Asymptotic behaviour of some nonlocal diffusion problems

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I Introduction

In any diffusion process – for heat or population – it is reasonable to admit that the diffusion velocity $\vec{v}$ is given at the point x by the Fourier law

$$\vec{v}(x) = -a\nabla u(x) \tag{1.1}$$

where u is the temperature or the density of population, ∇u is the usual gradient of u, a is a constant depending on the medium where the process is taking place. Of course the assumption that a is constant is a first approximation of the reality. For instance in material science it is clear that "physical constants" attached to a material will depend on its state, its temperature for example. In this note we would like to address the case where the constant a depends on nonlocal quantities. For example – in the case where one investigates the diffusion of a population (bacteria or else) – it is reasonable to assume that the mobility inside the medium depends on how crowded it is. Thus a could depend on

$$l(u) = \int_\Omega u(x)dx \tag{1.2}$$

the entire population in the domain Ω or else on the population of a preeminent group occupying some region $\Omega' \subset \Omega$ – i.e. a could be a function of

$$l(u) = \int_{\Omega'} u(x)dx\,. \tag{1.3}$$

This is this kind of dependence that we would like to study here.

So, let Ω be a connected bounded Lipschitz open set of $\mathbb{R}^n$. Of course $n = 2$ or 3 for the applications that we mentionned above. Let us denote by Γ the boundary of Ω and by Γ_0 a subset of Γ of $d\sigma$ measure positive. Then we set

$$V = \{v \in H^1(\Omega) \mid v \equiv 0 \ d\sigma - \text{a.e. on} \quad \Gamma_0\}. \tag{1.4}$$

We refer to [8], [9], [3] for an introduction to Sobolev spaces, $d\sigma$ denotes the surface measure on Γ. Let l be an application from V into $I\!R$ then we would like to consider the parabolic problem

$$\begin{cases} u_t - a(l(u))\Delta u = f \text{ in } \Omega \times (0,T)\,, \\ u(\cdot,t) \in V,\ t \in (0,T)\,, \\ u(\cdot,0) = u_0\,. \end{cases} \tag{1.5}$$

$f \in V'$ the strong dual of V, $u_0 \in L^2(\Omega)$ is an initial data. We are especially interested in the asymptotic behaviour of such a problem.

The paper is divided as follows. In section 2 we address the issue of existence of a solution. In section 3 we show some uniqueness results. In Section 4 we introduce the steady state problem and in Section 5 the convergence of the solution towards a steady state is investigated. This is done by using the dynamical systems point of view and extends previous results obtained in [5]. Finally in Section 6 we complete these results weakening our assumptions.

II Existence of a solution

Let a be a function from $I\!R$ into $I\!R_+$ such that

$$a \quad \text{is continuous}\,, \tag{2.1}$$

there exist two constant m, M such that

$$0 < m \le a(\xi) \le M, \quad \forall \xi \in I\!R\,. \tag{2.2}$$

Let V be the subspace of $H^1(\Omega)$ defined by (1.4). Then denote by l a mapping from $L^2(\Omega)$ into $I\!R$ such that

$$l \quad \text{is continuous from } L^2(\Omega) \text{ into } I\!R\,. \tag{2.3}$$

Then it holds:

Theorem 2.1 *Under the assumptions (2.1)–(2.3) let*

$$f \in L^2(0,T;V'), \quad u_0 \in L^2(\Omega)\,.$$

Then there exists u solution to

$$\begin{cases} u \in L^2(0,T;V) \cap C([0,T];L^2(\Omega), \quad u_t \in L^2(0,T;V'), \\ u(0) = u_0, \\ \dfrac{d}{dt}(u,v) + a(l(u)) \displaystyle\int_\Omega \nabla u \cdot \nabla v\, dx = \langle f, v\rangle \quad in \quad \mathcal{D}'(0,T),\ \forall v \in V\,. \end{cases} \tag{2.4}$$

In (2.4), (,) denotes the usual scalar product in $L^2(\Omega)$ and $\langle , \rangle$ the V', V-duality bracket. We refer to [6], [2] for the spaces introduced throughout the paper.

Proof. We are going to rely on the Schauder fixed point theorem. For this let $w \in L^2(0,T;L^2(\Omega))$. Then we claim that the mapping

$$t \longmapsto l(w(\cdot,t))$$

is measurable (see for instance [2] for this notion). So, the following mapping

$$t \longmapsto a(l(w(\cdot,t))) = a(l(w))$$

is also measurable and – by (2.2) – in $L^\infty(0,T)$. Then, it is well known – see for instance [6] that there exists a unique u solution to

$$\begin{cases} u \in L^2(0,T;V) \cap C([0,T],L^2(\Omega)),\ u_t \in L^2(0,T;V')\,, \\ u(0) = u_0\,, \\ \dfrac{d}{dt}(u,v) + a(l(w)) \displaystyle\int_\Omega \nabla u \cdot \nabla v dx = \langle f, v\rangle \text{ in } \mathcal{D}'(0,T),\ \forall v \in V\,. \end{cases} \tag{2.5}$$

Then, we would like to show that the mapping

$$w \longmapsto R(w) = u$$

from $L^2(0,T;L^2(\Omega))$ into itself has a fixed point which will be clearly a solution to (2.4). For that note first that the last equation of (2.5) reads also

$$\frac{du}{dt} - a(l(w))\Delta u = f \tag{2.6}$$

in $L^2(0,T;V')$. Hence it follows by taking the V', V-duality bracket with u

$$\langle \frac{du}{dt}, u\rangle + a(l(w)) \int_\Omega |\nabla u|^2 dx = \langle f, u\rangle \text{ a.e. } t \in (0,T)\,. \tag{2.7}$$

Without loss of generality we can assume that V is equipped with the norm given by (see [12])

$$|u|_V^2 = \int_\Omega |\nabla u|^2\, dx \tag{2.8}$$

so that from (2.7) one derives – thanks to (2.2) –

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + m|u|_V^2 \le |f|_*|u|_V \quad \text{a.e. } t \in (0,T)\,. \tag{2.9}$$

($|\quad|_2$ is the usual norm in $L^2(\Omega)$, $|\quad|_*$ the strong dual norm on V'). Using now the Young inequality

$$ab \le \frac{1}{2m}a^2 + \frac{m}{2}b^2$$

one easily gets

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \frac{m}{2}|u|_V^2 \leq \frac{1}{2m}|f|_* \quad \text{a.e } t \in (0,T)\,. \tag{2.10}$$

Integrating between 0 and T we obtain for a.e. $t \in (0,T)$

$$\frac{1}{2}|u|_2^2 + \frac{m}{2}\int_0^T |u|_V^2\,dt \leq \frac{1}{2}|u_0|_2^2 + \frac{1}{2m}\int_0^T |f|_*\,dt \tag{2.11}$$

and so

$$|u|_{L^2(0,T;V)},\ |u|_{L^2(0,T;L^2(\Omega))} \leq C \tag{2.12}$$

for some constant C independent of w. Using now (2.6) we obtain also

$$|u_t|_{L^2(0,T;V')} \leq C \tag{2.13}$$

where C is also independent of w. We consider then

$$B = B(0,C) = \{w \in L^2(0,T;L^2(\Omega))\ /\ |u|_{L^2(0,T;L^2(\Omega))} \leq C\}\,, \tag{2.14}$$

where C is the constant in (2.12). Then, clearly, R is a mapping from B into itself. Moreover, due to (2.13), (2.14), $R(B)$ is relatively compact in B (see for instance [10]). Thus in order to be able to apply the Schauder fixed point theorem it is enough to show that R is continuous. For that let w_n be a sequence sucht that

$$w_n \longrightarrow w \quad \text{in } B \text{ or } \quad L^2(0,T;L^2(\Omega))\,.$$

Let us denote by u_n the solution to (2.5) corresponding to w_n – i.e. $u_n = R(w_n)$. Up to the extraction of a subsequence one can assume that

$$w_n(\cdot,t) \longrightarrow w(\cdot,t) \quad \text{in} \quad L^2(\Omega), \quad \text{for a.e. } t \in (0,T)$$

and thus

$$a(l(w_n)) \longrightarrow a(l(w)) \quad \text{for a.e. } t \in (0,T)\,. \tag{2.15}$$

