

# A geometric inequality for stable solutions of semilinear elliptic problems in the Engel group

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## Abstract

We prove that, if  $\mathbb{E}$  is the Engel group and  $u$  is a stable solution of  $\Delta_{\mathbb{E}}u = f(u)$ , then

$$\int_{\{|\nabla_{\mathbb{E}}u| \neq 0\}} \left[ |\nabla_{\mathbb{E}}u|^2 \left\{ \left( p + \frac{\langle (Hu)^T \nu, \nu \rangle}{|\nabla_{\mathbb{E}}u|} \right)^2 + h^2 \right\} - \mathcal{J} \right] \eta^2 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}}\eta|^2 |\nabla_{\mathbb{E}}u|^2$$

for any test function  $\eta \in C_0^\infty(\mathbb{E})$ .

Here above,  $h$  is the horizontal mean curvature,  $p$  is the imaginary curvature and

$$\mathcal{J} := 2(X_3X_2uX_1u - X_3X_1uX_2u) + (X_4u)(X_1u - X_2u)$$

This can be interpreted as a geometric Poincaré inequality, extending the work of [21, 22, 13] to stratified groups of step 3.

As an application, we provide a non-existence result.

## 1 Introduction

The Engel group is the Lie group  $\mathbb{E}$  having Lie algebra  $\mathfrak{e}$  spanned by the vector fields  $X_1, X_2, X_3, X_4$  subject to the commuting relations

$$[X_1, X_2] = X_3, [X_1, X_3] = [X_2, X_3] = X_4.$$

The subRiemannian geometry of  $\mathbb{E}$  is the one in which the horizontal vectors in  $V_1 := \text{span}(X_1, X_2)$  play a distinguished role. The geometry is not trivial because  $V_1, V_2 := [V_1, V_1]$  and  $V_3 := [V_1, V_2]$  span  $\mathfrak{e}$ . The essence of subRiemannian geometry is that its information is carried out only by  $V_1$ , but in a way which might differ from the Riemannian case. For instance, to better understand the difficulties hidden by the mysterious geometry of the Engel group before going into the details of the semilinear equations on it let us consider the case of geodesics. The distance between two points  $P$  and  $Q$  in  $\mathbb{E}$  can be computed

as the infimum of the subRiemannian length of smooth curves joining them. Namely, we require that  $\dot{\gamma} = a_1 X_1 + a_2 X_2 \in V_1$  and compute the length as

$$\text{length}(\gamma) = \int \sqrt{a_1^2 + a_2^2} dt.$$

By definition there are  $\infty^1$  choices for the (unit) tangent vector of a geodesic at a point  $P$  of  $\mathbb{E}$ , but there are  $\infty^4$  points in  $\mathbb{E}$  and (Chow's Theorem) they can all be reached from  $P$  along geodesics. Having each geodesic  $\infty^1$  points, there must be two "hidden" real parameters to account for the  $\infty^3$  multiplicity to be reached starting from each tangent vector at  $P$ . In particular, let us consider a smooth three-dimensional hypersurface  $S \subset \mathbb{E}$  with  $P \in \mathbb{E}$  and consider the problem of finding the geodesic starting at  $P$  such that, for small  $t$ , which is the only length minimizing geodesic between  $P$  and  $\gamma(t)$ . Surely,  $\dot{\gamma}(t) := \nu \in V_1$  must be the vector normal to  $S$  in the sub-Riemannian geometry, but this leaves us with  $\infty^2$  geodesics to choose from. In [1] and [2] the problem was considered in the Heisenberg group, where the only missing parameter was rescued in terms of the "imaginary curvature" of  $S$ . In [3] the same problem was considered in groups of any step, although most applications were proved in the step-two case only. We refer to Section 2 for a more detailed treatment of the Engel group.

We now introduce the problem we study and present our main result. Given a domain  $\Omega \subset \mathbb{E}$  we consider solutions  $u$  of the following semilinear equation in  $\mathbb{E}$ ,

$$\Delta_{\mathbb{E}} u = f(u). \tag{1}$$

For simplicity, we assume  $f \in C^\infty(\mathbb{R})$ : in this way,  $u \in C^\infty(\Omega)$  by the regularity theory of [18] (the case of less regular nonlinearities  $f$  may be treated analogously, with only minor modifications).

Moreover, we assume that  $u$  is stable, that is

$$0 \leq \int_{\mathbb{E}} \langle \nabla_{\mathbb{E}} \eta, \nabla_{\mathbb{E}} \eta \rangle_{\mathbb{E}} + \int_{\mathbb{E}} \dot{f}(u) \eta^2 \tag{2}$$

for every  $\eta \in C_0^\infty(\Omega)$ .

The stability condition in (2) has been widely studied in the calculus of variation setting: indeed, it states that the second variation of the energy functional associated to (1) is nonnegative at the critical point  $u$  – hence, for instance, minimal solutions are always stable, but, in principle, stability is a weaker condition than minimality.

Equation (1) is called semilinear, since the only nonlinearity depends on the solution  $u$  (not on the space, neither on the derivatives of  $u$ ): such kind of equations have been studied in detail in the Euclidean framework, and in the subRiemannian one as well (see, e.g. [5, 6, 7, 17]), and they possess the remarkable geometric property that the operator is constant along the level sets of the solution.

At any point of

$$\mathbb{E}_0 := \{x \in \Omega \mid \nabla_{\mathbb{E}} u \neq 0\}$$

we denote by

$$\nu = \frac{\nabla_{\mathbb{E}} u}{|\nabla_{\mathbb{E}} u|}$$

the intrinsic unit normal to the level set of  $u$ , to wit  $\nu$  is the normalized projection on the horizontal fiber of the Riemannian normal.

We shall also consider the intrinsic tangent direction to the level set of  $u$

$$v := \frac{X_2 u}{|\nabla_{\mathbb{E}} u} X_1 - \frac{X_1 u}{|\nabla_{\mathbb{E}} u} X_2 \quad (3)$$

Let us observe that  $\forall p \in \mathbb{E}_0$

$$\langle \nu(p), v(p) \rangle_{p, \mathbb{E}} = 0$$

where  $\langle \cdot, \cdot \rangle_{p, \mathbb{E}}$  is the standard scalar product on the fiber  $H_p \mathbb{E}$ . We denote by  $Hu$  the intrinsic Hessian matrix, i.e.

$$Hu := \begin{pmatrix} X_1 X_1 u & X_2 X_1 u \\ X_1 X_2 u & X_2 X_2 u \end{pmatrix}$$

As usual, we define

$$(Hu)^2 := (Hu)(Hu)^T$$

and

$$|Hu| := \sqrt{|\nabla_{\mathbb{E}} X_1 u|^2 + |\nabla_{\mathbb{E}} X_2 u|^2}$$

Also, in  $\mathbb{E}_0$ , following an analogy in the Heisenberg group (see [1, 2, 19, 20]), we define the horizontal mean curvature

$$h := \operatorname{div}_{\mathbb{E}} \nu \quad (4)$$

and the imaginary curvature

$$p := -\frac{X_3 u}{|\nabla_{\mathbb{E}} u} \quad (5)$$

Let also

$$\mathcal{J} := 2(X_3 X_2 u X_1 u - X_3 X_1 u X_2 u) + (X_4 u)(X_1 u - X_2 u)$$

With this notation, we have:

**Theorem 1.1.**

$$\int_{\mathbb{E}_0} \left[ |\nabla_{\mathbb{E}} u|^2 \left\{ \left( p + \frac{\langle (Hu)^T \nu, v \rangle}{|\nabla_{\mathbb{E}} u} \right)^2 + h^2 \right\} - \mathcal{J} \right] \eta^2 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \eta|^2 |\nabla_{\mathbb{E}} u|^2$$

for any  $\eta \in C_0^\infty(\Omega)$ .

