Energy release rate along a kinked path

Matteo Negri

Dipartimento di Matematica - Università degli Studi di Pavia
Via A. Ferrata 1 - 27100 Pavia - Italy

e-mail: `matteo.negri@unipv.it`

Abstract. Considering anti-plane elasticity we provide an existence result for the energy release rate along a piecewise $C^{1,1}$ path that admits kinks. We provide two representations: an asymptotic one in terms of the stress intensity factor and an integral one in terms of the Eshelby tensor. Both the formulas make use of an implicit coefficient, depending on the crack path and obtained by a minimum problem.

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1 Introduction

In fracture mechanics the driving force for crack propagation is the so called energy release rate $\mathcal{G}$, i.e. the amount of elastic energy released by the body when the crack itself advances. According to Griffith's theory [6] the quasi-static propagation of a brittle crack is indeed governed by the balance between energy release rate and fracture toughness. This concept has been the subject of several fundamental works in fracture mechanics among which a couple of landmarks should be cited: the representation of $\mathcal{G}$ in terms of the stress intensity factors, due to Irwin [8], and the representation in terms of the J-integral, due to Rice [13].

On the mathematical side a fundamental result, concerning the energy release rate and its representations, is that of Destuynder and Djaoua [5]. In this work the above mentioned results of Irwin and Rice are settled in a rigorous way, i.e. in the framework of Sobolev spaces, considering the simple setting of a straight crack. The technical tools provided in [5] are enough to study the quasi-static evolution in the case of a predefined crack path of class C^2 (the reader interested in evolution problems may find in [12] an overview of the most recent results). Eventough straight cracks occur frequently, especially in material testing, it seems more realistic to study the propagation without any geometrical restriction on the fracture. Several results have been achieved in this perspective, employing however an evolution model not fully consistent with Griffith's theory (the interested reader will find a complete survey in [1]). At the present stage, understanding the phenomenon of rupture in its full generality is probably out of the grasp: it is not completely clear how to deal with branching and even which criteria performs better, among maximum energy release rate, principle of local symmetry, maximum hoop stress, vectorial J-integral etc. To pursue this target it seems fundamental to provide first a set of basic technical tools, more flexible and general than those developed in [5]. In this perspective, focusing on the energy release rate, there are among the others four recent results that deserve to be mentioned. In the first, Brokate and Khludnev [2] assume that the initial crack is rectilinear and prove that the classical integral representation formula with the Eshelby tensor holds true for every extention in $W^{2,p}$, with $p > 2$. In the second, due to Lazzaroni and Toader [10], the hypothesis on the initial crack is dropped, assuming this time that the path is in $C^{1,1}$, and thus in $W^{2,\infty}$. Using again a rectilinear crack, Knees and Mielke [9] provide an integral formula for $\mathcal{G}$ in the case of finite elasticity. In the above papers the regularity of the path does not allow for kinks, which are however an intriguing phenomenon in crack propagation. This feature has been taken into account in a paper by Chambolle, Francofort and Marigo [3]. Using an original approach, based on a duality argument, they assume that the initial crack is straight and that the extention are rectifiable sets. Under these assumptions they compute the energy release rate and provide a formula in terms of the stress intensity factors.

Our result follows the direction of these papers. We assume that the crack is of class $C^{1,1}$ apart from a point, the tip of the initial fracture, where we allow for kinks. Adapting the approach of [5] based on mappings, we compute the energy release rate, first in terms of the stress intensity factor. Then, taking advantage of the anti-plane setting we provide an integral representation with the Eshelby tensor. In both the formulas, as in the one of [3], there are some implicit coefficients which depend on the kink angle.

We conclude our introduction with a technical consideration. In [5], the energy release rate is represented first as volume integral, then as a path integral (the J-integral) and finally in terms of the stress intensity factors. More and more regularity is needed to pass from a representation to the following one. In our setting, the presence of the kink makes it more convenient to somehow invert the above scheme: with a direct proof, first comes the representation with the stress intensity factors, then a simple manipulation allows to write $\mathcal{G}$ as the product of a scalar coefficient, depending on the kink angle, with the stress intensity factor, for which the volume integral representation holds.

2 Elastic energy for the initial configuration

We assume that the reference domain Ω is open, bounded and Lipschitz regular. Since our argument is local, it does not seem too restrictive to assume from the very beginning that the initial crack K_0 can be represented as the graph of a function f, e.g. in the form $(t, f(t))$ with $t \in [-1, 0]$ and $f(0) = 0$. For technical reasons it will be necessary to assume that f is at least of class $C^{1,1}$ in $[-1, 0]$.

Given $\partial_D\Omega$ relatively open in $\partial\Omega$ and $g \in H^1(\Omega)$ the set of admissible configurations and variations will be respectively

$$\mathcal{U}(\Omega \setminus K_0) = \{u \in H^1(\Omega \setminus K_0) : u = g \ \partial_D\Omega\},$$

$$\mathcal{V}(\Omega \setminus K_0) = \{v \in H^1(\Omega \setminus K_0) : v = 0 \ \partial_D\Omega\}.$$

Now, for $u \in \mathcal{U}(\Omega \setminus K_0)$ the elastic energy is given by

$$E_0(u) = \frac{1}{2} \int_{\Omega \setminus K_0} \mu |\nabla u|^2 \, dx .$$

Since $\Omega \setminus \Gamma_0$ is connected, there exists a unique minimizer u_0 in $\mathcal{U}(\Omega \setminus K_0)$. Under these assumptions we know from [10] and [7] that there exists a unique value $k \in \mathbb{R}$ such that, in a small neighborhood U of the origin,

$$u_0 = k\hat{s} + \bar{u} , \tag{1}$$

where $\bar{u} \in H^2(U \setminus K_0)$ and $\hat{s} = \rho^{1/2} \sin((\theta - \alpha)/2)$ with $\alpha = \arctan f'_-(0)$.

3 Elastic energy for the incremental configuration

Consider an extention of f to $[0, 1]$ of class $C^{1,1}$ in $[0, 1]$. Note that f is continuous but in general it is not of class $C^{1,1}$ in the whole $[-1, 1]$, in particular $f'_-(0)$ and $f'_+(0)$ exist but they may not coincide (in such a case there is a kink in the origin).

Then for $h \in (0, 1]$ let the crack K_h be defined by $K_h = \{(t, f(t)) : t \in [-1, h]\}$. In this case, the set of configurations and variations are respectively

$$\mathcal{U}(\Omega \setminus K_h) = \{u \in H^1(\Omega \setminus K_h) : u = g \ \partial_D\Omega\},$$

$$\mathcal{V}(\Omega \setminus K_h) = \{v \in H^1(\Omega \setminus K_h) : v = 0 \ \partial_D\Omega\}.$$

For $u \in \mathcal{U}(\Omega \setminus K_h)$ the elastic energy is now

$$E_h(u) = \frac{1}{2} \int_{\Omega \setminus K_h} \mu |\nabla u|^2 \, dx \, .$$

Clearly there exists a unique minimizer u_h of E_h in the space $\mathcal{U}(\Omega \setminus K_h)$.