Moreover, since u_n satisfies the estimates (2.12), (2.13) one can find u^∞ such that – up to a subsequence – it holds

$$\begin{cases} u_n \rightharpoonup u^\infty & \text{in } L^2(0,T;V)\,, \\ u_n \longrightarrow u^\infty & \text{in } L^2(0,T;L^2(\Omega))\,, \\ u_{n,t} \rightharpoonup u_t^\infty & \text{in } L^2(0,T;V')\,. \end{cases} \tag{2.16}$$

We consider then the last equation of (2.5) corresponding to $w = w_n$ – i.e. for every $v \in V,\ \varphi \in \mathcal{D}(0,T)$ it holds

$$-\int_0^T\int_\Omega u_n v\,\varphi'\,dx\,dt + \int_0^T\int_\Omega a(l(w_n))\nabla u_n \nabla v\,\varphi\,dx\,dt = \int_0^T \langle f,v\rangle\,\varphi\,dt\,. \tag{2.17}$$

By (2.15) and the Lebesgue dominated convergence theorem one has

$$a(l(w_n))\varphi v \longrightarrow a(l(w))\varphi v$$

in $L^2(0,T;V)$ and passing to the limit in (2.17) one obtains that u^∞ satisfies for every $v \in V$

$$\frac{d}{dt}(u^\infty, v) + a(l(w)) \int_\Omega \nabla u^\infty \, \nabla v \, dx = \langle f, v \rangle \quad \text{in} \quad \mathcal{D}'(0,T)\,. \tag{2.18}$$

Next, one notices that it holds (see [6])

$$(u_n(t), v) - (u_0, v) = \int_0^T \langle u_{n,t}\,,\, v \rangle, \quad \text{for a.e. } t \in (0,T), \quad \forall v \in V. \tag{2.19}$$

Due to (2.16)– without loss of generality – one can assume that

$$u_n(t) \longrightarrow u^\infty(t) \quad \text{in} \quad L^2(\Omega), \quad \text{for a.e. } t \in (0,T)\,.$$

Passing to the limit in (2.19) it holds

$$(u^\infty(t), v) - (u_0, v) = \int_0^T \langle u_t^\infty, v \rangle = (u^\infty(t), v) - (u^\infty(0), v), \quad \text{for a.e. } t, \quad \forall v \in V.$$

(Note that due to (2.12), (2.13), (2.16), $u^\infty, u_n \in C([0,T]; L^2(\Omega))$). Thus it follows that

$$u^\infty(0) = u_0$$

and $u^\infty = u$ is solution to (2.5). Since u_n has only one possible limit, the whole sequence u_n converges toward u in B. This completes the proof of the continuity of R and the proof of the theorem □

III Uniqueness

In this section we would like to address the question of uniqueness of a solution to (2.4). For this purpose we assume that a and l are locally Lipschitz continuous – i.e. we assume that for any bounded interval $[-M, M]$ of $\mathbb{R}$ and any bounded set Q of $L^2(\Omega)$ it holds:

$$|a(\xi) - a(\xi')| \leq A_M \, |\xi - \xi'| \quad \forall \xi,\ \xi' \in [-M, M] \tag{3.1}$$

$$|l(u) - l(v)| \leq L_Q \, |u - v|_2 \quad \forall u, v \in Q\,, \tag{3.2}$$

where A_M, L_Q are two positive constants, $|\quad|_2$ denotes the usual $L^2(\Omega)$-norm. Then we have:

Theorem 3.1 *Under the assumptions of Theorem 2.1 if in addition we assume that (3.1), (3.2) holds then the solution u to (2.4) is unique.*

Proof. Let u_1, u_2 be two solutions to (2.4). By (2.6) it holds in $L^2(0,T;V')$

$$\frac{du_1}{dt} - a(l(u_1))\Delta u_1 = \frac{du_2}{dt} - a(l(u_2))\Delta u_2 \,.$$

This leads to

$$\frac{d}{dt}(u_1 - u_2) - a(l(u_1))\Delta(u_1 - u_2) = -(a(l(u_2)) - a(l(u_1)))\Delta u_2 \,.$$

Taking the V', V-duality bracket with $u_1 - u_2$ it comes

$$\begin{aligned}\langle \frac{d}{dt}(u_1 - u_2), u_1 - u_2\rangle + a(l(u_1)) \int_\Omega |\nabla(u_1 - u_2)|^2 \, dx \\ = a(l(u_2)) - a(l(u_1)) \int_\Omega \nabla u_2 \nabla(u_1 - u_2) dx \,.\end{aligned}$$

Since u_1, $u_2 \in C([0,T]; L^2(\Omega))$, there exists a bounded set Q of $L^2(\Omega)$ such that, for any $t \in [0,T]$, $u_1(t), u_2(t) \in Q$ and thus $(l(u_1(t)),\, l(u_2(t))) \in [-M, M]^2$ for some $M > 0$. Using (2.2), (3.1), (3.2) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + m|u_1 - u_2|_V^2 &\leq A_M L_Q \, |u_1 - u_2|_2 \int_\Omega |\nabla u_2| \; |\nabla(u_1 - u_2)| dx \\ &\leq A_M L_Q \, |u_1 - u_2|_2 \; |u_2|_V |u_1 - u_2|_V\end{aligned}$$

by the Cauchy-Schwarz inequality. By the Young inequality it follows that

$$\frac{1}{2}\frac{d}{dt}|u_1 - u_2|_2^2 + m|u_1 - u_2|_V^2 \leq \frac{m}{2}|u_1 - u_2|_V^2 + \frac{A_M^2 L_Q^2 \, |u_2|_V^2}{2m}|u_1 - u_2|_2^2 \,.$$

This implies that

$$\frac{d}{dt}|u_1 - u_2|_2^2 \leq \frac{A_M^2 L_Q^2 \, |u_2|_V^2}{m}|u_1 - u_2|_2^2 = C(t) \, |u_1 - u_2|_2^2 \,,$$

where $C(.) \in L^2(0,T)$. This reads also

$$\frac{d}{dt}\Big\{\exp\Big(-\int_0^t C(s)\, ds\Big) \, |u_1 - u_2|_2^2\Big\} \leq 0 \,,$$

and since the above function is non increasing and vanishes at 0 it vanishes identically. This completes the proof of uniqueness. □

Remark 3.1 Let us assume that $f \in L^\infty(0, +\infty; V')$. Then, from

$$\frac{du}{dt} - a(l(u))\Delta u = f$$

one derives after multiplication by u

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + a(l(u)) \int_\Omega |\nabla u|^2 dx = \langle f, u\rangle \,.$$

This implies

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + m|u|_V^2 \leq |f|_*|u|_V \leq \frac{m}{2}|u|_V^2 + \frac{1}{2m}|f|_*^2$$

by the Young inequality. Thus we obtain

$$\frac{d}{dt}|u|_2^2 + m|u|_V^2 \leq \frac{1}{m}|f|_*^2 \leq \frac{1}{m}|f|^2_{L^\infty(0,+\infty;V')}\,.$$

Since for some constant C it holds

$$C|u|_2 \leq |u|_V \quad \forall u \in V$$

one obtains for some other constant K

$$\frac{d}{dt}|u|_2^2 + K|u|_2^2 \leq \frac{1}{m}|f|^2_{L^\infty(0,+\infty;V')} = F$$

, i.e.

$$\frac{d}{dt}(e^{Kt}|u|_2^2) \leq e^{Kt}F\,.$$

Integrating one gets

$$e^{Kt}|u|_2^2 - |u_0|_2^2 \leq \int_0^t e^{Ks}F ds \leq \frac{e^{Kt}}{K}F\,.$$

Hence it follows that

$$|u|_2^2 \leq |u_0|_2^2 + \frac{1}{mK}|f|^2_{L^\infty(0,+\infty;V')}\,. \tag{3.3}$$

Thus u remains a priori bounded. This allows to consider cases where a is not defined on the whole real line.

Remark 3.2 As we will see below, even so uniqueness holds, monotonicity results do not hold as for the analoguous local case.

