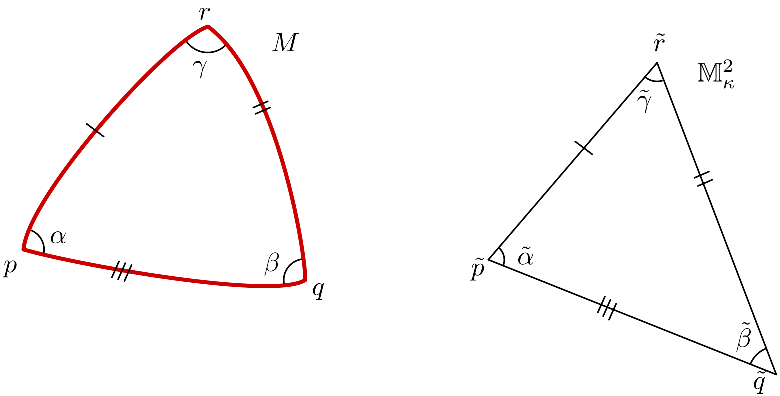
Spaces with curvature bounded below

Vitali Kapovitch

University of Toronto

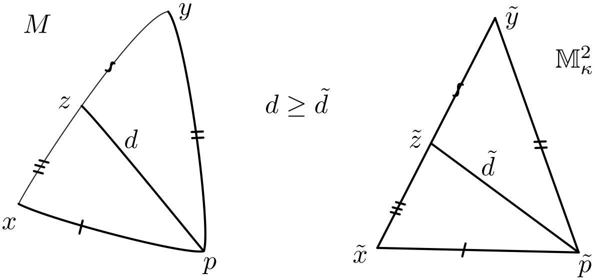
Theorem (Toponogov comparison)

Let (M^n, g) be a complete Riemannian manifold of $K_{sec} \geq \kappa$. Let Δpqr be a geodesic triangle in M and let $\Delta\tilde{p}\tilde{q}\tilde{r} = \tilde{\Delta}^\kappa pqr$ be a comparison triangle in $\mathbb{M}^n_\kappa$ Then $\alpha \geq \tilde{\alpha}, \beta \geq \tilde{\beta}, \gamma \geq \tilde{\gamma}$.

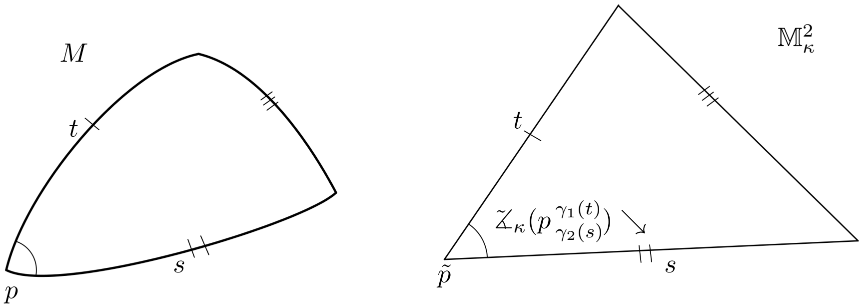


Equivalent formulations

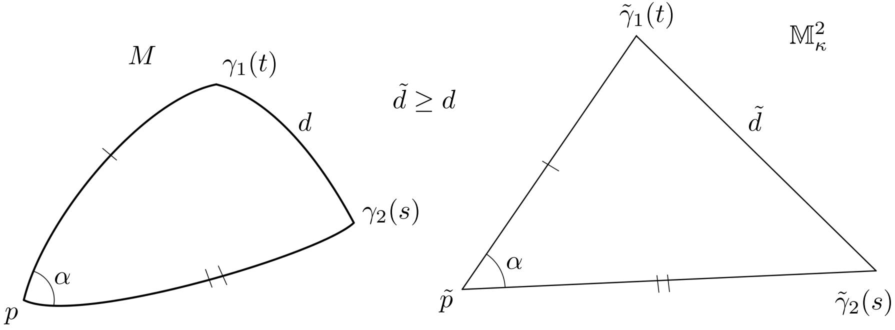
- **Point-on-a-side comparison.** For any geodesic $[xy]$ and $z \in]xy[$, we have
$$|pz| \geq |\tilde{p}\tilde{z}|$$
where $\Delta\tilde{p}\tilde{x}\tilde{y} = \tilde{\Delta}^\kappa pxy$ and $|xz| = |\tilde{x}\tilde{z}|$.



- **Comparison angles are monotone.**
  Given two shortest unit speed geodesics $\gamma_1(t), \gamma_2(s)$ with $p = \gamma_1(0), \gamma_2(0)$ in M the function $(t,s) \mapsto \tilde{\measuredangle}_\kappa(p\,{}^{\gamma_1(t)}_{\gamma_2(s)})$ is monotone non-increasing in both t and s.

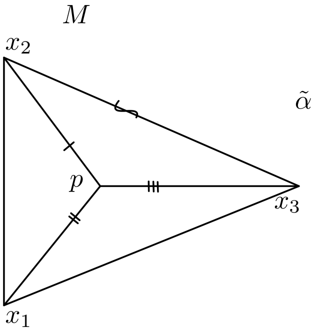


- **Hinge comparison** Given two shortest unit speed geodesics $\gamma_1(t), \gamma_2(s)$ with $p = \gamma_1(0), \gamma_2(0)$ in M let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be two geodesics in $\mathbb{M}^2_\kappa$ with $\tilde{p} = \tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$ and $\angle\gamma_1'(0)\gamma_2'(0) = \angle\tilde{\gamma}_1'(0)\tilde{\gamma}_2'(0)$.
Then $|\gamma_1(t), \gamma_2(s)| \leq |\tilde{\gamma}_1(t), \tilde{\gamma}_2(s)|$.

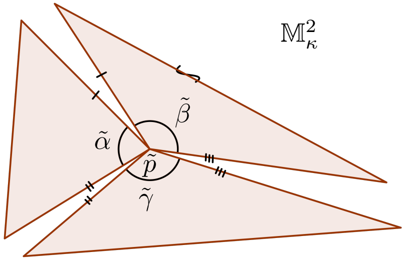


- **(1+3)-comparison** For any $p, x_1, x_2, x_3 \in M$ we have

$$\tilde{\measuredangle}_\kappa(p\,{}^{x_1}_{x_2}) + \tilde{\measuredangle}_\kappa(p\,{}^{x_2}_{x_3}) + \tilde{\measuredangle}_\kappa(p\,{}^{x_3}_{x_1}) \leq 2\pi$$



$$\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 2\pi$$



Rauch Comparison implies that it holds locally i.e. for every point in M with $\sec \geq \kappa$ there is a small neighbourhood where Toponogov comparison (in any of its equivalent formulations) holds.

Rauch Comparison implies that it holds locally i.e. for every point in M with $\sec \geq \kappa$ there is a small neighbourhood where Toponogov comparison (in any of its equivalent formulations) holds.
Globalization theorem implies that if comparison holds locally then it holds globally .

Alexandrov spaces

Definition

A complete intrinsic metric space $\mathcal{L}$ is called an Alexandrov space with curvature $\geq \kappa$ (briefly, $\mathcal{L} \in \mathrm{CBB}_\kappa$) if any quadruple $p, x^1, x^2, x^3 \in \mathcal{L}$ satisfies (1+3)-point comparison.

Alexandrov spaces

Definition

A complete intrinsic metric space $\mathcal{L}$ is called an Alexandrov space with curvature $\geq \kappa$ (briefly, $\mathcal{L} \in \mathrm{CBB}_\kappa$) if any quadruple $p, x^1, x^2, x^3 \in \mathcal{L}$ satisfies (1+3)-point comparison.

Theorem (Globalization Theorem)

Let $\mathcal{L}$ be a complete intrinsic metric space such that for any point p there exists $R > 0$ such that (1+3)-comparison with respect to κ holds for any $x_1, x_2, x_3, x_4 \in B(p, R)$.

Alexandrov spaces

Definition

A complete intrinsic metric space $\mathcal{L}$ is called an Alexandrov space with curvature $\geq \kappa$ (briefly, $\mathcal{L} \in \mathrm{CBB}_\kappa$) if any quadruple $p, x^1, x^2, x^3 \in \mathcal{L}$ satisfies (1+3)-point comparison.