Definition 3.1 *We define the energy release rate as*

$$\mathcal{G}(u_0) = - \lim_{h \to 0^+} \frac{E_h(u_h) - E_0(u_0)}{\mathcal{H}^1(K_h \setminus K_0)} \tag{2}$$

(we will prove existence in Theorem 11.4 and give a representation in Theorem 12.1).

In the above definition the increment in crack length can be computed easily as

$$\mathcal{H}^1(K_h \setminus K_0) = \int_{(0,h)} \left(1 + |f'(s)|^2\right)^{1/2} ds \, .$$

Clearly, being f of class $C^{1,1}$ in $[0,1]$ the above term is of order h and, more precisely,

$$\lim_{h \to 0} h^{-1} \int_{(0,h)} \left(1 + |f'(s)|^2\right)^{1/2} ds = \left(1 + |f'_+(0)|^2\right)^{1/2} .$$

Hence we can re-write the energy release rate as

$$\mathcal{G}(u_0) = \frac{-1}{(1 + |f'_+(0)|^2)^{1/2}} \lim_{h \to 0^+} \frac{E_h(u_h) - E_0(u_0)}{h} \, . \tag{3}$$

In order to compute the above limit we will make some changes of variable. For sake of clarity they are splitted in three steps.

4 Linearization of the initial crack

The first change of variable linearizes the initial crack, at least in the vicinity of the tip. Let $\eta \in C_0^\infty(\Omega)$ with $\eta = 1$ in a (small) neighborhood of the origin. Consider an extention $\tilde{f}$ of f in $C^{1,1}(-1,1)$ and define $\tau(x_1) = x_1 f'_-(0) - \tilde{f}(x_1)$.

Let us consider the map $\Upsilon : \Omega \to \Omega$ given by

$$\Upsilon(x) = x + \tau(x_1)\eta(x) \, \hat{e}_2 \, .$$

Υ is a $C^{1,1}$ map of Ω in itself, moreover

$$D\Upsilon = \begin{pmatrix} 1 & 0 \\ \tau'(x_1)\eta(x) + \tau(x_1)\eta_{,1}(x) & 1 + \tau(x_1)\eta_{,2}(x) \end{pmatrix} ,$$

$$\det D\Upsilon = 1 + \tau(x_1)\eta_{,2}(x) \, ,$$

hence upon a suitable choice of η it turns out that Υ is a diffeomorphism of Ω into itself. For convenience we will denote $\Gamma_0 = \Upsilon(K_0)$ and $\Gamma_h = \Upsilon(K_h)$. Note that the fracture sets Γ_h, for $h \geq 0$, are contained in the graph of the function

$$\bar{f}(x_1) = f(x_1) + \tau(x_1)\eta(x_1, f(x_1)) \, ,$$

but only Γ_0 is linear in a (small) neighborhood of the crack tip.

In this setting the functional spaces in the reference configuration $\Omega \setminus \Gamma_0$ becomes

$$\mathcal{U}(\Omega \setminus \Gamma_0) = \{ u \in H^1(\Omega \setminus \Gamma_0) : u = g \ \partial_D \Omega \} \, , \tag{4}$$

$$\mathcal{V}(\Omega \setminus \Gamma_0) = \{ v \in H^1(\Omega \setminus \Gamma_0) : v = 0 \ \partial_D \Omega \} , \tag{5}$$

while the energy becomes

$$E(u) = \frac{1}{2} \int_{\Omega \setminus \Gamma_0} \nabla u A \nabla u^T \, dx \tag{6}$$

with

$$A = \mu D\Upsilon D\Upsilon^T \mathrm{det} D\Upsilon^{-1} .$$

Note that the coefficients of A are of class $C^{0,1}$ in the whole Ω. As before, there exists a unique minimizer in $\mathcal{U}(\Omega \setminus \Gamma_0)$. By abuse of notation we will denote it again by u_0. Clearly u_0 solves also the variational problem

$$\int_{\Omega \setminus \Gamma_0} \nabla u_0 A \nabla v^T \, dx = 0 , \qquad \text{for every } v \in \mathcal{V}(\Omega \setminus \Gamma_0). \tag{7}$$

It is proved in [10] that u_0 can be represented again in the form

$$u_0 = k \hat{s} + \bar{u} , \tag{8}$$

where both k and $\hat{s}$ are the same as in (1) while $\bar{u}$ belongs to $H^2(U \setminus \Gamma_0)$. Before proceeding, let us state a Lemma which, despite its simplicity, will be fundamental in the sequel.

Lemma 4.1 *Let u_0 be the solution of* (7). *Then*

$$\int_{B_r} |\nabla u_0|^2 \, dx = O(r) .$$

Proof. By the representation (8) we can write

$$\int_{B_r} |\nabla u_0|^2 \, dx \leq 2k^2 \int_{B_r} |\nabla \hat{s}|^2 \, dx + 2 \int_{B_r} |\nabla \bar{u}|^2 \, dx .$$

First, note that $|\nabla \hat{s}|^2 = (1/4)\rho^{-1}$, hence

$$2 \int_{B_r} |\nabla \hat{s}|^2 \, dx = \pi r .$$

Then, by Sobolev inclusions $\nabla \bar{u} \in L^p$ for every $p < \infty$, hence, denoting by χ_r the characteristic function of B_r, we can write the estimate

$$\int_{B_r} |\nabla \bar{u}|^2 \, dx = \int_\Omega \chi_r |\nabla \bar{u}|^2 \, dx \leq \Big| \int_\Omega |\nabla \bar{u}|^{2p} \, dx \Big|^{1/p} |B_r|^{1/p'} = O(r^{2/p'}) .$$

The thesis follows upon choosing $p' > 2$. ∎

Finally, let us consider the incremental configuration. With this change of variable the spaces becomes

$$\mathcal{U}(\Omega \setminus \Gamma_h) = \{ u \in H^1(\Omega \setminus \Gamma_h) : u = g \ \partial_D \Omega \} . \tag{9}$$

$$\mathcal{V}(\Omega \setminus \Gamma_h) = \{ v \in H^1(\Omega \setminus \Gamma_h) : v = 0 \ \partial_D \Omega \} , \tag{10}$$

while the elastic energy becomes

$$E_h(u) = \frac{1}{2} \int_{\Omega \setminus \Gamma_h} \nabla u A \nabla u^T \, dx .$$

The unique minimizer in $\mathcal{U}(\Omega \setminus \Gamma_h)$ will be denoted again by u_h.

