+\infty$.

We denote by I_{k-1} the k-mesh curve lying in $\{(k-1)\Delta x\leqslant x\leqslant k\Delta x\}$, that is, the curve which is composed of all segments lying in $\{(k-1)\Delta x\leqslant x\leqslant k\Delta x\}$ and joining the mesh points as above.

It is obvious that for any $0<k<+\infty$ each k-mesh curve I divides the R^2 into I^+ part and I^- part, the I^- being the one containing the set $\{x<0\}$. As in [25] we also partially order these mesh curves by saying that $J_1>J_2$ if every point of the mesh

curve J_1 is either on J_2 or contained in J_2^+, and call J an immediate successor to I if $J > I$ and every mesh point of J except one is on I. Here J_1 (or J_2, I, J, resp.) is a j_1-mesh curve (or j_2-, i-, j-mesh curve, resp.).

4.2. Glimm scheme

In addition to (4.2), suppose that

$$\sum_{j=0}^{+\infty} |\omega(A_j)| < \delta(2). \tag{4.3}$$

Then we define the difference scheme in $\Omega_{\Delta x}$, that is, define the global approximate solution $U_{\Delta x,\theta}$ in $\Omega_{\Delta x}$ for any $\theta = (\theta_0, \theta_1, \theta_2, \dots)$. This can be done by carrying out the following steps inductively:

For $k = 0$, $U_{\Delta x,\theta}$ can be defined in $\{0 \leqslant x < \Delta x\} \cap \Omega_{\Delta x}$ with $U_{\Delta x,\theta}|_{x=0,y<0} = U_\infty$ by a shock polar.

Inductively, assume that the approximate solution $U_{\Delta x,\theta}$ has been constructed for $\{0 \leqslant x < k\Delta x\}$, then we will define the $U_{\Delta x,\theta}$ in $\{k\Delta x \leqslant x < (k+1)\Delta x\}$ by solving the following problems.

Set

$$U_{k,n} = U_{\Delta x,\theta}(k\Delta x-, a_{k,n}), \quad n \leqslant -1$$

and for $n \leqslant -1$ define

$$U_k^0(y) = U_{\Delta x,\theta}(k\Delta x-, a_{k,n}), \quad \text{if } y \in [y_k + 2n\Delta y, y_k + 2(n+1)\Delta y).$$

First to define $U_{\Delta x,\theta}$ in rhombus $T_{k,0}$ whose vertices are $((k+1)\Delta x, y_{k+1})$, $((k+1)\Delta x, -\Delta y + y_{k+1})$, $(k\Delta x, y_k)$, $(k\Delta x, -\Delta y + y_k)$, we have to solve the following mixed problem in $T_{k,0}$:

$$\begin{cases} W(U_k)_x + H(U_k)_y = 0 \quad \text{in } T_{k,0}, \\ U_k|_{x=k\Delta x} = U_k^0, \\ U_k \cdot \vec{n}_k|_{\Gamma_k} = 0, \end{cases} \tag{4.4}$$

where $\vec{n}_k$ is the outer normal vector to Γ_k. If problem (4.4) is solvable, then define $U_{\Delta x,\theta} = U_k$ in $T_{k,0}$. To solve it we need to consider the following two cases:

Case $(\mathrm{i})_k$: $U_{k,-1} \in B$. Then by Proposition 3.1, problem (4.4) admits a unique admissible solution U_k consisting of one weak 1-wave, that is, there exist a unique $\varepsilon_{k,0,1}$ and a constant state $U_{k,0}$ such that

$$\{U_{k,-1}, U_{k,0}\} = (\varepsilon_{k,0,1}, 0), \tag{4.5}$$

$$U_{k,0} \cdot \vec{n}_k = 0 \tag{4.6}$$

and

$$U_k = U_{k,0} \quad \text{in some neighborhood of } \Gamma_k. \tag{4.7}$$

Case $(\text{ii})_k$: $U_{k,-1} = U_\infty$. Then problem (4.4) can be solved by the shock polar or Proposition 3.2, that is, there exist a unique $\sigma_{(k)}$ and a constant state $U_{k,0}$ near the state U_0 such that

$$\{U_{k,-1}, U_{k,0}\} = (\sigma_{(k)}, 0) \tag{4.8}$$

and (4.6), (4.7) hold.

Secondly, to define $U_{\Delta x,\theta}$ in each rhombus $T_{k,n}$ $(n \leqslant -1)$ whose vertices are $(k\Delta x, (2n-1)\Delta y + y_k)$, $(k\Delta x, (2n+1)\Delta y + y_k)$, $((k+1)\Delta x, (2n-1)\Delta y + y_{k+1})$ and $((k+1)\Delta x, (2n+1)\Delta y + y_{k+1})$, we have to solve the following Riemann problem in each $T_{k,n}$ $(n \leqslant -1)$:

$$\begin{cases} W(U_k)_x + H(U_k)_y = 0 & \text{in } T_{k,n}, \\ U_k|_{x=k\Delta x} = U_k^0. \end{cases} \tag{4.9}$$

If problem (4.9) is solvable, then define $U_{\Delta x,\theta} = U_k$ in $T_{k,n}(n \leqslant -1)$. To solve problem (4.9), we just need to consider the following three cases:

Case $(\text{iii})_k$: $U_{k,n-1} = U_\infty$ and $U_{k,n} \in B$. Then by Proposition 3.3, problem (4.9) admits a unique admissible solution U_k consisting of a weak 2-wave and a Majda stable strong 1-shock, that is, there exists a unique $(\sigma_{(k)}, \varepsilon_{k,n,2})$ such that

$$\{U_{k,n-1}, U_{k,n}\} = (\sigma_{(k)}, \varepsilon_{k,n,2}), \tag{4.10}$$

where $\sigma_{(k)}$ is the shock speed of the strong shock and $\varepsilon_{k,n,2}$ is the magnitude of the weak 2-wave.

Case $(\text{iv})_k$: Both $U_{k,n-1}, U_{k,n} \in B$. Then by Lemma 3.1, problem (4.9) admits a unique admissible solution U_k consisting of two weak waves, that is, there exists a unique $(\varepsilon_{k,n,1}, \varepsilon_{k,n,2})$ such that

$$\{U_{k,n-1}, U_{k,n}\} = (\varepsilon_{k,n,1}, \varepsilon_{k,n,2}). \tag{4.11}$$

Case $(\text{v})_k$: $U_{k,n-1} = U_{k,n} = U_\infty$. Then $U_k = U_\infty$.