Theorem (Globalization Theorem)

Let $\mathcal{L}$ be a complete intrinsic metric space such that for any point p there exists $R > 0$ such that (1+3)-comparison with respect to κ holds for any $x_1, x_2, x_3, x_4 \in B(p, R)$.
Then $\mathcal{L} \in \mathrm{CBB}_\kappa$.

Alexandrov spaces

Definition

A complete intrinsic metric space $\mathcal{L}$ is called an Alexandrov space with curvature $\geq \kappa$ (briefly, $\mathcal{L} \in \mathrm{CBB}_\kappa$) if any quadruple $p, x^1, x^2, x^3 \in \mathcal{L}$ satisfies (1+3)-point comparison.

Theorem (Globalization Theorem)

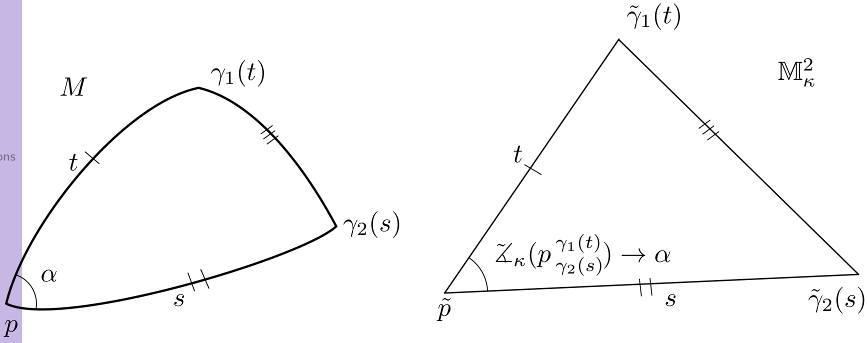
Let $\mathcal{L}$ be a complete intrinsic metric space such that for any point p there exists $R > 0$ such that (1+3)-comparison with respect to κ holds for any $x_1, x_2, x_3, x_4 \in B(p, R)$.
Then $\mathcal{L} \in \mathrm{CBB}_\kappa$.

We will only consider finite dimensional Alexandrov spaces.
They are proper and in particular geodesic.

Definition

Let $\mathcal{L} \in \mathrm{CBB}_\kappa$ and let $\gamma_1(t), \gamma_2(s)$ be geodesics with $\gamma_1(0) = \gamma_2(0) = p$.
Then the angle α between them is defined as

$$\alpha \overset{def}{:=} \lim_{t,s\to 0} \tilde\measuredangle_\kappa(p\,{}^{\gamma_1(t)}_{\gamma_2(s)})$$



Examples

- If M^n is a complete Riemannian manifold with $\sec \geq \kappa$ then $M \in \mathrm{CBB}_\kappa$.

Examples

- If M^n is a complete Riemannian manifold with $\sec \geq \kappa$ then $M \in \mathrm{CBB}_\kappa$.
- If $\mathcal{L} \in \mathrm{CBB}_\kappa$ and $U \subset \mathcal{L}$ is a closed convex subset then $U \in \mathrm{CBB}_\kappa$

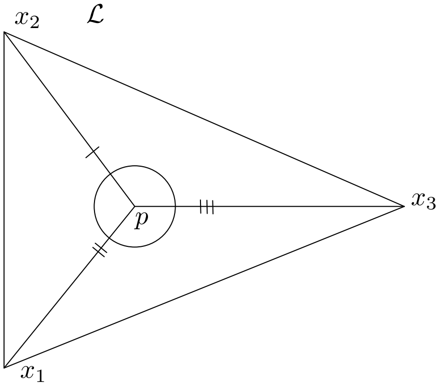
Examples

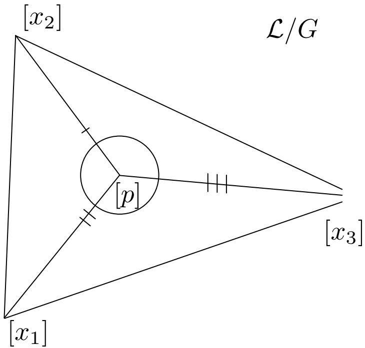
- If M^n is a complete Riemannian manifold with $\sec \geq \kappa$ then $M \in \mathrm{CBB}_\kappa$.
- If $\mathcal{L} \in \mathrm{CBB}_\kappa$ and $U \subset \mathcal{L}$ is a closed convex subset then $U \in \mathrm{CBB}_\kappa$
- If $(X, d_X), (Y, d_Y)$ are in CBB_κ with $\kappa \leq 0$ then $(X \times Y, d) \in \mathrm{CBB}_\kappa$ where $d((x,y),(x',y')) = \sqrt{d_X^2(x,x') + d_Y^2(y,y')}$

Examples

- If M^n is a complete Riemannian manifold with $\sec \geq \kappa$ then $M \in \mathrm{CBB}_\kappa$.
- If $\mathcal{L} \in \mathrm{CBB}_\kappa$ and $U \subset \mathcal{L}$ is a closed convex subset then $U \in \mathrm{CBB}_\kappa$
- If $(X, d_X), (Y, d_Y)$ are in CBB_κ with $\kappa \leq 0$ then $(X \times Y, d) \in \mathrm{CBB}_\kappa$ where $d((x,y),(x',y')) = \sqrt{d_X^2(x,x') + d_Y^2(y,y')}$
- If Σ has $curv \geq 1$ then $C(\Sigma)$ - the Euclidean cone over Σ has $curv \geq 0$. Here $d((t,v),(s,u)) := \sqrt{t^2 + s^2 - 2ts\cos d_\Sigma(u,v)}$.

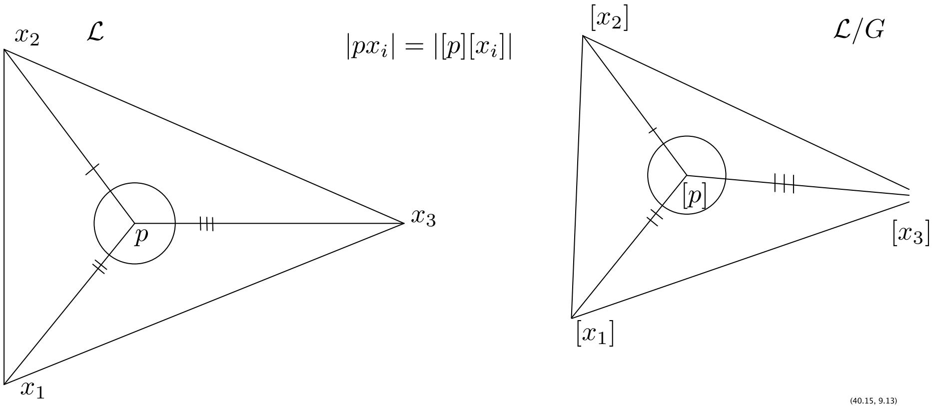
- Let $\mathcal{L} \in \mathrm{CBB}_\kappa$ and let G be a compact group acting on $\mathcal{L}$ by isometries. Then $\mathcal{L}/G$ is in CBB_κ too.





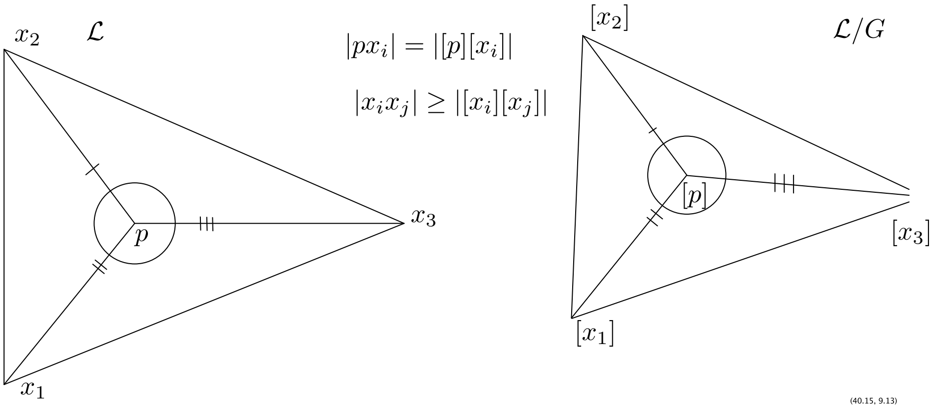
(40.15, 9.13)

- Let $\mathcal{L} \in \mathrm{CBB}_\kappa$ and let G be a compact group acting on $\mathcal{L}$ by isometries. Then $\mathcal{L}/G$ is in CBB_κ too.