With the piece of notation introduced in this section, the definition of energy release rate will be given again by (3).

5 Linearization of the incremental crack

The next change of variable completes the linearization of crack, transforming Γ_h into a small extention of Γ_0, that will remain unchanged. Clearly this mapping cannot be as regular as the previous one: remember that Γ_h is contained in the graph of $\bar{f}$ and that in general $\bar{f}$ is not of class $C^{1,1}$ in $[-1,1]$.

Since the idea is to map the graph of $\bar{f}$ on the graph of $x_1 f'_-(0)$, at least in the vicinity of the tip, let us introduce the function $\xi(x_1) = x_1 f'_-(0) - \bar{f}(x_1)$. Note that ξ is a Lipschitz function in $[-1,1]$, of class $C^{1,1}$ in $[-1,0]$ and in $[0,1]$. Then, for $1 < l < r$ let $\phi \in C^\infty(0,+\infty)$ with $\phi = 1$ in $(0,l]$, $\phi = 0$ in $[m,+\infty)$ and such that $\|\phi'\|_\infty < 1/2$, $\|\phi'\|_\infty \|\xi'\|_\infty < 1$. For $h \ll 1$ we can define the map $\Phi_h : \Omega \to \Omega$ as

$$\Phi_h(x) = x + \xi(x_1)\phi(|x|/h)\,\hat{e}_2\,.$$

Remark 5.1 *It is important to stress that the support of ϕ is scaled by h, so that $\Phi_h(x) = x$ in $\Omega \setminus B_{rh}$. Note that this is not so for the cut off function η appearing in Υ. Thanks to this fact the translation term $\xi(x_1)\phi(|x|/h)$ is of order h.*

It is clear that Φ_h is a Lipschitz map of Ω in itself. Let us check that it is indeed a diffeomorphism. First, note that for x_1 negative and sufficiently small we have $\xi(x_1) = 0$, hence if h is sufficiently small $\Phi_h(x) = x$ for every $x \in \Omega$ with $x_1 < 0$ and in particular $\Phi_h(\Gamma_0) = \Gamma_0$. Therefor, it is sufficient to consider the case $x_1 > 0$. Dropping the variables and denoting $\cos = x_1/|x|$, $\sin = x_2/|x|$, we can write a.e. in Ω

$$D\Phi_h = \begin{pmatrix} 1 & 0 \\ \xi'\phi + \xi\phi' \cos/h & 1 + \xi\phi' \sin/h \end{pmatrix}$$

and $\det D\Phi_h = 1 + \xi\phi' \sin/h$. Being ξ of class $C^{1,1}$ in $[0,1]$ with $\xi(0) = 0$ we have

$$|\xi\phi' \cos/h| = |\xi(x_1)/h|\,|\phi'(|x|/h)|\,|x_1/|x|| \le \|\xi'\|_\infty \|\phi'\|_\infty < 1\,.$$

Therefore Φ_h is a diffeomorphism. Arguing as above it is easy to see that $\|D\Phi_h\|_\infty$ is bounded uniformly with respect to h.

For $u \in \mathcal{U}(\Omega \setminus \Gamma_h)$ the change of variable allows to write the energy as

$$E_h(u) = \frac{1}{2}\int_{\Omega\setminus\Phi_h(\Gamma_h)} \nabla(u \circ \Phi_h^{-1}) A_h \nabla(u \circ \Phi_h^{-1})^T\,dx\,, \tag{11}$$

where

$$A_h = D\Phi_h (A \circ \Phi_h^{-1}) D\Phi_h^T \det D\Phi_h^{-1}\,.$$

By the properties of A and Φ_h it follows that A_h is uniformly bounded and that $A_h = A$ in $\Omega \setminus B_{rh}$, in particular A_h converges pointwise to A.

6 Local translation

Let ϕ be as in the previous section. Define $\psi_h(x) = \phi(|x|/h)$. For $h \ll 1$ let

$$\Psi_h(x) = x + h\psi_h(x)\,\hat{e}\,, \tag{12}$$

where $\hat{e}$ denotes the unit vector $(1, f'_-(0))/(1 + f'_-(0)^2)^{1/2}$. Note that the vertical component of Φ_h has the same order as the component of Ψ_h along $\hat{e}$.

Then,

$$D\Psi_h = I + h\hat{e} \otimes \nabla\psi_h\,,$$

$$\det D\Psi_h = 1 + h\,\mathrm{tr}(\hat{e} \otimes \nabla\psi_h) + h^2 \det(\hat{e} \otimes \nabla\psi_h)\,.$$

Since $\|h\nabla\psi_h\|_\infty \le \|\phi'\|_\infty < 1/2$, it is easy to check that for h small enough Ψ_h is a diffeomorphism of $\Omega \setminus \Gamma_0$ into $\Omega \setminus \Phi(\Gamma_h)$.

For $u \in \mathcal{U}(\Omega \setminus \Gamma_h)$ the function $u \circ \Phi_h^{-1} \circ \Psi_h$ belongs to $\mathcal{U}(\Omega \setminus \Gamma_0)$ and by (11) the energy is

$$E_h(u) = \frac{1}{2} \int_{\Omega \setminus \Gamma_0} \nabla (u \circ \Phi_h^{-1} \circ \Psi_h) A_{h,0} \nabla (u \circ \Phi_h^{-1} \circ \Psi_h)^T \, dx \,, \tag{13}$$

where

$$A_{h,0} = D\Psi_h^{-1} (A_h \circ \Psi_h) D\Psi_h^{-T} \det D\Psi_h \,.$$

For later convenience, let us define also the matrix $A'_h = A_{h,0} - A$. The main properties of $A_{h,0}$ are contained in the next Lemma. Its proof is left to the reader.

Lemma 6.1 *$A_{h,0}$ is symmetric, uniformly elliptic and uniformly bounded. Moreover $A_{h,0} = A$ in $\Omega \setminus B_{rh}$, in particular $A_{h,0} \to A$ pointwise in Ω.*

Finally, for $u \in \mathcal{U}(\Omega \setminus \Gamma_0)$ define

$$E_{h,0}(u) = \frac{1}{2} \int_{\Omega \setminus \Gamma_0} \nabla u A_{h,0} \nabla u^T \, dx \,.$$

Clearly $u_{h,0} = u_h \circ \Phi_h^{-1} \circ \Psi_h$ is the minimizer of $E_{h,0}$ in $\mathcal{U}(\Omega \setminus \Gamma_0)$. Clearly $u_{h,0}$ is also the solution of the variational problem

$$\int_{\Omega \setminus \Gamma_0} \nabla u_{h,0} A_{h,0} \nabla v^T \, dx = 0 \,, \qquad \text{for every } v \in \mathcal{V}(\Omega \setminus \Gamma_0). \tag{14}$$

7 Expansion and convergence of the displacements

In order to study the convergence of $u_{h,0}$ to u_0 let $u'_h \in \mathcal{V}(\Omega \setminus \Gamma_0)$ be defined by

$$u_{h,0} = u_0 + h^{1/2} u'_h \,. \tag{15}$$

The main property of u'_h is stated in the next Lemma.