Finally we define $U_k(k\Delta x, a_{k,n}) = U_{k,0}$ if $n \geqslant 0$ for simplification. Then it is obvious that $\varepsilon_{k,n,1} = \varepsilon_{k,n,2} = 0$ for $n \geqslant 0$ and $k \geqslant 0$.

4.3. Decreasing of Glimm functional

In this subsection we will show that under suitable conditions the approximate solution can be well defined in $\Omega_{\Delta x}$ by the steps in Section 4.2.

First by direct computation, we deduce a lemma related to the L^∞-estimates, as follows:

Lemma 4.1. (1) *If* $\{U_l, U_r\} = (\alpha, \beta)$, *with* $U_r, U_l \in B$, *then*

$$|U_l - U_r| \leqslant s_1(|\alpha| + |\beta|).$$

Here $s_1 = \max\{|\frac{\partial}{\partial\alpha}\Phi(\beta, \alpha, U)|, |\frac{\partial}{\partial\beta}\Phi(\beta, \alpha, U)| \,|\, U \in V_0, |\beta| + |\alpha| \leqslant \delta_4'\}$.

(2) *For any* $\sigma \in B_2$, *there holds*

$$|G(\sigma) - G(\sigma_0)| \leqslant s_2|\sigma - \sigma_0|.$$

Here $s_2 = \max\{|G_\sigma'(\tau)|, \tau \in B_2\}$.

Next, we will prove that under the suitable conditions $U_{\Delta x,\theta}$ can be globally defined. Inductively, we assume that $U_{\Delta x,\theta}$ is defined in $\{x < k\Delta x\} \cap \Omega_{\Delta x}$ by the steps in Section 4.2, and satisfies the following:

C-1(k-1) In each $\Omega_{\Delta x,j}$ $(0 \leqslant j \leqslant k-1)$ there is a Majda stable strong 1-shock of $U_{\Delta x,\theta}$: $S_*(\sigma_{(j)})$, with the speed $\sigma_{(j)} \in B_2$, which divides $\Omega_{\Delta x,j}$ into two parts: $\Omega^+_{\Delta x,j}$ and $\Omega^-_{\Delta x,j}$, where $\Omega^+_{\Delta x,j}$ is the part bounded by $S_*(\sigma_{(j)})$ and $\Gamma_j = \{y = b(x, j, \Delta x)\}$.

C-2(k-1) $U_{\Delta x,\theta}|_{\Omega^+_{\Delta x,j}} \in B$, $U_{\Delta x,\theta}|_{\Omega^-_{\Delta x,j}} = U_\infty$, $0 \leqslant j \leqslant k-1$.

C-3(k-1) $\{S_*(\sigma_{(j)}),\ j = 0, \ldots, k-1\}$ form an approximate 1-characteristic $\chi_{\Delta x,\theta}$: $y = \chi_{\Delta x,\theta}(x)$, which issues from the origin.

Here and in sequel $S_*(\sigma_{(j)})$ denotes the strong 1-shock or the strong 1-shock front with the speed $\sigma_{(j)}$. Then we will prove that under suitable conditions $U_{\Delta x,\theta}$ can be defined in $\Omega_{\Delta x,k}$ and satisfies (C-1(k)), (C-2(k)) and (C-3(k)).

Indeed, by the induction hypotheses: (C-1(k-1)), (C-2(k-1)) and (C-3(k-1)), we can first define $U_{\Delta x,\theta}$ and the Majda stable strong 1-shock $S_*(\sigma_{(k)})$ in $\Omega_{\Delta x,k}$ by the steps in Section 4.2. Moreover, due to the construction in Section 4.2, there exists a diamond $\Lambda_{k,n(k)}$ such that $S_*(\sigma_{(k-1)})$ enters $\Lambda_{k,n(k)}$ and $S_*(\sigma_{(k)})$ issues from the centre of $\Lambda_{k,n(k)}$. Therefore extend $\chi_{\Delta x,\theta}$ to $\Omega_{\Delta x,k}$ such that $\chi_{\Delta x,\theta} = S_*(\sigma_{(k)})$ in $\Omega_{\Delta x,k}$, and define $\Omega^-_{\Delta x,k}$ and $\Omega^+_{\Delta x,k}$ in the same way as in (C-1(k-1)). Then it suffices to impose some suitable conditions so that there will hold (C-2(k)) and $\sigma_k \in B_2$. To this aim we will introduce a Glimm functional.

We first present here some notations that will be used in the proof. In the sequel, we use the Greek letters except σ to denote the weak waves and denote by α_j (or β_j, etc., resp.) the jth weak wave from weak wave α (or β, etc., resp.). Moreover, without confusion, we also use α_j (or β_j, etc., resp.) to denote the magnitude of α_j (or β_j, etc., resp.).

Let J be a k-mesh curve. Then $U_{\Delta x,\theta}|_J$ consists of a strong shock wave and various weak waves.

Definition 4.2.

$$L_j(J) = \sum \{|\alpha_j|\colon \alpha_j \text{ crosses } J\}, \quad j = 1, 2,$$

$$L_0(J) = \sum \{|\omega(A)|\colon A \in \Omega_J\},$$

$$Q_j(J) = \sum \{\Delta(\alpha_j, \beta_j)\colon \text{ both } \alpha_j \text{ and } \beta_j \text{ cross } J, \text{ and } \alpha_j \text{ lies below } \beta_j \text{ on } J\},$$

$$j = 1, 2,$$

$$Q_{21}(J) = \sum \{|\alpha_2|\,|\beta_1|\colon \text{ both } \alpha_2 \text{ and } \beta_1 \text{ cross } J, \text{ and } \alpha_2 \text{ lies below } \beta_1 \text{ on } J\},$$

where Ω_J is the set of corner points A_n lying in J^+, that is,

$$\Omega_J = \{A_n | A_n \in J^+ \cap \partial\Omega_{\Delta x}, A_n = (n\Delta x, y_n),\ n \geqslant 0\},$$

and by α_j and β_j we mean a weak j-wave from α and a weak j-wave from β, respectively.