Theorem 1.1 is a sort of geometric weighted Poincaré inequality, in the sense that the weighted  $L^2$ -norm of any test function is bounded by a weighted  $L^2$ -norm of its gradient, and the weights are built with geometric objects.

In the Euclidean case, the analogue of Theorem 1.1 was established in [21, 22], and recently many extensions have been performed (see, in particular, [11, 12]). As far as we know, the first applications in the subRiemannian setting, were performed in [4, 13] for the Heisenberg group and in [14] for the Grushin plane. In several cases, these type of geometric weighted inequalities lead to rigidity results (such as classification, symmetry, or non existence, of solutions). Differently from the Euclidean case, the weight on the left hand side of the inequality does not need to be positive in general, due to the presence of  $\mathcal{J}$ . Thus, the presence of noncommuting vector fields, complicates the geometry of the level sets via the sign of  $\mathcal{J}$ . Indeed, if  $\mathcal{J} \leq 0$ , when the right hand side of the inequality in Theorem 1.1 vanishes, one obtains that the level sets of  $u$  satisfy the geometric equations, see Corollary 4.2,

$$\begin{cases} p + \frac{\langle (Hu)^T \nu, v \rangle}{|\nabla_{\mathbb{E}} u|} = 0 \\ h = 0 \end{cases} \quad (6)$$

Also, a more geometric interpretation of the quantity  $\mathcal{J}$ , in dependence of the intrinsic tangent and normal vectors, will be given in Lemma 3.8.

In this paper, Theorem 1.1 is a first attempt to adapt these geometric weighted inequalities to stratified groups of step higher than 2 (for other different weighted inequalities in  $\mathbb{R}^4$  obtained via the Engel group, see Theorem 1.4 of [15]).

The higher the step of the group, the more complicated are the combinatorics occurring in the inequality, and the more difficult is the geometric interpretation of the quantities involved. Nevertheless, the Engel group still maintains a reasonable level of geometric insight and provides a challenging source of problems for this approach. For instance, we think that it would be interesting to investigate whether or not rigidity results and geometric properties of stable solutions may be obtained from these kind of Poincaré inequalities (or by other methods as well).

In this spirit we are able to prove a first non-existence result for semilinear equations in the Engel group in Theorem 4.7.

To conclude this introduction we think that is interesting to propose some suggestion for further research. In particular we think that it could be interesting to see if a result like the one proved in Theorem 1.1 holds in more general Carnot groups. Moreover, we think that it could be stimulating to study the geometric nature of  $\mathcal{J}$ , in particular it would be desirable to relate  $\mathcal{J}$  with the sub-Riemannian metric structure of  $\mathbb{E}$ .

The organization of this paper is as follows. In Section 2, we recall what the Engel group is and what its basic properties are. Then, we prove Theorem 1.1

in Section 3. Finally, in Section 4, we derive the geometric equations (6) in Corollary 4.2, and we provide a non-existence result in Theorem 4.7.

## 2 The Engel group

We recall the basic definitions and properties of the Engel group.

**Definition 2.1.** *The Engel algebra is the finite dimensional Lie algebra  $\mathfrak{e}$  with basis  $(X_1, X_2, X_3, X_4)$  where the only nonvanishing commutators relationship among the generators are*

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = [X_2, X_3] = X_4 \quad (7)$$

**Remark 2.2.** It is easy to see that the Engel algebra is stratified of step 3, i.e.

$$\mathfrak{e} = \mathfrak{e}_1 \oplus \mathfrak{e}_2 \oplus \mathfrak{e}_3$$

where  $\mathfrak{e}_1 := \text{span}\{X_1, X_2\}$ ,  $\mathfrak{e}_2 := \text{span}\{X_3\}$  and  $\mathfrak{e}_3 := \text{span}\{X_4\}$  and

$$[\mathfrak{e}_1, \mathfrak{e}_1] = \mathfrak{e}_2, \quad [\mathfrak{e}_1, \mathfrak{e}_2] = \mathfrak{e}_3, \quad [\mathfrak{e}_1, \mathfrak{e}_3] = \{0\} \quad (8)$$

**Definition 2.3.** *The Engel group, denoted by  $\mathbb{E}$ , is the simply connected nilpotent Lie group associated to  $\mathfrak{e}$ .*

Since  $\mathbb{E}$  is a Carnot group we can represent it by means of graded coordinates associated to the basis  $(X_1, X_2, X_3, X_4)$ ; it follows that  $\mathbb{E} = (\mathbb{R}^4, \cdot, \{\delta_r\}_{r>0})$ , where  $\forall (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{E}$

$$\begin{aligned} (x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) &:= \\ &= \left( x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1), \right. \\ &x_4 + y_4 + \frac{1}{2}[(x_1 y_3 - x_3 y_1) + (x_2 y_3 - x_3 y_2)] + \\ &\left. + \frac{1}{12}[(x_1 - y_1 + x_2 - y_2)(x_1 y_2 - x_2 y_1)] \right) \end{aligned} \quad (9)$$

and the homogeneous dilations on  $\mathbb{E}$  are

$$\delta_r(x_1, x_2, x_3, x_4) := (rx_1, rx_2, r^2 x_3, r^3 x_4) \quad r > 0$$

The representation of the basis  $(X_1, X_2, X_3, X_4)$  on the graded coordinates gives

$$\begin{aligned} X_1(x_1, x_2, x_3, x_4) &= \partial_1 - \frac{x_2}{2} \partial_3 - \left( \frac{x_3}{2} + \frac{x_2}{12}(x_1 + x_2) \right) \partial_4 \\ X_2(x_1, x_2, x_3, x_4) &= \partial_2 + \frac{x_1}{2} \partial_3 - \left( \frac{x_3}{2} - \frac{x_1}{12}(x_1 + x_2) \right) \partial_4 \\ X_3(x_1, x_2, x_3, x_4) &= \partial_3 + \frac{1}{2}(x_1 + x_2) \partial_4 \\ X_4(x_1, x_2, x_3, x_4) &= \partial_4 \end{aligned}$$

We fix a left invariant Riemannian metric on  $\mathbb{R}^4$  that makes the above vector fields orthonormal, i.e. at every  $p \in \mathbb{E}$  we give a scalar product  $\langle \cdot, \cdot \rangle_{p, \mathbb{E}}$  such that

$$\langle X_i(p), X_i(p) \rangle_{p, \mathbb{E}} = 1 \quad i \in \{1, 2, 3, 4\}$$

and

$$\langle X_i(p), X_j(p) \rangle_{p, \mathbb{E}} = 0 \quad \text{if } i \neq j$$

As usual in the context of Carnot groups, for every  $p \in \mathbb{E}$  we define the Horizontal fiber at  $p$  as the subspace of the tangent space at  $p$  generated by  $X_1(p)$  and  $X_2(p)$ , i.e

$$H_p \mathbb{E} := \text{span}\{X_1(p), X_2(p)\} \subset T_p \mathbb{E}$$

and the horizontal subbundle of the tangent bundle associated to  $\mathbb{E}$  as

$$H\mathbb{E} := \bigcup_{p \in \mathbb{E}} H_p \mathbb{E}$$

Finally, we briefly recall the notion of Carnot-Carathéodory distance on  $\mathbb{E}$ , see [7] for a more detailed treatment.