IV Steady states

For simplicity we suppose that in (1.5) f is independent of t i.e. $f \in V'$ the dual of V. Moreover, in addition to (2.3), we suppose that l is linear (for other cases we refer to [11], [4]). Then we consider the problem of finding a solution to (1.5) independent of t – i.e. the problem of finding a weak solution to

$$\begin{cases} -a(l(u))\Delta u = f & \text{in } \Omega, \\ u \in V\,. \end{cases} \tag{4.1}$$

As we are about to see the solution of such a problem relies on a fixed point argument in $I\!R$ – and not in an infinite dimensional space as it would be the case for a local problem. To see this, let us first introduce φ the weak solution to

$$\begin{cases} -\Delta\varphi = f \text{ in } \Omega, \\ \varphi \in V\,. \end{cases} \tag{4.2}$$

The existence and uniqueness of a solution – i.e. the existence and uniqueness of a φ such that

$$\varphi \in V, \qquad \int_\Omega \nabla\varphi \cdot \nabla v \, dx = \langle f, v \rangle \,, \quad \forall\, v \in V \tag{4.3}$$

is a direct consequence of the Lax–Milgram theorem due to the fact that V is a Hilbert space for the norm

$$|v|_V = \Big(\int_\Omega |\nabla v|^2 \, dx \Big)^{1/2} \tag{4.4}$$

(see [12] – note that we suppose Γ_0 of positive measure).

Then we can prove:

Theorem 4.1 *Let $a(\cdot)$ be a mapping from $I\!R$ into $(0, +\infty)$. The problem (4.1) has as much solutions as the problem in $I\!R$*

$$a(\mu)\mu = l(\varphi). \tag{4.5}$$

Proof. Let us first consider u solution to (4.1). Since $a(l(u))$ is a constant the first equation of (4.1) reads also

$$-\Delta(a(l(u))u) = f \quad \text{in } \Omega.$$

Thus, by (4.2) one has

$$a(l(u))u = \varphi.$$

Taking now l of both sides one obtains, due to the linearity of l,

$$a(l(u))l(u) = l(\varphi)$$

i.e. $l(u) \in I\!R$ is solution to (4.5).

Conversely, let μ be a solution to (4.5). Then there exists a unique weak solution to

$$\begin{cases} -a(\mu)\Delta u = f \text{ in } \Omega, \\ u \in V. \end{cases} \tag{4.6}$$

Of course due to the uniqueness of the solution to (4.2) one has also

$$a(\mu)u = \varphi.$$

Then, taking l of both sides it comes

$$a(\mu)l(u) = l(\varphi) = a(\mu)\mu$$

and thus since $a > 0$

$$l(u) = \mu.$$

Going back to (4.6) we see that u is solution to (4.1). This completes the proof of the theorem. □

Of course many situations can then occur. Let us list few of them. First if $l(\varphi) = 0$ the only solution to (4.5) is $\mu = 0$ and thus the only solution to (4.1) is

$$u = \frac{\varphi}{a(0)}. \tag{4.7}$$

If a is a continuous function such that

$$0 < m \leq a(\xi) \quad \forall \xi \in I\!R \tag{4.8}$$

then the range of the mapping

$$\mu \to a(\mu)\mu$$

is the whole real line and by the intermediate value theorem one is always insured of the existence of a solution to (4.1). To fix the ideas let us now assume that

$$l(\varphi) > 0. \tag{4.9}$$

In this situation, any μ such that

$$a(\mu) = \frac{l(\varphi)}{\mu}$$

will produce a solution to (4.1). In other words any point at the intersection of the graph of a and the hyperbola $\mu \mapsto \frac{l(\varphi)}{\mu}$ will provide a solution to (4.1). Thus, in the case for instance of the figure below – see Figure 1 –the problem (4.1) has three solutions. One can also have a complete continuum of solutions as shown on Figure 2.

Remark 4.1 As announced in the Remark 3.2 monotonicity with respect to the initial data does not hold for this kind of problems. Indeed, consider for instance

$$\Omega = (0,1), \quad l(u) = \int_\Omega u(x)\,dx, \quad f \in V', \quad f > 0.$$

Let us asume

$$a(\xi) = \xi \tag{4.10}$$

and denote by φ the solution to (4.2) i.e. more precisely the solution to:

$$-\varphi_{xx} = f \text{ in } \Omega, \quad \varphi \in H_0^1(\Omega). \tag{4.11}$$

(Due to the maximum principle note that $\varphi > 0$). Then – see (4.5) – the solution to (4.1) is given by

$$u = \left(\int_\Omega \varphi(x)\,dx\right)^{-\frac{1}{2}} \varphi \tag{4.12}$$

Note that (2.2) does not hold for (4.10) but since

$$\int_\Omega \varphi(x)\,dx > 0,$$

one can modify $a(\cdot)$ given by (4.10) in such a way that (2.2) holds. Let v denote a nonnegative smooth function with compact support in $(0, 1/2)$. Then, consider the problem (1.5) with initial data

$$u_0^1 = u, \quad u_0^2 = u + v.$$

A monotonicity principle would imply that the solutions u^1, u^2 of (1.5) corresponding respectively to the initial data u_0^1 and u_0^2 satisfy

$$u^2(t) \geq u^1(t) = u, \quad \forall t \geq 0. \tag{4.13}$$

But on $(1/2, 1)$ one has

$$\frac{d}{dt} u^2/_{t=0} = a(l(u^2))u^2_{xx}/_{t=0} - a(l(u))u_{xx} = \Big(a(l(u+v)) - a(l(u))\Big)u_{xx} < 0.$$

It follows that for t small enough

$$u^2(t) < u \ \text{ on } (\frac{2}{3}, \frac{3}{4})$$

which contradicts (4.13).

V Asymptotic behaviour

In this section we would like to study the asymptotic behaviour of the solution to (1.5) in different cases.

5.1 A collection of results on dynamical systems

We introduce here various classical results that will be useful later on. We refer for instance to [2] for proofs and complements.

Let us denote by (X, d) a complete metric space equipped with the distance d.

Definition 5.1 *A dynamical system on X is a family of mapping $\{S(t)\}_{t\geq 0}$ from X into itself such that*

(i) $S(t) : X \to X$ *is continuous*

(ii) $S(0)x = x \ \forall\, x \in X$,

(iii) $S(t+s) = S(t) \circ S(s) \ \forall\, s, t \geq 0$,

(iv) $\forall\, x \in X$, $t \mapsto S(t)x$ *is continuous from* $[0, \infty)$ *into* X.

For any $x \in X$, the curve

$$t \mapsto S(t)x$$

is called the *trajectory issued from* x. Next, with the asymptotic behaviour in mind, we introduce the notion of *ω-limit set.* More precisely one has:

Definition 5.2 *Let $x \in X$. The set*

$$\begin{aligned} \omega(x) &:= \{\, y \in X \mid \exists\, t_n \to +\infty \text{ such that } y = \lim_{n\to+\infty} S(t_n)x \,\} \\ &= \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \{S(t)x\}}. \end{aligned} \tag{5.1}$$

is called ω-limit set of x.

Then one can show if $\{S(t)\}_{t\geq 0}$ is a dynamical system on X:

Theorem 5.1 *For any $x \in X$, $t \geq 0$ it holds*

$$\omega(S(t)x) = \omega(x), \tag{5.2}$$

$$S(t)\omega(x) \subset \omega(x). \tag{5.3}$$

Moreover, if $\bigcup_{t\geq 0}\{S(t)x\}$ is relatively compact in X it holds

$$S(t)\omega(x) = \omega(x) \neq \emptyset, \tag{5.4}$$

$$\omega(x) \text{ is compact, connected in } X, \tag{5.5}$$

$$d(S(t)x, \omega(x)) \to 0 \text{ when } t \to +\infty, \tag{5.6}$$

the distance between two sets A, B being defined as

$$d(A,B) = \inf_{a\in A,\ b\in B} d(a,b). \tag{5.7}$$

Let (X,d) be a complete metric space and $\{S(t)\}_{t\geq 0}$ a dynamical system on X.

Definition 5.3 *A continuous function $\Phi : X \to I\!R$ is called a Lyapunov function for $\{S(t)\}_{t\geq 0}$ if it holds*

$$\Phi(S(t)x) \leq \Phi(x) \quad \forall\, x \in X,\ \forall\, t \geq 0.$$

In particular for any $x \in X$ the function

$$t \mapsto \Phi(S(t)x)$$

is nonincreasing.