Moreover, by the induction hypotheses: (C-1(k-1)) and (C-2(k-1)), and by (4.1) and (4.3), we can choose positive constants $K_{j*} > 0$ $(j = 0, 1, 2)$ and $K_* > 0$ such that the following inequalities hold for any $U \in \bar{B}$, $\omega \in \overline{B_1}$ and $\sigma \in \bar{B}_2$:

$$K_{j*} - |K_j| > K_* \quad (j = 1, 2), \tag{4.12}$$

$$1 - K_{1*}K_{2*} > K_* \tag{4.13}$$

and

$$K_{0*} - \max\{|K_0|,\ |K_0''|, |K_{01}'''|,\ |K_{02}'''|,\ |K_3|\} > K_*, \tag{4.14}$$

where K_j $(j = 0, 1, 2)$ and K_0'', K_{01}''', K_{02}''', K_3 are coefficients given in the propositions and lemmas in Section 3.

Let σ^J be the speed of the strong shock crossing J. For any constants $C_1 > 0, C > 0$ and $K > 0$, we define the following.

Definition 4.3.

$$L(J) = K_{0*}L_0(J) + L_1(J) + K_{1*}L_2(J),$$

$$L_s(J) = |\sigma^J - \sigma_0| + C_1 L(J),$$

$$Q(J) = Q_2(J) + Q_1(J) + Q_{21}(J),$$

$$Q_*(J) = |L(J)|^2,$$

$$F(J) = L(J) + K\{Q(J) + CQ_*(J)\},$$

$$F_s(J) = |\sigma^J - \sigma_0| + C_1 F(J).$$

Let

$$\delta(3) = \frac{1}{1 + K_{1*} + K_{0*}} \min(\delta(1)/8(s_1 + s_2), \delta(2)/8, \delta'(2)/8),$$

then we have the following:

Proposition 4.1. *Suppose that the function b satisfies* (4.2) *and* (4.3). *Let I and J be two k-mesh curves such that J is an immediate successor to I, and suppose that*

$$\sigma^I \in B_2, \quad U_{\Delta x,\theta}|_{I \cap (\Omega^+_{\Delta x,k-1} \cup \Omega^+_{\Delta x,k})} \in B.$$

There exist constants $\delta' > 0$, $K > 0$, $C > 0$ *and* $C_1 > 1$, *depending only on the system in* (1.1), *the state* U_∞ *and the state* U_0, *such that if* $F_s(I) \leqslant \delta'$ *then*

$$U_{\Delta x,\theta}|_{J \cap (\Omega^+_{\Delta x,k-1} \cup \Omega^+_{\Delta x,k})} \in B, \quad \sigma^J \in B_2 \tag{4.15}$$

and

$$F_s(I) \geqslant F_s(J). \tag{4.16}$$

Proof. Let Λ be the diamond between I and J. The proof of the proposition is divided into four cases:

Case 1: Λ covers a part of $\partial\Omega_{\Delta x}$ and Λ covers no part of $\chi_{\Delta x,\theta}$. Then Ω_I and Ω_J differ by the vertex point A_k, that is, $\Omega_J = \Omega_I \backslash \{A_k\}$ with $A_k = (k\Delta x, y_k)$. Moreover, $\sigma^I = \sigma^J$.

Denote $\omega = \omega(A_k)$. Let $I = I_0 \cup I'$ and $J = I_0 \cup J'$ such that $\partial\Lambda = I' \cup J'$. Let ε_1 be the weak 1-wave crossing J', and let γ_1 and α_2 be the weak 1-wave and the weak 2-wave crossing I', respectively, with α_2 lying below γ_1 on I (see Fig. 9). Here and throughout the proof, if a rarefaction wave crossing I is split into parts that cross I_0 and I' then these parts are considered to be two different rarefaction waves as in

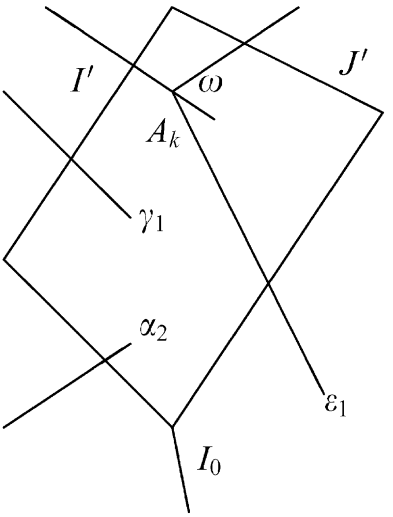

Fig. 9. Case 1.

[6,24]. And let $\gamma_1 = 0$ (or $\alpha_2 = 0$, resp.) if there is no 1-weak wave (or no weak 2-wave, resp.) entering Λ.

Define

$$L_j(I_0) = \sum \{|\beta_j|,\ \beta_j \in I_{0,(j)}\},$$

$$\Delta(I_{0,(j)}, \gamma) = \sum \{\Delta(\beta_j, \gamma),\ \beta_j \in I_{0,(j)}\},$$

$$\Delta(I_{0,(j)}, \omega) = \sum \{\Delta(\beta_j, \omega),\ \beta_j \in I_{0,(j)}\}$$

where by $\beta_j \in I_{0,(j)}$ we mean that a weak j-wave β_j with magnitude β_j crosses I_0.

Now we can carry out the proof. By Proposition 3.1, we have

$$\varepsilon_1 = \gamma_1 + K_1\alpha_2 + K_0\omega + O(1)Q'(\Lambda), \tag{4.17}$$

where

$$Q'(\Lambda) = |\gamma_1|\,|\alpha_2| + \Delta(\gamma_1, \omega) + |\alpha_2|\,|\omega| + |\alpha_2|^2 + |\omega|^2.$$

Therefore

$$L(J) \leqslant L(I) - (K_{0*} - K_0)|\omega| - (K_{1*} - K_1)|\alpha_2| + O(1)Q'(\Lambda), \tag{4.18}$$

which gives

$$\begin{aligned}|L(J)|^2 \leqslant |L(I)|^2 &- 2\{(K_{0*}-K_0)|\omega| + (K_{1*}-K_1)|\alpha_2|\}\{L_1(I_0)+K_{1*}L_2(I_0)\}\\ &- \{(K_{0*}+K_0)|\omega| + 2|\gamma_1| + (K_{1*}+K_1)|\alpha_2|\}\\ &\cdot \{(K_{0*}-K_0)|\omega| + (K_{1*}-K_1)|\alpha_2|\}\\ &+ O(1)L(I)Q'(\Lambda). \end{aligned} \tag{4.19}$$