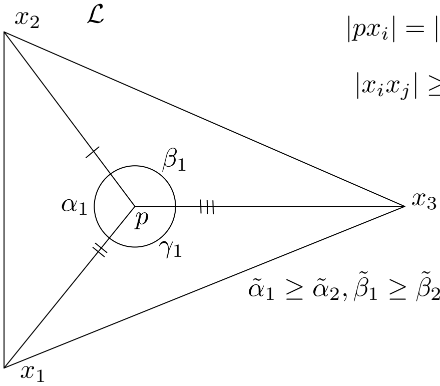
(40.15, 9.13)

- Let $\mathcal{L} \in \mathrm{CBB}_\kappa$ and let G be a compact group acting on $\mathcal{L}$ by isometries. Then $\mathcal{L}/G$ is in CBB_κ too.



(40.15, 9.13)

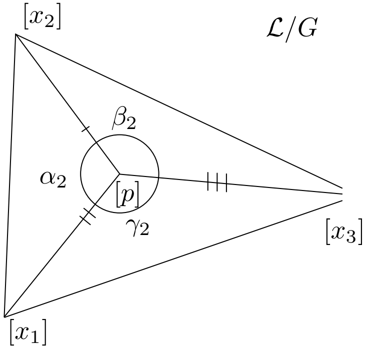
- Let $\mathcal{L} \in \mathrm{CBB}_\kappa$ and let G be a compact group acting on $\mathcal{L}$ by isometries. Then $\mathcal{L}/G$ is in CBB_κ too.



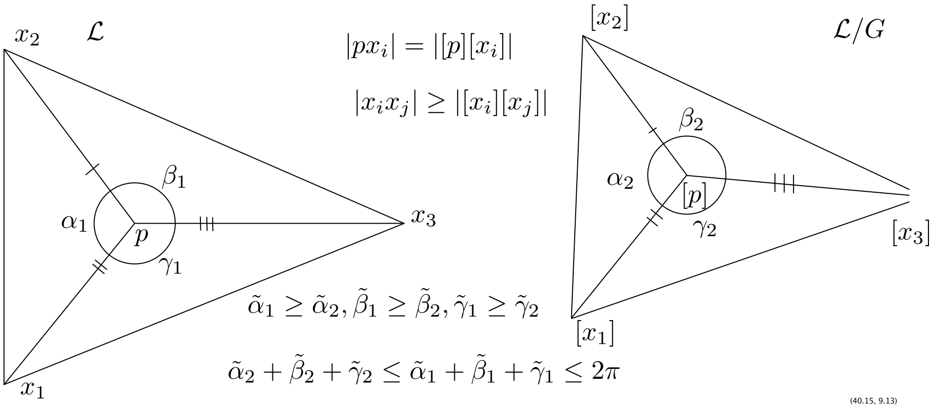
$$|px_i| = |[p][x_i]|$$

$$|x_i x_j| \geq |[x_i][x_j]|$$

$$\tilde{\alpha}_1 \geq \tilde{\alpha}_2, \tilde{\beta}_1 \geq \tilde{\beta}_2, \tilde{\gamma}_1 \geq \tilde{\gamma}_2$$



- Let $\mathcal{L} \in \mathrm{CBB}_\kappa$ and let G be a compact group acting on $\mathcal{L}$ by isometries. Then $\mathcal{L}/G$ is in CBB_κ too.



(40.15, 9.13)

Gromov-Hausdorff convergence

Definition

Let X, Y be two compact inner metric spaces. A map $f\colon X \to Y$ is called an ϵ-Hausdorff approximation if

- $||f(p)f(q)| - |pq|| \le \epsilon$ *for any* $p, q \in X$;
- *For any* $y \in Y$ *there exists* $p \in X$ *such that* $|f(p)y| \le \epsilon$

We define the Gromov-Hausdorff distance between X and Y as $d_{G-H}(X, Y) = \inf \epsilon$ such that there exist ϵ-Hausdorff approximation from X to Y.

Gromov-Hausdorff convergence

Definition

Let X, Y be two compact inner metric spaces. A map $f\colon X \to Y$ is called an ϵ-Hausdorff approximation if

- $||f(p)f(q)| - |pq|| \le \epsilon$ *for any* $p, q \in X$;
- *For any* $y \in Y$ *there exists* $p \in X$ *such that* $|f(p)y| \le \epsilon$

We define the Gromov-Hausdorff distance between X and Y as $d_{G-H}(X, Y) = \inf \epsilon$ such that there exist ϵ-Hausdorff approximation from X to Y.

Gromov-Hausdorff distance (symmetrized) turns out to be a distance on the set of isometry classes of compact inner metric spaces.

Gromov-Hausdorff convergence

Definition

Let X, Y be two compact inner metric spaces. A map $f\colon X \to Y$ is called an ϵ-Hausdorff approximation if

- $||f(p)f(q)| - |pq|| \le \epsilon$ *for any* $p, q \in X$;
- *For any* $y \in Y$ *there exists* $p \in X$ *such that* $|f(p)y| \le \epsilon$

We define the Gromov-Hausdorff distance between X and Y as $d_{G-H}(X, Y) = \inf \epsilon$ such that there exist ϵ-Hausdorff approximation from X to Y.

Gromov-Hausdorff distance (symmetrized) turns out to be a distance on the set of isometry classes of compact inner metric spaces.

Remark

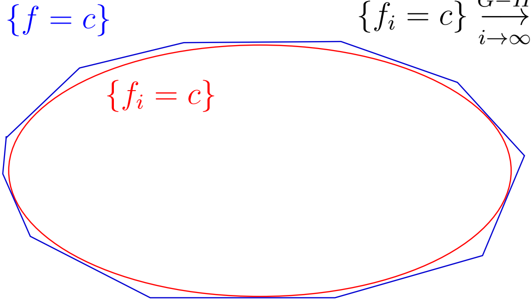
If $f\colon X \to Y$ is an ϵ-Hausdorff approximation then there exist $g\colon Y \to X$ which is a 2ϵ-Hausdorff approximation.

Gromov-Hausdorff convergence

Example

Let $\sec N \geq \kappa$ and let $f\colon N \to \mathbb{R}$ be convex. Let $f_i\colon N \to \mathbb{R}$ be smooth convex functions converging to f. Let c be any value in the range of f different from $\max f$. Then $\{f_i = c\}$ is a smooth manifold of $\sec \geq \kappa$ and $\{f_i = c\}$ Gromov-Hausdorff converges to $\{f = c\}$ with respect to the intrinsic metrics. (This is not obvious). In particular $\{f = c\}$ is in CBB_κ.

$$\{f_i = c\} \xrightarrow[i\to\infty]{G-H} X = \{f = c\}$$



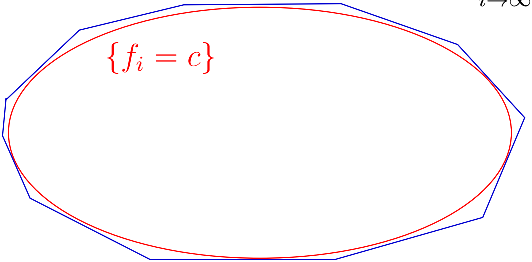
Gromov-Hausdorff convergence

Example

Let $\sec N \geq \kappa$ and let $f\colon N \to \mathbb{R}$ be convex. Let $f_i\colon N \to \mathbb{R}$ be smooth convex functions converging to f. Let c be any value in the range of f different from $\max f$. Then $\{f_i = c\}$ is a smooth manifold of $\sec \geq \kappa$ and $\{f_i = c\}$ Gromov-Hausdorff converges to $\{f = c\}$ with respect to the intrinsic metrics. (This is not obvious). In particular $\{f = c\}$ is in CBB_κ.