**Definition 2.4.** A locally Lipschitz curve  $\lambda : [0, T] \rightarrow \mathbb{R}^4$  is said to be horizontal if there are  $c_1, c_2 \in L^\infty([0, T])$  such that

$$\dot{\lambda}(t) = c_1(t)X_1(\lambda(t)) + c_2(t)X_2(\lambda(t)) \quad \mathcal{L}^1 - q.o \ t \in [0, T]$$

and

$$c_1(t)^2 + c_2(t)^2 \leq 1 \quad \text{in } [0, T]$$

The CC-distance between two points  $p, q \in \mathbb{E}$  is defined as follows

$$d_{CC}(x, y) := \inf\{T > 0 \mid \exists \lambda : [0, T] \rightarrow \mathbb{R}^4 \text{ horizontal s.t } \lambda(0) = x, \lambda(T) = y\}$$

**Remark 2.5.** It is a classical result that  $d_{CC}$  is a distance on  $\mathbb{E}$ , see [9]. Moreover, we have

$$d_{CC}(x, y) = d_{CC}(z \cdot x, z \cdot y) \quad \forall x, y, z \in \mathbb{E}$$

and

$$d_{CC}(\delta_r(x), \delta_r(y)) = r d_{CC}(x, y) \quad \forall x, y \in \mathbb{E}, \forall r \in (0, \infty)$$

Finally, let  $K \Subset \Omega$  be a compact set then there exists a constant  $\alpha > 0$  such that

$$d_{CC}(x, y) \geq \alpha |x - y| \tag{10}$$

for all  $x, y \in K$ .

**Definition 2.6.** Let  $u : \Omega \rightarrow \mathbb{R}$  be a  $C^1$  map then the horizontal gradient of  $u$  is defined as follows

$$\nabla_{\mathbb{E}} u(x) := X_1 u(x) X_1 + X_2 u(x) X_2$$

moreover, if  $u \in C^1(\Omega, \mathbb{R}^2)$  the horizontal divergence of  $u$  is

$$\text{div}_{\mathbb{E}} u(x) := X_1 u_1(x) + X_2 u_2(x)$$

Finally, if  $u : \Omega \rightarrow \mathbb{R}$  is  $C^2$  the horizontal Laplacian of  $u$  is

$$\Delta_{\mathbb{E}} u(x) := X_1 X_1 u(x) + X_2 X_2 u(x)$$

**Remark 2.7.** Let us explicitly observe that the intrinsic gradient does not depend of the basis  $(X_1, X_2)$ .

**Remark 2.8.** If, in Theorem 1.1,  $u$  does not depend on  $x_4$ , then the situation boils down to the one in the Heisenberg group (note indeed that  $X_3X_1u = X_1X_3u$ , so Theorem 1.1 reduces to Theorem 2.3 in [13]).

### 3 Proof of Theorem 1.1

The proof of our first result needs some preliminary, technical computations, by which we obtain some useful identities.

**Lemma 3.1.** *Let  $j \in \{1, 2\}$ . If  $u \in C^\infty(\Omega)$  then in  $\mathbb{E}_0$  we have*

$$X_j|\nabla_{\mathbb{E}}u| = \langle X_j(\nabla_{\mathbb{E}}u), \nu \rangle_{\mathbb{E}}. \quad (11)$$

Moreover, for each  $\eta \in C_0^\infty(\Omega)$ ,

$$\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta) = \frac{\eta}{|\nabla_{\mathbb{E}}u|} (Hu)^T \nabla_{\mathbb{E}}u + |\nabla_{\mathbb{E}}u| \nabla_{\mathbb{E}}\eta \quad (12)$$

and

$$|\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)|^2 = \frac{\eta^2}{|\nabla_{\mathbb{E}}u|^2} |(Hu)^T \nabla_{\mathbb{E}}u|^2 + 2\eta \langle \nabla_{\mathbb{E}}u, (Hu) \nabla_{\mathbb{E}}\eta \rangle_{\mathbb{E}} + |\nabla_{\mathbb{E}}\eta|^2 |\nabla_{\mathbb{E}}u|^2 \quad (13)$$

*Proof.* Equation (11) is straightforward. Also, the proof of (12) follows from the following simple calculation:

$$\begin{aligned} \nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta) &= \eta \nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|) + |\nabla_{\mathbb{E}}u| \nabla_{\mathbb{E}}\eta = \\ &= \frac{\eta}{|\nabla_{\mathbb{E}}u|} (Hu)^T \nabla_{\mathbb{E}}u + |\nabla_{\mathbb{E}}u| \nabla_{\mathbb{E}}\eta \end{aligned} \quad (14)$$

Furthermore

$$\begin{aligned} |\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)|^2 &= \langle \nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta), \nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta) \rangle_{\mathbb{E}} = \\ &= \left\langle \frac{\eta}{|\nabla_{\mathbb{E}}u|} (Hu)^T \nabla_{\mathbb{E}}u, \frac{\eta}{|\nabla_{\mathbb{E}}u|} (Hu)^T \nabla_{\mathbb{E}}u \right\rangle_{\mathbb{E}} + \\ &+ 2 \left\langle \frac{\eta}{|\nabla_{\mathbb{E}}u|} (Hu)^T \nabla_{\mathbb{E}}u, |\nabla_{\mathbb{E}}u| \nabla_{\mathbb{E}}\eta \right\rangle_{\mathbb{E}} + \\ &+ \langle |\nabla_{\mathbb{E}}u| \nabla_{\mathbb{E}}\eta, |\nabla_{\mathbb{E}}u| \nabla_{\mathbb{E}}\eta \rangle_{\mathbb{E}}. \end{aligned}$$

Hence

$$\begin{aligned} |\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)|^2 &= \left( \frac{\eta}{|\nabla_{\mathbb{E}}u|} \right)^2 |(Hu)^T \nabla_{\mathbb{E}}u|^2 + 2\eta \langle (Hu)^T \nabla_{\mathbb{E}}u, \nabla_{\mathbb{E}}\eta \rangle_{\mathbb{E}} + \\ &+ |\nabla_{\mathbb{E}}\eta|^2 |\nabla_{\mathbb{E}}u|^2 \\ &= \left( \frac{\eta}{|\nabla_{\mathbb{E}}u|} \right)^2 |(Hu)^T \nabla_{\mathbb{E}}u|^2 + 2\eta \langle \nabla_{\mathbb{E}}u, (Hu) \nabla_{\mathbb{E}}\eta \rangle_{\mathbb{E}} + \\ &+ |\nabla_{\mathbb{E}}\eta|^2 |\nabla_{\mathbb{E}}u|^2 \end{aligned}$$

and this proves (13).  $\square$

**Lemma 3.2.** *Let  $u \in C^\infty(\Omega)$  then*

$$|Hu|^2 - \langle (Hu)^2 \nu, \nu \rangle_{\mathbb{E}} = |(Hu)^T v|^2. \quad (15)$$

Moreover, in  $\mathbb{E}_0$

$$|(Hu)^T v|^2 = |\nabla_{\mathbb{E}} u|^2 \left\{ \left( p + \frac{\langle (Hu)^T \nu, v \rangle}{|\nabla_{\mathbb{E}} u|} \right)^2 + h^2 \right\}. \quad (16)$$

*Proof.* We note that for each  $p \in \mathbb{E}$   $(\nu(p), v(p))$  is an orthonormal basis of  $H_p \mathbb{E}$ . Then (15) follows, for instance, from Lemma 3 in [4].