Lemma 7.1 *Let u'_h be defined by* (15)*. Then $u'_h \rightharpoonup 0$ in $\mathcal{V}(\Omega \setminus \Gamma_0)$.*

Proof. From the variation formulation (7) we can write that

$$\int_{\Omega \setminus \Gamma_0} \nabla u_0 A_{h,0} \nabla v^T \, dx = - \int_{\Omega \setminus \Gamma_0} \nabla u_0 (A - A_{h,0}) \nabla v^T \, dx \,, \qquad \text{for every } v \in \mathcal{V}(\Omega \setminus \Gamma_0).$$

Then, from (14) we get

$$\int_{\Omega \setminus \Gamma_0} (\nabla u_{h,0} - \nabla u_0) A_{h,0} \nabla v^T \, dx = \int_{\Omega \setminus \Gamma_0} \nabla u_0 A'_h \nabla v^T \, dx \,, \qquad \text{for every } v \in \mathcal{V}(\Omega \setminus \Gamma_0).$$

Hence $u'_h \in \mathcal{V}(\Omega \setminus \Gamma_0)$ solves

$$\int_{\Omega \setminus \Gamma_0} \nabla u'_h A_{h,0} \nabla v^T \, dx = h^{-1/2} \int_{\Omega \setminus \Gamma_0} \nabla u_0 A'_h \nabla v^T \, dx \,, \qquad \text{for every } v \in \mathcal{V}(\Omega \setminus \Gamma_0). \tag{16}$$

As A'_h is supported in B_{rh} by Lemma 4.1 we deduce that $h^{-1/2} \nabla u_0 A'_h$ is uniformly bounded in L^2. Hence

$$h^{-1/2} \int_{\Omega \setminus \Gamma_0} \nabla u_0 A'_h \nabla v^T \, dx \leq c \, \| v \|_{\mathcal{V}(\Omega \setminus \Gamma_0)} \,,$$

where c is independent of h. Then by Lemma 6.1 and by the Lax-Milgram Theorem $\| u'_h \|_{\mathcal{V}(\Omega \setminus \Gamma_0)}$ is bounded. Hence, up to subsequences $u'_h \rightharpoonup u'_0$ in $\mathcal{V}(\Omega \setminus \Gamma_0)$. Passing to the limit in (16) yields

$$\int_{\Omega \setminus \Gamma_0} \nabla u'_0 A \nabla v^T \, dx = 0 \,, \qquad \text{for every } v \in \mathcal{V}(\Omega \setminus \Gamma_0),$$

from which the thesis follows. ∎

Remark 7.2 *We remark that the expansion of* (15)*, here suggested by Lemma 4.1, is actually the same appearing in [11] and [3]. As we will see, this is not in contrast with the existence of $\mathcal{G}$ that requires a variation of energy of order h. To conclude, it is worth to mention that in general the convergence of Lemma 7.1 is not strong.*

8 Re-writing the energy release rate

For $u' \in \mathcal{V}(\Omega \setminus \Gamma_0)$ let $u = u_0 + h^{1/2}u'$ and consider the functional

$$\mathcal{G}_h(u') = \frac{E_{h,0}(u) - E_0(u_0)}{h} = \frac{E_{h,0}(u_0 + h^{1/2}u') - E_0(u_0)}{h}.$$

Clearly u'_h, defined by (15), is the minimizer of $\mathcal{G}_h$ in $\mathcal{V}(\Omega \setminus \Gamma_0)$. Let us write $\mathcal{G}_h(u')$ more explicitly. Using the representation $u = u_0 + h^{1/2}u'$ we get

$$\nabla u A_{h,0} \nabla u^T = \nabla u_0 A_{h,0} \nabla u_0^T + 2h^{1/2} \nabla u' A_{h,0} \nabla u_0^T + h \nabla u' A_{h,0} \nabla u'^{\,T}.$$

Writing $A_{h,0} = A + A'_h$ the first term above reads

$$\nabla u_0 A_{h,0} \nabla u_0^T = \nabla u_0 A \nabla u_0^T + \nabla u_0 A'_h \nabla u_0^T.$$

Then

$$\begin{aligned} \mathcal{G}_h(u') &= \frac{1}{2} \int_{\Omega \setminus \Gamma_0} h^{-1} (\nabla u A_{h,0} \nabla u^T - \nabla u_0 A \nabla u_0^T)\, dx \\ &= \frac{1}{2} \int_{\Omega \setminus \Gamma_0} h^{-1} \nabla u_0 A'_h \nabla u_0^T + 2h^{-1/2} \nabla u' A_{h,0} \nabla u_0^T + \nabla u' A_{h,0} \nabla u'^{\,T}\, dx. \end{aligned}$$

Since A'_h is supported in B_{rh} we can write

$$\int_{\Omega \setminus \Gamma_0} \nabla u_0 A'_h \nabla u_0^T\, dx = \int_{B_{rh} \setminus \Gamma_0} \nabla u_0 A'_h \nabla u_0^T\, dx.$$

For the same reason, being $u' \in \mathcal{V}(\Omega \setminus \Gamma_0)$ the variational formulation (7) for u_0 allows to write

$$\int_{\Omega \setminus \Gamma_0} \nabla u' A_{h,0} \nabla u_0^T\, dx = \int_{\Omega \setminus \Gamma_0} \nabla u' A'_h \nabla u_0^T\, dx = \int_{B_{rh} \setminus \Gamma_0} \nabla u' A'_h \nabla u_0^T\, dx.$$

Therefore

$$\begin{aligned} \mathcal{G}_h(u') \;=\; & \frac{1}{2} \int_{B_{rh} \setminus \Gamma_0} h^{-1} \nabla u_0 A'_h \nabla u_0^T + \\ & \frac{1}{2} \int_{\Omega \setminus \Gamma_0} \nabla u' A_{h,0} \nabla u'^{\;T}\, dx + \int_{B_{rh} \setminus \Gamma_0} h^{1/2} \nabla u' A'_h \nabla u_0^T\, dx. \end{aligned}$$

Next, we introduce the functionals

$$\mathcal{I}_h(u_0) = \frac{1}{2} \int_{B_{rh} \setminus \Gamma_0} h^{-1} \nabla u_0 A'_h \nabla u_0^T\, dx \tag{17}$$

and

$$\mathcal{J}_h(u') = \frac{1}{2} \int_{\Omega \setminus \Gamma_0} \nabla u' A_{h,0} \nabla u'^{\;T}\, dx + \int_{B_{rh} \setminus \Gamma_0} h^{-1/2} \nabla u' A'_h \nabla u_0^T\, dx. \tag{18}$$

Clearly

$$\mathcal{G}_h(u') = \mathcal{I}_h(u_0) + \mathcal{J}_h(u')$$

and $u'_h \in \operatorname{argmin}\{\mathcal{J}_h(u') : u' \in \mathcal{V}(\Omega \setminus \Gamma_0)\}$ where u'_h has been defined in (15).