$\{f = c\}$

$$\{f_i = c\} \xrightarrow[i\to\infty]{G-H} X = \{f = c\}$$



Gromov-Hausdorff convergence

- If $\mathcal{L}_i \xrightarrow[i\to\infty]{G-H} \mathcal{L}$ and $\mathcal{L}_i \in \mathrm{CBB}_\kappa$ for all i then $\mathcal{L} \in \mathrm{CBB}_\kappa$ too.

Gromov-Hausdorff convergence

- If $\mathcal{L}_i \xrightarrow[i\to\infty]{G-H} \mathcal{L}$ and $\mathcal{L}_i \in \mathrm{CBB}_\kappa$ for all i then $\mathcal{L} \in \mathrm{CBB}_\kappa$ too.

Exercise

Let $f_i\colon [0,1] \to \mathbb{R}$ be a sequence of convex functions converging to a convex function f. Then the lengths of the graphs also converge, i.e.

$$L(\Gamma_{f_i}) \to L(\Gamma_f)$$

Gromov precompactness criterion

Theorem (Gromov)

Let $\mathfrak{M}$ be a class of compact (inner) metric spaces satisfying the following property. There exists a function $N\colon (0,\infty) \to (0,\infty)$ such that
for any $\delta > 0$ and any $X \in \mathfrak{M}$ there are at most $N(\delta)$ points in X with pairwise distances $\geq \delta$.

Gromov precompactness criterion

Theorem (Gromov)

Let $\mathfrak{M}$ be a class of compact (inner) metric spaces satisfying the following property. There exists a function $N\colon (0,\infty) \to (0,\infty)$ such that
for any $\delta > 0$ and any $X \in \mathfrak{M}$ there are at most $N(\delta)$ points in X with pairwise distances $\geq \delta$.
Then $\mathfrak{M}$ is precompact in the Gromov-Hausdorff topology.

Gromov precompactness criterion

Theorem (Gromov)

Let $\mathfrak{M}$ be a class of compact (inner) metric spaces satisfying the following property. There exists a function $N\colon (0,\infty) \to (0,\infty)$ such that
for any $\delta > 0$ and any $X \in \mathfrak{M}$ there are at most $N(\delta)$ points in X with pairwise distances $\geq \delta$.
Then $\mathfrak{M}$ is precompact in the Gromov-Hausdorff topology.

Corollary

The class $\mathfrak{M}_{\mathrm{sec}}(n,\kappa,D)$ of complete n-manifolds with $\mathrm{sec} \geq \kappa, \mathrm{diam} \leq D$ is precompact in the the Gromov-Hausdorff topology.

Bishop-Gromov volume comparison

Theorem (Bishop-Gromov's Relative Volume Comparison)

Suppose M^n has $\mathrm{Ric}_M \geq (n-1)k$. Then

(1) $$\frac{\mathrm{Vol}\,(\partial B_r(p))}{\mathrm{Vol}(\partial B_r^k(0))} \text{ and } \frac{\mathrm{Vol}\,(B_r(p))}{\mathrm{Vol}(B_r^k(0))} \text{ are nonincreasing in } r.$$

In particular,

(2) $$\mathrm{Vol}\,(B_r(p)) \leq \mathrm{Vol}(B_r^k(0)) \quad \text{for all } r > 0,$$

(3) $$\frac{\mathrm{Vol}\,(B_r(p))}{\mathrm{Vol}\,(B_R(p))} \geq \frac{\mathrm{Vol}(B_r^k(0))}{\mathrm{Vol}(B_R^k(0))} \quad \text{for all } 0 < r \leq R,$$

Here $B_r^k(0)$ is the ball of radius r in the n-dimensional simply connected space of constant curvature k.

Bishop-Gromov volume comparison

Theorem (Bishop-Gromov's Relative Volume Comparison)

Suppose M^n has $\mathrm{Ric}_M \geq (n-1)k$. Then

(1) $$\frac{\mathrm{Vol}\,(\partial B_r(p))}{\mathrm{Vol}(\partial B_r^k(0))} \text{ and } \frac{\mathrm{Vol}\,(B_r(p))}{\mathrm{Vol}(B_r^k(0))} \text{ are nonincreasing in } r.$$

In particular,

(2) $$\mathrm{Vol}\,(B_r(p)) \leq \mathrm{Vol}(B_r^k(0)) \quad \text{for all } \ r > 0,$$

(3) $$\frac{\mathrm{Vol}\,(B_r(p))}{\mathrm{Vol}\,(B_R(p))} \geq \frac{\mathrm{Vol}(B_r^k(0))}{\mathrm{Vol}(B_R^k(0))} \quad \text{for all } \ 0 < r \leq R,$$

Here $B_r^k(0)$ is the ball of radius r in the n-dimensional simply connected space of constant curvature k.

Note that this implies that if the volume of a big ball has a lower bound, then all smaller balls also have lower volume bounds

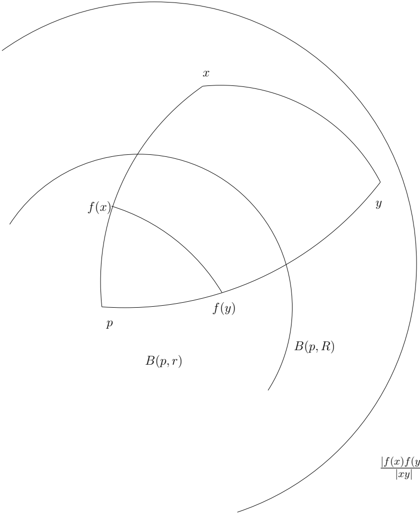
Idea of the proof for $K_{\sec} \geq 0$

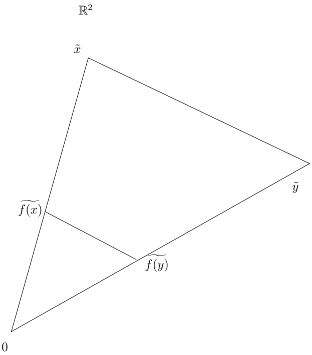
Consider the following *contraction* map $f\colon B_R(p) \to B_r(p)$. Given $x \in B_R(p)$ let $\gamma\colon [0,1] \to M$ be a shortest geodesic from p to x. Define $f(x) = \gamma(\frac{r}{R})$. At points where the geodesics are not unique we choose any one. By Toponogov comparison we have that

$$|f(x)f(y)| \geq \frac{r}{R}|xy|$$

Therefore

$$\mathrm{Vol} B_r(p) \geq \mathrm{Vol} f(B_R(p)) \geq (\frac{r}{R})^n \mathrm{Vol} B_R(p)$$





$$\frac{|f(x)f(y)|}{|xy|} \geq \frac{|\tilde{f(x)}\tilde{f(y)}|}{|\tilde{x}\tilde{y}|} = \frac{r}{R}$$

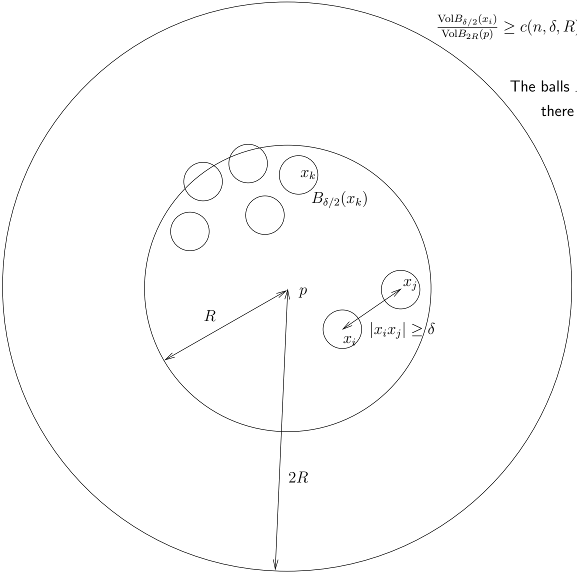
Corollary (δ-separated net bound)

Let (M^n, g) *have* $\mathrm{Ric} \geq \kappa(n-1)$. *Let* $p \in M, 0 < \delta < R/2$. *Suppose* $x_1, \dots x_N$ *is a* δ*-separated net in* $B_R(p)$. *Then*

$$N \leq C(n, R, \delta)$$

Proof.

