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The influence of nonlocal nonlinearities on the long time behavior of solutions of diffusion problems

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Abstract

In this paper we study a nonlocal diffusion problem. The existence is proved using the Schauder fixed point theorem. The convergence of the solution towards a steady state is investigated by using the dynamical systems point of view.

Keywords: Nonlocal diffusion problems; Global existence; Asymptotic behavior

1. Introduction

In this paper, we consider a nonlocal diffusion problem. The main questions we here address are the global existence, uniqueness of a solution, and the convergence towards a steady state.

As typical examples of parabolic equations with nonlocal nonlinearities, let us mention the following:

- Equations with space integral term, of the form

$$u_t - \Delta u = g\left(\int_\Omega f(u(t,y))\, dy \right). \tag{1.1}$$

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Some problems involving both local and nonlocal terms, of the type

$$u_t - \Delta u = \int_\Omega f(u(t,y))\,dy + h(u(t,x)). \tag{1.2}$$

- Equations with localized source, of the form

$$u_t - \Delta u = f(u(t, x_0(t))). \tag{1.3}$$

- Equations with space–time integral, of the form

$$u_t - \Delta u = f\left(\int_0^t \int_\Omega \beta(y) g(u(s,y))\,dy\,ds\right). \tag{1.4}$$

  Each equation is considered in a bounded domain with homogeneous Dirichlet boundary conditions.

Problems of these types arise in various models in physics and engineering have been studied by a number of authors. To cite just a few, problems of type (1.1) or (1.2) are related to some ignition models for compressible reactive gases. For problems of these types and some of their variants, the blow-up of solutions was studied, among others, by Bebernes et al. [1], Deng et al. [5], Chadam et al. [3], Wang and Wang [11].

Eq. (1.3) describes physical phenomena where the reaction is driven by the temperature at a single site. This equation was studied by Cannon and Yin [2], Chadam et al. [3], Wang and Chen [10], in the case $x_0(t) = \text{Const.}$, and by Souplet [8] for variable $x_0(t)$.

Last, problems of type (1.4) play an important role in the theory of nuclear reactor dynamics. The blow-up of solutions was studied by Souplet [8], Pao [7] and Guo and Su [6].

Recently, Souplet [9] determined the rate and profile of blow-up of solutions for large classes of nonlocal problems of each type above. He proved that the solutions have global blow-up, and that the blow-up rate is uniform in all compact subsets of the domain.

In any diffusion process, the diffusion velocity $\vec{v}$ is given at the point x by the Fourier law $\vec{v}(x) = -a\nabla u(x)$ where u is the temperature and a is a constant depending on the medium where the process is taking place. The assumption a is constant is, in fact, a first approximation of the reality. For instance, in material science it is clear that physical constants attached to a material will depend on its state, its temperature for example. In this paper, we would like to address the case where the constant a depends on nonlocal quantities. Thus, a could depend on

$$g(u) = \int_\Omega u(x)\,dx.$$

So, let Ω be a connected bounded Lipschitz open set of $\mathbb{R}^n$. We denote by Γ the boundary of Ω and by $\{\Gamma_0, \Gamma_1\}$ a partition of it. Set

$$V = H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_0\},$$

and $g : V \to \mathbb{R},\ f \in V'$, where V' denotes the strong dual of V.

Consider the parabolic problem

$$\begin{cases} u_t - a(g(u))\Delta u = f & \text{in } \Omega \times (0, T), \\ u(\cdot, t) \in V, \ \ t \in (0, T), \\ u(\cdot, 0) = u_0 \in L^2(\Omega). \end{cases} \tag{1.5}$$

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of a solution to (1.5). In Section 3, we introduce the steady-state problem, and in Section 4 we study the convergence of the solution towards a steady state.

2. Existence and uniqueness

Without loss of generality, we can assume that V is equipped with the norm

$$||u||_V^2 = \int_\Omega |\nabla u|^2\, dx$$

and we denote by $\langle \cdot, \cdot \rangle$ the $V' - V$ duality bracket.

The existence result reads as follows:

Theorem 2.1. *Assume that $a : \mathbb{R} \to \mathbb{R}_+$ is continuous, $0 < a_0 \leqslant a(x) \leqslant a_1$ for all $x \in \mathbb{R}$ and $g : L^2(\Omega) \to \mathbb{R}$ is continuous. Then for any $f \in L^2(0, T; V')$ and $u_0 \in L^2(\Omega)$ there exists u with*

$$u \in L^2(0, T; V) \cap C([0, T], L^2(\Omega)), \quad u_t \in L^2(0, T; V')$$

solution to

$$\begin{cases} \frac{d}{dt}(u, v) + a(g(u)) \int_\Omega \nabla u \cdot \nabla v\, dx = \langle f, v \rangle & \text{in } \mathscr{D}'(0, T), \quad \forall v \in V, \\ u(0) = u_0. \end{cases} \tag{2.1}$$

Proof. Let $w \in L^2(0, T; L^2(\Omega))$, then thanks to Dautray–Lions [4], the problem

$$\begin{cases} u \in L^2(0, T; V) \cap C([0, T], L^2(\Omega)), \quad u_t \in L^2(0, T; V'), \\ \frac{d}{dt}(u, v) + a(g(w)) \int_\Omega \nabla u \cdot \nabla v\, dx = \langle f, v \rangle \quad \text{in } \mathscr{D}'(0, T), \quad \forall v \in V, \\ u(0) = u_0 \end{cases} \tag{2.2}$$

has a unique solution. Indeed, the mapping $t \mapsto g(w(\cdot, t))$ is measurable, and hence the mapping $t \mapsto a(g(w(\cdot, t))) = a(g(w))$ is also measurable and belongs to $L^\infty(0, T)$.

Whence, to prove that (2.1) has a solution, it suffices to prove that the mapping

$$h : L^2(0, T; L^2(\Omega)) \to L^2(0, T; L^2(\Omega)),$$

$$w \mapsto h(w) = u$$

has a fixed point. This will be done with the help of the Schauder fixed point theorem.