Then by (4.12), (4.14) and (4.19), we have

$$\begin{aligned}Q_*(I) - Q_*(J) \geqslant 2K_*\{|\omega| + |\alpha_2|\}\{L_1(I_0) + K_{1*}L_2(I_0)\} + O(1)L(I)Q'(\Lambda)\\ + K_*\min(K_*, 2)\{|\omega| + |\alpha_2|\}\{|\omega| + |\alpha_2| + |\gamma_1|\}. \end{aligned} \tag{4.20}$$

On the other hand, from (4.17) we can deduce that

$$\begin{aligned}Q(J) \leqslant Q(I) &- \Delta(\alpha_2, I_{0,(2)}) + K_1|\alpha_2|\,|L_2(I_0)| + K_1\Delta(\alpha_2, I_{0,(1)})\\ &+ K_0\Delta(\omega, I_{0,(1)}) + K_0|\omega|\,|L_2(I_0)|\\ &- |\gamma_1|\,|\alpha_2| + O(1)L(I_0)Q'(\Lambda). \end{aligned} \tag{4.21}$$

Then by (4.20) and (4.21), we can find constants C_* and $\delta'(1)$ depending only on K_{j*} $(j = 0, 1, 2)$ and $O(1)$ such that if $C \geqslant C_*$ and $L(I) \leqslant \delta'(1)$ then

$$\frac{K_*}{2}\{|\gamma_1| + |\omega| + |\alpha_2|\}\{|\omega| + |\alpha_2|\} \leqslant Q(I) + CQ_*(I) - (Q(J) + CQ_*(J)). \tag{4.22}$$

Thus by (4.18) and (4.22) we can find constants $C_1^* > 1$ and $K' > 0$ depending only on K_* and $O(1)$ such that (4.16) holds for any $C_1 \geqslant C_1^*$ and $K \geqslant K'$.

Let $\delta' = \min(\delta'(1),\ \delta(3)/(K_{0*} + K_{1*} + 1))$. If $F_s(I) \leqslant \delta'$, then

$$L(I) \leqslant \frac{1}{C_1}\delta' \leqslant \delta'.$$

Therefore we have

$$F_s(I) \geqslant F_s(J).$$

Moreover, we have

$$|\sigma^J - \sigma_0| + L(J) \leqslant \left(1 + \frac{1}{C_1}\right)F_s(J) \leqslant 2\delta',$$

which yields (4.15) by Lemma 4.1.

Case 2: Λ covers a part of $\partial\Omega_{\Delta x}$, and $S_*(\sigma_{(k-1)})$ issues from A_{k-1} and enters Λ. Then from the construction it follows that $\Omega_I = \Omega_J \cup \{A_k\}$ and $S_*(\sigma_{(k)})$ issues from A_k and crosses J. Moreover, there is no weak wave crossing I or J.

Denote $\omega = \omega(A_k)$ as in Case 1. Then

$$F(I) - F(J) \geqslant K_{0*}|\omega| + KCK_{0*}^2|\omega|^2.$$

Let δ' be given as in Case 1, and let K, C, C_1 be any constants with $C_1 \geqslant 1$ and $KCK_{0*}^2 \geqslant$ bounds of $|O(1)|$, where $O(1)$ is given by Proposition 3.2.

If $F_s(I) \leqslant \delta'$, then by direct computation we can obtain (4.15) and (4.16).

Case 3 (*Weak–strong interaction*): Λ lies in the interior of $\Omega_{\Delta x}$ and the strong shock $S_*(\sigma_{(k-1)})$ enters Λ. Then $S_*(\sigma_{(k)})$ issues from the center of Λ, and $\sigma^I = \sigma_{(k-1)}$, $\sigma^J = \sigma_{(k)}$.

Let $I = I_0 \cup I'$ and $J = I_0 \cup J'$ such that $\partial\Lambda = I' \cup J'$. Let γ_1 and α_2 be the weak 1-wave and the weak 2-wave, respectively, crossing I' with α_2 lying below γ_1 on I, and let ε_2 be the weak 2-wave crossing J' (see Fig. 10). In this case, denote $Q'(\Lambda) = |\gamma_1|^2 + |\alpha_2|\,|\gamma_1|$, and by Proposition 3.3 we have

$$\varepsilon_2 = \alpha_2 + K_2\gamma_1 + O(1)Q'(\Lambda),$$

$$\sigma_{(k)} = \sigma_{(k-1)} + K_3\gamma_1 + O(1)Q'(\Lambda).$$

Therefore

$$L(J) \leqslant L(I) - (1 - |K_2|K_{1*})|\gamma_1| + O(1)Q'(\Lambda), \tag{4.23}$$

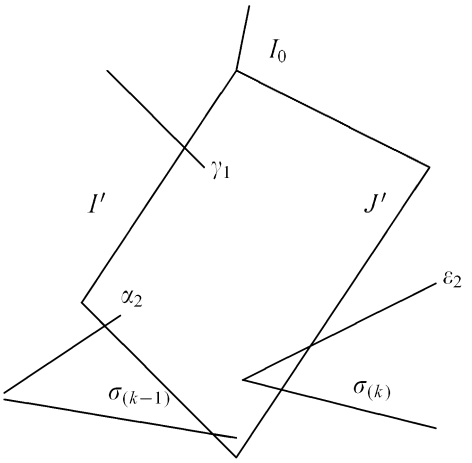

Fig. 10. Case 3.

which gives

$$|L(J)|^2 \leqslant |L(I)|^2 - 2(1-|K_2|K_{1*})|\gamma_1|\{K_{0*}L_0(I) + L_1(I_0) + K_{1*}L_2(I_0)\}$$
$$- (1-|K_2|K_{1*})|\gamma_1|\{(1+|K_2|K_{1*})|\gamma_1| + 2K_{1*}|\alpha_2|\} + O(1)L(I)Q'(\Lambda).$$

Then by (4.12) and the contraction inequality (4.13) we can deduce that

$$Q_*(I) - Q_*(J) \geqslant 2K_*|\gamma_1|L(I_0) + K_*|\gamma_1|\{|\gamma_1| + K_{1*}|\alpha_2|\} + O(1)L(I)Q'(\Lambda).$$

Thus in the same way as in the proof of Case 1, we can choose positive constants δ', C_*, C_1^* and K' such that if $F_s(I) \leqslant \delta'$ then there hold (4.15) and (4.16) for any $C > C_*$, $C_1 > C_1^*$ and $K > K'$.