In order to prove (16), we begin observing that

$$\begin{aligned} (Hu)^T &= \begin{pmatrix} X_1 X_1 u & X_1 X_2 u \\ X_2 X_1 u & X_2 X_2 u \end{pmatrix} = \\ &= \begin{pmatrix} X_1 X_1 u & X_2 X_1 u \\ X_1 X_2 u & X_2 X_2 u \end{pmatrix} + \begin{pmatrix} 0 & X_1 X_2 u - X_2 X_1 u \\ X_2 X_1 u - X_1 X_2 u & 0 \end{pmatrix} = \\ &= Hu + \begin{pmatrix} 0 & X_3 u \\ -X_3 u & 0 \end{pmatrix}. \end{aligned} \quad (17)$$

Now we define

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let also  $\mathcal{Z}$  and  $H_\nu \in \text{Mat}(\mathbb{R}, 2 \times 2)$  be defined as

$$\mathcal{Z}_{ij} := \nu_i \langle (Hu)^T \nu, \nu \rangle_j \quad \text{and} \quad (H_\nu)_{ij} := X_j(\nu_i)$$

for  $i, j \in \{1, 2\}$ . So, we use (11) to obtain that

$$\begin{aligned} \mathcal{Z}_{ij} + |\nabla_{\mathbb{E}} u| (H_\nu)_{ij} &= \\ &= \nu_i \langle X_j(\nabla_{\mathbb{E}} u), \nu \rangle_{\mathbb{E}} + |\nabla_{\mathbb{E}} u| X_j \left( \frac{X_i u}{|\nabla_{\mathbb{E}} u|} \right) = \\ &= \frac{X_i u}{|\nabla_{\mathbb{E}} u|} \langle X_j(\nabla_{\mathbb{E}} u), \nu \rangle_{\mathbb{E}} + X_j X_i u - \frac{X_i u}{|\nabla_{\mathbb{E}} u|} X_j |\nabla_{\mathbb{E}} u| = \\ &= X_j X_i u \end{aligned}$$

that is

$$\mathcal{Z} + |\nabla_{\mathbb{E}} u| H_\nu = Hu$$

Hence, we can rewrite (17) in the following way

$$(Hu)^T = (X_3 u) J + \mathcal{Z} + |\nabla_{\mathbb{E}} u| H_\nu \quad (18)$$

Furthermore

$$Jv = -\nu \quad (19)$$



and

$$\begin{aligned} (\mathcal{Z}v)_i &= \sum_{j=1}^2 \mathcal{Z}_{ij}v_j = \\ &= \sum_{j=1}^2 \nu_i \langle (Hu)^T \nu \rangle_j v_j = \nu_i \langle (Hu)^T \nu, v \rangle_{\mathbb{E}} \end{aligned}$$

that is

$$\mathcal{Z}v = \langle (Hu)^T \nu, v \rangle_{\mathbb{E}} \nu \quad (20)$$

By plugging (19) and (20) into (18), we conclude that

$$(Hu)^T v = \left( -X_3 u + \langle (Hu)^T \nu, v \rangle_{\mathbb{E}} \right) \nu + |\nabla_{\mathbb{E}} u| H_{\nu} v \quad (21)$$

and so

$$\begin{aligned} |(Hu)^T v|^2 &= \langle (Hu)^T v, (Hu)^T v \rangle_{\mathbb{E}} = \\ &= \left( -X_3 u + \langle (Hu)^T \nu, v \rangle_{\mathbb{E}} \right)^2 + |\nabla_{\mathbb{E}} u|^2 |H_{\nu} v|^2 + \\ &+ 2 \left( -X_3 u + \langle (Hu)^T \nu, v \rangle_{\mathbb{E}} \right) |\nabla_{\mathbb{E}} u| \langle H_{\nu} v, \nu \rangle_{\mathbb{E}} \end{aligned}$$

From this and the definitions in (4) and (5), we obtain that the proof of (16) is completed if we prove that

$$\langle H_{\nu} v, \nu \rangle_{\mathbb{E}} = 0 \quad (22)$$

and that

$$|H_{\nu} v| = |\operatorname{div}_{\mathbb{E}} \nu| \quad (23)$$

To this end, let us observe that, by (21),

$$|\nabla_{\mathbb{E}} u| \langle H_{\nu} v, \nu \rangle_{\mathbb{E}} = \langle (Hu)^T v, \nu \rangle_{\mathbb{E}} - \langle (Hu)v, \nu \rangle_{\mathbb{E}} + X_3 u \quad (24)$$

Now, by (17),

$$(Hu)^T v - (Hu)v = \begin{pmatrix} -\frac{X_1 u}{|\nabla_{\mathbb{E}} u|} X_3 u \\ -\frac{X_2 u}{|\nabla_{\mathbb{E}} u|} X_3 u \end{pmatrix} = -(X_3 u) \nu$$

hence

$$\langle (Hu)^T v - (Hu)v, \nu \rangle_{\mathbb{E}} = -X_3 u$$

By plugging this into (24), we obtain (22).

To obtain (23), we argue as follows. By (22), we know that  $H_{\nu} v$  is parallel (or antiparallel) to  $\nu$ , therefore

$$H_{\nu} v = \pm |H_{\nu} v| \nu$$

Hence, by (21),

$$\begin{aligned} \pm |\nabla_{\mathbb{E}} u| |H_{\nu} v| &= \langle |\nabla_{\mathbb{E}} u| H_{\nu} v, v \rangle_{\mathbb{E}} = \\ &= \langle (Hu)^T v, v \rangle_{\mathbb{E}} = \sum_{i,j=1}^2 (X_i X_j u) v_i v_j \end{aligned} \quad (25)$$

Now, we remark that

$$\nu_i^2 = 1 - v_i^2 \quad (26)$$

To prove this, we take  $i = 1$  (the case  $i = 2$  being analogous), and we observe that

$$\nu_1^2 = v_2^2 = 1 - v_1^2$$

which establishes (26).

On the other hand, if  $i \neq j$ ,

$$\nu_i \nu_j = \nu_1 \nu_2 = (-v_2)(v_1) = -v_i v_j \quad (27)$$

So, by (11), (26) and (27), we obtain

$$\begin{aligned} |\nabla_{\mathbb{E}} u| |\operatorname{div}_{\mathbb{E}} \nu| &= |\nabla_{\mathbb{E}} u| \sum_{i=1}^2 X_i \left( \frac{X_i u}{|\nabla_{\mathbb{E}} u|} \right) = \\ &= \sum_{i=1}^2 X_i X_i u - \frac{X_i u}{|\nabla_{\mathbb{E}} u|} \langle X_i (\nabla_{\mathbb{E}} u), \nu \rangle_{\mathbb{E}} = \\ &= \sum_{i=1}^2 X_i X_i u - \sum_{i,j=1}^2 (X_i X_j u) \nu_i \nu_j = \\ &= \sum_{i=1}^2 X_i X_i u - \sum_{i=1}^2 (X_i X_i u) \nu_i^2 - \sum_{i \neq j=1}^2 (X_i X_j u) \nu_i \nu_j = \\ &= \sum_{i=1}^2 X_i X_i u - \sum_{i=1}^2 (X_i X_i u) (1 - v_i^2) + \sum_{i \neq j=1}^2 (X_i X_j u) v_i v_j = \\ &= \sum_{i=1}^2 (X_i X_i u) v_i^2 + \sum_{i \neq j=1}^2 (X_i X_j u) v_i v_j = \\ &\quad \sum_{i,j=1}^2 (X_i X_j u) v_i v_j \end{aligned} \quad (28)$$