From (2.2) we have for $v = u$

$$\left\langle \frac{du}{dt}, u \right\rangle + a(g(w)) \int_\Omega |\nabla u|^2 \, dx = \langle f, u \rangle, \quad t \in (0, T)$$

and consequently

$$\frac{1}{2} \frac{d}{dt} ||u||^2 + a_0 ||u||_V^2 \leqslant ||f||_{V'} ||u||_V, \quad t \in (0, T). \tag{2.3}$$

Now, since

$$||f||_{V'} ||u||_V \leqslant \frac{1}{2a_0} ||f||_{V'}^2 + \frac{a_0}{2} ||u||_V^2,$$

we deduce from (2.3) that

$$\frac{1}{2} \frac{d}{dt} ||u||^2 + \frac{a_0}{2} ||u||_V^2 \leqslant \frac{1}{2a_0} ||f||_{V'}^2, \quad t \in (0, T).$$

Integration over $(0, T)$ yields

$$\frac{1}{2} ||u||^2 + \frac{a_0}{2} \int_0^T ||u||_V^2 \, dt \leqslant \frac{1}{2} ||u_0||^2 + \frac{1}{2a_0} \int_0^T ||f||_{V'}^2 \, dt,$$

and hence

$$||u||_{L^2(0,T;V)}, \quad ||u||_{L^2(0,T;L^2(\Omega))} \leqslant c_1, \tag{2.4}$$

where $c_1 > 0$ is a positive constant independent of w.

Since $u_t - a(g(w))\Delta u = f$, we deduce the existence of a positive constant $c_2 > 0$ independent of w such that

$$||u_t||_{L^2(0,T;V')} \leqslant c_2. \tag{2.5}$$

Hence, h maps B into itself and $h(B)$ is relatively compact in B, where we set

$$B = \{ w \in L^2(0, T; L^2(\Omega)) \colon ||u||_{L^2(0,T;L^2(\Omega))} \leqslant c_1 \}.$$

To apply the Schauder fixed point theorem, we just need to show that h is continuous. Let w_n be a sequence such that

$$w_n \to w \quad \text{in } B \text{ or } L^2(0,T;L^2(\Omega))$$

and $u_n = h(w_n)$.

For a subsequence, still denoted by the same symbol, we have

$$w_n(\cdot,t) \to w(\cdot,t) \quad \text{in } L^2(\Omega), \quad t \in (0,T)$$

and also

$$a(g(w_n)) \to a(g(w)), \quad t \in (0,T). \tag{2.6}$$

As a consequence of (2.4) and (2.5), there exists $\tilde{u}$ such that

$$\begin{cases} u_n \rightharpoonup \tilde{u} & \text{in } L^2(0,T;V), \\ u_n \to \tilde{u} & \text{in } L^2(0,T;L^2(\Omega)), \\ (u_n)_t \rightharpoonup (\tilde{u})_t & \text{in } L^2(0,T;V'). \end{cases} \tag{2.7}$$

For every $v \in V, \varphi \in \mathscr{D}(0,T)$, it holds that

$$-\int_0^T \int_\Omega u_n v \varphi' \, dx\, dt + \int_0^T \int_\Omega a(g(w_n)) \nabla u_n \nabla v \varphi \, dx\, dt = \int_0^T \langle f, v \rangle \varphi \, dt. \tag{2.8}$$

Thanks to (2.6) and to the Lebesgue dominated convergence theorem, we have

$$a(g(w_n))\varphi v \to a(g(w))\varphi v \quad \text{in } L^2(0,T;V)$$

and passing to the limit in (2.8) we deduce that for every $v \in V$:

$$\frac{d}{dt}(\tilde{u}, v) + a(g(w)) \int_\Omega \nabla \tilde{u} \ \nabla v \, dx = \langle f, v \rangle \quad \text{in } \mathscr{D}'(0,T).$$

Now since

$$(u_n(t), v) - (u_0, v) = \int_0^T \langle (u_n)_t, v \rangle, \quad t \in (0,T), \ \forall v \in V, \tag{2.9}$$

$$u_n(t) \to \tilde{u}(t) \quad \text{in } L^2(\Omega), \quad t \in (0,T),$$

then by passing to the limit in (2.9) we get

$$(\tilde{u}(t), v) - (u_0, v) = \int_0^T \langle \tilde{u}_t, v \rangle = (\tilde{u}(t), v) - (\tilde{u}(0), v), \quad t \in (0,T), \ \forall v \in V$$

and consequently

$$\tilde{u}(0) = u_0 \quad \text{and} \quad \tilde{u} = u.$$

Thus, u_n converges toward u in B, which completes the proof. □

Now, we are interested to know whether the solution given in Theorem 2.1 is unique. We have

Theorem 2.2. *In addition to the hypotheses of Theorem* 2.1, *assume that a and g are locally Lipschitz continuous, then the solution u is unique.*

Proof. If u_1, u_2 are two solutions, then we have

$$\frac{d}{dt}(u_1 - u_2) - a(g(u_1))\Delta(u_1 - u_2) = -(a(g(u_2)) - a(g(u_1)))\Delta u_2,$$

and consequently

$$\left\langle \frac{d}{dt}(u_1 - u_2), u_1 - u_2 \right\rangle + a(g(u_1)) \int_\Omega |\nabla(u_1 - u_2)|^2 \, dx$$
$$= a(g(u_2)) - a(g(u_1)) \int_\Omega \nabla u_2 \nabla(u_1 - u_2) \, dx.$$

Since $u_1, u_2 \in C([0, T], L^2(\Omega))$, then there exists a bounded set $S \subset L^2(\Omega)$ such that

$$u_1(t), u_2(t) \in S, \quad \forall t \in [0, T]$$

and thus for some $A > 0$

$$(g(u_1(t)), g(u_2(t))) \in [-A, A] \times [-A, A].$$

As a and g are locally Lipschitz continuous, and if we denote by $c(S), c(A)$ the Lipschitz constants, then we have

$$\frac{1}{2}\frac{d}{dt}||u_1 - u_2||^2 + a_0||u_1 - u_2||_V^2 \leqslant c(A)c(S)||u_1 - u_2|| \int_\Omega |\nabla u_2| \, |\nabla(u_1 - u_2)| dx$$
$$\leqslant c(A)c(S)||u_1 - u_2|| \; ||u_2||_V ||u_1 - u_2||_V.$$