By Bishop-Gromov we have $\mathrm{Vol} B_{\delta/2}(x_i) \geq c(n, R, \delta) \mathrm{Vol} B_{2R}(p)$. Since the balls $B_{\delta/2}(x_i)$ are disjoint we have $\mathrm{Vol} B_{2R}(p) \geq \sum_i \mathrm{Vol} B_{\delta/2}(x_i) \geq N \cdot c(n, R, \kappa, \delta) \mathrm{Vol} B_{2R}(p)$ and the result follows. □



$\frac{\mathrm{Vol} B_{\delta/2}(x_i)}{\mathrm{Vol} B_{2R}(p)} \geq c(n, \delta, R)$

The balls $B_{\delta/2}(x_i)$ are disjoint therefore there can only be so many of them.

Example

Let G be a compact connected Lie group with a biinvariant Riemannian metric g_{bi}. It has $\sec \geq 0$.

Example

Let G be a compact connected Lie group with a biinvariant Riemannian metric g_{bi}. It has $\sec \geq 0$.
Suppose G acts isometrically on a manifold (M^n, g) with $\sec \geq \kappa$ with $\kappa \leq 0$.
Let $M_\varepsilon = (G, \varepsilon g_{bi}) \times (M, g))/G$ with the induced Riemannian metric g_ε.

Example

let $G = S^1, M = S^3$. For the action $\lambda(z_1, z_2) = (\lambda z_1, z_2)$ we have that (S^3, g_ε) has $\sec \geq 0$ and $(S^3, g_\varepsilon) \to S^2_+$. For the diagonal Hopf action $\lambda(z_1, z_2) = (\lambda z_1, z_2)$ we have that (S^3, g_ε) converges to S^2.

Example

Let G be a compact connected Lie group with a biinvariant Riemannian metric g_{bi}. It has $\sec \geq 0$.
Suppose G acts isometrically on a manifold (M^n, g) with $\sec \geq \kappa$ with $\kappa \leq 0$.
Let $M_\varepsilon = (G, \varepsilon g_{bi}) \times (M, g))/G$ with the induced Riemannian metric g_ε.
Then $\sec(M_\varepsilon) \geq \kappa$ and $M_\varepsilon \to M/G$ as $\varepsilon \to 0$.

Example

let $G = S^1, M = S^3$. For the action $\lambda(z_1, z_2) = (\lambda z_1, z_2)$ we have that (S^3, g_ε) has $\sec \geq 0$ and $(S^3, g_\varepsilon) \to S^2_+$. For the diagonal Hopf action $\lambda(z_1, z_2) = (\lambda z_1, z_2)$ we have that (S^3, g_ε) converges to S^2.

Theorem (Yamaguchi)

Let $M_i^n \to N^m$ as $i \to \infty$ in Gromov-Hausdorff topology. where $\sec(M_i^n) \ge k$ and N is a smooth manifold. Then for all large i there exists a fiber bundle $F_i \to M_i^n \to N$.

Theorem (Yamaguchi)

Let $M_i^n \to N^m$ as $i \to \infty$ in Gromov-Hausdorff topology. where $\sec(M_i^n) \geq k$ and N is a smooth manifold. Then for all large i there exists a fiber bundle $F_i \to M_i^n \to N$.

Theorem (Perelman's stability theorem)

Let $\mathcal{X}_i^n$ be a sequence of n-dimensional Alexandrov spaces with $curv \geq \kappa$ Gromov-Hausdorff converging to $\mathcal{X}$ where $\dim \mathcal{X} = n$.

Theorem (Yamaguchi)

Let $M_i^n \to N^m$ as $i \to \infty$ in Gromov-Hausdorff topology. where $\sec(M_i^n) \geq k$ and N is a smooth manifold. Then for all large i there exists a fiber bundle $F_i \to M_i^n \to N$.

Theorem (Perelman's stability theorem)

Let $\mathcal{X}_i^n$ be a sequence of n-dimensional Alexandrov spaces with $curv \geq \kappa$ Gromov-Hausdorff converging to $\mathcal{X}$ where $\dim \mathcal{X} = n$. Then $\mathcal{X}_i$ is homeomorphic to $\mathcal{X}$ for all large i.

Space of directions

Definition

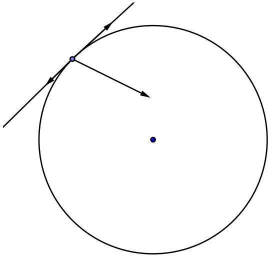
Let $X \in \mathrm{CBB}^n_\kappa$ and $p \in X$ be a point. Let S_pX be the space of directions of geodesics starting at p with angle as the distance on S_pX. We define the space of directions to X at p as the metric completion of S_pX

$$\Sigma_p X :\stackrel{def}{=} \bar{S}_p X$$

- For a smooth manifold M^n with $\sec \geq k$ we have that $S_pM = \Sigma_pM \cong \mathbb{S}^{n-1}$ for any $p \in M$.

- If $\mathcal{X} = \bar{B}(0,1) \subset \mathbb{R}^2$. then for any $p \in \partial B(0,1)$ we have that

$$S_p\mathcal{X} \cong (0,\pi) \qquad \Sigma_p\mathcal{X} \cong [0,\pi]$$



- If G is a compact Lie group acting isometrically on M^n and $\mathcal{X} = M/G$ then for any $p \in M$

$$\Sigma_{[p]}X \cong \mathbb{S}^{k-1}/G_p$$

where k is the codimension of the orbit Gp in M and G_p is the isotropy subgroup of p.

- If G is a compact Lie group acting isometrically on M^n and $\mathcal{X} = M/G$ then for any $p \in M$

$$\Sigma_{[p]}X \cong \mathbb{S}^{k-1}/G_p$$

where k is the codimension of the orbit Gp in M and G_p is the isotropy subgroup of p. In particular, Let Z_2 act on $\mathbb{R}^n$ by reflection in the origin. Then $X = \mathbb{R}^n/G = C(\mathbb{RP}^{n-1})$ and $\Sigma_{[0]}\mathcal{X} \cong \mathbb{RP}^{n-1}$

- If G is a compact Lie group acting isometrically on M^n and $\mathcal{X} = M/G$ then for any $p \in M$

$$\Sigma_{[p]}X \cong \mathbb{S}^{k-1}/G_p$$

where k is the codimension of the orbit Gp in M and G_p is the isotropy subgroup of p. In particular, Let Z_2 act on $\mathbb{R}^n$ by reflection in the origin. Then $X = \mathbb{R}^n/G = C(\mathbb{RP}^{n-1})$ and $\Sigma_{[0]}\mathcal{X} \cong \mathbb{RP}^{n-1}$

Theorem

Let $X \in \mathrm{CBB}^n_\kappa$ and $p \in X$ be a point. Then $(\lambda X, p) \to C(\Sigma_p \mathcal{X})$ as $\lambda \to \infty$. In particular, $C(\Sigma_p \mathcal{X})$ has $curv \geq 0$ and hence Σ has $curv \geq 1$.

$$T_p\mathcal{X} \stackrel{def}{=} C(\Sigma_p \mathcal{X})$$

is called the tangent space to $\mathcal{X}$ at p.

Theorem (Perelman)

Let $\mathcal{X} \in \mathrm{CBB}^n_\kappa$. Then for any $p \in \mathcal{X}$ there is a small neighbourhood of p homeomorphic to $T_p\mathcal{X}$.

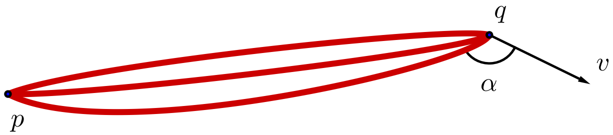
Theorem (Perelman)

Let $\mathcal{X} \in \mathrm{CBB}^n_\kappa$. Then for any $p \in \mathcal{X}$ there is a small neighbourhood of p homeomorphic to $T_p\mathcal{X}$. By induction on dimension this implies that X is a stratified manifold.