By comparing (25) and (28), we see that

$$\pm |\nabla_{\mathbb{E}} u| |H_{\nu} v| = |\nabla_{\mathbb{E}} u| |\operatorname{div}_{\mathbb{E}} \nu|$$

which implies (23), as desired.  $\square$

**Lemma 3.3.** *For each  $u \in C^{\infty}(\Omega)$  it holds that*

$$X_1 \Delta_{\mathbb{E}} u = \Delta_{\mathbb{E}} X_1 u + 2X_3 X_2 u + X_4 u; \quad (29)$$

$$X_2 \Delta_{\mathbb{E}} u = \Delta_{\mathbb{E}} X_2 u - 2X_1 X_3 u + X_4 u; \quad (30)$$

$$X_3 \Delta_{\mathbb{E}} u = \Delta_{\mathbb{E}} X_3 u - 2X_4 X_1 u - 2X_4 X_2 u; \quad (31)$$

$$\text{and } X_4 \Delta_{\mathbb{E}} u = \Delta_{\mathbb{E}} X_4 u. \quad (32)$$

*Proof.* For the first equality

$$\begin{aligned}
X_1\Delta_{\mathbb{E}}u &= X_1(X_1X_1u) + X_1(X_2X_2u) \\
&= \Delta_{\mathbb{E}}X_1u + X_1X_2X_2u - X_2X_2X_1u \\
&= \Delta_{\mathbb{E}}X_1u + X_3X_2u + X_2X_1X_2u - X_2X_2X_1u \\
&= \Delta_{\mathbb{E}}X_1u + 2X_3X_2u + X_2X_3u - X_3X_2u \\
&= \Delta_{\mathbb{E}}X_1u + 2X_3X_2u + X_4u.
\end{aligned}$$

The second and the third equality follow in a similar way, indeed

$$\begin{aligned}
X_2\Delta_{\mathbb{E}}u &= X_2(X_1X_1u) + X_2(X_2X_2u) \\
&= \Delta_{\mathbb{E}}X_2u - X_1X_1X_2u + X_2X_1X_1u \\
&= \Delta_{\mathbb{E}}X_2u - X_3X_1u + X_1X_2X_1u - X_1X_1X_2u \\
&= \Delta_{\mathbb{E}}X_2u - X_3X_1u - X_1X_3u \\
&= \Delta_{\mathbb{E}}X_2u - X_3X_1u + X_1X_3u - 2X_1X_3u \\
&= \Delta_{\mathbb{E}}X_2u - 2X_1X_3u + X_4u
\end{aligned}$$

and

$$\begin{aligned}
X_3\Delta_{\mathbb{E}}u &= X_3(X_1X_1u) + X_3(X_2X_2u) \\
&= X_1X_3X_1u + X_2X_3X_2u - X_4X_1u - X_4X_2u \\
&= \Delta_{\mathbb{E}}X_3u - 2X_4X_1u - 2X_4X_2u.
\end{aligned}$$

The last is a direct consequence of  $X_1X_4u = X_4X_1u$  and  $X_2X_4u = X_4X_2u$ .  $\square$

Using Lemma 3.3, we obtain

**Corollary 3.4.** *Let  $u \in C^\infty(\Omega)$  be a solution of (1) then*

$$\begin{aligned}
\Delta_{\mathbb{E}}X_1u + 2X_3X_2u + X_4u &= \dot{f}(u)X_1u \\
\Delta_{\mathbb{E}}X_2u - 2X_1X_3u + X_4u &= \dot{f}(u)X_2u \\
\Delta_{\mathbb{E}}X_3u - 2X_4X_1u - 2X_4X_2u &= \dot{f}(u)X_3u.
\end{aligned}$$

Now, some observation related to the Coarea formula, in order to reduce the computations in the whole of  $\mathbb{E}$  to the one in  $\mathbb{E}_0$ .

**Lemma 3.5.** *If  $u : \Omega \rightarrow \mathbb{R}$  is Lipschitz with respect to the  $d_{CC}$  distance then, for every  $c \in \mathbb{R}$ , the set  $\{x \in \Omega \mid \nabla_{\mathbb{E}}u(x) \neq 0\} \cap \{x \in \Omega \mid u(x) = c\}$  has zero Lebesgue measure.*

*Proof.* For every  $f \in L^1(\Omega)$  by the Coarea formula proved in [19] we have

$$\int_{\Omega} f|\nabla_{\mathbb{E}}u|dx = \int_{-\infty}^{+\infty} \left( \int_{\{x \in \Omega \mid u(x)=t\}} f d|\partial E_t|_{\mathbb{E}} \right) dt$$

where  $|\partial E_t|_{\mathbb{E}}$  is the Engel group perimeter (see [7, 16, 17]).

If we take as  $f$  the characteristic function of the set  $U \cap \{u = c\}$  where  $U \subseteq \Omega$  is a bounded domain, then

$$\int_{\{x \in \Omega \mid u(x) = t\}} f d|\partial E_t|_{\mathbb{E}} = 0 \quad \forall t \neq c$$

hence

$$\int_{U \cap \{u=c\}} |\nabla_{\mathbb{E}} u| dx = 0$$

that implies the desired result.  $\square$

Using Lemma 3.5 and (10) it easily follows the following

**Corollary 3.6.** *If  $u \in Lip_{loc}(\Omega)$  then, for every  $c \in \mathbb{R}$ , the set  $\{x \in \Omega \mid \nabla_{\mathbb{E}} u(x) \neq 0\} \cap \{x \in \Omega \mid u(x) = c\}$  has zero Lebesgue measure.*

With this, we are in the position of proving the following geometric inequality:

**Proposition 3.7.** *Let  $u \in C^\infty(\Omega)$  be a stable weak solution of (1). Then, for each  $\eta \in C_0^\infty(\Omega)$ ,*

$$\begin{aligned} & \int_{\mathbb{E}_0} [|Hu|^2 - \langle (Hu)^2 \nu, \nu \rangle_{\mathbb{E}}] \eta^2 - 2 \int_{\mathbb{E}_0} (X_3 X_2 u X_1 u - X_3 X_1 u X_2 u) \eta^2 - \\ & - \int_{\mathbb{E}_0} (X_4 u)(X_1 u - X_2 u) \eta^2 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \eta|^2 |\nabla_{\mathbb{E}} u|^2 \end{aligned}$$

*Proof.* Multiplying by  $(X_1 u) \eta^2$  equation (29) in Corollary 3.4 and by  $(X_2 u) \eta^2$  equation (30) and then integrating by parts we obtain

$$\begin{aligned} & - \int_{\mathbb{E}} \langle \nabla_{\mathbb{E}} X_1 u, \nabla_{\mathbb{E}} (X_1 u \eta^2) \rangle_{\mathbb{E}} + 2 \int_{\mathbb{E}} X_3 X_2 u (X_1 u) \eta^2 + \int_{\mathbb{E}} X_4 u (X_1 u) \eta^2 = \\ & = \int_{\mathbb{E}} \dot{f}(u) (X_1 u)^2 \eta^2 \\ & - \int_{\mathbb{E}} \langle \nabla_{\mathbb{E}} X_2 u, \nabla_{\mathbb{E}} (X_2 u \eta^2) \rangle_{\mathbb{E}} - 2 \int_{\mathbb{E}} X_1 X_3 u (X_2 u) \eta^2 + \int_{\mathbb{E}} X_4 u (X_2 u) \eta^2 = \\ & = \int_{\mathbb{E}} \dot{f}(u) (X_2 u)^2 \eta^2 \end{aligned}$$