Case 4 (*Weak–weak interaction*): Λ lies in the interior of $\Omega_{\Delta x}$ and covers no part of $\chi_{\Delta x,\theta}$. Then no strong shock enters Λ. Carrying out the same step as in the standard case (see [9,25]), we can choose for any constant $C > 0$ a suitable constant $\delta''(1) > 0$ such that if $L(I) < \delta''(1)$, then

$$Q(I) + CQ_*(I) - Q(J) - CQ_*(J) \geqslant \tfrac{1}{4}Q(\Lambda).$$

Here $Q(\Lambda)$ denotes the quadratic term of interactions in Λ as in [9,17,25]. Thus we can choose K large enough such that (4.15) and (4.16) hold.

Then from the discussion of the above four cases, we can choose δ', C, C_1 and K such that the proposition holds. This completes the proof. □

From Proposition 4.1 we can deduce the following for any $k \geqslant 1$:

Theorem 4.1. *Suppose that the function b satisfies* (4.2) *and* (4.3), *and let* δ', K, C *and* C_1 *be the constants given in Proposition* 4.1. *If the induction hypotheses* (C-1(k-1)), (C-2(k-1)) *and* (C-3(k-1)) *hold, and if* $F_s(I_{k-1}) \leqslant \delta'$, *then*

$$U_{\Delta x,\theta}|_{\Omega^+_{\Delta x,k}} \in B, \quad U_{\Delta x,\theta}|_{\Omega^-_{\Delta x,k}} = U_\infty, \quad \sigma_k \in B_2 \tag{4.24}$$

and

$$F_s(I_{k-1}) \geqslant F_s(I_k). \tag{4.25}$$

Lemma 4.2. *Suppose that assumption* (A2) *holds. Then for any* $0 \leqslant x_1 < x_2$, *there exists an* $x_0 \in [x_1, x_2]$ *such that* $\frac{b(x_2)-b(x_1)}{x_2-x_1} \in [b'_-(x_0), b'_+(x_0)]$ *if* $b'_-(x_0) \leqslant b'_+(x_0)$, *or* $\frac{b(x_2)-b(x_1)}{x_2-x_1} \in [b'_+(x_0), b'_-(x_0)]$ *if* $b'_-(x_0) \geqslant b'_+(x_0)$. *Here* $b'_-(x)$ *and* $b'_+(x)$ *denote the left derivate and the right derivate of b at the point x, respectively.*

Proof. Let

$$g(x) = b(x) - \frac{b(x_2) - b(x_1)}{x_2 - x_1}(x - x_1) - b(x_1),$$

and choose an $x_0 \in (x_1, x_2)$ such that $g(x_0)$ is the maximum or minimum of $g(x)$ in $[x_1, x_2]$, then it follows the desired result. □

For any function g and any interval $E \subset R^1$, denote by $\mathrm{TV}\{g; E\}$ the total variation of g on E and denote $g(x+) = \lim_{\substack{y \to x \\ y > x}} g(y)$ and $g(x-) = \lim_{\substack{y \to x \\ y < x}} g(y)$. As $b'(0+) = 0$, by Lemma 4.2 we have

$$\sum_{j=0}^{\infty} |\omega(A_j)| \leqslant \mathrm{TV}\{b'_+; [0, \infty)\}.$$

Thus we can choose a $\delta^* > 0$, depending only on $\delta(2)$, $\min_{V_0} |\lambda_{1,2}|$ and $\min_{\sigma \in B_2} |\sigma|$, such that if $\mathrm{TV}\{b'_+; [0, \infty)\} < \delta^*$, then (4.2) and (4.3) hold. Moreover, by Lemma 4.2 and Theorem 4.1, we can deduce the following:

Theorem 4.2. *There exists a $\delta'_0 \in (0, \delta^*)$ such that if*

$$\mathrm{TV}\{b'_+; [0, +\infty)\} < \delta'_0,$$

then, for all $\theta \in \prod_{k=0}^{+\infty} (-1, 1)$ and every $\Delta x > 0$, the modified Glimm Scheme in Section 4.2 *defines an approximate solution $U_{\Delta x, \theta}$ and its approximate strong* 1*-shock front $\chi_{\Delta x, \theta}$ in $\Omega_{\Delta x}$, which satisfy* (C-1(k-1)), (C-2(k-1)), (C-3(k-1)) *and* (4.25) *for any $k \geqslant 1$. In addition,*

$$\mathrm{TV}\{U_{\Delta x, \theta}(k\Delta x-, \cdot); (-\infty, y_k]\} < 4C_2\delta'_0$$

for any $k \geqslant 0$ and

$$|\chi_{\Delta x, \theta}(x + h) - \chi_{\Delta x, \theta}(x)| \leqslant (|\sigma_0| + C_3)|h| + 2\Delta x$$

for any $x \geqslant 0$ and $h > 0$, where the constants C_2 and C_3 depend only on K_{j*} ($j = 0, 1, 2$), K_*, C, C_1, K and the bound of $O(1)$.

4.4. Estimates on the approximate shock front

For any $k \geqslant 1$ and any interaction diamond $\Lambda \subset \{(k-1)\Delta x \leqslant x \leqslant (k+1)\Delta x\}$, we use the same notations as given in the proof of Proposition 4.1 to define the

following:

$$Q'_{\Delta x,\theta}(\Lambda) = \begin{cases} |\gamma_1|\,|\alpha_2| + \Delta(\gamma_1,\omega) + |\alpha_2|\,|\omega| + |\alpha_2|^2 + |\omega|^2, & \text{case 1,} \\ |\omega|^2, & \text{case 2,} \\ |\gamma_1|^2 + |\alpha_2|\,|\gamma_1|, & \text{case 3,} \\ Q(\Lambda), & \text{case 4,} \\ 0, & \text{other cases} \end{cases}$$

and

$$E_{\Delta x,\theta}(\Lambda) = \begin{cases} |\omega| + |\alpha_2|, & \text{case 1,} \\ |\omega|, & \text{case 2,} \\ |\gamma_1|, & \text{case 3,} \\ 0, & \text{case 4 or other cases.} \end{cases}$$

Then $Q'_{\Delta x,\theta}(\Lambda)$ is the interaction potential in the Λ and $E_{\Delta x,\theta}(\Lambda)$ is the sum of the strengths of waves that interact with the boundary or the strong shock. In the same way as in proving Theorem 4.1, we can get the following (see also [17]):

Theorem 4.3. *Let $U_{\Delta x,\theta}$ be an approximate solution given by Theorem* 4.2. *There exists a constant $M>0$ independent of $U_{\Delta x,\theta}$, θ and Δx such that*

$$\sum_{\Lambda} Q'_{\Delta x,\theta}(\Lambda) \leqslant M, \tag{4.26}$$

$$\sum_{\Lambda} E_{\Delta x,\theta}(\Lambda) \leqslant M. \tag{4.27}$$

Here each summation is over all the diamonds.