Finally, we can re-define the energy release rate as

$$\mathcal{G}(u_0) = \frac{-1}{(1+|f'_+(0)|^2)^{1/2}} \lim_{h\to 0} \mathcal{G}_h(u'_h) \,.$$

In order to study the limit of $\mathcal{G}_h$ it seems natural to find the Γ-limit [4] of $\mathcal{G}_h$ in $\mathcal{V}_0(\Omega \setminus \Gamma_0)$. To this end, the main difficulty come from $\mathcal{J}_h$ and in particular from the second term because $h^{-1/2} A_{h,0} \nabla u_0 \rightharpoonup 0$ in L^2 but in general this convergence is not strong. Choosing the strong topology it is easy to see that the Γ-limit of $\mathcal{J}_h$ would be

$$\frac{1}{2} \int_{\Omega\setminus\Gamma_0} \nabla u' A \nabla u'^{\,T} \, dx \,,$$

however u'_h is not strongly compact in $\mathcal{V}_0(\Omega \setminus \Gamma_0)$. Conversely, employing the weak topology the compactness of u'_h is easily proved, by Lemma 7.1, but it is harder to characterize a Γ-limit. In fact, the div-curl Lemma seems not useful to treat the term $h^{-1/2}\nabla u'_h A'_h \nabla u_0$ which is concentrated around the crack tip, i.e. on the boundary of $\Omega\setminus\Gamma_0$. For this reasons it is convenient to blow up the functionals before studying their limits. The blow up will also highlight the dependence on $f'_\pm(0)$.

9 Blow up

Consider $R > r$, let α_h and β_h be defined as

$$\alpha_h(s) = \begin{cases} h & 0 \le s < r \\ (R-hr)/(R-r) & r \le s \le R \\ 1 & s > R \,, \end{cases} \qquad \beta_h(s) = \begin{cases} 0 & 0 \le s < r \\ Rr(h-1)/(R-r) & r \le s \le R \\ 0 & s > R \,. \end{cases}$$

Define the piecewise affine function $\lambda_h(s) = \alpha_h(s)s + \beta_h(s)$ and the map $\Lambda_h(x) = x\lambda_h(|x|)/|x|$. It is not difficult to see that Λ_h is a diffeomorphism in the plane. Let us also define Λ_0 as the pointwise limit of Λ_h.

Now, let us apply this blow-up. As $\Lambda_h(x) = hx$ in B_{rh} a simple computation gives

$$\mathcal{I}_h(u_0) = \frac{1}{2} \int_{B_r\setminus\Gamma_0} h^{-1} \nabla(u_0 \circ \Lambda_h) B'_h \nabla(u_0\circ\Lambda_h)^T \, dx \,, \tag{19}$$

where $B'_h = A'_h \circ \Lambda_h$. Similarly, the second integral in $\mathcal{J}_h$ becomes

$$\int_{B_r\setminus\Gamma_0} h^{-1/2} \nabla(u'\circ\Lambda_h) B'_h \nabla(u_0\circ\Lambda_h)^T \, dx \,.$$

Finally, the first integral in $\mathcal{J}_h$ becomes

$$\int_{\Omega\setminus\Gamma_0} \nabla(u'\circ\Lambda_h) B_{h,0} \nabla(u'\circ\Lambda_h)^T \,,$$

where $B_{h,0} = D\Lambda_h^{-1}(A_{h,0}\circ\Lambda_h) D\Lambda_h^{-T} \det D\Lambda_h$. Hence

$$\mathcal{J}_h(u') = \frac{1}{2} \int_{\Omega\setminus\Gamma_0} \nabla(u'\circ\Lambda_h) B_{h,0} \nabla(u'\circ\Lambda_h)^T + \int_{B_r\setminus\Gamma_0} h^{-1/2} \nabla(u'\circ\Lambda_h) B'_h \nabla(u_0\circ\Lambda_h)^T \, dx \,.$$

Then, for $w \in \mathcal{V}(\Omega\setminus\Gamma_0)$ let

$$\widehat{\mathcal{J}}_h(w) = \frac{1}{2} \int_{\Omega\setminus\Gamma_0} \nabla w B_{h,0} \nabla w^T + \int_{B_r\setminus\Gamma_0} h^{-1/2} \nabla w B'_h \nabla(u_0\circ\Lambda_h)^T \, dx \tag{20}$$

and let $w_h = u'_h \circ \Lambda_h$. Then $w_h \in \operatorname{argmin}\{\widehat{\mathcal{J}}_h(w) : w \in \mathcal{V}(\Omega\setminus\Gamma_0)\}$. Finally let

$$\widehat{\mathcal{G}}_h(w) = \mathcal{I}_h(u_0) + \widehat{\mathcal{J}}_h(w) \,.$$

Obviously $w_h \in \text{argmin}\{\widehat{\mathcal{G}}_h(w) : w \in \mathcal{V}(\Omega \setminus \Gamma_0)\}$, hence $\mathcal{G}_h(u'_h) = \widehat{\mathcal{G}}_h(w_h)$ and

$$\mathcal{G}(u_0) = \frac{-1}{(1+|f'_+|^2(0))^{1/2}} \lim_{h\to 0} \widehat{\mathcal{G}}_h(w_h)\,. \tag{21}$$

Note that $w_h = u'_h$ in $\Omega \setminus B_R$ by the definition of Λ_h.

Let us give a closer look to the limits of the matrices B'_h and $B_{h,0}$ appearing in (19) and (20).