First Variation Formula

Let $X \in \mathrm{CBB}^n_\kappa$ and $p \in X$ be a point. Let $f(x) = |xp|$. Then f has directional derivatives at every pout $q \in p$ given by the following formula. Let $\Uparrow_q^p = \{u \in \Sigma_q\mathcal{X} |$ such that u is a direction of a shortest geodesic from q to $p\}$. Then

$$D_v f(q) = -\cos\alpha \text{ where } \alpha = \inf_{u\in\Uparrow_q^p} \angle uv$$

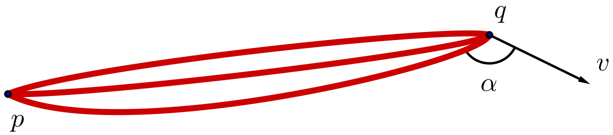


First Variation Formula

Let $X \in \mathrm{CBB}^n_\kappa$ and $p \in X$ be a point. Let $f(x) = |xp|$. Then f has directional derivatives at every pout $q \in p$ given by the following formula. Let $\Uparrow_q^p = \{u \in \Sigma_q \mathcal{X} |$ such that u is a direction of a shortest geodesic from q to $p\}$. Then

$$D_v f(q) = -\cos\alpha \text{ where } \alpha = \inf_{u \in \Uparrow_q^p} \angle uv$$

$$D_v f(q) = \inf_{u \in \Uparrow_q^p} \langle u, v \rangle$$



Application to Erdos problem

Theorem (Erdos-Perelman)

Let Γ be a discrete group of isometries of $\mathbb{R}^n$. Then Γ has no more than 2^n isolated singular orbits.

Application to Erdos problem

Theorem (Erdos-Perelman)

Let Γ be a discrete group of isometries of $\mathbb{R}^n$. Then Γ has no more than 2^n isolated singular orbits.

Let $\mathcal{X} = \mathbb{R}^n/\Gamma$. Then isolated singular orbits project to points p_i in $\mathcal{X}$ such that $\Sigma_{p_i}\mathcal{X}$ has diameter $\leq \pi/2$ (exercise). Now the above result follows from

Application to Erdos problem

Theorem (Erdos-Perelman)

Let Γ be a discrete group of isometries of $\mathbb{R}^n$. Then Γ has no more than 2^n isolated singular orbits.

Let $\mathcal{X} = \mathbb{R}^n/\Gamma$. Then isolated singular orbits project to points p_i in $\mathcal{X}$ such that $\Sigma_{p_i}\mathcal{X}$ has diameter $\leq \pi/2$ (exercise). Now the above result follows from

Theorem (Perelman)

Let $\mathcal{X} \in \mathrm{CBB}_0^n$. Let $p_1, \ldots p_N \in \mathcal{X}$ satisfy $\operatorname{diam} \Sigma_{p_i} \leq \pi/2$. Then $N \leq 2^n$.

Application to Erdos problem

Theorem (Erdos-Perelman)

Let Γ be a discrete group of isometries of $\mathbb{R}^n$. Then Γ has no more than 2^n isolated singular orbits.

Let $\mathcal{X} = \mathbb{R}^n/\Gamma$. Then isolated singular orbits project to points p_i in $\mathcal{X}$ such that $\Sigma_{p_i}\mathcal{X}$ has diameter $\leq \pi/2$ (exercise). Now the above result follows from

Theorem (Perelman)

Let $\mathcal{X} \in \mathrm{CBB}_0^n$. Let $p_1, \dots p_N \in \mathcal{X}$ satisfy $\operatorname{diam} \Sigma_{p_i} \leq \pi/2$. Then $N \leq 2^n$.

This bound is sharp (e.g. for a cube).

Proof

Assume $\mathcal{X}$ is compact. Consider the Voronoi cells $V_i = \{x \in \mathcal{X}|$ such that $|xp_i| \le |xp_j|$ for all $j \neq i\}$.
For any i and let $U_i = \{x \in \mathcal{X}|$ such that x is the middle of a shortest geodesic $[py(x)]$ for some (necessarily uniquely defined) $y(x)$.
Look at the map $f_i \colon U_i \to \mathcal{X}$ given by $x \mapsto y(x)$.
Then f is onto and, by Toponogov comparison, f is 2-Lipschitz. therefore,

$$\operatorname{Vol} U_i \ge \frac{1}{2^n} \operatorname{Vol} \mathcal{X}$$

Claim:

$$U_i \subset V_i$$

Let $x \in U_i$. then there is y such that x is the midpoint of $[p_i y]$. Let $a = |p_i x| = |xy| = \frac{1}{2}|p_i y|$. Let $j \neq i$.
We have $\alpha \leq \pi/2$ because $\operatorname{diam} \Sigma_{p_j} \mathcal{X} \leq \pi/2$. Consider the triangle $\Delta p_i y p_j$ and its comparison triangle $\Delta \tilde{p}_i \tilde{y} \tilde{p}_j$ in $\mathbb{R}^2$ Then $\tilde{\alpha} \leq \alpha \leq \pi/2$ and hence $\tilde{p}_j$ lies outside the circle of radius a centred at $\tilde{x}$. Hence $|\tilde{p}_j \tilde{x}| \geq a$.
By triangle comparison $|p_j x| \geq |\tilde{p}_j \tilde{x}|$. Therefore

$$|p_j x| \geq |\tilde{p}_j \tilde{x}| \geq a = |p_i x|$$

Since j was arbitrary this means that $U_i \subset V_i$ which proves the claim.

Therefore

$$\operatorname{Vol} V_i \geq \operatorname{Vol} U_i \geq \frac{1}{2^n} \operatorname{Vol} \mathcal{X}$$

Since $\sum_{i=1}^{N} \operatorname{Vol} V_i = \operatorname{Vol} \mathcal{X}$ this implies that $N \leq 2^n$. □

Therefore

$$\mathrm{Vol}\, V_i \geq \mathrm{Vol}\, U_i \geq \frac{1}{2^n} \mathrm{Vol}\, \mathcal{X}$$

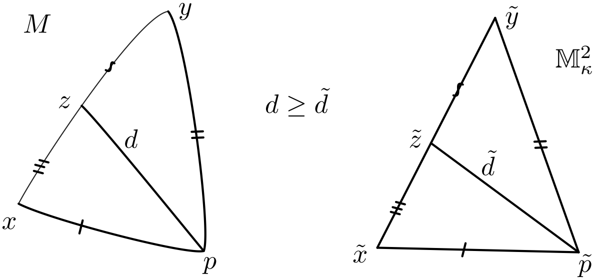
Since $\sum_{i=1}^{N} \mathrm{Vol}\, V_i = \mathrm{Vol}\, \mathcal{X}$ this implies that $N \leq 2^n$. □
If $\mathcal{X}$ is non compact then using Busemann functions one can construct a proper convex function $f\colon \mathcal{X} \to \mathbb{R}$.
Let $Y = \{f \leq c\}$ be a sublevel set containing all singular points of $\mathcal{X}$ in its interior. Then the double of Y along its boundary is still an Alexandrov space of $curv \geq 0$. By the result for the compact case Y has at most 2^n singular points.

Concavity of distance functions

- **Point-on-a-side comparison.** Recall that in a space of $curv \geq \kappa$ for any geodesic $[xy]$ and $z \in]xy[$, we have

$$|pz| \geq |\tilde{p}\tilde{z}|$$

where $\Delta\tilde{p}\tilde{x}\tilde{y} = \tilde{\Delta}^{\kappa}pxy$ and $|xz| = |\tilde{x}\tilde{z}|$.



How concave is that?

This means that distance function to a point is more concave than the distance function to a point in the space of constant curvature. How concave is that?

Definition

Define $\mathrm{md}_k(r)$ *by the formula*

$$\mathrm{md}_k(r) = \begin{cases} \frac{r^2}{2} \text{ if } r = 0 \\ \frac{1}{k}(1 - \cos(\sqrt{k}r)) \text{ if } k > 0 \\ \frac{1}{k}(1 - \cosh(\sqrt{|k|}r)) \text{ if } k < 0 \end{cases}$$

How concave is that?

This means that distance function to a point is more concave than the distance function to a point in the space of constant curvature. How concave is that?

