Smooth and creased equilibria for elastic-plastic plates and beams

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Abstract. We show that minimizers of elastic-plastic energies dependent on jump integrals are smooth provided a smallness condition is fulfilled by the load. We examine also the structure of extremals when this smallness condition is violated.

1. Introduction

The problem of modelling elastic-plastic plates and beams has been widely studied in the literature ([12],[7],[6]). In last years several results in the derivation of these models by variational limits of thin 3D elastic plastic plates has been achieved in [8], [9],[10]. The resulting functional contains a *volume* term which is responsible of the elastic energy released in the deformation and a *surface* term which represents the cost of formation of free *plastic hinges*: plasticity occurs along free *yield lines* whose location satisfies a variational principle ([8],[9],[10]).
Here we deal with the consistency of these models: in particular it would be quite natural to expect that if external loads are small then solutions have no creases, while it should be possible exhibiting a threshold and a transverse load with total mass above this threshold such that the corresponding solution has at least one plastic hinge.
This fact, at least in the case of beams, is strictly related to the structure of the Green function of the operator d^4/dx^4 and to the best constant in *Poincarè inequality* (see Lemma 3.1). However when load distribution is symmetric with respect to the center of the beam then the solution is always regular provided a safe load condition is satisfied, say total mass of load is less than 8 times the yielding constant normalized by the length of the beam.
We can prove that, for generic transverse load acting on elastic-plastic plates or beams, the behavior of the material remains elastic as long as the maximum stress

does not exceed a critical value (see Theorem 2.2 and Theorem 3.7) while, beyond this value we can exhibit examples undergoing formation of (no more than two) plastic hinges, at least in the case of beams (see Theorem 4.1). Here we mention only the case of homogeneous Dirichlet boundary conditions; detailed proofs for homogeneous and non homogeneous boundary data are given in [11].

2. Regular minimizers for clamped elastic-plastic plates

Let $\Sigma \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary, $\mu > 0, \beta > 0, \gamma > 0$ and $\lambda \in \mathbb{R}$ be given constants such that $3\lambda + 2\mu > 0$ and σ be a finite Radon measure such that $\sigma = f\,dx + \sigma_s$, $\operatorname{spt} f \subset \overline{\Sigma}, \operatorname{spt}\sigma_s \subset\subset \Sigma$.
According to the variational model derived in [9], we study the functional

$$\begin{aligned} \mathcal{P}(w) = \quad & \frac{2}{3}\mu \int_\Sigma (|\nabla^2 w|^2 + \frac{\lambda}{\lambda + 2\mu} |\Delta_a w|^2)\, dx + \\ & + \beta\,\mathcal{H}^1(S_{\nabla w}) + \gamma \int_{S_{\nabla w}} \left| \left[\frac{\partial w}{\partial \nu_{\nabla w}} \right] \right| \, d\mathcal{H}^1 - \int_\Sigma w\, d\sigma \end{aligned} \tag{2.1}$$

to be minimized among scalar functions $w \in SBH(\mathbb{R}^2)$ such that $\operatorname{spt} w \subset \overline{\Sigma}$.

Here $SBH(\mathbb{R}^2)$ is the space of $W^{1,1}(\mathbb{R}^2)$ functions whose Hessian is a (matrix valued) measure without Cantor part and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. If $w \in SBH(\mathbb{R}^2)$, then $S_{\nabla w}$ denotes the set of jump points of ∇w, $\nu_{\nabla w}$ its normal unit vector, $\nabla^2 w$ denotes the absolutely continuous part of $D^2 w$ and $\Delta_a w$ the absolutely continuous part of Δw, that is $\Delta_a w = \operatorname{Tr} \nabla^2 w$. The total variation in $\mathbb{R}$ of σ will be denoted with $|\sigma|_T$.

We remark explicitly that, in general, a minimizer of $\mathcal{P}$ exists whenever a smallness condition on the total mass of σ is satisfied, namely the following result holds (see [2],[3],[4],[5][9]):

Theorem 2.1. *Assume that*

$$|\sigma|_T < 4\gamma. \qquad \textbf{(safe load condition)} \tag{2.2}$$

Then $\mathcal{P}$ achieves a finite minimum.