Consequently, by summing term by term, we get

$$\begin{aligned} & - \int_{\mathbb{E}} (|\nabla_{\mathbb{E}} X_1 u|^2 + |\nabla_{\mathbb{E}} X_2 u|^2) \eta^2 - \\ & - \int_{\mathbb{E}} \langle \nabla_{\mathbb{E}} X_1 u, \nabla_{\mathbb{E}} \eta^2 \rangle_{\mathbb{E}} X_1 u - \int_{\mathbb{E}} \langle \nabla_{\mathbb{E}} X_2 u, \nabla_{\mathbb{E}} \eta^2 \rangle_{\mathbb{E}} X_2 u + \end{aligned} \quad (33)$$

$$\begin{aligned} & + 2 \int_{\mathbb{E}} (X_3 X_2 u X_1 u - X_1 X_3 u X_2 u) \eta^2 + \int_{\mathbb{E}} (X_4 u)(X_1 u + X_2 u) \eta^2 = \\ & = \int_{\mathbb{E}} \dot{f}(u) |\nabla_{\mathbb{E}} u|^2 \eta^2 \end{aligned} \quad (34)$$

On the other hand, since  $u$  is stable, by choosing  $|\nabla_{\mathbb{E}}u|\eta$  as a test function in (2) we obtain

$$0 \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta)|^2 + \int_{\mathbb{E}} \dot{f}(u)|\nabla_{\mathbb{E}}u|^2\eta^2 \quad (35)$$

By Corollary 3.6 we have that  $\nabla_{\mathbb{E}}(|\nabla_{\mathbb{E}}u|\eta) = 0$  almost everywhere outside  $\mathbb{E}_0$ ; hence making use of (13) we obtain from (35) that

$$\begin{aligned} 0 \leq & \int_{\mathbb{E}_0} \left( \frac{\eta^2}{|\nabla_{\mathbb{E}}u|^2} |(Hu)^T \nabla_{\mathbb{E}}u|^2 + 2\eta \langle (Hu)^T \nabla_{\mathbb{E}}u, \nabla_{\mathbb{E}}\eta \rangle_{\mathbb{E}} + |\nabla_{\mathbb{E}}\eta|^2 |\nabla_{\mathbb{E}}u|^2 \right) + \\ & + \int_{\mathbb{E}} \dot{f}(u)|\nabla_{\mathbb{E}}u|^2\eta^2 \end{aligned}$$

So, noticing that  $2\eta\nabla_{\mathbb{E}}\eta = \nabla_{\mathbb{E}}\eta^2$ , and using (33), after a simplification we obtain that

$$\begin{aligned} & \int_{\mathbb{E}} |Hu|^2\eta^2 - \int_{\mathbb{E}_0} \frac{\eta^2}{|\nabla_{\mathbb{E}}u|^2} |(Hu)^T \nabla_{\mathbb{E}}u|^2 - 2 \int_{\mathbb{E}} (X_3X_2uX_1u - X_1X_3uX_2u)\eta^2 - \\ & - \int_{\mathbb{E}} (X_4u)(X_1u + X_2u)\eta^2 \leq \int_{\mathbb{E}_0} |\nabla_{\mathbb{E}}\eta|^2 |\nabla_{\mathbb{E}}u|^2. \end{aligned}$$

Recalling that

$$X_1X_3u = X_3X_1u + X_4u$$

we get the thesis.  $\square$

Then, from Proposition 3.7 and Lemma 3.2 we immediately obtain Theorem 1.1.

We end this section by giving some more geometric insight on the quantity  $\mathcal{J}$ , in relation with the intrinsic normal and tangent vectors:

**Lemma 3.8.** *For every  $u \in C^2(\Omega)$  and every  $x \in \mathbb{E}_0$  it holds*

$$\mathcal{J}(x) = -|\nabla_{\mathbb{E}}u|(x) \langle \nabla_{\mathbb{E}}X_3u(x), v(x) \rangle_{\mathbb{E}} - |\nabla_{\mathbb{E}}u|(x)^2 \langle X_3\nu(x), v(x) \rangle_{\mathbb{E}} \quad (36)$$

*Proof.* By definition in  $\mathbb{E}_0$

$$\langle \nabla_{\mathbb{E}}X_3u, v \rangle_{\mathbb{E}} = \frac{1}{|\nabla_{\mathbb{E}}u|} (X_1X_3uX_2u - X_2X_3uX_1u) \quad (37)$$

and using (7) we obtain

$$\langle \nabla_{\mathbb{E}}X_3u, v \rangle_{\mathbb{E}} = \frac{1}{|\nabla_{\mathbb{E}}u|} \left[ (X_3X_1uX_2u - X_3X_2uX_1u) + X_4u(X_2u - X_1u) \right]. \quad (38)$$

Moreover, in  $\mathbb{E}_0$ ,

$$\langle X_3\nu, v \rangle_{\mathbb{E}} = \frac{1}{|\nabla_{\mathbb{E}}u|^2} (X_3X_1uX_2u - X_3X_2uX_1u) \quad (39)$$

hence adding (38) and (39) we get the thesis.  $\square$

Using Theorem 1.1 and Lemma 3.8 it immediately follows that

**Corollary 3.9.** *Let  $u \in C^\infty(\Omega)$  be a stable weak solution of (1). Then, for each  $\eta \in C_0^\infty(\Omega)$ ,*

$$\begin{aligned} & \int_{\mathbb{E}_0} \left( |\nabla_{\mathbb{E}} u|^2 \left\{ \left( p + \frac{\langle (Hu)^T \nu, v \rangle_{\mathbb{E}}}{|\nabla_{\mathbb{E}} u|} \right)^2 + h^2 \right\} + |\nabla_{\mathbb{E}} u| \langle \nabla_{\mathbb{E}} X_3 u, v \rangle_{\mathbb{E}} + |\nabla_{\mathbb{E}} u|^2 \langle X_3 \nu, v \rangle_{\mathbb{E}} \right) \eta^2 \\ & \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \eta|^2 |\nabla_{\mathbb{E}} u|^2 \end{aligned} \quad (40)$$

## 4 Some applications to entire stable solutions: geometric equations and non-existence results

From now on, for every  $x = (x_1, x_2, x_3, x_4) \in \mathbb{E}$ , we denote by

$$|x| := \left( (x_1^2 + x_2^2)^6 + x_3^6 + x_4^4 \right)^{\frac{1}{12}}$$

the standard gauge norm in  $\mathbb{E}$  (see [7], [10]), and we denote by

$$B(0, R) := \{x \in \mathbb{E} \mid |x| < R\}$$

the gauge open ball centered at 0 of radius  $R$ .

The following Lemma is proved in [13].