Lemma 9.1 *In the ball B_r*

$$B_{h,0} \to \mu (FD\Psi)^{-1}(FD\Psi)^{-T}\det(FD\Psi)\,, \tag{22}$$

where F is uniformly bounded and depends on f only through $f'_\pm(0)$. In $\Omega \setminus B_r$

$$B_{h,0} \to D\Lambda_0^{-1}(A \circ \Lambda_0) D\Lambda_0^{-T}\det D\Lambda_0\,. \tag{23}$$

Proof. Remember that $A_{h,0} = D\Psi_h^{-1}(A_h \circ \Psi_h)D\Psi_h^{-T}\det D\Psi_h = A + A'_h$ and that in B_r we have $\Lambda_h(x) = hx$. The term $h\nabla\psi_h(hx)$ appearing in $D\Phi_h \circ \Lambda_h$ becomes simply $\nabla\psi(x)$ and thus $D\Psi_h \circ \Lambda_h = D\Psi$ is independent of h. Moreover $D\Lambda_h = hI$, it follows that

$$B_{h,0} = A_{h,0} \circ \Lambda_h = D\Psi^{-1}(A_h \circ \Psi_h \circ \Lambda_h) D\Psi^{-T}\det D\Psi\,.$$

As $\Psi_h(x) = x + h\psi_h(x)\,\hat{e}$ we can write

$$\Psi_h \circ \Lambda_h(x) = h(x + \psi(x)\hat{e}) = h\zeta(x)\,.$$

Remember that $A_h = D\Phi_h(A \circ \Phi_h^{-1})D\Phi_h^T \det D\Phi_h^{-1}$ where

$$D\Phi_h = \begin{pmatrix} 1 & 0 \\ \xi'\phi + \xi\phi' \cos/h & 1 + \xi\phi' \sin/h \end{pmatrix}.$$

Hence $D\Phi_h \circ \Psi_h \circ \Lambda_h$ has the same form as above adopting the notation

$$\xi = \xi(h\zeta_1)\,, \quad \xi' = \xi'(h\zeta_1)\,, \quad \phi' = \phi'(|\zeta|)\,, \quad \cos = \zeta_1/|\zeta|\,, \quad \sin = \zeta_2/|\zeta|\,.$$

We recall that $\xi(x_1) = x_1 f'_-(0) - \bar{f}(x_1)$ and that $\bar{f}(x_1) = f(x_1) + (x_1 f'_-(0) - \tilde{f}(x_1))$ for $|x_1| \ll 1$; hence for $h \ll 1$ we can write

$$\xi(x_1) = \tilde{f}(x_1) - f(x_1)\,.$$

Let us also recall that $\tilde{f}$ is a $C^{1,1}$ extention, hence $\tilde{f}'_\pm(0) = f'_-(0)$. It follows that $\xi(0) = 0$, $\xi'_-(0) = 0$ and $\xi'_+(0) = f'_-(0) - f'_+(0)$. Then, we have $\lim_{h\to 0}\xi(h\zeta_1) = 0$, and

$$\lim_{h\to 0} \xi'(h\zeta_1) = \begin{cases} \xi'_-(0) & \text{if } \zeta_1 < 0 \\ \xi'_+(0) & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } \zeta_1 < 0 \\ f'_-(0) - f'_+(0) & \text{otherwise.} \end{cases}$$

In a similar way

$$\lim_{h\to 0} \xi(h\zeta_1)/h = \begin{cases} 0 & \text{if } \zeta_1 < 0 \\ \zeta_1(f'_-(0) - f'_+(0)) & \text{otherwise.} \end{cases}$$

Therefore the pointwise limit F of the matrix $D\Phi_h \circ \Psi_h \circ \Lambda_h$ is bounded and depends on f only by means of $f'_\pm(0)$. Now, let us consider the limit of $A \circ \Phi_h^{-1} \circ \Psi_h \circ \Lambda_h$. Clearly $\Phi_h \circ \Psi_h \circ \Lambda_h \to 0$ in B_r, hence by continuity

$$A \circ \Phi_h^{-1} \circ \Psi_h \circ \Lambda_h \to A(0)\,.$$

In order to conclude, it is sufficient to note that $A(0) = \mu I$, then

$$A_h \circ \Psi_h \circ \Lambda_h \to \mu F^{-1}F^{-T}\det F$$

and hence

$$B_{h,0} = A_{h,0} \circ \Lambda_h \to \mu(FD\Psi)^{-1}(FD\Psi)^{-T}\det(FD\Psi)\,.$$

In $\Omega \setminus B_{rh}$ we have both $A_h = A$ and $\Psi_h = \mathrm{id}$, hence $A_h \circ \Psi_h = A$; moreover $D\Psi_h = I$, hence $A_{h,0} = A$. Since $\Lambda_h(\Omega \setminus B_r) = \Omega \setminus B_{rh}$ it follows that in $\Omega \setminus B_r$ we have $A_{h,0} \circ \Lambda_h = A \circ \Lambda_h$. In conclusion

$$B_{h,0} = D\Lambda_h^{-1}(A \circ \Lambda_h) D\Lambda_h^{-T} \det D\Lambda_h \,.$$

As $D\Lambda_h \to D\Lambda_0$ pointwise in Ω by the continuity of A we get that in $\Omega \setminus B_r$

$$B_{h,0} \to D\Lambda_0^{-1}(A \circ \Lambda_0) D\Lambda_0^{-T} \det D\Lambda_0 \,,$$

which concludes the proof. ■

Lemma 9.2 *In the ball B_r*

$$B'_h \to B_0 - \mu I \,. \tag{24}$$

(Remember that $B'_h = 0$ in $\Omega \setminus B_r$).

Proof. Remember that $A_{h,0} = A + A'_h$ in B_r, hence $B_{h,0} = A \circ \Lambda_h + A'_h \circ \Lambda_h = A \circ \Lambda_h + B'_h$. Since $A \circ \Lambda_h \to \mu I$ the thesis follows by the previous Lemma. ■

Definition 9.3 *The previous Lemmas suggests to define*

$$B_0 = \begin{cases} \mu (FD\Psi)^{-1}(FD\Psi)^{-T}\det(FD\Psi) & \text{in } B_r \\ D\Lambda_0^{-1}(A \circ \Lambda_0) D\Lambda_0^{-T} \det D\Lambda_0 & \text{in } B_R \setminus B_r . \\ A & \text{in } \Omega \setminus B_R . \end{cases}$$

$$B'_0 = \begin{cases} B_0 - \mu I & \text{in } B_r \\ 0 & \text{in } \Omega \setminus B_r . \end{cases}$$

Clearly $B_{h,0} \to B_0$ and $B'_h \to B'_0$ pointwise in Ω.