Our analysis shows that when the maximum stress does not exceed a critical value depending on the material, then the behavior of the material itself remains elastic (see [11] for a detailed proof).

Theorem 2.2. *Let u be the unique solution of*

$$u \in H^2(\mathbb{R}^2)\,, \qquad u \equiv 0 \ \text{in}\ \mathbb{R}^2 \setminus \overline{\Sigma}\,, \qquad \Delta^2 u \ = \ \frac{3(\lambda + 2\mu)}{8\mu(\lambda + \mu)}\,\sigma \quad \text{in}\ \Sigma\,. \tag{2.3}$$

If

$$\left\| D^2 u \ + \ \frac{\lambda}{\lambda + 2\mu}\,(\Delta u)\,\mathbb{I} \right\|_{L^\infty(\Sigma)} \ \le \ \frac{3\gamma}{4\mu} \tag{2.4}$$

then u is the unique minimizer of $\mathcal{P}$.

We notice that standard estimates on the solution of (2.3) show that when $|\sigma|_T$ is sufficiently small then (2.4) is satisfied.

3. The clamped elastic-plastic beam

Analogous properties can be proved for beams; moreover, in dimension one a lot of additional information about creased solution can be stated. We summarize the main results in the present and in the next section, where the following assumptions are always understood: $L > 0$, $\beta > 0$, $\gamma > 0$ are given constants and $\sigma = f\,dx + \sigma_s$ is a Radon measure in $\mathbb{R}$ such that $\operatorname{spt} f \subset [0, L]$ and $\operatorname{spt} \sigma_s \subset\subset (0, L)$.
According with the beam model obtained in [8] and [10] we want to minimize the functional

$$\mathcal{F}(w) \;=\; \frac{1}{2}\int_{\mathbb{R}} |\ddot{w}|^2\;dx - \int_{\mathbb{R}} w\,d\sigma \;+ \beta\,\sharp(S_{\dot{w}}) \;+\; \gamma \sum_{S_{\dot{w}}} \;|[\dot{w}]| \tag{3.1}$$

among scalar functions w such that $w \in SBH(\mathbb{R})$, $\operatorname{spt} w \subset [0, L]$.
Here and in the following: $\sharp$ is the counting measure; if $v \in L^1_{loc}(\mathbb{R})$ then $v', \dot{v}$ denote respectively the distributional derivative of v and its absolutely continuous part; $SBH(\mathbb{R})$ denotes the space of $W^{1,1}(\mathbb{R})$ functions such that $v'' = (v')'$ is a finite Radon measure without Cantor part; for every $v \in SBH(\mathbb{R})$, $\dot{v} \equiv v'$ holds true, $S_{\dot{v}}$ is the set of jump points of $\dot{v}$ and $\ddot{v}$ denotes the absolutely continuous part of $v'' = (\dot{v})'$.

3.1. Existence of minimizers

Analogously to the case of elastic-plastic plates, existence of minimizers of (3.1) depends upon an estimate of the embedding constant: in this special case we have the following sharp result for the optimal embedding constant.

Lemma 3.1. *(Poincarè inequality* [11]*) Let $z \in SBH(\mathbb{R})$ with $\operatorname{spt} z \subset [0, L]$. Then*

$$\|z\|_{L^\infty} \leq \frac{L}{8}|z''|_T$$

and we notice that equality holds when $z(x) = \frac{L}{2} - |x - \frac{L}{2}|$ for $x \in [0, L]$.

Starting from the above Poincarè inequality we can prove that a smallness condition (safe load condition) on $|\sigma|_T$ entails the existence of minimizers.

Lemma 3.2. *Assume that*

$$|\sigma|_T \;<\; \frac{8\,\gamma}{L} \qquad \textit{(}\textbf{safe load condition)} \tag{3.2}$$

then $\mathcal{F}$ achieves a finite minimum.