By Young's inequality we infer

$$\frac{1}{2}\frac{d}{dt}||u_1 - u_2||^2 + a_0||u_1 - u_2||_V^2 \leqslant \frac{a_0}{2}||u_1 - u_2||_V^2 + \frac{c^2(S)c^2(A)||u_2||_V^2}{2a_0}||u_1 - u_2||^2$$

which implies that

$$\frac{d}{dt}||u_1 - u_2||^2 \leqslant \frac{c^2(S)c^2(A)||u_2||_V^2}{a_0}||u_1 - u_2||^2$$

and also

$$\frac{d}{dt}\left\{\exp\left(-\int_0^t \frac{c^2(S)c^2(A)||u_2||_V^2}{a_0}\,ds\right)||u_1 - u_2||^2\right\} \leqslant 0.$$

Since the above function is nonincreasing and vanishes at 0, it vanishes in fact everywhere. Hence we have uniqueness. □

Remark 2.3. If $f \in L^\infty(\mathbb{R}_+, V')$, then we can consider cases where a is not defined on the whole real line.

Indeed, we have from the proof of Theorem 2.1.

$$\frac{d}{dt}||u||^2 + c_3||u||^2 \leqslant \frac{d}{dt}||u||^2 + a_0||u||_V^2 \leqslant \frac{1}{a_0}||f||^2_{L^\infty(\mathbb{R}_+, V')},$$

where $c_3 > 0$ is a positive constant.

Hence, by integration we obtain

$$||u||^2 \leqslant ||u_0||^2 + \frac{1}{a_0 c_4}||f||^2_{L^\infty(\mathbb{R}_+, V')},$$

$c_4 > 0$ is a positive constant. Thus u remains a priori bounded.

3. Steady states

Assume, for simplicity, that f is independent of time, that is $f \in V'$ the dual of V and that g is linear. In this section we are interested in finding the weak solutions to the problem

$$\begin{cases} -\Delta(a(g(u))u) = f & \text{in } \Omega, \\ u \in V. \end{cases} \tag{3.1}$$

The main result here is

Theorem 3.1. *Let $a(\cdot) : \mathbb{R} \to \mathbb{R}_+$. Then, problem* (3.1) *has as much solutions as the problem*

$$a(\phi)\phi = g(\psi) \quad \textit{in } \mathbb{R}, \tag{3.2}$$

where ψ *is the unique solution of*

$$\begin{cases} -\Delta\psi = f & \textit{in } \Omega, \\ \psi \in V. \end{cases} \tag{3.3}$$

Proof. If u is a solution to (3.1), then we have

$$-\Delta(a(g(u))u) = f \quad \text{in } \Omega.$$

Hence, by (3.3) we get

$$a(g(u))u = \psi.$$

Applying g to both sides yields

$$a(g(u))g(u) = g(\psi).$$

This means that $g(u) \in \mathbb{R}$ is solution to (3.2).

Now, if ϕ solves (3.2). Then, there exists a unique weak solution to

$$\begin{cases} -a(\phi)\Delta u = f & \text{in } \Omega, \\ u \in V. \end{cases} \tag{3.4}$$

Since the solution of (3.3) is unique, we have

$$a(\phi)u - \psi$$

and then by applying g to both sides we get

$$a(\phi)g(u) = g(\psi) = a(\phi)\phi.$$

Consequently, as $a > 0$, we get $g(u) = \phi$. Going back to (3.4), we obtain that u is solution to (3.1).

Corollary 3.2. (i) *If* $g(\psi) = 0$, *then the only solution to* (3.1) *is* $u = \frac{\psi}{a(0)}$.

(ii) *If* a *is a continuous function such that* $0 < a_0 \leqslant a(x)$ *for all* $x \in \mathbb{R}$, *then problem* (3.1) *admits always a solution*.

Proof. (i) If $g(\psi) = 0$, the only solution to (3.2) is $\phi = 0$, and hence the only solution to (3.1) is $u = \frac{\psi}{a(0)}$.

(ii) The mapping $\phi \mapsto a(\phi)\phi$ has $\mathbb{R}$ for range, and thanks to the intermediate value theorem, there exists always a solution to (3.1).

If $g(\psi) > 0$, then any ϕ such that $a(\phi) = \frac{g(\psi)}{\phi}$ gives a solution to (3.1). Roughly speaking, any point at the intersection of the graph of a and the hyperbola $\phi \mapsto \frac{g(\psi)}{\phi}$ provides a solution to (3.1). □

4. Convergence results

In this section we assume that a is locally Lipschitz continuous so that by Theorem 2.2 problem (2.1) has a unique weak solution.

We begin by proving two preliminary results.

Lemma 4.1. *Let $u_0^n \in L^2(\Omega)$ be a sequence such that*

$$u_0^n \rightharpoonup u_0 \quad \text{in } L^2(\Omega) \quad \text{as } n \to +\infty. \tag{4.1}$$

If, u^n, u are the solutions to (2.1) *with initial date u_0^n, u_0, respectively. Then,*

$$u^n(t) \rightharpoonup u(t), \qquad \forall t \geqslant 0 \quad \text{in } L^2(\Omega). \tag{4.2}$$

