Arbeitsgemeinschaft mit aktuellem Thema: CONVEX INTEGRATION

Mathematisches Forschungsinstitut Oberwolfach

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Organizers:

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Introduction:

Convex integration was introduced by M. Gromov in 1973 as one of the methods for solving partial differential equations and inequalities of geometric nature. The idea of the method was extracted by Gromov from J. Nash's approach for proving his famous C^1-isometric embedding theorem. The method proved to be very powerful for proving results in the spirit of the so-called h-principle and allowed to obtain surprising existence results in geometry, topology and singularity theory. In the context of partial differential equations the method turned out to be especially effective for finding solutions of minimal smoothness (i.e. C^k-solutions of k-th order partial differential equations) of underdetermined highly non-linear systems of equations. More recently, a new version of the convex integration was found which turned out to have interesting applications in the calculus of variations (counterexamples to elliptic regularity, existence of mappings with incompatible gradients, fluid like behaviour of shape memory alloys).

Although the two versions of the method are similar in spirit, there are huge technical differences obstructing the exchange of ideas between this two fields:

- in geometry one is looking for C^1 or smoother solutions and naturally considers only connected partial differential relations;) applications to open differential relations are studied to a larger extend; solutions of closed relations are obtained as limits of solutions of appropriate open relations, and the convergence often is ensured (in a uniform way) by some intrinsic strict convexity of the relation.

- in the Calculus of Variations the set of admissible gradients is often a priori disconnected, so one can just look for solutions that are only Lipschitz, and thus additional effort/techniques are needed to enforce convergence of gradients. In several applications the set of admissible gradients is very small and which leads to additional constraints (e. g. prescribed determinant), which even the approximating open relations have to satisfy.

The Arbeitsgemeinschaft should bring together people interested in all possible applications of convex integration, although not necessarily familiar with all the techniques used on both sides. In this way it should enhance the communication between this two groups.

Talks:

1. **Introduction to the h-principle: topological methods for solving differential equations and inequalities**
   The language of jets and the geometric point of view on partial differential relations, necessary homotopical conditions for solvability of PDR, the h-principle: the first examples.
   References: [15, 10, 11].

2. **Differential inclusions in the Calculus of Variations**
   In the CalcVar context we have to deal with 3 problems - which is done in different ways that will be described in the lectures to follow:

   a) Find a strategy to modify the gradient of approximate solutions, i.e. along which directions the gradient can be moved towards the admitted gradients for the differential inclusion. Here the most important feature is that these directions are not always obvious -

eg. if the solutions set (=set of admissible gradients) contains no rank-one connections. This is eg. the case for the configurations found by Scheffer [31] and Tartar [36], and leads to the notion of rank-one convexity and its hull-operator.

b) Techniques to modify the gradients in the desired way, it is to split its distribution (understood as a measure on the space of all matrices) along the chosen directions in order to get better and better approximate solutions. Often one has to respect additional constraints (prescribed determinant, symmetricity etc) - since they hold for the solution set of the differential relation, due to c) they must be satisfied also by all approximate solutions.

c) How to ensure that the sequence of approximate solutions, whose gradients will stay close and closer (in measure) to the admissible once, is convergent in a sufficiently strong sense so that the gradient of the limit takes only admissible values. Here it turns out to be crucial that the improved approximate solutions is just a local modification of its predecessor.

In the course of the lectures we will first discuss how to ensure c), as this will force us to treat the b) in a more specific way. In this particular lecture, just give a first glance on the problems: Rank-one connections and rigidity versus the Tartar configuration, approximate solutions and semiconvex hulls of the admissible gradients.
References [19]: Section 1.1, Chapter 3: Intro and first three sections, [20]: Section 1 and 2, [25]. See also [5, 23] for a general introduction.

3. **Methods for proving the h-principle 1: Holonomic approximation** Applications: Smale-Hirsch immersion theory, Gromov's h-principle for invariant open differential relations over open manifolds, directed embeddings.
References: [15, 10, 11].

4. **Methods for proving the h-principle 2: Removal of singularities** Applications: construction of non-singular mappings.
References:[15, 16, 17]

5. **Convergence of Gradients I: Controlled L^∞-convergence** S.Müller & V. Šverák's modification of Gromov's arguments, M.Sytchev's later

refinements as given in [29] and the earlier work by Müller,V. Šverák[24], [27].

6. **Convergence of Gradients II: The Baire category** Dacorogna and Marcellini's work in [6], [7]. Focus on the point of continuity approach from ([18],[19] Section 3.3).

7. **Method of convex integration for solving partial differential inequalities 1:** Description of the method.
References: [14, 15, 10, 34].

8. **Modification of gradients I: Existence of functions with prescribed jacobian determinant**
Arguments from [8] (up to Lemma 4 there), Section 6.3 in [27] and briefly discuss the Appendix A there.

9. **Modification of gradients II: Respecting the determinant constraint**
as treated in [27] (based on the previous Lecture) - present the existence results in [27] and discuss the relativization of rank-one convexity to a minor constraint (Sect. 3 in [27]).

10. **Method of convex integration for solving partial differential inequalities 2:** Geometric applications: totally real embeddings, applications to symplectic geometry.
References: [15, 14, 10, 34].

11. **Modification of gradients III: Other constraints**
To handle other constraints seems technically much simpler, so here one could treat piecewise affine solutions to Partial Differential Inclusions, higher order PDI, modification of symmetric gradients as in [7],[19]: Section 3.1 and 4.4

12. **Convex integration as a method of finding C^1-solutions of first order PDE's**
References: [15, 14, 10]

13. **Failure of regularity for elliptic systems**
Present the counterexample constructed in [26]: describe how to rewrite the second order Euler-Lagrange equation as a first order differential

inclusion. Then explain the stable embedding of the T_4-configuration in this situation and how it allows to construct the in-approximations which lead to the existence of wild solutions, i.e. lipschitz but nowhere C^1-solutions of a nonlinear elliptic second order equation with x and u independence.

14. **Failure of regularity II** Briefly explain the modification of the foregoing approach to the parabolic case ([28]) and some open problems remaining there. Then discuss some present limits of the methods as investigated in [20], in particular Chapter 5 and Section 6.3 there.

15. **Nash-Kuiper C^1-isometric embedding theorem.(2 Lectures!)** References:[15, 10].

16. **Applications in material science** Shape memory alloys as treated in [9], in particular introduce the two, three and n-well problem as a partial differential inclusion arising from the energy landscape of these crystall and the different situation for elastic nematomers - [1, 3]

    Briefly explain how the corresponding rank-one convex hull can be computed in 2×2 dimensions and show how this allows to construct solutions of this PDI in these low dimensions, see [2]. If time permits, mention the existence of piecewise affine solutions to the unconstrained 2 well problem (see Section 4.4 in [19]). Sketch a few basic relations to homogenisation and the theory of composite materials, see eg. [22].

17. **The N-gradient problem** Which finite number of gradients can can be joint into the gradient of one lipschitz map, what are the compatibility conditions? Present the regularity result obtained in [4] that no 4 incomptabile gradients can together create a mapping. Explain the counterexample using 5 pairwise not rank-one connected gradients as given in [19], and briefly mention the piecewise affine counterexample obtained also there (Section 4.4).

18. **Convex integration of higher order PDE**
    References:[15, 34]

19. **The best regularity class of solutions of PDE produced by the convex integration method**
    References:[15, 33, 34]

References

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Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`kirchhei@maths.ox.ac.uk`

by January 31, 2003 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.