Define

$$\sigma_{\Delta x,\theta}(x) = \sigma_{(k)}, \quad \text{if } x \in [k\Delta x, (k+1)\Delta x).$$

Then from Proposition 3.3 and Theorem 4.3, we can deduce the following:

Lemma 4.3. *There exists a constant M_1 independent of Δx, θ and $U_{\Delta x,\theta}$ such that*

$$\mathrm{TV}\{\sigma_{\Delta x,\theta};\ [0,+\infty)\} = \sum_{k=0}^{+\infty} |\sigma_{(k+1)} - \sigma_{(k)}| \leqslant M_1. \tag{4.28}$$

Denote $\Gamma_{\mathrm{b}} = \bigcup_{k=0}^{+\infty} \overline{\Lambda_{k,0}}$, $\Gamma_{\mathrm{s}} = \bigcup_{k=0}^{+\infty} \overline{\Lambda_{k,n(k)}}$, where $\Lambda_{k,0}$ $(k \geqslant 0)$ and $\Lambda_{k,n(k)}$ $(k \geqslant 0)$ are the diamonds whose centres lie on the boundary $\partial\Omega_{\Delta x}$ and the shock front $\chi_{\Delta x,\theta}$,

respectively. Let $L_{\Delta x,\theta}(\Gamma_{\rm b})$ (or $L_{\Delta x,\theta}(\Gamma_{\rm s})$, resp.) be the summation of the strengths of the weak waves leaving $\Gamma_{\rm b}$ (or leaving $\Gamma_{\rm s}$, resp.) and let

$$Q'_{\Delta x,\theta}(R) = \sum_{\Lambda\subset R} Q'_{\Delta x,\theta}(\Lambda),$$

$$E_{\Delta x,\theta}(R) = \sum_{\Lambda\subset R} E_{\Delta x,\theta}(\Lambda),$$

where $R = \Gamma_{\rm b}$ or $\Gamma_{\rm s}$. Then by the results in Section 3, we have the following:

Lemma 4.4. *There exists a constant M_3 independent of $U_{\Delta x,\theta}$, Δ, θ such that*

$$L_{\Delta x,\theta}(\Gamma_{\rm s}) \leqslant M_3(E_{\Delta x,\theta}(\Gamma_{\rm s}) + Q'_{\Delta x,\theta}(\Gamma_{\rm s})), \tag{4.29}$$

$$L_{\Delta x,\theta}(\Gamma_{\rm b}) \leqslant M_3(E_{\Delta x,\theta}(\Gamma_{\rm b}) + Q'_{\Delta x,\theta}(\Gamma_{\rm b})). \tag{4.30}$$

5. Global weak solution

5.1. Convergence of the approximate solution

According to the above discussion we can extend $U_{\Delta x,\theta}$ by the constant $U_{k,0}$ continuously across the boundary to whole strip $\{k\Delta x < x < (k+1)\Delta x\}$ for every $k \geqslant 0$.

Let the line $\{x = a\}$, with $a > 0$, intersect $\partial\Omega_{\Delta x} = \cup\{A_{k-1}A_k, k \geqslant 1\}$ at the point $(a, p_a^{\Delta x,})$. In the same way as in [27], by Theorem 4.2 we can prove the following:

Lemma 5.1. *The following inequality*

$$\int_{-\infty}^{0} |U_{\Delta x,\theta}(x+h, y+p_{x+h}^{\Delta x}) - U_{\Delta x,\theta}(x, y+p_x^{\Delta x})|\, dy \leqslant M_1|h| \tag{5.1}$$

holds for any $h > 0$ and $x \geqslant 0$, where the constant M_1 is independent of Δx, θ and h.

Set

$$J(\theta, \Delta x, \phi) = \sum_{k=1}^{+\infty} \int_{-\infty}^{0} \phi(k\Delta x, y+y_k) \cdot [U_{\Delta x,\theta}]|_{x=k\Delta x}\, dy \tag{5.2}$$

with $\phi = (\phi_1, \phi_2) \in C_c^\infty(R^2, R^2)$ and

$$[U_{\Delta x,\theta}]|_{x=k\Delta x} = U_{\Delta x,\theta}(k\Delta x+, y) - U_{\Delta x,\theta}(k\Delta x-, y).$$

Then carrying out the same step as in [25], we have the following:

Lemma 5.2. *There are a null set* $N \subset \prod_{k=0}^{+\infty}(-1,1)$ *and a subsequence of* $\{\Delta x\}$ *denoted by* $\{\Delta_j'\}_{j=1}^{\infty}$, *which has the limit* 0, *such that*

$$J(\theta, \Delta_j', \phi) \xrightarrow[\Delta_j' \to 0]{} 0$$

for any $\theta \in (\prod_{k=0}^{+\infty}(-1,1)) \backslash N$ *and* $\phi_1, \phi_2 \in C_c^{\infty}(R^2)$.

To establish the main result, we need to estimate the jumps of the approximate shock front.

Let

$$d_k = \frac{\sigma_{(k-1)}\Delta x - (y_k - y_{k-1}) + \Delta y}{\Delta y}.$$

Then by the choice of Δx and $\{y_k\}$, and by Lemma 2.4, we have $d_k \in (0,1)$. Moreover, d_k depends only on $\{\theta_l,\ 0 \leqslant l \leqslant k-1\}$. Thus define

$$I_k(\Delta x, \theta) = \mathbf{1}_{(-1,d_k)}(\theta_k) \cdot (d_k - 1)\Delta y + \mathbf{1}_{(d_k,1)}(\theta_k) \cdot (d_k + 1)\Delta y,$$

$$I(x, \Delta x, \theta) = \sum_{k=1}^{[x/\Delta x]} I_k(\Delta x, \theta),$$

where $\mathbf{1}_A$ denotes the character function of the set A and $[x/\Delta x]$ denotes the maximal integer that does not exceed $x/\Delta x$. Then $I_k(\Delta x, \theta)$ is the jump of the function $y = \chi_{\Delta x,\theta}(x)$ at $x = k\Delta x$, and is a measurable function of (θ, x), which depends only on $U_{\Delta x,\theta}|_{\{0 \leqslant x < k\Delta x\}}$ and $\{\theta_l,\ 0 \leqslant l \leqslant k\}$.