Proof. By (4.1), u_0^n is bounded in $L^2(\Omega)$ and thanks to (2.4) and (2.5) there exists a positive constant $c > 0$ independent of n such that

$$||u^n||_{L^2(0,T;V)} \leqslant c,$$

$$||u^n||_{L^\infty(0,T;L^2(\Omega))} \leqslant c,$$

$$||u_t^n||_{L^2(0,T;V')} \leqslant c.$$

Consequently, for a subsequence, still denoted by the same symbol, we obtain

$$\begin{cases} u^n \rightharpoonup \tilde{u} & \text{in } L^2(0,T;V), \\ u^n \to u & \text{in } L^2(0,T;L^2(\Omega)), \\ u^n \rightharpoonup \tilde{u} & \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak-star}, \\ u_t^n \rightharpoonup \tilde{u}_t & \text{in } L^2(0,T;V'). \end{cases} \tag{4.3}$$

For every $v \in V$ and $\varphi \in \mathcal{D}(0,T)$ we have by definition

$$-\int_0^T \int_\Omega u^n v \varphi'(t)\, dx\, dt + \int_0^T \int_\Omega a(g(u^n))(\nabla u^n \cdot \nabla v)\varphi(t)\, dx\, dy$$
$$= \int_0^T \langle f, v \rangle \varphi(t)\, dt. \tag{4.4}$$

From (4.3) we have for almost every $t \in (0, T)$

$$g(u^n) \to g(\tilde{u}) \quad \text{in } L^2(0, T)$$

and then by the Lebesgue theorem it holds that

$$a(g(u^n))\varphi\nabla v \to a(g(\tilde{u}))\varphi\nabla v \quad \text{in } L^2(0, T; L^2(\Omega))$$

and passing to the limit in (4.4), we get that for every $v \in V$

$$\frac{d}{dt}(\tilde{u}, v) + a(g(\tilde{u}))\int_\Omega \nabla\tilde{u}\cdot\nabla v\,dx = \langle f, v\rangle \quad \text{in } \mathscr{D}'(0, T).$$

Now, for every $v \in V$:

$$(u^n(t), v) - (u_0^n, v) = \int_0^t \langle u_t^n, v\rangle\,dt$$

for almost every t, and since $u^n(t) \to \tilde{u}(t)$ in $L^2(\Omega)$, we obtain

$$(\tilde{u}(t), v) - (u_0, v) = \int_0^t \langle \tilde{u}_t, v\rangle\,dt = (\tilde{u}(t), v) - (\tilde{u}(0), v).$$

Thus, $\tilde{u}(0) = u_0$ and then $\tilde{u} = u$. Consequently,

$$u^n \rightharpoonup u \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad \text{weak-star}$$

and in particular for every $v \in V$

$$(u^n(t), v) \rightharpoonup (u(t), v) \quad \text{in } L^\infty(0, T) \quad \text{weak-star.}$$

The sequence of functions $((u^n(t), v))_n$ is equicontinuous and thus relatively compact in $C([0, T])$. Indeed, for every $t_1, t_2 \in [0, T]$, $t_2 > t_1$, we have

$$\begin{aligned}(u^n(t_2), v) - (u^n(t_1), v) &= \int_{t_1}^{t_2} \langle u_t^n, v\rangle\,dt \\ &\leqslant \int_{t_1}^{t_2} \|u_t^n\|_{V'}\|v\|_V \\ &\leqslant \sqrt{t_2 - t_1}\|v\|_V\|u_t^n\|_{L^2(0,T;V')}, \\ &\leqslant c\sqrt{t_2 - t_1}.\end{aligned}$$

Hence, we deduce that

$$(u^n(t), v) \to (u(t), v) \quad \text{in } C([0, T]), \quad \forall v \in V.$$

Since V is dense in $L^2(\Omega)$ and $u^n(t)$ is bounded, then we deduce that

$$(u^n(t), v) \to (u(t), v) \quad \forall v \in L^2(\Omega), \quad \forall t \geqslant 0$$

which completes the proof. $\square$

Proposition 4.2. *If* $a(\cdot) \geqslant a_0 > 0$ *is continuous in* $\mathbb{R}_+$, *and if* w *is solution to*

$$\begin{cases} w \in C([0,T], L^2(\Omega)) \cap L^2(0,T;V), \ \ w_t \in L^2(0,T;V'), \\ w(0) \leqslant 0, \ \ w(0) \not\equiv 0, \\ \frac{d}{dt}(w,v) + a(t)\int_\Omega \nabla w \cdot \nabla v \, dx \leqslant 0 \quad \text{in } \mathscr{D}'(0,T), \ \forall v \in V, \ v \geqslant 0. \end{cases} \tag{4.5}$$

Then, we have

$$w(x,t) < 0 \quad \forall t > 0, \quad a.e. \ x \in \Omega. \tag{4.6}$$

Proof. Let Ω' be a smooth subdomain of Ω large enough so that $\int_{\Omega'} |v(0)| dx = \int_{\Omega'} |w(0)| dx > 0$, where v is the weak solution to

$$\begin{cases} v_t - \Delta v = 0 \quad \text{in } \Omega' \times (0, T^*), \\ v(0) = w(0) \quad \text{in } \Omega', \\ v(\cdot, t) \in H_0^1(\Omega') \ \forall t \in (0, T^*). \end{cases}$$

Thanks to Dautray–Lions [4], we know that for every $\varepsilon > 0$

$$v \in C^\infty((\varepsilon, T^*) \times \Omega')$$

and

$$v(x,t) < 0, \quad \forall (x,t) \in \Omega' \times (0, T^*]. \tag{4.7}$$

Next, by the weak maximum principle, we have

$$w(t) \leqslant 0 \quad \text{a.e. in } \Omega, \quad \forall t \geqslant 0.$$

Moreover, setting

$$\bar{v}(\cdot, t) = v\left(\cdot, \int_0^t a(s)\, ds\right) \tag{4.8}$$

we have, in a weak sense:

$$\bar{v}_t = v_t \cdot a(t) = a(t)\Delta v = a(t)\Delta \bar{v},$$

$$\bar{v}(0) = v(0) = w(0)|_{\Omega'}.$$

The first above equation holds in $\Omega' \times (0,T)$ with $\int_0^T a(s)\,ds = T^*$. But T^* and thus T can be chosen arbitrarily, hence by the weak maximum principle we get

$$w \leqslant \bar{v} \quad \text{a.e. in } \Omega', \quad \forall t \in [0,T].$$