Then one has the following result known as the *invariance principle of LaSalle:*

Theorem 5.2 *Let $\{S(t)\}_{t\geq 0}$ be a dynamical system on X, $x \in X$ such that*

$$\bigcup_{t\geq 0}\{S(t)x\} \text{ is relatively compact in } X \tag{5.8}$$

Let Φ be a Lyapunov function for this dynamical system, then:

$$\begin{aligned} &\text{(i)} && \text{there exists } C \text{ such that } \lim_{t\to+\infty} \Phi(S(t)x) = C, \\ &\text{(ii)} && \Phi(y) = C \quad \forall\, y \in \omega(x), \\ &\text{(iii)} && \Phi(S(t)y) = \Phi(y) = C \quad \forall\, y \in \omega(x)\ \forall\, t \geq 0. \end{aligned} \tag{5.9}$$

5.2 Asymptotic analysis

In this section we suppose that we are under the assumptions of Theorem 3.1. In particular we suppose that a is locally Lipschitz continuous so that (1.5) admits a unique weak solution.

Let us start with the following continuity result:

Lemma 5.1 *Let $u_0^n \in L^2(\Omega)$ be a sequence such that when $n \to +\infty$*

$$u_0^n \rightharpoonup u_0 \quad \text{in } L^2(\Omega). \tag{5.10}$$

Let u^n, u the solution to (2.4) corresponding to the initial data u_0^n, u_0 respectively. Then it holds

$$u^n(t) \rightharpoonup u(t) \quad \forall\, t \geq 0 \quad \text{in } L^2(\Omega). \tag{5.11}$$

(One denotes by $u^n(t)$, $u(t)$ the functions $u^n(\cdot, t)$, $u(\cdot, t)$ respectively).

Proof. Since $u_0^n \rightharpoonup u_0$ in $L^2(\Omega)$, u_0^n is bounded in $L^2(\Omega)$ independently of n and from (2.12)–(2.13) one deduces that for some constant C independent of n it holds:

$$|u^n|_{L^2(0,T;V)} \leq C, \tag{5.12}$$

$$|u^n|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{5.13}$$

$$|u_t^n|_{L^2(0,T;V')} \leq C. \tag{5.14}$$

Thus, one can extract a subsequence from n – that we still label n – such that, when $n \to +\infty$:

$$\begin{array}{ll} u^n \rightharpoonup u^\infty & \text{in } L^2(0,T;V), \\ u^n \to u^\infty & \text{in } L^2(0,T;L^2(\Omega)), \\ u^n \rightharpoonup u^\infty & \text{in } L^\infty(0,T;L^2(\Omega)) \ *\text{-weak}, \\ u_t^n \rightharpoonup u_t^\infty & \text{in } L^2(0,T;V'). \end{array} \tag{5.15}$$

(We used the compactness of the embedding from $H^1(0,T;V,V')$ into $L^2(0,T;L^2(\Omega))$). By definition u^n satisfies for every $v \in V$:

$$\begin{aligned} -\int_0^T \int_\Omega u^n v\, \varphi'(t)\, dx\, dt \quad &+ \quad \int_0^T \int_\Omega a(l(u^n))(\nabla u^n \cdot \nabla v)\, \varphi(t)\, dx\, dy \\ &= \quad \int_0^T \langle f, v\rangle\, \varphi(t)\, dt, \qquad \forall\, \varphi \in \mathcal{D}(0,T). \end{aligned} \tag{5.16}$$

Clearly, from (5.15),

$$l(u^n) \to l(u^\infty) \quad \text{in } L^2(0,T)$$

and – up to a subsequence – one can assume that this convergence holds for a.e. $t \in (0,T)$. By the Lebesgue theorem one has then that

$$a(l(u^n))\varphi\nabla v \to a(l(u^\infty))\varphi\nabla v \tag{5.17}$$

in $L^2(\Omega \times (0,T)) = L^2(0,T;L^2(\Omega))$ and passing to the limit in (5.16) one gets that u^∞ satisfies for every $v \in V$:

$$\frac{d}{dt}(u^\infty, v) + a(l(u^\infty)) \int_\Omega \nabla u^\infty \cdot \nabla v \, dx = \langle f, v \rangle \quad \text{in } \mathcal{D}'(0,T).$$

Moreover, it holds for every $v \in V$:

$$(u^n(t), v) - (u_0^n, v) = \int_0^t \langle u_t^n, v \rangle \, dt \qquad \text{a.e. } t.$$

Since, up to a subsequence, one can assume that for almost every t

$$u^n(t) \to u^\infty(t) \quad \text{in } L^2(\Omega)$$

passing to the limit in the above inequality leads to

$$(u^\infty(t), v) - (u_0, v) = \int_0^t \langle u_t^\infty, v \rangle \, dt = (u^\infty(t), v) - (u^\infty(0), v).$$

Thus $u^\infty(0) = u_0$ and by uniqueness of the solution to (2.4) it follows that

$$u^\infty = u.$$

By uniqueness of the limit one obtains then that the whole sequence u^n satisfies (5.15) with $u^\infty = u$. Thus we have

$$u^n \rightharpoonup u \quad \text{in } L^\infty(0,T;L^2(\Omega)) \ *\text{-weak}$$

and in particular for every $v \in V$

$$(u^n(t), v) \rightharpoonup (u(t), v) \quad \text{in } L^\infty(0,T) \ *\text{-weak}.$$

Now, for every $t_1, t_2 \in [0,T]$, $t_2 > t_1$ it holds

$$\begin{aligned}
(u^n(t_2), v) - (u^n(t_1), v) &= \int_{t_1}^{t_2} \langle u_t^n, v \rangle \, dt \\
&\leq \int_{t_1}^{t_2} |u_t^n|_* |v|_V \\
&\leq (t_2 - t_1)^{\frac{1}{2}} |v|_V |u_t^n|_{L^2(0,T;V')} \\
&\leq C(t_2 - t_1)^{\frac{1}{2}}.
\end{aligned}$$

It follows that the sequence of functions $(u^n(t), v)$ is equicontinuous and thus relatively compact in $C([0,T])$. By uniqueness of the possible limit, one deduces that

$$(u^n(t), v) \to (u(t), v))$$

in $C([0,T])$ for every $v \in V$. V being dense in L^2 and $u^n(t)$ being bounded it follows easily that

$$(u^n(t), v) \to (u(t), v) \quad \forall\, v \in L^2(\Omega),\ \forall\, t \geq 0$$

which gives (5.11) and completes the proof of the lemma. $\square$

Let us now suppose that l is nonnegative in the sense that

$$\begin{cases} l(v) \geq 0 \quad \forall\, v \geq 0,\ v \in L^2(\Omega), \\ l \not\equiv 0\,. \end{cases} \tag{5.18}$$

For f, we recall that f is independent of t and that

$$\begin{cases} f \in V', \quad \langle f, v\rangle \geq 0 \quad \forall\, v \geq 0,\ v \in V, \\ f \not\equiv 0\,. \end{cases} \tag{5.19}$$

In short we suppose $f \geq 0$ and $f \not\equiv 0$. Going back to (4.2) it results from the maximum principle that

$$\varphi > 0 \quad \text{in } \Omega$$

and thus

$$l(\varphi) > 0\,. \tag{5.20}$$

(Note that one can write for some $g \in L^2(\Omega)$, $l(v) = \int_\Omega g(x)v(x)\,dx$).

Then we consider μ_1, μ_2 two intersection points of the graph of a with the graph of the hyperbola $y = l(\varphi)/\mu$ and we suppose that we are in one of the cases described by Figure 1 or 2.

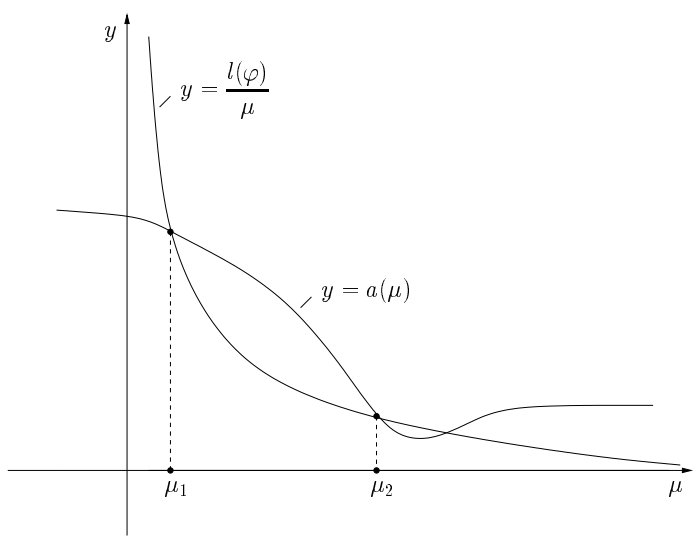

Figure 1.