Lemma 5.3. (1) *For any* $x \geqslant 0,\ \Delta x > 0$ *and* $\theta \in \prod_{k=0}^{+\infty}(-1,1)$,

$$\chi_{\Delta x,\theta}(x) = I(x, \Delta x, \theta) + \int_0^x \sigma_{\Delta x,\theta}(s)\, ds.$$

(2) *There exist a null set* N_1 *and a subsequence of* $\{\Delta_j'\}$ *denoted by* $\{\Delta_j''\}_{j=0}^{\infty}$, *which has the limit* 0, *such that there holds the following*:

$$\int_0^{+\infty} e^{-x} |I(x, \Delta_j'', \theta)|^2\, dx \xrightarrow[\Delta_j'' \to 0]{} 0$$

for any $\theta \in \prod_{k=0}^{+\infty}(-1,1) \backslash N_1$. *Here* $\{\Delta_j'\}$ *is given by Lemma* 5.2.

Proof. Part (1) follows by the direct computation. It suffices to prove part (2).

As in [25], let

$$d\theta = \prod_{j=0}^{\infty} (d\theta_j/2).$$

Then for any $k>j$, we have

$$\int I_k I_j \, d\theta = \int \prod_{i=1}^{k-1} d\theta_i \left(I_j \int I_k \, d\theta_k \right) = 0.$$

Therefore, we can deduce the following:

$$\int |I(x,\Delta x,\theta)|^2 d\theta = \sum_{k=1}^{[x/\Delta x]} \int |I_k(\Delta x,\theta)|^2 \, d\theta \leqslant 4\left|\frac{\Delta y}{\Delta x}\right|^2 x\Delta x.$$

Then by choosing a subsequence of $\{\Delta_j'\}$, $\{\Delta_j''\}$ such that $\sum_{j=0}^{+\infty} \Delta_j'' < +\infty$, we can get (2). This completes the proof. □

Then by Lemmas 5.1–5.3, 4.3 and Theorem 4.2, we can get the following:

Theorem 5.1. *Suppose assumptions* (A1)–(A3) *hold, then there exists a* $\delta>0$ *such that if* $\mathrm{TV}\{b_+'; [0,\infty)\}<\delta$ *then for each* $\theta \in (\prod_{k=0}^{+\infty}(-1,1)) \backslash (N \cup N_1)$, *there exist a sequence of mesh sizes,* $\{\Delta_k\}_k$ *with* $\Delta_k \to 0$ *as* $k \to +\infty$, *and a pair of functions* $U_\theta \in L^\infty(\Omega, B)$, $\chi_\theta \in \mathrm{Lip}([0,+\infty))$ *with* $\chi_\theta(0)=0$, *such that*

(i) $\{U_{\Delta_k,\theta}(x,\cdot)\}$ *is convergent in* $L^1(-\infty, b(x))$ *to* $U_\theta(x,\cdot)$ *for every* $x>0$, *and* U_θ *is a global weak solution of problem* (1.1) *in* Ω, *with*

$$\mathrm{TV}\,\{U_\theta(x,\cdot); (-\infty, b(x)]\} < M_2$$

for every $x \in [0,+\infty)$, *where* M_2 *is a constant depending only on the system, the function* b *and* $\mathrm{TV}\{b_+', [0,+\infty)\}$;

(ii) $\{\chi_{\Delta_k,\theta}\}$ *is convergent uniformly to* χ_θ *in any bounded* x*-interval*;

(iii) $\{\sigma_{\Delta_k,\theta}\}$ *is convergent in* $L^1_{\mathrm{loc}}([0,+\infty))$ *and almost everywhere to* $\sigma_\theta \in \mathrm{BV}([0,\infty), B_2)$ *and* $\chi_\theta(x) = \int_0^x \sigma_\theta(t)\, dt$.

In addition, if θ *is equidistributed, then* $\chi_\theta(x)<b(x)$ *for any* $x>0$, *with*

$$U_\theta|_{\Omega_\infty} = U_\infty, \quad \sqrt{u_\theta^2+v_\theta^2}|_{\Omega_0} < \sqrt{u_\infty^2+v_\infty^2}.$$

Moreover, Rankine–Hugoniot relation holds almost everywhere along the set $\{y = \chi_\theta(x)\}$. *Here* $\Omega_\infty = \{(x,y) | y<\chi_\theta(x)\}$ *and* $\Omega_0 = \{(x,y) | \chi_\theta(x)<y<b(x)\}$.

The proof of (i), (ii) and the proof of the convergence of $\{\sigma_{\Delta_k,\theta}\}$ in (iii) can be carried out in the same way as in the standard case (see [9,10,27]). The equality in (iii), $\chi_\theta(x) = \int_0^x \sigma_\theta(t)\,dt$, can be deduced from Lemma 5.3 and the result on convergence of $\{\chi_{\Delta_k,\theta}\}$ and $\{\sigma_{\Delta_k,\theta}\}$. Moreover, by the construction of the solution and by the results in [26], we can prove the remaining part of Theorem 5.1. Therefore, the proof is complete.

5.2. Asymptotic behaviour of the strong shock

Let $\theta \in \prod_{k=0}^{\infty} (-1,1)\backslash(N_1 \cup N)$ be given in Theorem 5.1 and be equidistributed, and let U_θ and χ_θ be the solution and its shock front given in Theorem 5.1, respectively. By Theorem 5.1 and the results in [10,18], it follows that the solution U_θ contains at most countable shock fronts and countable points of waves interactions. Moreover, we can modify the solution U_θ such that U_θ is continuous outside the shock curves and the points of waves interactions. Then,

Lemma 5.4.

$$\mathrm{TV}\{U_\theta(x,\cdot);(\chi_\theta(x),b(x))\} \xrightarrow[x\to+\infty]{} 0. \tag{5.3}$$

Proof. Let $\{\Delta_l\}$ be the sequence given in Theorem 5.1. Let $Q'_{l,\theta}(\Lambda) = Q'_{\Delta_l,\theta}(\Lambda)$ and let $E_{l,\theta}(\Lambda) = E_{\Delta_l,\theta}(\Lambda)$. As in [10], we denote by $dQ'_{l,\theta}$ and $dE_{l,\theta}$ the measures assigning the quantities $Q'_{l,\theta}(\Lambda)$ and $E_{l,\theta}(\Lambda)$ to the centre of Λ respectively.