Hence, (4.6) follows from (4.7) to (4.8). $\square$

Now, we study the asymptotic behavior of the solution u to (2.1). We suppose that g satisfies

$$\begin{cases} g(v) \geqslant 0 \quad \forall v \geqslant 0, \ v \in L^2(\Omega), \\ g \not\equiv 0, \end{cases} \tag{4.9}$$

and that

$$\begin{cases} f \in V', \ \langle f, v \rangle \geqslant 0, \ \forall v \geqslant 0, \ v \in V, \\ f \not\equiv 0. \end{cases} \tag{4.10}$$

We define ϕ_1, ϕ_2 as the two intersection points of the graph of a with the graph of the hyperbola $\frac{g(\psi)}{\phi}$. To ϕ_1, ϕ_2 correspond two stationary points given by

$$\phi_1 = \frac{\psi}{a(\phi_1)} < \phi_2 = \frac{\psi}{a(\phi_2)}.$$

There are two cases to be distinguished:

Case 1:

$$a(\phi_2) \leqslant a(\phi) \leqslant a(\phi_1), \quad \forall \phi \in [\phi_1, \phi_2]. \tag{4.11}$$

In this case, we have two subcases

Subcase 1:

$$\frac{g(\psi)}{\phi} < a(\phi) \quad \forall \phi \in (\phi_1, \phi_2), \quad \text{cf. Theorem 4.6.} \tag{4.12}$$

Subcase 2:

$$\frac{g(\psi)}{\phi} = a(\phi) \quad \forall \phi \in (\phi_1, \phi_2), \quad \text{cf. Theorem 4.8.} \tag{4.13}$$

Case 2:

$$a(\phi_1) \leqslant a(\phi) < \frac{g(\psi)}{\phi}, \quad \forall \phi \in \left[\frac{g(\psi)}{\max_{\phi \in [\phi_1,\phi_2]} a(\phi)}, \phi_1\right), \quad \max_{\phi \in [\phi_1,\phi_2]} a(\phi) > a(\phi_1), \tag{4.14}$$

cf. Theorem 4.9.

First, we consider case 1, and study the asymptotic behavior of u the solution to (2.1). We restrict ourselves to the case where

$$u_1 \leqslant u_0 \leqslant u_2 \quad \text{a.e. in } \Omega. \tag{4.15}$$

Theorem 4.3. *Assume that* (4.9)–(4.11), (4.15) *hold. Then, the weak solution u of* (2.1) *satisfies*

$$u_1 \leqslant u(\cdot, t) \leqslant u_2 \quad \textit{a.e. in } \Omega, \quad \forall t \geqslant 0. \tag{4.16}$$

Proof. We assume first that $u_1 < u_0 < u_2$ a.e. in Ω, then by (4.9) we have

$$\phi_1 = g(u_1) < g(u_0) < g(u_2) = \phi_2.$$

We claim that

$$0 < t^* := \sup\{t > 0:\ g(u(\cdot, s)) \in [\phi_1, \phi_2],\ \forall s \in [0, t]\} = +\infty. \tag{4.17}$$

Indeed, if this is not the case, and since $u \in C(\mathbb{R}_+, L^2(\Omega))$ we have

$$g(u(\cdot, t^*)) = \phi_1 \ \text{ or } \ \phi_2.$$

Without loss of generality, we assume that $g(u(\cdot, t^*)) = \phi_2$. Then, for almost every t we have

$$\frac{du}{dt} - a(g(u))\Delta u = f = -a(\phi_2)\Delta u_2 \quad \text{in } L^2(0, t^*; V')$$

or equivalently,

$$\frac{d}{dt}(u - u_2) - a(g(u))\Delta(u - u_2) = \frac{a(\phi_2) - a(g(u))}{a(\phi_2)} \cdot f \quad \text{in } L^2(0, t^*; V').$$

Since $g(u) \in [\phi_1, \phi_2]$, we deduce that

$$\frac{d}{dt}(u - u_2) - a(g(u))\Delta(u - u_2) \leqslant 0 \quad \text{in } V', \quad \text{for a.e. } t \in (0, t^*).$$

Hence, if we set $w = u - u_2$ we get

$$\frac{dw}{dt} - a(g(u))\Delta w \leqslant 0,$$

$$w(0) = u_0 - u_2 < 0$$

and then by Proposition 4.2

$$w(t^*) < 0,$$

which contradicts $g(u(\cdot, t^*)) = \phi_2$. Thus, $t^* = +\infty$ and

$$u_1 < u(t) < u_2 \quad \text{a.e. in } \Omega, \quad \forall t \geqslant 0.$$

Now, if u_0 satisfies

$$\frac{\psi}{a(\phi_1)} = u_1 \leqslant u_0 \leqslant u_2 = \frac{\psi}{a(\phi_2)}$$

then as $n \to +\infty$, it holds that

$$u_1 < u_0^n = \max\left(\min\left(\left(u_1 + \frac{\psi}{n}\right), u_0\right), \left(u_2 - \frac{\psi}{n}\right)\right) < u_2$$

and thus if u^n is the solution to (2.1) with initial data u_0^n, then for any t

$$u^n(t) \in X := \{v \in L^2(\Omega) \colon u_1 \leqslant v \leqslant u_2 \text{ a.e. in } \Omega\}.$$

The set X is closed and convex in $L^2(\Omega)$, and by Lemma 4.1 is also weakly closed and

$$u(t) \in X, \quad \forall t > 0.$$

Proposition 4.4. (i) *For $u_0 \in X$, we define $S(t)u_0 = u(t)$ where $u(t) = u(\cdot, t)$ denotes the solution of* (2.1). *Then $(S(t))_{t \geqslant 0}$ is a dynamical system on X.*

(ii) *The mapping $u \mapsto (\Phi, u)$ is a Lyapounov function on X, where Φ is the weak solution to*

$$\begin{cases} -\Delta \Phi = g & \text{in } \Omega, \\ \Phi \in V. \end{cases}$$

Proof. (i) From Theorem 4.3 we deduce that $S(t)$ maps X into X. It is easy to check that it is a dynamical system thanks to Lemma 4.1 and since $u \in C([0, T], L^2(\Omega))$.