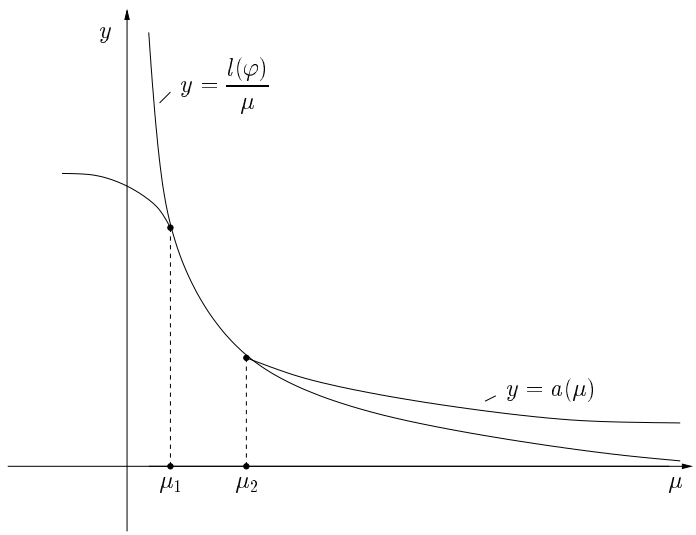

Figure 2.

In other words we assume that

$$\begin{array}{rcll} a(\mu_2) & \leq & a(\mu) \leq a(\mu_1) & \forall\, \mu \in [\mu_1, \mu_2], \\ \dfrac{l(\varphi)}{\mu} & < & a(\mu) & \forall\, \mu \in (\mu_1, \mu_2) \quad \text{in case of Figure 1}, \\ \dfrac{l(\varphi)}{\mu} & = & a(\mu) & \forall\, \mu \in [\mu_1, \mu_2] \quad \text{in case of Figure 2}. \end{array} \tag{5.21}$$

To μ_1, μ_2 correspond two stationary points (see Theorem 4.1) given by

$$\mu_1 = \frac{\varphi}{a(\mu_1)} < \mu_2 = \frac{\varphi}{a(\mu_2)}. \tag{5.22}$$

We would like now to analyze the asymptotic behaviour of u the solution to (2.4). We will restrict ourselves to the case where

$$u_1 \leq u_0 \leq u_2 \quad \text{a.e. in } \Omega. \tag{5.23}$$

First let us show:

Theorem 5.3 *Let $a(.) \geq m > 0$ be a continuous function on $\mathbb{R}_+$. Let w solution to*

$$\begin{cases} w \in C([0,T], L^2(\Omega)) \cap L^2(0,T;V), \ w_t \in L^2(0,T;V'), \\ w(0) \leq 0, \quad w(0) \not\equiv 0, \\ \dfrac{d}{dt}(w,v) + a(t) \displaystyle\int_\Omega \nabla w \cdot \nabla v \, dx \leq 0 \quad \textit{in } \mathcal{D}'(0,T), \ \forall\, v \in V, \ v \geq 0. \end{cases} \tag{5.24}$$

Then, one has:

$$w(x,t) < 0 \quad \forall t > 0, \ \textit{a.e. } x \in \Omega. \tag{5.25}$$

Proof. Let Ω' be a smooth subdomain of Ω and v be the weak solution to

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \Omega' \times (0, T^*), \\ v(0) = w(0) & \text{in } \Omega', \\ v(\cdot, t) \in H_0^1(\Omega') & \forall t \in (0, T^*). \end{cases} \tag{5.26}$$

(See [6]). We assume Ω' large enough so that $\int_{\Omega'} |v(0)|\, dx = \int_{\Omega'} |w(0)|\, dx > 0$.

It is well known (see [6], [1]) that for any $\varepsilon > 0$

$$v \in C^\infty((\varepsilon, T^*) \times \Omega').$$

Moreover,

$$v(x,t) < 0, \quad \forall\, (x,t) \in \Omega' \times (0, T^*]. \tag{5.27}$$

To see this last point one notices first that by the weak maximum principle:

$$v(x,t) \leq 0 \quad \text{a.e. in } \Omega,\ \forall t \geq 0.$$

Moreover, for ε small enough

$$|v(\cdot, \varepsilon)|_{2,\Omega'} > 0.$$

This is due to the fact that $v(0) \not\equiv 0$ and $v \in C([0, T^*], L^2(\Omega'))$. The usual maximum principle applied to the domain $\Omega' \times [\varepsilon, T^*]$ implies then (5.27) – see [6]. Next, by the weak maximum principle again, one has:

$$w(t) \leq 0 \quad \text{a.e. in } \Omega,\ \forall t \geq 0.$$

Moreover, setting

$$\tilde{v}(\cdot, t) = v\Big(\cdot, \int_0^t a(s)\, ds\Big) \tag{5.28}$$

one has in a weak sense

$$\tilde{v}_t = v_t \cdot a(t) = a(t)\Delta v = a(t)\Delta \tilde{v},$$

with

$$\tilde{v}(0) = v(0) = w(0)_{/\Omega'}.$$

(The first above equation holds in $\Omega' \times (0, T)$ with $\int_0^T a(s)\, ds = T^*$ but T^* and thus T can be chosen arbitrarily). Then, the weak maximum principle leads to

$$w \leq \tilde{v} \quad \text{a.e. in } \Omega', \quad \forall t \in [0, T]$$

from which it follows (recall that Ω' is arbitrary and $a(.) \geq m > 0$) that (5.25) holds due to (5.27)-(5.28). This completes the proof of the theorem. □

Then we turn to the following stability result that holds true when one is under the assumptions described by Figure 1 or 2. Thus we have:

Theorem 5.4 *Under the assumptions above, in particular if (5.18), (5.19), (5.21), (5.23) hold then, if u denotes the weak solution to (2.4), it holds:*

$$u_1 \leq u(t) = u(\cdot, t) \leq u_2 \quad a.e. \ in \ \Omega, \ \forall t \geq 0. \tag{5.29}$$

Proof. Let us assume first that

$$u_1 < u_0 < u_2 \quad \text{a.e. in } \Omega. \tag{5.30}$$

Then, due to (5.18), one has

$$\mu_1 = l(u_1) < l(u_0) < l(u_2) = \mu_2. \tag{5.31}$$

Let us then denote by t_* ($t_* > 0$ by (5.31)) the supremum

$$t^* = \sup\{\, t > 0 \mid l(u(\cdot, s)) \in [\mu_1, \mu_2] \ \ \forall\, s \in [0, t]\,\}. \tag{5.32}$$

We claim that $t^* = +\infty$. Indeed, if not, then one has

$$l(u(\cdot, t^*)) = \mu_1 \quad \text{or} \quad \mu_2$$

(recall that $u \in C([0, +\infty); L^2(\Omega))$). Suppose for instance that

$$l(u(\cdot, t^*)) = \mu_2 \tag{5.33}$$

(the proof in the case of equality with μ_1 would be the same). Then, for almost every t, it holds in $L^2(0, t^*; V')$

$$\begin{aligned} \frac{du}{dt} - a(l(u))\Delta u = f &= -a(\mu_2)\Delta u_2 \\ \iff \frac{d}{dt}(u - u_2) - a(l(u))\Delta(u - u_2) &= (a(\mu_2) - a(l(u)))(-\Delta u_2) \\ \iff \frac{d}{dt}(u - u_2) - a(l(u))\Delta(u - u_2) &= \frac{(a(\mu_2) - a(l(u))}{a(\mu_2)} \cdot f. \end{aligned}$$

Since $l(u) \in [\mu_1, \mu_2]$, by (5.19), (5.21) one deduces that

$$\frac{d}{dt}(u - u_2) - a(l(u))\Delta(u - u_2) \leq 0 \quad \text{in } V'$$

for a.e. $t \in (0, t^*)$. Setting $w = u - u_2$ it comes

$$\frac{dw}{dt} - a(l(u))\Delta w \leq 0, \qquad w(0) = u_0 - u_2 < 0. \tag{5.34}$$

By Theorem 5.3 one deduces that

$$w(t^*) < 0$$

and a contradiction to (5.33). One has thus $t^* = +\infty$ and by the above argument

$$u_1 < u(t) < u_2 \quad \text{a.e. in } \Omega, \ \forall\, t \geq 0. \tag{5.35}$$