**Lemma 4.1.** *Let  $g \in L_{loc}^\infty(\mathbb{R}^n, [0, +\infty))$  and let  $q > 0$ . Let also, for any  $\tau > 0$ ,*

$$\eta(\tau) := \int_{B(0, \tau)} g(x) dx \quad (41)$$

*Then, for every  $0 < r < R$ ,*

$$\int_{B(0, R) \setminus B(0, r)} \frac{g(x)}{|x|^q} dx \leq q \int_r^R \frac{\eta(\tau)}{\tau^{q+1}} d\tau + \frac{1}{R^q} \eta(R)$$

**Corollary 4.2.** *Let  $u$  be a stable solution of  $\Delta_{\mathbb{E}} u = f(u)$  in the whole of  $\mathbb{E}$  with*

$$\mathcal{J} \leq 0 \quad \text{in } \mathbb{E}_0 \quad (42)$$

*For any  $\tau > 0$  and any  $x = (x_1, x_2, x_3, x_4) \in \mathbb{E}$ , let us define*

$$\eta(\tau) := \int_{B(0, \tau)} |\nabla_{\mathbb{E}} u(x)|^2 dx \quad (43)$$

*If*

$$\liminf_{R \rightarrow \infty} \frac{\int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3} d\tau + \frac{\eta(R)}{R^2}}{(\log R)^2} = 0 \quad (44)$$

then, the level set of  $u$  in the proximity of noncharacteristic points are such that

$$\operatorname{div}_{\mathbb{E}} \nu = 0 \quad (45)$$

and on such sets the following equation holds

$$p = -\frac{1}{|\nabla_{\mathbb{E}} u|} \langle Hu\nu, \nu \rangle_{\mathbb{E}} \quad (46)$$

*Proof.* This is a modification of the proof of Corollary 3.2 of [13], where we take into account the more complicated algebraic calculations of the Engel group. Given  $R > 1$ , we define

$$\phi_R(x) := \begin{cases} 1 & \text{if } x \in B(0, \sqrt{R}) \\ 2(\log R)^{-1} \log(R/|x|) & \text{if } x \in B(0, R) \setminus B(0, \sqrt{R}) \\ 0 & \text{if } x \in \mathbb{E} \setminus B(0, R) \end{cases}$$

We observe that

$$\begin{aligned} X_1|x|^{12} &= 12(x_1^2 + x_2^2)^5 x_1 - Ax_2 - 2x_3x_4^3 \\ \text{and } X_2|x|^{12} &= 12(x_1^2 + x_2^2)^5 x_2 + Ax_1 - 2x_3x_4^3 \end{aligned}$$

with  $A := 3x_3^5 + (1/3)(x_1 + x_2)x_4^3$ . Since  $|x_1| \leq |x|$ ,  $|x_2| \leq |x|$ ,  $|x_3| \leq |x|^2$  and  $|x_4| \leq |x|^3$ , we conclude that  $|A| \leq C_1|x|^{10}$  and so

$$|\nabla_{\mathbb{E}}|x|^{12}| \leq C_2|x|^{11}$$

for some for some  $C_1, C_2 > 0$ .

Notice also that, in  $B(0, R) \setminus B(0, \sqrt{R})$ ,

$$\phi_R(x) = C(R) - (1/6)(\log R)^{-1} \log|x|^{12}$$

for some  $C(R) \in \mathbb{R}$ , thus

$$|\nabla_{\mathbb{E}}\phi_R(x)| = (1/6)(\log R)^{-1}|x|^{-12}|\nabla_{\mathbb{E}}|x|^{12}| \leq C_3(\log R)^{-1}|x|^{-1}$$

in  $B(0, R) \setminus B(0, \sqrt{R})$ , for some  $C_3 > 0$ . Therefore, by (42) and Theorem 1.1,

$$\begin{aligned} & \int_{\mathbb{E}_0} \left[ |\nabla_{\mathbb{E}} u|^2 \left\{ \left( p + \frac{\langle (Hu)^T \nu, \nu \rangle}{|\nabla_{\mathbb{E}} u|} \right)^2 + h^2 \right\} \right] \phi_R^2 \\ & \leq \int_{\mathbb{E}} |\nabla_{\mathbb{E}} \phi_R|^2 |\nabla_{\mathbb{E}} u|^2 \leq C_4 (\log R)^{-2} \int_{B(0, R) \setminus B(0, \sqrt{R})} \frac{|\nabla_{\mathbb{E}} u|^2}{|x|^2} \end{aligned}$$

for some  $C_4 > 0$ . On the other hand, by Lemma 4.1,

$$\int_{B(0, R) \setminus B(0, \sqrt{R})} \frac{|\nabla_{\mathbb{E}} u|^2}{|x|^2} \leq 2 \int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3} d\tau + \frac{1}{R^2} \eta(R)$$

All in all,

$$\begin{aligned} & \int_{\mathbb{E}_0} \left[ |\nabla_{\mathbb{E}} u|^2 \left\{ \left( p + \frac{\langle (Hu)^T \nu, v \rangle}{|\nabla_{\mathbb{E}} u|} \right)^2 + h^2 \right\} \right] \phi_R^2 \\ & \leq 2C_4 (\log R)^{-2} \left[ \int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3} d\tau + \frac{1}{R^2} \eta(R) \right] \end{aligned}$$

Then the claim follows by sending  $R \rightarrow \infty$ , thanks to (44).  $\square$

**Remark 4.3.** Recalling Lemma 3.8, we observe that (42) is implied by the following monotonicity conditions:

$$\langle X_3 \nu, v \rangle_{\mathbb{E}} \geq 0 \quad \text{and} \quad \langle \nabla_{\mathbb{E}} X_3 u, v \rangle_{\mathbb{E}} \geq 0$$

**Remark 4.4.** Condition (44) may be seen as a bound on the energy growth: for instance, it is satisfied if  $\eta(R)/R^2$  stays bounded for large  $R$ , i.e. if the energy in  $B(0, R)$  does not grow more than  $R^2$ . Of course, this is quite a strong assumption on the decay of  $\nabla_{\mathbb{E}} u$  in the variables  $(x_3, x_4)$  and it would be desirable to investigate in which way such condition may be weakened.

**Remark 4.5.** We stress that equations (45) and (46) may be seen as geometric equations along the level sets of the solution  $u$ . In particular, (45) may be stated as saying that the level set is a minimal surface for the Engel framework (in analogy with the Euclidean setting and in the terminology of [19]). Also, (46) is a prescription on the imaginary curvature  $p$ , in relation with the Hessian, the normal, and the tangent vectors.

**Remark 4.6.** Let us observe that if  $u$  is solution of (1.1) that do not depend on  $x_3$  and  $x_4$  then  $u$  satisfies  $\Delta u = f(u)$ , where  $\Delta$  is the classical Euclidean Laplacian. Moreover, by [13, Remark 3.4], every bounded stable solution of (1.1) that do not depend on the last two coordinates and satisfies (44) has to be constant.

**Theorem 4.7.** *There exists no stable solution of  $\Delta_{\mathbb{E}} u = f(u)$  satisfying*

- i. The zeros of  $\check{f}$  (if any) are isolated;*
- ii.  $\{x \in \mathbb{E} \mid \nabla_{\mathbb{E}} u(x) = 0\} = \emptyset$ ;*
- iii.  $u \in L^\infty(\mathbb{E})$ ;*
- iv.  $\langle X_3 \nu, v \rangle_{\mathbb{E}} \geq 0$  in  $\mathbb{E}$ ;*
- v.  $\langle \nabla_{\mathbb{E}} X_3 u, v \rangle_{\mathbb{E}} \geq 0$  in  $\mathbb{E}$ ;*
- vi. the set  $\{(X_1 u + X_2 u) = 0\}$  has zero Lebesgue measure;*

$$\text{vii. } \liminf_{R \rightarrow \infty} \frac{\int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3} d\tau + \frac{\eta(R)}{R^2}}{(\log R)^2} = 0;$$



where  $\eta$  is as in Corollary 4.2.