10 Localization of $\widehat{\mathcal{J}}_h$

Before proving the existence of the energy release rate it is convenient to 'localize' $\widehat{\mathcal{J}}_h$ in the ball B_R. Consider the space

$$\mathcal{W}_{h,R} = \{ w \in H^1(B_R \setminus \Gamma_0) : w = w_h = u'_h \text{ in } \partial B_R \}$$

and define for $w \in \mathcal{W}_{h,R}$ the localized functional

$$\widehat{\mathcal{J}}_{h,R}(w) = \frac{1}{2} \int_{B_R \setminus \Gamma_0} \nabla w B_{h,0} \nabla w^T + \int_{B_r \setminus \Gamma_0} h^{-1/2} \nabla w B'_h \nabla (u_0 \circ \Lambda_h) \,. \tag{25}$$

Consider $\widehat{\mathcal{J}}_{h,R} = +\infty$ in $H^1 \setminus \mathcal{W}_{h,R}$. Clearly we have $w_h \in \operatorname{argmin}\{\widehat{\mathcal{J}}_{h,R}(w) : w \in \mathcal{W}_{h,R}\}$. In a similar way consider the space

$$\mathcal{W}^c_{h,R} = \{ w \in H^1(\Omega \setminus (\bar{B}_R \cup \Gamma_0)) : w = u'_h \text{ in } \partial B_R \cup \partial_D \Omega \,, w = 0 \text{ in } \partial_D \Omega \}$$

and the energy

$$\widehat{\mathcal{J}}^c_{h,R}(w) = \frac{1}{2} \int_{\Omega \setminus (\bar{B}_R \cup \Gamma_0)} \nabla w B_{h,0} \nabla w^T \, dx = \frac{1}{2} \int_{\Omega \setminus (\bar{B}_R \cup \Gamma_0)} \nabla w A \nabla w^T \, dx \,. \tag{26}$$

Clearly $w_h \in \operatorname{argmin}\{\widehat{\mathcal{J}}^c_{h,R}(w) : w \in \mathcal{W}^c_{h,R}\}$ and $w_h = u_h$ in $\Omega \setminus \bar{B}_R$.

For later convenience let us introduce also the space

$$\mathcal{W}_{0,R} = \{ w \in H^1(B_R \setminus \Gamma_0) : w = 0 \text{ in } \partial B_R \}$$

and define for $w \in \mathcal{W}_{0,R}$ the localized functional

$$\widehat{\mathcal{J}}_{0,R}(w) = \frac{1}{2} \int_{B_R \setminus \Gamma_0} \nabla w B_0 \nabla w^T + k \int_{B_r \setminus \Gamma_0} \nabla w B'_0 \nabla \hat{s} \,. \tag{27}$$

Consider $\widehat{\mathcal{J}}_{0,R} = +\infty$ in $H^1 \setminus \mathcal{W}_{0,R}$ and let $w_0 \in \operatorname{argmin}\{\widehat{\mathcal{J}}_R(w) : w \in \mathcal{W}_{0,R}\}$.

11 Existence of the energy release rate

In this section we will prove the existence of the energy release rate. For convenience the proof is splitted in three lemmas.

Lemma 11.1 *Let $\mathcal{I}$ be defined in* (19)*. Then*

$$\mathcal{I}(u_0) = \lim_{h\to 0} \mathcal{I}_h(u_0) = k^2 \frac{1}{2} \int_{B_r \setminus \Gamma_0} \nabla \hat{s} B_0' \nabla \hat{s} \, dx \,. \tag{28}$$

Proof. Let $\hat{s} = \rho^{1/2} \sin((\theta - \alpha)/2)$ and $u_0 = k\hat{s} + \bar{u}$. Then $\hat{s} \circ \Lambda_h = h^{1/2} \hat{s}$ in B_r. Thus

$$\begin{aligned} \mathcal{I}_h(u_0) &= h^{-1} \frac{1}{2} \int_{B_r \setminus \Gamma_0} \nabla (u_0 \circ \Lambda_h) B_h' \nabla (u_0 \circ \Lambda_h) \, dx \\ &= \frac{1}{2} \int_{B_r \setminus \Gamma_0} k^2 \nabla \hat{s} B_h' \nabla \hat{s} \, dx + \\ & \quad \frac{1}{2} \int_{B_r \setminus \Gamma_0} 2kh^{-1/2} \nabla \hat{s} B_h' \nabla (\bar{u} \circ \Lambda_h) + h^{-1} \nabla (\bar{u} \circ \Lambda_h) B_h' \nabla (\bar{u} \circ \Lambda_h) \, dx \,. \end{aligned}$$

Moreover, denoting by χ_s the characteristic function of the ball B_s, we can write

$$\int_\Omega \chi_r |\nabla (\bar{u} \circ \Lambda_h)|^2 \, dx = \int_\Omega \chi_{rh} |\nabla \bar{u}|^2 \, dx \,.$$

Arguing as in the proof of Lemma 4.1 we have $h^{-1/2} \chi_r \nabla (\bar{u} \circ \Lambda_h) \to 0$ strongly in L^2, from which (28) is proved. ■

Lemma 11.2 *The functionals $\widehat{\mathcal{J}}_{h,R}$ Γ-converge to $\widehat{\mathcal{J}}_{0,R}$ with respect to the weak topology of H^1 and $\widehat{\mathcal{J}}_{h,R}(w_h) \to \widehat{\mathcal{J}}_{0,R}(w_0)$.*

Proof. Let us begin with the liminf-inequality. First, it is better to re-write $\widehat{\mathcal{J}}_{h,R}$ in a more convenient way. Let $u_0 = k\hat{s} + \bar{u}$ with $\hat{s} = \rho^{1/2} \sin((\theta - \alpha)/2)$. Then $\hat{s} \circ \Lambda_h = h^{1/2} \hat{s}$ in B_r and thus

$$\begin{aligned} \widehat{\mathcal{J}}_{h,R}(w) &= \frac{1}{2} \int_{B_R \setminus \Gamma_0} \nabla w B_{h,0} \nabla w^T \, dx + \\ & \quad \frac{1}{2} \int_{B_r \setminus \Gamma_0} \nabla w B_h' \big(k \nabla \hat{s} + h^{-1/2} \nabla (\bar{u} \circ \Lambda_h) \big) \, dx \,. \end{aligned}$$

Now, let $w_h \rightharpoonup w$ in $H^1_0(B_R)$. Then by lower-semicontinuity we can write

$$\int_{B_R \setminus \Gamma_0} \nabla w B_0 \nabla w^T \, dx \le \liminf_{h \to 0} \int_{B_R \setminus \Gamma_0} \nabla w_h B_{h,0} \nabla w_h^T \, dx \,.$$

Since $k \nabla \hat{s} + h^{-1/2} \chi_r \nabla (\bar{u} \circ \Lambda_h) \to k \nabla \hat{s}$ strongly in L^2 (see the proof of Lemma 11.1) we have

$$k \int_{B_r \setminus \Gamma_0} \nabla w B_0' \nabla \hat{s} \, dx = \lim_{h \to 0} \int_{B_r \setminus \Gamma_0} \nabla w_h B_h' \big(k \nabla \hat{s} + h^{-1/2} \nabla (\bar{u} \circ \Lambda_h) \big) \, dx \,.$$

We can conclude that $\widehat{\mathcal{J}}_{0,R}(w) \le \liminf_{h \to 0} \widehat{\mathcal{J}}_{h,R}(w_h)$.