If now u_0 satisfies

$$\frac{\varphi}{a(\mu_1)} = u_1 \leq u_0 \leq u_2 = \frac{\varphi}{a(\mu_2)}$$

then, for $n \to +\infty$, it holds

$$u_1 < u_0^n = \left(u_1 + \frac{\varphi}{n}\right) \wedge u_0 \vee \left(u_2 - \frac{\varphi}{n}\right) < u_2$$

($\vee$ denotes the maximum, $\wedge$ the minimum of two functions). Thus, if u^n is the solution to (2.4) corresponding to u_0^n, one has

$$u^n(t) \in X = \{\, v \in L^2(\Omega) \mid u_1 \leq v \leq u_2 \text{ a.e. in } \Omega \,\} \tag{5.36}$$

for any t. X convex, closed in $L^2(\Omega)$ is also weakly closed and by (5.11), $u(t) \in X \ \forall\, t > 0$. This completes the proof of the theorem. □

In order to study the asymptotic behaviour of the solution of (2.4), we will rely on the theory of dynamical systems. In what follows we assume that we are under the assumptions described in Figure 1 or 2. Moreover, X denotes the set given by (5.36). It is clear that X is a closed, bounded, convex set of $L^2(\Omega)$. One can equip it with the weak topology. This topology is metrizable (see for instance [7]) for some distance d for which X is a complete and compact space. For $u_0 \in X$, one defines then

$$S(t)u_0 = u(t) \tag{5.37}$$

where $u(t) = u(\cdot, t)$ denotes the solution to (2.4). One has:

Theorem 5.5 *$\{S(t)\}_{t\geq 0}$ defined by (5.37) is a dynamical system on X.*

Proof. First the fact that $S(t)$ maps X into itself follows from Theorem 5.4. The point (i) of Definition 5.1 follows from Lemma 5.1. The points (ii), (iii) are easy to check. Finally the last point follows from the fact that

$$u \in C([0,T], L^2(\Omega)).$$

This completes the proof of the theorem. □

Let us now find out a Lyapunov function for $\{S(t)\}_{t\geq 0}$. For this purpose we introduce Φ the weak solution to

$$\begin{cases} -\Delta\Phi = l & \text{in } \Omega, \\ \Phi \in V. \end{cases} \tag{5.38}$$

Since $l \in L^2(\Omega) \subset V'$ it is clear that (5.38) admits a unique solution. Moreover due to (5.18) one has

$$\Phi > 0 \quad \text{a.e. in } \Omega. \tag{5.39}$$

If now one takes $v = \Phi$ in the last equation of (2.4) it comes

$$\frac{d}{dt}(\Phi, u) + a(l(u)) \int_\Omega \nabla u \cdot \nabla \Phi \, dx = \langle f, \Phi \rangle = \langle -\Delta \varphi, \Phi \rangle.$$

Then, (4.2) and (5.38) imply that it holds:

$$\frac{d}{dt}(\Phi, u) = l(\varphi) - a(l(u))l(u)\,, \qquad \text{for a.e. } t \in (0, +\infty)\,. \tag{5.40}$$

If we choose $u_0 \in X$ it is clear that by Theorem 5.4 one has

$$\frac{d}{dt}(\Phi, u) \leq 0 \tag{5.41}$$

in the case of the Figure 1 and, in the case of Figure 2,

$$\frac{d}{dt}(\Phi, u) = 0. \tag{5.42}$$

Thus $u \mapsto (\Phi, u)$ is a Lyapunov function on X.

Let us now deduce from all of this the asymptotic behaviour of u solution to (2.4). Let us assume first that we are in the case of the Figure 1. Then one has:

Theorem 5.6 *Under the assumptions described in Figure 1, in particular if (5.21) holds, then, for $u_0 \in X$, $u_0 \neq u_2$, it holds:*

$$u(t) = S(t)u_0 \to u_1 \tag{5.43}$$

in $L^2(\Omega)$ when $t \to +\infty$.

Proof. Let us denote by $\omega(u_0)$ the ω-limit set of u_0. By (5.40) and Theorem 5.4 one has (see (5.38) for the definition of Φ)

$$\frac{d}{dt}(\Phi, u) = l(\varphi) - a(l(u))l(u) \leq 0. \tag{5.44}$$

Thus the scalar product with Φ is a Lyapunov function for our problem. Clearly, by (5.29), one has:

$$(\Phi, u(t)) \geq (\Phi, u_1). \tag{5.45}$$

Moreover, since by (5.44) the function (Φ, u) is nonincreasing, there exists C such that

$$\lim_{t \to +\infty} (\Phi, u(t)) = C = (\Phi, w) \tag{5.46}$$

for any $w \in \omega(u_0)$ – (see Theorem 5.2).

Going back to (5.44) for any $w \in \omega(u_0)$ it holds by (5.9)

$$\frac{d}{dt}(\Phi, S(t)w) = 0 = l(\varphi) - a(l(S(t)w))l(S(t)w)$$

i.e. one has due to the continuity of the map $t \mapsto l(S(t)w)$

$$l(S(t)w) = \mu_1 \quad \forall\, t \quad \text{or} \quad l(S(t)w) = \mu_2 \quad \forall\, t.$$

This implies that

$$l(w) = \mu_1 \quad \text{or} \quad \mu_2 \quad \forall\, w \in \omega(u_0).$$

Set

$$\omega_i = \{\, w \in \omega(u_0) \mid l(w) = \mu_i \,\} \quad i = 1, 2,$$

w_i are two disjoint closed subsets of X such that

$$\omega(u_0) = \omega_1 \cup \omega_2.$$

It follows – see (5.5) – that

$$\omega(u_0) = \omega_1 \quad \text{or} \quad \omega_2$$

i.e. if u denotes the solution to (2.4) one has as $t \to +\infty$

$$l(u(t)) \to \mu_1 \quad \text{or} \quad l(u(t)) \to \mu_2.$$

We can then show the following lemma:

Lemma 5.2 *Let u be the weak solution to (2.4). If*

$$l(u(t)) \to \mu_i \quad (i = 1, 2) \tag{5.47}$$

when $t \to +\infty$ then

$$u(t) \to u_i \quad (i = 1, 2) \tag{5.48}$$

in $L^2(\Omega)$ strong.

Thus, if $l(u(t)) \to \mu_2$, one has $u(t) \to u_2$. Since $(\Phi, u(t))$ is nonincreasing and $\Phi > 0$, this implies that

$$(\Phi, u(t)) = (\Phi, u_2) \quad \forall\, t.$$

Hence, by (5.39), $u(t) = u_2\ \forall\, t$ which contradicts $u_0 \neq u_2$. One has so:

$$l(u(t)) \to \mu_1$$

and by Lemma 5.2 the result follows. □

Proof of Lemma 5.2 ¿From (1.5) and the definition of u_i one has for a.e. $t \geq 0$, and ¿in V',

$$\begin{aligned} \frac{d}{dt}(u - u_i) - a(l(u))\Delta u &= -a(\mu_1)\Delta u_i \\ \iff \frac{d}{dt}(u - u_i) - a(l(u))\Delta(u - u_i) &= -(a(\mu_i) - a(l(u)))\Delta u_i. \end{aligned}$$

Taking the V', V-duality bracket with $u - u_i$ and setting

$$\varepsilon(t) = |a(\mu_i) - a(l(u))|$$

we get by (2.2):

$$\frac{1}{2}\frac{d}{dt}|u - u_i|_2^2 + m\int_\Omega |\nabla(u - u_i)|^2\,dx \le \varepsilon(t)\int_\Omega |\nabla u_i||\nabla(u - u_i)|\,dx. \tag{5.49}$$

By the Young inequality one obtains then for any $\delta > 0$

$$\int_\Omega |\nabla u_i||\nabla(u - u_i)|\,dx \le \frac{\delta}{2}\int_\Omega |\nabla(u - u_i)|^2\,dx + \frac{1}{2\delta}\int_\Omega |\nabla u_i|^2\,dx. \tag{5.50}$$

Since ε is continuous and bounded one can choose δ such that

$$\delta|\varepsilon|_\infty = m$$

where $|\varepsilon|_\infty = \sup_{t\ge 0}\varepsilon(t)$. Then, the conjunction of (5.49), (5.50) leads to

$$\frac{1}{2}\frac{d}{dt}|u - u_i|_2^2 + \frac{m}{2}\int_\Omega |\nabla(u - u_i)|^2\,dx \le C\varepsilon(t)$$

for some constant C independent of t. So, one obtains for some constant c

$$\frac{d}{dt}|u - u_i|_2^2 + c|u - u_i|_2^2 \le C\varepsilon(t) \tag{5.51}$$

(this is due to the Poincaré inequality).