Since Theorem 4.3 implies the compactness of $\{dQ'_{l,\theta}\}$ and $\{dE_{l,\theta}\}$, we can select subsequences of $\{dQ'_{l,\theta}\}$, $\{dE_{l,\theta}\}$ and $\{\Delta_l\}$, which we still use the same notations to denote, so that $\Delta_l \to 0$ and so that the limits

$$dQ'_{l,\theta} \to dQ'_\theta$$

and

$$dE_{l,\theta} \to dE_\theta$$

exist in the W^* topology for measures. Moreover, $Q'_\theta(\Omega) < \infty$ and $E_\theta(\Omega) < \infty$. Therefore, for any $\varepsilon > 0$ we can choose a $x_\varepsilon > 0$ independent of $\{U_{\Delta_l,\theta}\}$ such that for any $l > 0$,

$$\sum_{k \geqslant [x_\varepsilon/\Delta x]} Q'_{l,\theta}(\Lambda_{k,n}) < \varepsilon, \tag{5.4}$$

$$\sum_{k \geqslant [x_\varepsilon/\Delta x]} E_{l,\theta}(\Lambda_{k,n}) < \varepsilon. \tag{5.5}$$

Moreover, let $X_\varepsilon^1 = (x_\varepsilon, y_\varepsilon^1)$ (or $X_\varepsilon^2 = (x_\varepsilon, y_\varepsilon^2)$) be the point lying in the $\chi_{\Delta_l,\theta}$ (or $\partial\Omega_{\Delta_k}$ reps.), then we can find $x_\varepsilon^0 > x_\varepsilon$ independent of $\{\Delta x\}$ and $\{U_{\Delta_l,\theta}\}$ such that the approximate 2-characteristic issuing from X_ε^1 intersects $\partial\Omega_{\Delta_l}$ at some point in $\{x < x_\varepsilon^0\}$ and the approximate 1-characteristic issuing from X_ε^2 intersects $\chi_{\Delta_l,\theta}$ at some point in $\{x < x_\varepsilon^0\}$. Then by the approximate conservation laws and Lemma 4.4 we can deduce the following:

$$\mathrm{TV}\{U_{\Delta_l,\theta}(x-,\cdot); (\chi_{\Delta_l,\theta}(x), b(x))\} < O(1)\varepsilon$$

for $x > x_\varepsilon^0$, where the bound of $O(1)$ is independent of ε, x, $U_{\Delta_l,\theta}$ and Δ_l.

Thus, passing to the limit as $\Delta_l \to 0$, by Theorem 5.1 and regularity of U_θ we deduce the following:

$$\mathrm{TV}\{U_\theta(x,\cdot); (\chi_\theta(x), b(x))\} \leqslant O(1)\varepsilon$$

for $x > x_\varepsilon^0$, which completes the proof. □

Denote by $\{y = b_l(x)\}$ the boundary of Ω_{Δ_l}. Let

$$\sigma_\infty = \lim_{x\to+\infty} \sigma_\theta(x)$$

and

$$b'_\infty = \lim_{x\to+\infty} b'(x).$$

From Theorem 5.1 and (A2) we know that these equalities are well defined. Furthermore, from the choice of the neighbourhoods, we have

$$\arctan b'_\infty \in \left(\arctan\frac{v_\infty}{u_\infty} - \omega_{\mathrm{ext}}, \arctan\frac{v_\infty}{u_\infty}\right).$$

Theorem 5.2. *There exists a constant state $U^+ \in S_1^-(U_\infty)$ such that*

$$\lim_{x\to+\infty} \sup\{|U_\theta(x,y) - U^+| \,|\chi_\theta(x) < y < b(x)\} = 0. \tag{5.6}$$

Moreover, the pair (U^+, σ_∞) is the solution to the following equations:

$$G(\sigma) = U, \tag{5.7}$$

$$G(\sigma)\cdot(-b'_\infty, 1) = 0, \tag{5.8}$$

where the function G is given in Proposition 2.1.

Proof. Let $U_{l,\theta} = U_{\Delta_l,\theta}$, $\sigma_{l,\theta} = \sigma_{\Delta_l,\theta}$ and $\chi_{l,\theta} = \chi_{\Delta_l,\theta}$. Then according to the construction of the approximate solutions, for every $x>0$ we have

$$\begin{aligned}&|G(\sigma_{l,\theta}(x-)) \cdot (-b_l'(x-), 1)| + \sup_{\chi_{l,\theta}(x)<y<b_l(x)} |G(\sigma_{l,\theta}(x-)) - U_{l,\theta}(x-, y)| \\ &\quad \leqslant O(1)\mathrm{TV}\{U_{\Delta_l,\theta}(x-,\cdot); (\chi_{\Delta_l,\theta}(x), b_l(x))\},\end{aligned}$$

where the bound of $O(1)$ is independent of $\{\Delta_l\}_l$, θ, x and $\{U_{l,\theta}\}_l$. Then, passing to the limit as $\Delta_l \to 0$, by Theorem 5.1 and regularity of U_θ we can get the following:

$$\begin{aligned}&|G(\sigma_\theta(x-)) \cdot (-b'(x-), 1)| + \sup_{\chi_\theta(x)<y<b(x)} |G(\sigma_\theta(x-)) - U_\theta(x-, y)| \\ &\quad \leqslant O(1)\mathrm{TV}\{U_\theta(x-,\cdot); (\chi_\theta(x), b(x))\}\end{aligned} \tag{5.9}$$

for every $x>0$.

Moreover, by Theorem 5.1 we have $G(\sigma_\theta) \in \mathrm{BV}([0, \infty), B)$. Let

$$U^+ = \lim_{x\to+\infty} G(\sigma_\theta(x)),$$

then the result follows by Lemma 5.4 and (5.9). □

Acknowledgments

This work was partially supported by NSFC Project 19701009 and the Ministry of Education of China. Part of this work was done during author's stay in Osaka University for a program sponsored by the Monbusho of Japanese government. The author is deeply indebted to Professor Shuxing Chen for his leading me to the problems of this category and also for his encouragement. My thanks also goes to Professor Matsumura and his seminar group for their hospitality, and to the referee for insightful comments.

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