Let us prove the limsup-inequality. Let $\mathcal{L} : H^{1/2}(\partial B_R) \to H^1(B_R \setminus \Gamma_0)$ denote a bounded lifting operator and consider the sequence $\mathcal{L}(w_h)$. Since $u_h' \rightharpoonup 0$ in $H^1(\Omega \setminus \Gamma_0)$ then $u_h' \to 0$ in $H^{1/2}(\partial B_R)$ and hence $\mathcal{L}(w_h) \to 0$ in $H^1(B_R \setminus \Gamma_0)$. Given $w \in \mathcal{W}_{0,R}$, let us write it as $w = \mathcal{L}(w_h) - z_h$. Then $z_h \in \mathcal{W}_{h,R}$ and $z_h \to w$ in $H^1(B_R \setminus \Gamma_0)$. It is easy to check that $\lim_{h \to 0} \widehat{\mathcal{J}}_{h,R}(z_h) = \widehat{\mathcal{J}}_{0,R}(w)$.

Finally, let us check the convergence of minimizers. Remember that the minimizer of $\widehat{\mathcal{J}}_{h,R}$ is $w_h = u_h' \circ \Lambda_h$ and that u_h' is bounded in $H^1(B_R \setminus \Gamma_0)$. Using again the change of variable, it is easy to see that w_h is bounded in $H^1(B_R \setminus \Gamma_0)$ and thus weakly compact. By a standard result on Γ-convergence [4] it follows that $\widehat{\mathcal{J}}_{h,R}(w_h) \to \widehat{\mathcal{J}}_{0,R}(w_0)$. ■

Lemma 11.3 *Let* $\widehat{\mathcal{J}}^c_{h,R}$ *the functional defined in* (26). *Then*

$$\lim_{h\to 0} \widehat{\mathcal{J}}^c_{h,R}(w_h) = 0\,.$$

Proof. It is sufficient to remember that $w_h = u'_h \in \operatorname{argmin}\{\widehat{\mathcal{J}}^c_{h,R}(w) : w \in \mathcal{W}^c_{h,R}\}$ and that $u'_h \to 0$ in $H^{1/2}(\partial B_R)$. ■

Writing $\widehat{\mathcal{G}}_h(w_h) = \mathcal{I}_h(u_0) + \widehat{\mathcal{J}}_{h,R}(w_h) + \widehat{\mathcal{J}}^c_{h,R}(w_h)$ and combining the three previous Lemmas follows the existence of the energy release rate, as stated in the next Theorem.

Theorem 11.4 *The energy release rate* (2) *exists and is given by*

$$\mathcal{G}(u_0) = -\frac{\mathcal{I}(u_0) + \widehat{\mathcal{J}}_{0,R}(w_0)}{(1+|f'_+(0)|^2)^{1/2}}\,,$$

where

$$\mathcal{I}(u_0) = \frac{k^2}{2}\int_{B_r\setminus\Gamma_0} \nabla\hat{s}B'_0\nabla\hat{s}\,dx\,,$$

$$\widehat{\mathcal{J}}_{0,R}(w) = \frac{1}{2}\int_{B_R\setminus\Gamma_0}\nabla w B_0 \nabla w^T\,dx + k\int_{B_r\setminus\Gamma_0}\nabla w B'_0\nabla\hat{s}\,dx\,.$$

and $w_0 \in \operatorname{argmin}\{\widehat{\mathcal{J}}_{0,R}(w) : w \in \mathcal{W}_{0,R}\}$.

We will see how to re-write the above formula in a more convenient way in the next section. It is worth to stress the fact that $\mathcal{G}$ depends on u_0 only by means of the coefficient k.

12 Representation of the energy release rate

Let us denote by $\hat{w}$ the minimizer with $k = 1$. Thank to the fact that $w_0 = 0$ in ∂B_R it follows easily that $w_0 = k\hat{w}$ is the minimizer of $\widehat{\mathcal{J}}_{0,R}$. Therefore, using also the fact that B'_0 is supported in B_r, we can write

$$\begin{aligned}\widehat{\mathcal{J}}_{0,R}(w) &= \frac{k^2}{2}\int_{B_R\setminus\Gamma_0}\nabla\hat{w}B_0\nabla\hat{w}^T\,dx + k^2\int_{B_r\setminus\Gamma_0}\nabla\hat{w}B'_0\nabla\hat{s}^T\,dx\\ &= \frac{k^2}{2}\int_{B_R\setminus\Gamma_0}\nabla\hat{w}B_0\nabla\hat{w}^T + 2\nabla\hat{w}B'_0\nabla\hat{s}^T\,dx\end{aligned}$$

and

$$\mathcal{I}(u_0) = \frac{k^2}{2}\int_{B_R\setminus\Gamma_0}\nabla\hat{s}B'_0\nabla\hat{s}\,dx\,.$$

As a consequence $\mathcal{G}(u_0)$ can be represented as in the following Theorem.

Theorem 12.1 *The energy release rate can be represented as*

$$\mathcal{G}(u_0) = \frac{-k^2}{2(1+|f'_+|^2(0))^{1/2}}\int_{B_R\setminus\Gamma_0}(\nabla\hat{w}B_0\nabla\hat{w}^T + 2\nabla\hat{w}B'_0\nabla\hat{s}^T + \nabla\hat{s}B'_0\nabla\hat{s})\,dx\,. \tag{29}$$

It seem particularly usefull in view of the applications to crack propagation to have at one's disposal an integral representation formula for the energy release rate. When there are no kinks the classical representation with the Eshelby tensor holds true for $C^{1,1}$ cracks [10]. To this end, let η be a cut off function with $\eta = 1$ is a (small) neighborhood of the origin and consider the tangent map

$$\mathcal{T}(x) = (1, \tilde{f}'(x_1))\,\eta(x)\,,$$

where $\tilde{f}$ is again a $C^{1,1}$ extension of f. Under these assumptions we have

$$k^2 = \frac{4}{\pi}\int_{\Omega\setminus K_0}\mathbb{E}(u_0)\cdot D\mathcal{T}\,dx\,,$$

where $\mathbb{E}$ is the Eshelby tensor. Combining the last formula with (29) allows to write the energy release rate with a volume integral.

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