Setting

$$y(t) = |u - u_i|_2^2$$

the above inequality reads

$$\iff \begin{array}{rl} y'(t) + cy(t) & \le C\varepsilon(t) \\ (e^{ct}y(t))' & \le C\varepsilon(t)e^{ct}. \end{array}$$

Integrating between t_0 and t, it comes

$$\iff \begin{array}{rcl} e^{ct}y(t) - e^{ct_0}y(t_0) & \le & C\displaystyle\int_{t_0}^t \varepsilon(s)e^{cs}\,ds \\ y(t) & \le & y(t_0)e^{c(t_0-t)} + C\displaystyle\int_{t_0}^t \varepsilon(s)e^{c(s-t)}\,ds. \end{array}$$

$\varepsilon_0 > 0$ being given one chooses t_0 large enough such that

$$\varepsilon(s) \le \frac{c}{C}\frac{\varepsilon_0}{2} \quad \forall\, s \ge t_0.$$

Then, for $t > t_0$, it holds:

$$\begin{aligned} y(t) &\leq y(t_0)e^{c(t_0-t)} + c\tfrac{\varepsilon_0}{2}\int_{t_0}^{t} e^{c(s-t)}\,ds \\ &\leq y(t_0)e^{c(t_0-t)} + c\tfrac{\varepsilon_0}{2}\left.\frac{e^{c(s-t)}}{c}\right|_{t_0}^{t} \\ &\leq y(t_0)e^{c(t_0-t)} + \tfrac{\varepsilon_0}{2}. \end{aligned}$$

Choosing now t large enough one gets clearly

$$y(t) \leq \varepsilon_0$$

which completes the proof of the lemma. □

Let us now assume that we are under the assumptions of Figure 2. For any $\mu \in [\mu_1, \mu_2]$

$$\frac{\varphi}{a(\mu)} \tag{5.52}$$

is a stationary point. Let us consider $u_0 \in X$. Then, by Theorem 5.4, one has

$$u(t) \in X, \quad \forall\, t \in (0, +\infty)$$

and by (5.40)

$$\frac{d}{dt}(\Phi, u) = 0 \quad \Leftrightarrow \quad (\Phi, u(t)) = (\Phi, u_0) \quad \forall\, t \in (0, +\infty). \tag{5.53}$$

¿From (5.53) it is clear that a natural candidate for the limit of $u(t)$ ¿is given by (5.52) with $a(\mu)$ such that

$$\frac{(\Phi, \varphi)}{a(\mu)} = (\Phi, u_0)$$

i.e. a natural limit of $u(t)$ is

$$u_\infty = \frac{(\Phi, u_0)}{(\Phi, \varphi)}\,\varphi. \tag{5.54}$$

This is what we would like to establish now.

First, we remark that since $\omega(u_0)$ is compact in X equipped with the weak topology, there exists $w_0 \in \omega(u_0)$ such that

$$a(l(w_0)) = \min_{w \in \omega(u_0)} a(l(w)). \tag{5.55}$$

(Recall that a and l are continuous).

Let us set

$$u^* = \frac{\varphi}{a(l(w_0))}. \tag{5.56}$$

Then we have:

Lemma 5.3 *For any $w \in \omega(u_0)$ it holds:*

$$w \leq u^* \quad a.e. \text{ in } \Omega. \tag{5.57}$$

Proof. Let $w \in \omega(u_0)$, $v(t) = S(t)w$. One has in $L^2(0,T;V')$ – recall that by (5.56) $-a(l(w_0))\Delta u^* = -\Delta\varphi = f$:

$$\frac{d}{dt}(v - u^*) - a(l(v))\Delta(v - u^*) = -(a(l(w_0)) - a(l(v))\Delta u^*.$$

Taking the V', V-duality bracket with $(v - u^*)^+$ it comes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(v-u^*)^+|_2^2 + a(l(v))\int_\Omega |\nabla(v-u^*)^+|^2\,dx &= \Big(a(l(w_0)) - a(l(v))\Big)\langle -\Delta u^*, (v-u^*)^+\rangle \\ &\leq 0.\end{aligned}$$

This is indeed due to the fact that $v \in \omega(u_0)$, (5.55), $-\Delta u^* \geq 0$. So, for some constant c it comes

$$\begin{aligned} & \frac{d}{dt}|(v-u^*)^+|_2^2 + c|(v-u^*)^+|_2^2 \leq 0 \\ \Longleftrightarrow \quad & \frac{d}{dt}\{e^{ct}|(v-u^*)^+|_2^2\} \leq 0 \\ \Longrightarrow \quad & |(v-u^*)^+|_2^2 \leq e^{-ct}|(v(0)-u^*)^+|_2^2, \end{aligned}$$

$$\text{i.e.} \qquad \begin{aligned} |(S(t)w-u^*)^+|_2^2 &\leq e^{-ct}|(w-u^*)^+|_2^2 \\ &\leq e^{-ct}\sup_{w\in\omega(u_0)}|(w-u^*)^+|_2^2 \\ &\leq Ke^{-ct}. \end{aligned}$$

Due to (5.4) any $z \in \omega(u_0)$ can be written under the form

$$z = S(t)w$$

for t arbitrary. It follows that $(z - u^*)^+ = 0\ \forall\, z \in \omega(u_0)$ – which is (5.57). □

We can now show:

Theorem 5.7 *Under the assumptions described in Figure 2, in particular if*

$$a(\mu) = \frac{l(\varphi)}{\mu} \quad \forall\, \mu \in [\mu_1, \mu_2], \tag{5.58}$$

then for $u_0 \in X$ it holds in $L^2(\Omega)$

$$u(t) \to u_\infty \tag{5.59}$$

where u_∞ is given by (5.54).

Proof. We claim first that

$$u^* \leq u_\infty \tag{5.60}$$

where u_∞ is given by (5.54). Indeed, if not – since both functions are multiple of $\varphi > 0$ – it holds

$$u^* > u_\infty.$$

Let $w \in \omega(u_0)$. Set $w(t) = S(t)w$. By (5.54), (5.9) one has then

$$(\Phi, w) = (\Phi, u_0) = (\Phi, u_\infty) < (\Phi, u^*). \tag{5.61}$$

Moreover $v = w - u^*$ satisfies

$$\begin{aligned} \frac{d}{dt}(w - u^*) = \frac{dw}{dt} &= a(l(w))\Delta w + f \\ &= a(l(w))\Delta(w - u^*) + (a(l(w)) - a(l(w_0))\Delta u^* \end{aligned}$$

weakly. But it holds (see (5.55))

$$a(l(w)) \geq a(l(w_0)).$$

Thus $v = w - u^*$ satisfies

$$\frac{dv}{dt} - a(l(w))\Delta v \leq 0, \qquad v(0) = w - u^* \leq 0, \qquad v(0) \not\equiv 0.$$

($v(0) \not\equiv 0$ by (5.61)). Using Theorem 5.3, it follows that

$$w(t) - u^* < 0 \quad \forall\, t > 0$$

this for any $w \in \omega(u_0)$. Since $S(t)\omega(u_0) = \omega(u_0)$, it follows that

$$l(w) < l(u^*), \quad \forall w \in \omega(u_0).$$

This contradicts

$$l(u^*) = \frac{l(\varphi)}{a(l(w_0))} = l(w_0),$$

and proves (5.60). Thus for any $w \in \omega(u_0)$ it holds

$$w \leq u^* \leq u_\infty.$$

Since $(\Phi, w) = (\Phi, u_\infty)$ this imposes $w = u_\infty$ $\forall\, w \in \omega(u_0)$ – i.e.

$$\omega(u_0) = \{u_\infty\}.$$

Thus $u(t)$ the solution to (2.4) satisfies $u(t) \rightharpoonup u_\infty$ in $L^2(\Omega)$ and thus $a(l(u(t)) \to a(l(u_\infty))$. The result follows then from Lemma 5.2. □

VI Some complements

It is of course easy to obtain an analogous result to Theorem 5.6 when $a(.)$ is located below the hyperbola (see [5] for the type of results available). To end this note we would like to address a case where the assumption

$$a(\mu_2) \leq a(\mu) \leq a(\mu_1) \tag{6.1}$$

of (5.21) fails. So, we are going to assume that we are in the situation of Figure 3.