(ii) Since $g \in L^2(\Omega) \subset V'$, the problem

$$\begin{cases} -\Delta \Phi = g & \text{in } \Omega, \\ \Phi \in V \end{cases} \tag{4.18}$$

has a unique solution, and by (4.8) it satisfies $\Phi > 0$ a.e. in Ω.

If we choose $v = \Phi$ in (2.1), we get

$$\frac{d}{dt}(\Phi, u) + a(g(u)) \int_\Omega \nabla u \cdot \nabla \Phi \, dx = \langle f, \Phi \rangle.$$

Hence by (4.18) we deduce that

$$\frac{d}{dt}(\Phi, u) = g(\psi) - a(g(u))g(u) \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{4.19}$$

If we choose $u_0 \in X$, then by Theorem 4.3 we obtain

$$\frac{d}{dt}(\Phi, u) \leqslant 0$$

and in the case of (4.12) and (4.13)

$$\frac{d}{dt}(\Phi, u) = 0. \qquad \square$$

Lemma 4.5. *Let u be the weak solution to* (2.1). *Then, if*

$$g(u(t)) \to \phi_i \quad \textit{as } t \to +\infty, \quad i = 1, 2,$$

then

$$u(t) \to u_i \quad \textit{in } L^2(\Omega), \quad i = 1, 2.$$

Proof. By definition we have

$$\frac{d}{dt}(u - u_i) - a(g(u))\Delta u = -a(\phi_1)\Delta u_i \quad \text{in } V' \text{ for a.e } t \geqslant 0$$

or equivalently

$$\frac{d}{dt}(u - u_i) - a(g(u))\Delta(u - u_i) = -(a(\phi_1) - a(g(u)))\Delta u_i$$
$$\text{in } V' \text{ for a.e } t \geqslant 0.$$

Hence, when multiplying with $u - u_i$ we get

$$\frac{1}{2}\frac{d}{dt}||u - u_i||^2 + a_0 \int_\Omega |\nabla(u - u_i)|^2 \, dx$$
$$\leqslant |a(\phi_i) - a(g(u))| \int_\Omega |\nabla u_i| \, |\nabla(u - u_i)| \, dx.$$

Since for every $\varepsilon > 0$ we have

$$\int_\Omega |\nabla u_i|\,|\nabla(u-u_i)|\,dx \leqslant \frac{\varepsilon}{2}\int_\Omega |\nabla(u-u_i)|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega |\nabla u_i|^2\,dx$$

we obtain by choosing $\varepsilon\sup_{t\geqslant 0}|a(\phi_i)-a(g(u))| = a_0$ that

$$\frac{1}{2}\frac{d}{dt}||u-u_i||^2 + \frac{a_0}{2}\int_\Omega |\nabla(u-u_i)|^2\,dx \leqslant c_1|a(\phi_i)-a(g(u))|$$

where $c_1 > 0$ is a positive constant.

Hence, from the Poincaré inequality we deduce that

$$(||u-u_i||^2)'(t) + c_2||u-u_i||^2(t) \leqslant c_1|a(\phi_i)-a(g(u))|$$

and then integration between t_0 and t yields

$$||u-u_i||^2(t) \leqslant ||u-u_i||^2(t_0)e^{c_2(t_0-t)} + c_1\int_{t_0}^t |a(\phi_i) - a(g(u))|(s)e^{c_2(s-t)}\,ds.$$

If $\varepsilon > 0$ is a given positive real, let t_0 be such that

$$|a(\phi_i)-a(g(u))|(s) \leqslant \frac{c_2}{c_1}\frac{\varepsilon}{2} \quad \forall s \geqslant t_0,$$

then for $t > t_0$ we have

$$\begin{aligned}||u-u_i||^2(t) &\leqslant ||u-u_i||^2(t_0)e^{c_2(t_0-t)} + c_2\frac{\varepsilon}{2}\int_{t_0}^t e^{c_2(s-t)}\,ds\\ &\leqslant ||u-u_i||^2(t_0)e^{c_2(t_0-t)} + \frac{\varepsilon}{2}.\end{aligned}$$

Hence, if t is large enough we get

$$||u-u_i||^2(t) \leqslant \varepsilon$$

which completes the proof. □

Theorem 4.6. *Assume that* (4.11) *and* (4.12) *hold. Then, for* $u_0 \in X,\ u_0 \neq u_2$, *we have*

$$u(t) = S(t)u_0 \to u_1 \quad \text{in } L^2(\Omega) \quad \text{as } t \to +\infty.$$

Proof. From (4.19) we have

$$\frac{d}{dt}(\Phi, u) = g(\psi) - a(g(u))g(u) \leqslant 0 \tag{4.20}$$

and hence by Theorem 4.3 we get

$$(\Phi, u(t)) \geqslant (\Phi, u_1).$$

Moreover, since by (4.20) the function (Φ, u) is nonincreasing, there exists $c_3 > 0$ such that

$$\lim_{t \to +\infty} (\Phi, u(t)) = c_3 = (\Phi, w)$$

for any $w \in \omega(u_0)$ the ω-limit set of u_0.