*Proof.* By contradiction: let  $u \in C^3(\mathbb{E})$  be a stable solution of (1) and satisfying (i), (ii), (iii), (iv), (v), (vi) and (vii). By (iii) and [8, Th. 2.10] we have

$$|\nabla_{\mathbb{E}} u| \in L^\infty(\mathbb{E}) \quad (47)$$

We claim that

$$X_3 u = 0 \quad \text{in } \mathbb{E} \quad (48)$$

To this end, we argue by contradiction, supposing that there exists  $Q \in \mathbb{E}$  such that

$$X_3 u(Q) \neq 0. \quad (49)$$

Thus we consider the following Cauchy problem

$$\begin{cases} \phi'(s) = v(\phi(s)) \\ \phi(0) = Q \end{cases}$$

where  $v$  is as in (3). By (ii) and the fact that  $|v| = 1$  it follows that the solution exists and it is defined for any  $s \in \mathbb{R}$ . Moreover, by (ii)

$$u(\phi(s))' = \left\langle \nabla_{\mathbb{E}} u(\phi(s)), \phi'(s) \right\rangle_{\mathbb{E}} = |\nabla_{\mathbb{E}} u(\phi(s))| \langle \nu(\phi(s)), v(\phi(s)) \rangle_{\mathbb{E}} = 0 \quad \forall s \in \mathbb{R}$$

that is,  $\phi$  lies on the level set of  $u$ , namely

$$\phi(s) \in \{x \in \mathbb{E} \mid u(x) = u(Q)\} \quad \forall s \in \mathbb{R}.$$

Furthermore,

$$|\nabla_{\mathbb{E}} u(\phi(s))|' = \left\langle \nabla_{\mathbb{E}} |\nabla_{\mathbb{E}} u|(\phi(s)), \phi(s)' \right\rangle_{\mathbb{E}} \quad \forall s \in \mathbb{R}$$

and by (14) (applied here with  $\eta \equiv 1$ ) and Corollary 4.2 (recall also Remark 4.3) we get

$$\begin{aligned} |\nabla_{\mathbb{E}} u(\phi(s))|' &= \frac{1}{|\nabla_{\mathbb{E}} u(\phi(s))|} \left\langle (Hu)^T \nabla_{\mathbb{E}} u(\phi(s)), v(\phi(s)) \right\rangle_{\mathbb{E}} = \\ &= \langle \nu(\phi(s)), (Hu)v(\phi(s)) \rangle_{\mathbb{E}} = -|\nabla_{\mathbb{E}} u(\phi(s))| p(\phi(s)) = \\ &= X_3 u(\phi(s)) \quad \forall s \in \mathbb{R} \end{aligned} \quad (50)$$

which, via (49), implies

$$|\nabla_{\mathbb{E}} u(\phi(s))|'_{|s=0} \neq 0 \quad (51)$$

From (50) we deduce

$$\begin{aligned} |\nabla_{\mathbb{E}} u(\phi(s))|'' &= (X_3 u(\phi(s)))' = \left\langle \nabla_{\mathbb{E}} X_3 u(\phi(s)), \phi'(s) \right\rangle_{\mathbb{E}} = \\ &= \langle \nabla_{\mathbb{E}} X_3 u(\phi(s)), v(\phi(s)) \rangle_{\mathbb{E}} \quad \forall s \in \mathbb{R} \end{aligned} \quad (52)$$

and by (iv) we deduce also that

$$|\nabla_{\mathbb{E}}u(\phi(s))|'' \geq 0 \quad \forall s \in \mathbb{R} \quad (53)$$

Therefore, defining  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\Phi(s) := |\nabla_{\mathbb{E}}u(\phi(s))| - |\nabla_{\mathbb{E}}u(Q)|$$

we have that  $\Phi \in C^2(\mathbb{R})$ ,  $\Phi(0) = 0$ ,  $\Phi'(s) \neq 0 \quad \forall s \in \mathbb{R}$  and  $\Phi''(s) \geq 0 \quad \forall s \in \mathbb{R}$ , thanks to (51) and (53). It follows that

$$\sup_{\mathbb{R}} \Phi = +\infty$$

but this is in contradiction with (47), hence (48) is established. Now we claim that

$$X_4u = 0 \quad (54)$$

By Corollary 3.4, we have

$$\begin{aligned} \Delta_{\mathbb{E}}X_3u - 2X_4X_1u - 2X_4X_2u &= X_3\Delta_{\mathbb{E}}u = \\ &= X_3(f(u)) = \dot{f}(u)X_3u \end{aligned}$$

and so by (48) it follows that

$$X_4(X_1u + X_2u) = 0 \quad (55)$$

Moreover, by Corollary 3.4 and (48),

$$\begin{aligned} \Delta_{\mathbb{E}}X_1u - X_4u &= \Delta_{\mathbb{E}}X_1u - 2X_4u + X_4u = \\ &= \Delta_{\mathbb{E}}X_1u - 2(X_2X_3 - X_3X_2)u + X_4u = \Delta_{\mathbb{E}}X_1u + 2X_3X_2u + X_4u = \\ &= \dot{f}(u)X_1u \end{aligned} \quad (56)$$

and

$$\begin{aligned} \Delta_{\mathbb{E}}X_2u + X_4u &= \Delta_{\mathbb{E}}X_2u - 2X_1X_3u + X_4u = \\ &= \dot{f}(u)X_2u \end{aligned} \quad (57)$$

By adding (56) and (57) we obtain

$$\Delta_{\mathbb{E}}(X_1u + X_2u) = \dot{f}(u)(X_1u + X_2u) \quad (58)$$

and so, by Lemma 3.3,

$$\begin{aligned} \Delta_{\mathbb{E}}X_4(X_1u + X_2u) &= X_4\Delta_{\mathbb{E}}(X_1u + X_2u) = \\ &= X_4(\dot{f}(u)(X_1u + X_2u)) = \\ &= \ddot{f}(u)X_4u(X_1u + X_2u) + \dot{f}(u)X_4(X_1u + X_2u) \end{aligned} \quad (59)$$

Accordingly, using (59) and (55), we conclude that

$$\ddot{f}(u)X_4u(X_1u + X_2u) = 0 \quad \text{in } \mathbb{E}$$

Hence, by (vi)

$$\ddot{f}(u)X_4u = 0 \quad \text{almost everywhere in } \mathbb{E}$$

and so, by continuity,

$$\ddot{f}(u)X_4u = 0 \quad \text{everywhere in } \mathbb{E}$$

This implies that (54) holds at any point of the open set  $G := \{x \in \mathbb{E} \mid \ddot{f}(u(x)) \neq 0\}$ . So, by continuity, (54) holds at any point of its closure  $\overline{G}$ .

We show that (54) also holds at points of  $\mathbb{E} \setminus \overline{G}$  (if any). For this, let us take  $x_o \in \mathbb{E} \setminus \overline{G}$ . Since the latter is an open set, there exists an open neighborhood  $V$  such that

$$x_o \in V \subseteq (\mathbb{E} \setminus \overline{G}) \subseteq \mathbb{E} \setminus G = \{x \in \mathbb{E} \mid \ddot{f}(u(x)) = 0\}.$$

In particular,  $\ddot{f}(u(x)) = 0$  for any  $x \in V$ . Thus, by (i),  $u(x)$  must be constant for any  $x \in V$ . Therefore,  $X_4u(x) = 0$  for any  $x \in V$ , and, in particular,  $X_4u(x_o) = 0$ .

This shows that (54) holds at points of  $\mathbb{E} \setminus \overline{G}$  too, and so the proof of (54) is completed.

Now, by (48) and (54), we conclude that  $u$  does not depend on  $x_3$  and  $x_4$  and by Remark 4.6 we conclude that  $u$  is constant but this is impossible by (ii), which proves Theorem 4.7.  $\square$

**Remark 4.8.** Of course, we do not believe that our Theorem 4.7 is optimal: we just consider it a first attempt towards the understanding of semilinear equations in the Engel framework and, as far as we know, this is the first non-existence result in this setting. We think it would be interesting to develop a stronger theory and possibly to drop some structural assumptions in Theorem 4.7.

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