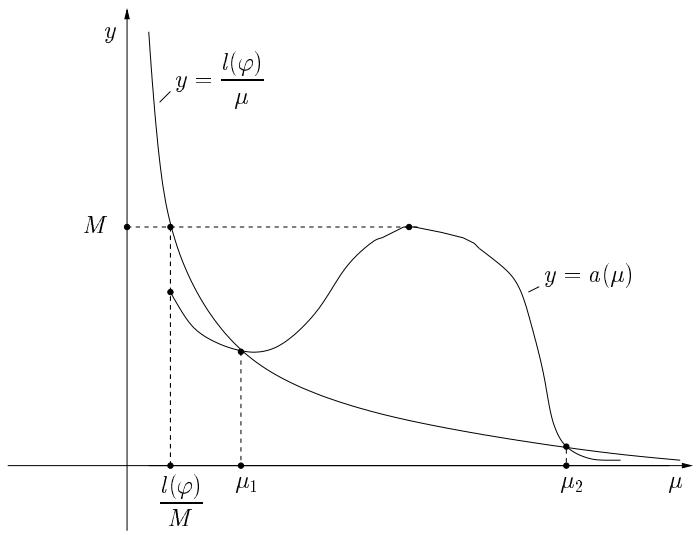

Figure 3.

In other words we set

$$M = \max_{\mu \in [\mu_1, \mu_2]} a(\mu) \tag{6.2}$$

and we assume that

$$M > a(\mu_1) \tag{6.3}$$

Moreover we suppose – under the assumptions of the preceding section – that

$$a(\mu_1) \leq a(\mu) < \frac{l(\varphi)}{\mu}, \quad \forall \mu \in \left[\frac{l(\varphi)}{M}, \mu_1\right[. \tag{6.4}$$

Then we can show

Theorem 6.1 *Under the assumptions of the preceding section and in the situation described by Figure 3 – i.e. if (6.2)-(6.4) hold – then for*

$$\frac{\varphi}{M} \leq u_0 \leq u_2, \quad u_0 \neq u_2 \tag{6.5}$$

and if u denotes the solution to (2.4) one has

$$\lim_{t\to+\infty} u(t) = u_1 \tag{6.6}$$

in $L^2(\Omega)$.

Proof. First proceeding exactly as in Theorem 5.4 one shows that

$$\frac{\varphi}{M} \le u(t) \le u_2, \quad \forall t \ge 0. \tag{6.7}$$

$\Big($ for the left-hand side inequality one uses the fact that

$$M \ge a(\mu), \quad \forall \mu \in \Big[\frac{l(\varphi)}{M}, \mu_2\Big] \tag{6.8}$$

i.e. setting $\varphi^* = \dfrac{\varphi}{M}$ one proceeds as in below (5.38) but using

$$\begin{aligned}\frac{d}{dt}(u-\varphi^*) - a(l(u))\Delta(u-\varphi^*) &= f + a(l(u))\Delta\varphi^* \\ &= -\Delta\varphi\Big(1 - \frac{a(l(u))}{M}\Big) \ge 0\Big).\end{aligned}$$

Next we introduce $w_0 \in \omega(u_0)$ as in (5.55) and u^* as in (5.56) (of course X is now the set $X := \{v \in L^2(\Omega) \,/\, \frac{\varphi}{M} \le v \le u_2\}$). Note that $u \mapsto (\Phi, u)$ is no more a priori a Lyapunov function here. We separate two cases:

<u>*Case (i):*</u> $a(l(w_0)) \ge a(l(u_1))$.

Then it holds

$$u^* = \frac{\varphi}{a(l(w_0))} \le \frac{\varphi}{a(l(u_1))} = u_1. \tag{6.9}$$

Due to Lemma 5.3 that holds without any change, one has

$$\frac{\varphi}{M} \le w \le u^* \le u_1, \quad \forall w \in \omega(u_0). \tag{6.10}$$

Thus for any $w \in \omega(u_0)$ it holds for $w(t) = S(t)w$

$$\frac{d}{dt}(w(t), \Phi) = l(\varphi) - a(l(w(t))\, l(w(t)) \ge 0, \quad \forall t \ge 0. \tag{6.11}$$

Let us introduce the set

$$\mathcal{A} := \{z \in \omega(u_0) \mid (\Phi, z) = \inf_{w\in\omega(u_0)} (w, \Phi) = c_* \}$$

which is nonempty since $\omega(u_0)$ is weakly compact in $L^2(\Omega)$. By (6.11), for any $z \in \omega(u_0)/\mathcal{A}$ one has

$$(\Phi, S(t)z) \ge (\Phi, z) > c_*, \quad t \ge 0. \tag{6.12}$$

Next we claim that $u_1 \in \mathcal{A}$. Indeed, otherwise by (6.10) and Theorem 5.3, then by (6.4) and (6.11), we would have for any $z \in \mathcal{A}$ and any $t > 0$

$$\begin{aligned} S(t)z < u_1 &\Longrightarrow l(S(t)z) < l(u_1) \\ &\Longrightarrow a(l(S(t)z))\, l(S(t)z) < l(\varphi) \\ &\Longrightarrow (\Phi, S(t)z) > (\Phi, z) = c_*. \end{aligned} \tag{6.13}$$

But (6.12)-(6.13) contradict $S(t)\omega(u_0) = \omega(u_0)$. Hence $u_1 \in \mathcal{A}$. So, for any $z \in \omega(u_0)$ one has $z \leq u_1$ and $(\Phi, z) \geq (\Phi, u_1)$ which clearly imply $z = u_1$, i.e. $\omega(u_0) = \{u_1\}$ and (6.6) then follows from Theorem 5.1 and Lemma 5.2.

Case (ii): $a(l(w_0)) < a(l(u_1))$.

Due to Lemma 5.3 one has

$$w_0 \leq u^* = \frac{\varphi}{a(l(w_0))}.$$

This leads to

$$a(l(w_0)) \leq \frac{l(\varphi)}{l(w_0)}$$

(recall that by (6.7), $l(w_0) > 0$).
Since we have assumed

$$a(l(w_0)) < a(l(u_1))$$

the only possibility is

$$l(w_0) = l(u_2) = \mu_2$$

(see Figure 3).
But then $u_2 \in \omega(u_0)$. Indeed if not, for any $w \in \omega(u_0)$ one would have $w \leq u_2$ and $w \neq u_2$. Thus, for any positive t

$$S(t)w < u_2$$

which implies $l(S(t)w) < l(u_2)$ and contradicts $S(t)\omega(u_0) = \omega(u_0)$.

To conclude, we now show that in fact u_2 cannot belong to $\omega(u_0)$ and thus *case (ii)* is impossible. To do this we introduce $\mathcal{O}(u_0)$ the weak closure of the trajectory of u_0 in $L^2(\Omega)$. On account of (5.40) and Figure 3, $t \mapsto (\Phi, S(t)u_0)$ is non increasing in a neighbourhood of t_1 whenever $l(S(t_1)u_0) \in]\mu_1, \mu_2]$ and therefore

$$\sup_{w \in \mathcal{O}(u_0)} (\Phi, w) = \max\Big((\Phi, u_0), \sup_{\substack{z \in \mathcal{O}(u_0) \\ l(z) \leq \mu_1}} (\Phi, z)\Big). \tag{6.14}$$

But the set $\mathcal{O}(u_0) \cap \{z \mid l(z) \leq \mu_1\}$ is weakly compact in $L^2(\Omega)$, thus there exists $z_* \in \mathcal{O}(u_0) \cap \{z / l(z) \leq \mu_1\}$ such that

$$(\Phi, z_*) = \sup_{\substack{z \in \mathcal{O}(u_0) \\ l(z) \leq \mu_1}} (\Phi, z)\,.$$

Since $l(z_*) \leq \mu_1 < l(u_2)$, one has $z_* \leq u_2$ and $z_* \neq u_2$. Hence $(\Phi, z_*) < (\Phi, u_2)$. It then follows from (6.14) that

$$\sup_{w \in \mathcal{O}(u_0)} (\Phi, w) < (\Phi, u_2)$$

and therefore

$$u_2 \notin \mathcal{O}(u_0) \supset \omega(u_0).$$

□

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