From (4.20) and for any $w \in \omega(u_0)$ we have

$$\frac{d}{dt}(\Phi, S(t)w) = 0 = g(\psi) - a(g(S(t)w))g(S(t)w)$$

that is, by the continuity of $t \mapsto g(S(t)w)$, that

$$g(S(t)w) = \phi_1 \quad \text{for all } t \text{ or } g(S(t)w) = \phi_2 \quad \text{for all } t.$$

Hence,

$$g(w) = \phi_1 \text{ or } \phi_2 \quad \text{for all } w \in \omega(u_0).$$

We have $\omega(u_0) = \omega_1 \cup \omega_2$ where

$$\omega_i = \{w \in \omega(u_0)\colon g(w) = \phi_i\}, \quad i = 1, 2.$$

Consequently, as $t \to +\infty$, we have

$$g(u(t)) \to \phi_1 \quad \text{or} \quad g(u(t)) \to \phi_2.$$

Thus, if $g(u(t)) \to \phi_2$, we have $u(t) \to u_2$. Since $(\Phi, u(t))$ is nonincreasing and $\Phi > 0$ this implies that

$$(\Phi, u(t)) = (\Phi, u_2) \quad \forall t.$$

Since $\Phi > 0$ a.e. in Ω, $u(t) = u_2$ for all t which contradicts $u_0 \neq u_2$. Thus,

$$g(u(t)) \to \phi_1$$

and Lemma 4.5 permits us to conclude. $\square$

Now, let us turn to the stability if the solution $u(t)$ to (2.1) under hypotheses (4.11) and (4.13). For any $\phi \in [\phi_1, \phi_2]$, then $\frac{\psi}{a(\phi)}$ is a stationary point and by (4.19) we have

$$\frac{d}{dt}(\Phi, u) = 0 \Leftrightarrow (\Phi, u(t)) = (\Phi, u_0) \quad \forall t \geqslant 0.$$

Hence, a natural candidate for the limit of $u(t)$ is

$$u_\infty = \frac{(\Phi, u_0)}{(\Phi, \psi)}\psi.$$

Since $\omega(u_0)$ is compact, then there exists $w_0 \in \omega(u_0)$ such that

$$a(g(w_0)) = \min_{w \in \omega(u_0)} a(g(w)). \tag{4.21}$$

First, we need a preliminary result:

Lemma 4.7. *For any $w \in \omega(u_0)$, it holds that*

$$w \leqslant \frac{\psi}{a(g(w_0))} \quad \text{a.e. in } \Omega.$$

Proof. Let $w \in \omega, v(t) = S(t)w$. Now

$$-a(g(w_0))\Delta\left(\frac{\psi}{a(g(w_0))}\right) = -\Delta\psi = f$$

and hence

$$\frac{d}{dt}\left(v - \frac{\psi}{a(g(w_0))}\right) - a(g(v))\Delta\left(v - \frac{\psi}{a(g(w_0))}\right) = -(a(g(w_0)) - a(g(v)))\Delta\left(\frac{\psi}{a(g(w_0))}\right).$$

Thus we have, since $v \in \omega(u_0), -\Delta(\frac{\psi}{a(g(w_0))}) \geqslant 0$:

$$\frac{1}{2}\frac{d}{dt}\left\|\left(v - \frac{\psi}{a(g(w_0))}\right)^+\right\|^2 + a(g(v))\int_\Omega \left|\nabla\left(v - \frac{\psi}{a(g(w_0))}\right)^+\right|^2 dx = (a(g(w_0)) - a(g(v)))\left\langle -\Delta\left(\frac{\psi}{a(g(w_0))}\right), \left(v - \frac{\psi}{a(g(w_0))}\right)^+\right\rangle \leqslant 0.$$

There exists a positive constant $c_3 > 0$ such that

$$\frac{d}{dt}\left\|\left(v - \frac{\psi}{a(g(w_0))}\right)^+\right\|^2 + c_3\left\|\left(v - \frac{\psi}{a(g(w_0))}\right)^+\right\|^2 \leqslant 0$$

and hence

$$\left\| \left(S(t)w - \frac{\psi}{a(g(w_0))} \right)^+ \right\|^2 \leqslant e^{-c_3 t} \left\| \left(w - \frac{\psi}{a(g(w_0))} \right)^+ \right\|^2 \leqslant c_4 e^{-c_3 t}.$$

For any $z \in \omega(u_0)$, we can write it as $z = S(t)w$, and hence

$$\left(z - \frac{\psi}{a(g(w_0))} \right)^+ = 0 \quad \forall z \in \omega(u_0).$$

Hence, the proof is by now complete. □

Theorem 4.8. *Assume that* (4.11) *and* (4.13) *hold. Then, for* $u_0 \in X$, *we have*

$$u(t) \to u_\infty \quad in\ L^2(\Omega).$$

Proof. Assume, by contradiction, that $\frac{\psi}{a(g(w_0))} > u_\infty$.

Let $w \in \omega(u_0)$ and set $w(t) = S(t)w$, then we have

$$(\Phi, w) = (\Phi, u_0) = (\Phi, u_\infty) < \left(\Phi, \frac{\psi}{a(g(w_0))} \right).$$

Since $a(g(w)) \geqslant a(g(w_0))$, then we get

$$\frac{d}{dt}\left(w - \frac{\psi}{a(g(w_0))} \right) - a(g(w))\Delta\left(w - \frac{\psi}{a(g(w_0))} \right) \leqslant 0,$$

$$\left(w - \frac{\psi}{a(g(w_0))} \right)(0) \leqslant 0,$$

$$\left(w - \frac{\psi}{a(g(w_0))} \right)(0) \not\equiv 0.$$

For any $w \in \omega(u_0)$, by Theorem 4.3 we have

$$w(t) - \frac{\psi}{a(g(w_0))} < 0 \quad \forall t > 0.$$

Since $S(t)\omega(u_0) = \omega(u_0)$, it follows that

$$g(w) < g\left(\frac{\psi}{a(g(w_0))} \right) \quad \forall w \in \omega(u_0)$$

which contradicts

$$g\left(\frac{\psi}{a(g(w_0))}\right) = \frac{g(\psi)}{a(g(w_0))} = g(w_0)$$

and proves that $\frac{\psi}{a(g(w_0))} \leqslant u_\infty$.

Thus, for any $w \in \omega(u_0)$ it holds

$$w \leqslant \frac{\psi}{a(g(w_0))} \leqslant u_\infty.$$

Since $(\Phi, w) = (\Phi, u_\infty)$ this imposes $w = u_\infty$ for all $w \in \omega(u_0)$, i.e., $\omega(u_0) = \{u_\infty\}$.