Definition

Define $\mathrm{md}_k(r)$ *by the formula*

$$\mathrm{md}_k(r) = \begin{cases} \frac{r^2}{2} \text{ if } r = 0 \\ \frac{1}{k}(1 - \cos(\sqrt{k}r)) \text{ if } k > 0 \\ \frac{1}{k}(1 - \cosh(\sqrt{|k|}r)) \text{ if } k < 0 \end{cases}$$

Then

$$\mathrm{md}_k(0) = 0, \mathrm{md}_k'(0) = 1 \text{ and } \mathrm{md}_k'' + k\,\mathrm{md}_k \equiv 1$$

Let $f(x) = \mathrm{md}_k(|xp|)$ where $p \in \mathbb{M}_k^n$ - simply connected space form of constant curvature k we have $\mathrm{Hess}_x f = (1 - kf(x))\mathrm{Id}$. In particular, for any unit speed geodesic $\gamma(t)$ we have that

$$f(\gamma(t))'' + kf(\gamma(t)) = 1$$

Let $f(x) = \mathrm{md}_k(|xp|)$ where $p \in \mathbb{M}^n_k$ - simply connected space form of constant curvature k we have $\mathrm{Hess}_x f = (1 - kf(x))\mathrm{Id}$. In particular, for any unit speed geodesic $\gamma(t)$ we have that

$$f(\gamma(t))'' + kf(\gamma(t)) = 1$$

Note that for $k = 0$ this means that $\mathrm{Hess}_x f = \mathrm{Id}$ and

$$f(\gamma(t))'' = 1$$

Theorem (Toponogov restated)

Let $\mathcal{X}$ *have* $curv \geq k$ *and* $p \in \mathcal{X}$. *Let* $f(x) = \mathrm{md}_k(|xp|)$. *Then*

$$\mathrm{Hess}_x f \leq (1 - kf(x))\mathrm{Id}$$

and

$$f(\gamma(t))'' + kf(\gamma(t)) \leq 1$$

For any unit speed geodesic γ.

Theorem (Toponogov restated)

Let $\mathcal{X}$ *have* $curv \geq k$ *and* $p \in \mathcal{X}$. *Let* $f(x) = \mathrm{md}_k(|xp|)$. *Then*

$$\mathrm{Hess}_x f \leq (1 - kf(x))\mathrm{Id}$$

and

$$f(\gamma(t))'' + kf(\gamma(t)) \leq 1$$

For any unit speed geodesic γ.

These inequalities can be understood in the barrier sense or in the following sense.

Definition

A function $f\colon M \to \mathbb{R}$ *is called* λ*-concave if for any unit speed geodesic* $\gamma(t)$ *we have*

$$f(\gamma(t)) + \frac{\lambda t^2}{2} \text{ is concave}$$

Why do we care?

This means that spaces with $curv \geq \kappa$ naturally possess a LOT of semiconcave functions. Why do we care? They provide a useful technical tools.

Specifically, the following simple observation is of crucial importance.

Theorem

Gradient flow of a concave function f on an Alexandrov space is 1-Lipshitz.

Proof.

Let $p, q \in \mathcal{X}$ and let $\gamma\colon [0, d] \to \mathcal{X}$ be a unit speed geodesic with $\gamma(0) = p, \gamma(d) = q$. Here $d = |pq|$. Let ϕ_t be the gradient flow of f.

Why do we care?

This means that spaces with $curv \geq \kappa$ naturally possess a LOT of semiconcave functions. Why do we care? They provide a useful technical tools.

Specifically, the following simple observation is of crucial importance.

Theorem

Gradient flow of a concave function f on an Alexandrov space is 1-Lipshitz.

Proof.

Let $p, q \in \mathcal{X}$ and let $\gamma\colon [0, d] \to \mathcal{X}$ be a unit speed geodesic with $\gamma(0) = p, \gamma(d) = q$. Here $d = |pq|$. Let ϕ_t be the gradient flow of f.

Why do we care?

This means that spaces with $curv \geq \kappa$ naturally possess a LOT of semiconcave functions. Why do we care? They provide a useful technical tools.
Specifically, the following simple observation is of crucial importance.

Theorem

Gradient flow of a concave function f on an Alexandrov space is 1-Lipshitz.

Proof.

Let $p, q \in \mathcal{X}$ and let $\gamma\colon [0, d] \to \mathcal{X}$ be a unit speed geodesic with $\gamma(0) = p, \gamma(d) = q$. Here $d = |pq|$. Let ϕ_t be the gradient flow of f.

By the first variation formula

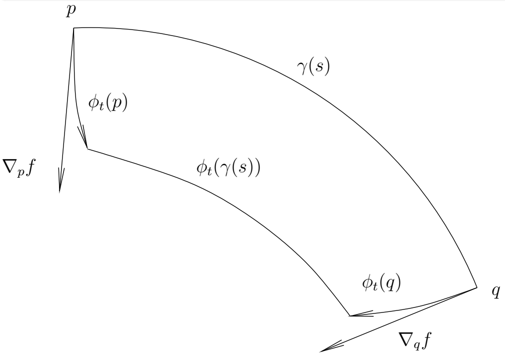
$$|\phi_t(p)\phi_t(q)|'_+(0) \leq \langle \nabla_q f, \gamma'(d)\rangle - \langle \nabla_p f, \gamma'(0)\rangle$$

By the first variation formula

$$|\phi_t(p)\phi_t(q)|'_+(0) \leq \langle \nabla_q f, \gamma'(d)\rangle - \langle \nabla_p f, \gamma'(0)\rangle$$

$$\langle \nabla_q f, \gamma'(d)\rangle - \langle \nabla_p f, \gamma'(0)\rangle = f(\gamma(d))' - f(\gamma(0))' \leq 0$$

since $f(\gamma(s))$ is concave. □



A similar argument shows that if f is λ-concave then ϕ_t is $e^{\lambda t}$-Lipschitz.
Yamaguchi's Fibration Theorem and gradient flows of semi-concave functions are key technical tools for proving topological results about manifolds with lower sectional curvature bounds. Typical application is the following

Let $X \in \mathrm{CBB}^n_\kappa$ and let $f\colon \mathcal{X} \to \mathbb{R}$ be λ-concave and locally Lipschitz. Let $p \in \mathcal{X}$. Then $df_p\colon T_p\mathcal{X} \to \mathbb{R}$ is Lipschitz and concave.

A similar argument shows that if f is λ-concave then ϕ_t is $e^{\lambda t}$-Lipschitz. Yamaguchi's Fibration Theorem and gradient flows of semi-concave functions are key technical tools for proving topological results about manifolds with lower sectional curvature bounds. Typical application is the following

Let $X \in \mathrm{CBB}^n_\kappa$ and let $f\colon \mathcal{X} \to \mathbb{R}$ be λ-concave and locally Lipschitz. Let $p \in \mathcal{X}$. Then $df_p\colon T_p\mathcal{X} \to \mathbb{R}$ is Lipschitz and concave.

Gradients of semiconcave functions

Definition

A vector in $T_p\mathcal{X}$ is called the gradient of f at p and denoted by $\nabla_p f$ is if it satisfies the following conditions

1. $df_p(\nabla_p f) = |(\nabla_p f|^2$

Gradients of semiconcave functions

Definition

A vector in $T_p\mathcal{X}$ is called the gradient of f at p and denoted by $\nabla_p f$ is if it satisfies the following conditions

1. $df_p(\nabla_p f) = |(\nabla_p f|^2$
2. $df_p(v) \leq \langle \nabla_p f, v \rangle$ *for any* $v \in T_p\mathcal{X}$

Gradients of semiconcave functions

Definition

A vector in $T_p\mathcal{X}$ is called the gradient of f at p and denoted by $\nabla_p f$ is if it satisfies the following conditions

1. $df_p(\nabla_p f) = |(\nabla_p f|^2$
2. $df_p(v) \le \langle \nabla_p f, v\rangle$ *for any* $v \in T_p\mathcal{X}$

Lemma

1. *Let $f\colon \mathcal{X} \to \mathbb{R}$ be λ-concave and Lipschitz and let $p \in \mathcal{X}$. Then $\nabla_p f$ exists and is unique.*

Gradients of semiconcave functions

Definition

A vector in $T_p\mathcal{X}$ is called the gradient of f at p and denoted by $\nabla_p f$ is if it satisfies the following conditions