Thus

$$u(t) \rightharpoonup u_\infty \quad \text{in } L^2(\Omega),$$

$$a(g(u(t)) \to a(g(u_\infty)),$$

which, by the help of Lemma 4.5, proves the theorem. □

Now, we study case (4.14). The result of convergence is

Theorem 4.9. *Under hypothesis* (4.14), *for*

$$\frac{\psi}{\max_{\phi \in [\phi_1, \phi_2]} a(\phi)} \leqslant u_0 \leqslant u_2, \quad u_0 \neq u_2,$$

we have

$$\lim_{t \to +\infty} u(t) = u_1 \quad \textit{in } L^2(\Omega).$$

Proof. Let

$$X = \left\{ v \in L^2(\Omega) \colon \frac{\psi}{\max_{\phi \in [\phi_1, \phi_2]} a(\phi)} \leqslant v \leqslant u_2 \right\},$$

then first note that $u \mapsto (\Phi, u)$ is no more a priori a Lyapounov function, and proceeding exactly as in Theorem 4.3 we can show that

$$\frac{\psi}{\max_{\phi \in [\phi_1, \phi_2]} a(\phi)} \leqslant u(t) \leqslant u_2 \quad \forall t \geqslant 0.$$

We distinguish two cases:

Case 1: $a(g(w_0)) \geqslant a(g(u_1))$. Then it holds

$$\frac{\psi}{a(g(w_0))} \leqslant \frac{\psi}{a(g(u_1))} = u_1.$$

Thanks to Lemma 4.7 we have

$$\frac{\psi}{\max_{\phi \in [\phi_1, \phi_2]} a(\phi)} \leqslant w \leqslant \frac{\psi}{a(g(w_0))} \leqslant u_1 \quad \forall w \in \omega(u_0). \tag{4.22}$$

Then, for any $w \in \omega(u_0)$, the function $w(t) = S(t)w$ satisfies

$$\frac{d}{dt}(w(t), \Phi) = g(\psi) - a(g(w(t))g(w(t)) \geqslant 0 \quad \forall t \geqslant 0. \tag{4.23}$$

Define

$$\mathscr{S} := \left\{ x \in \omega(u_0) \colon (\Phi, x) = \inf_{w \in \omega(u_0)} (w, \Phi) = m \right\}$$

which is nonempty since $\omega(u_0)$ is weakly compact in $L^2(\Omega)$.

For any $x \in \omega(u_0) \backslash \mathscr{S}$, we have thanks to (4.23)

$$(\Phi, S(t)x) \geqslant (\Phi, x) > m, \quad \forall t \geqslant 0. \tag{4.24}$$

Assume, by contradiction, that $u_1 \notin \mathscr{S}$, then for any $x \in \mathscr{S}$ and $t > 0$ we have

$$\begin{aligned} S(t)x < u_1 &\Rightarrow g(S(t)x) < g(u_1) \\ &\Rightarrow a(g(S(t)x))g(S(t)x) < g(\psi) \\ &\Rightarrow (\Phi, S(t)x) > (\Phi, x) = m \end{aligned}$$

which contradicts $S(t)\omega(u_0) = \omega(u_0)$. Hence, $u_1 \in \mathscr{S}$.

Thus, for any $x \in \omega(u_0)$ we have

$$x \leqslant u_1 \quad \text{and} \quad (\Phi, x) \geqslant (\Phi, u_1).$$

Consequently

$$x = u_1 \quad \text{and} \quad \omega(u_0) = \{u_1\}.$$

The conclusion follows from Lemma 4.5.

Case 2: $a(g(w_0)) < a(g(u_1))$. By Lemma 4.7 and since $g(w_0) > 0$ we get

$$w_0 \leqslant \frac{\psi}{a(g(w_0))}$$

and hence

$$a(g(w_0)) \leqslant \frac{g(\psi)}{g(w_0)}.$$

But, since $a(g(w_0)) < a(g(u_1))$, then we have

$$g(w_0) = g(u_2) = \phi_2.$$

As $S(t)\omega(u_0) = \omega(u_0)$, then we claim that $u_2 \in \omega(u_0)$, because otherwise we would have $w \leqslant u_2$ and $w \neq u_2$. Thus $S(t)w < u_2$ for any $t > 0$ and then $g(S(t)w) < g(u_2)$. Thus $u_2 \in \omega(u_0)$.

The rest of the proof consists in proving that u_2 cannot belong to $\omega(u_0)$ and that case 2 is in fact impossible.

The mapping $t \mapsto (\Phi, S(t)u_0)$ is nonincreasing in a neighborhood t_1 whenever $g(S(t_1)u_0) \in (\phi_1, \phi_2]$ and therefore

$$\sup_{w \in \mathscr{C}(u_0)} (\Phi, w) = \max\left((\Phi, u_0), \sup_{\substack{x \in \mathscr{C}(u_0) \\ g(x) \leqslant \phi_1}} (\Phi, x) \right),$$

where $\mathscr{C}(u_0)$ is the weak closure of the trajectory of u_0 in $L^2(\Omega)$.

Now, the set $\mathscr{C}(u_0) \cap \{x : g(x) \leqslant \phi_1\}$ is weakly compact in $L^2(\Omega)$, and consequently there exists $x_* \in \mathscr{C}(u_0) \cap \{x : g(x) \leqslant \phi_1\}$ such that

$$(\Phi, x_*) = \sup_{\substack{x \in \mathscr{C}(u_0) \\ g(x) \leqslant \phi_1}} (\Phi, x). \tag{4.25}$$

Since $g(x_*) \leqslant \phi_1 < g(u_2)$, we get $x_* \leqslant u_2$ and $x_* \neq u_2$. Thus $(\Phi, x_*) < (\Phi, u_2)$ and by (4.25) we have

$$\sup_{w \in \mathscr{C}(u_0)} (\Phi, w) < (\Phi, u_2)$$

and hence

$$u_2 \notin \mathscr{C}(u_0) \supset \omega(u_0).$$

This completes the proof of the theorem. □

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