1. $df_p(\nabla_p f) = |(\nabla_p f|^2$
2. $df_p(v) \le \langle \nabla_p f, v\rangle$ *for any* $v \in T_p\mathcal{X}$

Lemma

1. *Let $f\colon \mathcal{X} \to \mathbb{R}$ be λ-concave and Lipschitz and let $p \in \mathcal{X}$. Then $\nabla_p f$ exists and is unique.*
2. *$|\nabla_p f|$ is lower semicontinuous:*
   Let $(\mathcal{X}_i, p_i)$ be a sequence of Alexandrov spaces with $curv \ge \kappa$ converging to $(\mathcal{X}, p)$. And let $f_i\colon \mathcal{X}_i \to \mathbb{R}$ be L-Lipschitz and λ-concave and converge to $f\colon \mathcal{X} \to \mathbb{R}$. Then

$$|\nabla_p f| \le \liminf_{i\to\infty} |\nabla_{p_i} f_i|$$

Gradient curves of semiconcave functions

Definition

Let $\mathcal{X} \in \mathrm{CBB}_\kappa$ and let $f\colon \mathcal{X} \to \mathbb{R}$ be λ-concave and Lipschitz. A curve $\gamma\colon I \to \mathcal{X}$ is called a gradient curve of f if

$$\gamma'_+(t) = \nabla_{\gamma(t)} f \text{ for all } t$$

Gradient curves of semiconcave functions

Definition

Let $\mathcal{X} \in \mathrm{CBB}_\kappa$ and let $f\colon \mathcal{X} \to \mathbb{R}$ be λ-concave and Lipschitz. A curve $\gamma\colon I \to \mathcal{X}$ is called a gradient curve of f if

$$\gamma'_+(t) = \nabla_{\gamma(t)} f \text{ for all } t$$

Theorem

For any $p \in \mathcal{X}$ there exists unique gradient curve of f $\gamma\colon [0,\infty) \to \mathcal{X}$ such that $\gamma(0) = p$.

Gradient curves of semiconcave functions

Definition

Let $\mathcal{X} \in \mathrm{CBB}_\kappa$ and let $f\colon \mathcal{X} \to \mathbb{R}$ be λ-concave and Lipschitz. A curve $\gamma\colon I \to \mathcal{X}$ is called a gradient curve of f if

$$\gamma'_+(t) = \nabla_{\gamma(t)} f \text{ for all } t$$

Theorem

For any $p \in \mathcal{X}$ there exists unique gradient curve of f $\gamma\colon [0,\infty) \to \mathcal{X}$ such that $\gamma(0) = p$.

Remark

Gradient curves starting at different points can merge.

Theorem (Splitting theorem)

Let $\mathcal{X}^n$ be an Alexandrov space of $curv \geq 0$. Suppose $\mathcal{X}$ contains a line.
Then X is isometric to $Y \times \mathbb{R}$ for some Alexandrov space Y^{n-1} of $curv \geq 0$.

Theorem (Splitting theorem)

Let $\mathcal{X}^n$ be an Alexandrov space of $curv \geq 0$. Suppose $\mathcal{X}$ contains a line.
Then X is isometric to $Y \times \mathbb{R}$ for some Alexandrov space Y^{n-1} of $curv \geq 0$.

Corollary (Almost splitting)

Let $(M_i^n, p_i) \xrightarrow[i\to\infty]{G-H} (X, p)$ where $\sec_{M_i} \geq -\frac{1}{i}$, where p_i is the middle of a shortest geodesic of length $l_i \to \infty$. Then X is isometric to $Y \times \mathbb{R}$ for some nonnegatively curved space Y.

Theorem (Splitting theorem)

Let $\mathcal{X}^n$ be an Alexandrov space of $curv \geq 0$. Suppose $\mathcal{X}$ contains a line.
Then X is isometric to $Y \times \mathbb{R}$ for some Alexandrov space Y^{n-1} of $curv \geq 0$.

Corollary (Almost splitting)

Let $(M_i^n, p_i) \xrightarrow[i\to\infty]{G-H} (X, p)$ where $\sec_{M_i} \geq -\frac{1}{i}$, where p_i is the middle of a shortest geodesic of length $l_i \to \infty$. Then X is isometric to $Y \times \mathbb{R}$ for some nonnegatively curved space Y.

Theorem

Let $\mathcal{X}^n$ be a compact Alexandrov space of $curv \geq 0$. Then a finite cover of $\mathcal{X}$ is homeomorphic to $T^k \times \mathcal{Y}$ for some simply-connected nonnegatively curved Alexandrov space $\mathcal{Y}$. In particular, $\pi_1(X)$ is virtually abelian.

Definition

A closed smooth manifold M is called almost nonnegatively curved if it admits a sequence of Riemannian metrics g_i such that satisfy

$$\sec(M, g_i) \geq -1 \quad \textit{and} \quad (M, g_i) \to \{pt\} \textit{ as } i \to \infty.$$

by rescaling this is equivalent to
M admits a sequence of metrics g_i such that

$$\operatorname{diam}(M, g_i) \leq 1 \qquad \sec(M, g_i) \geq -1/i$$

Definition

A closed smooth manifold M is called almost nonnegatively curved if it admits a sequence of Riemannian metrics g_i such that satisfy

$$\sec(M, g_i) \geq -1 \quad \textit{and} \quad (M, g_i) \to \{pt\} \textit{ as } i \to \infty.$$

by rescaling this is equivalent to
M admits a sequence of metrics g_i such that

$$\operatorname{diam}(M, g_i) \leq 1 \qquad \sec(M, g_i) \geq -1/i$$

Example (Boring Example)

Let M^n be compact of $\sec \geq 0$. Then it is almost nonnegatively curved.

Example

Let N^3 be the space of real 3×3 of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

N^3 is a nilpotent Lie group. Let $\Gamma = N \cap \mathrm{SL}(3, \mathbb{Z})$. Then $M^3 = N/\Gamma$ admits almost nonnegative sectional curvature. But it does not admit nonnegative sectional curvature because Γ is not virtually abelian.

Example

Let N^3 be the space of real 3×3 of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

N^3 is a nilpotent Lie group. Let $\Gamma = N \cap \mathrm{SL}(3, \mathbb{Z})$. Then $M^3 = N/\Gamma$ admits almost nonnegative sectional curvature. But it does not admit nonnegative sectional curvature because Γ is not virtually abelian.

Theorem (Fukaya-Yamaguchi, Kapovitch-Petrunin-Tuschmann)

Let M^n be almost nonnegatively curved. Then $\pi_1(M)$ contains a nilpotent subgroup of index $\leq C(n)$.

Theorem (KPT)

Let M^n be almost nonnegatively curved. Then a finite cover of M is a fiber bundle over a nilmanifold with a simply connected fiber.

Theorem (KPT)

Let M^n be almost nonnegatively curved. Then a finite cover of M is a fiber bundle over a nilmanifold with a simply connected fiber.

Theorem (KPT)

Let M^n be almost nonnegatively curved. Then a finite cover of M' of M is nilpotent, i.e $\pi_1(M')$ is nilpotent and $\pi_1(M')$ acts on $\pi_k(M')$ nilpotently for all $k \geq 2$.

Example

Let $h\colon S^3 \times S^3 \to S^3 \times S^3$ be defined by

$$h : (x, y) \mapsto (xy, yxy).$$

This map is a diffeomorphism and the induced map h_* on $\pi_3(S^3 \times S^3)$ is given by the matrix $A_h = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Notice that the eigenvalues of A_h are different from 1 in absolute value.

Example

Let $h\colon S^3\times S^3\to S^3\times S^3$ be defined by

$$h:(x,y)\mapsto(xy,yxy).$$

This map is a diffeomorphism and the induced map h_* on $\pi_3(S^3\times S^3)$ is given by the matrix $A_h=\begin{pmatrix}1&1\\1&2\end{pmatrix}$. Notice that the eigenvalues of A_h are different from 1 in absolute value. Let M be the mapping cylinder of h. Clearly, M has the structure of a fiber bundle $S^3\times S^3\to M\to S^1$, and the action of $\pi_1(M)\cong\mathbb{Z}$ on $\pi_3(M)\cong\mathbb{Z}^2$ is generated by A_h. In particular, M is not a nilpotent space and hence, it does not admit almost nonnegative curvature.