Proof. By Lemma 3.1, (3.2) and Young inequality we get

$$
\begin{aligned}
\left|\int_{\mathbb{R}} w\, d\sigma\right| \;&\leq\; \frac{L}{8}|\sigma|_T |w''|_T \leq \frac{L}{8}|\sigma|_T \sum_{S_{\dot w}} |[\dot w]| + \gamma \int_0^L |\ddot w|\, dx \leq \\
&\leq \frac{L}{8}|\sigma|_T \sum_{S_{\dot w}} |[\dot w]| + \frac{1}{4}\int_0^L |\ddot w|^2\, dx + \gamma^2 L
\end{aligned}
\tag{3.3}
$$

for every $w \in SBH(0,L)$ such that $\operatorname{spt} w \subset [0,L]$. Hence

$$
\mathcal{F}(w) \geq \frac{1}{4}\int_0^L |\ddot w|^2\, dx + \beta\, \sharp(S_{\dot w}) + \left(\gamma - \frac{L}{8}|\sigma|_T\right) \sum_{S_{\dot w}} |[\dot w]| - \gamma^2 L
$$

and existence of minimizers follows by (3.2) with a standard compactness and l.s.c argument (see [1], [3], [4], [8], [9]). $\square$

Evaluation of the first variation of $\mathcal{F}$ yields (see [11]) the following statement.

Theorem 3.3. *(Euler equations) Let $w \in argmin\mathcal{F}$. Then*

$$
\begin{array}{llll}
(i) & (\ddot w)'' & = \sigma & \text{in } (0,L) \\
(ii) & \ddot w_- & = \gamma\, \operatorname{sign}([\dot w]) & \text{in } S_{\dot w} \cap (0,L] \\
(iii) & \ddot w_+ & = \gamma\, \operatorname{sign}([\dot w]) & \text{in } S_{\dot w} \cap [0,L)\,.
\end{array}
$$

In particular $\ddot w \in BH(0,L)$, hence $\ddot w$ is continuous in $[0,L]$ and $w'''' = \sigma$ in $(0,L) \setminus S_{\dot w}$.
We notice that if $\sigma_s \equiv 0$ then $\ddot w_- = \ddot w_+$ on the whole $(0,L)$.

Remark 3.4. We notice that if $w \in \operatorname{argmin}\mathcal{F}$ and $S_{\dot w} = \emptyset$, then $w(0) = w(L) = w'(0) = w'(L))$, $\ddot w = w''$ and condition i) of Theorem (3.3) entails $w'''' = \sigma$ in $(0,L)$: hence $w \equiv u$, say it is the solution of (3.4).

Another important consequence of Euler equations is the following statement.

Lemma 3.5. *(Compliance identity) Assume that w satisfies conditions (i),(ii),(iii) of Theorem* 3.3.*Then*

$$
\mathcal{F}(w) \;=\; -\frac{1}{2}\int_0^L |\ddot w|^2\, dx \;+\; \beta\, \sharp\, (S_{\dot w}).
$$

Proof. By i) we have $(\ddot w)'' = \sigma$. By ii) iii) $\ddot w$ is continuous in $(0,L)$. By taking into account $\operatorname{spt}\sigma_s \subset\subset (0,L)$, $\operatorname{spt} w \subset [0,L]$ and $w'' = \ddot w + \sum_{S_{\dot w}} [\dot w]\, d\, \sharp \llcorner S_{\dot w}$ we get

$$
\int_{\mathbb{R}} w\, d\sigma = \int_0^L w\, d\sigma = \int_0^L (\ddot w)'' w = -\int_0^L (\ddot w)' w' = \int_0^L \ddot w\, w'' = \int_0^L |\ddot w|^2 + \sum_{S_{\dot w}} \ddot w\, [\dot w]\,.
$$

Recalling that $\ddot{w} = \gamma \,\mathrm{sign}[\dot{w}]$

$$\int_0^L w\,d\sigma \;=\; \int_0^L |\ddot{w}|^2\,dx \;+\; \gamma \sum_{S_{\dot{w}}} |[\dot{w}]|$$

and the thesis follows by the definition of $\mathcal{F}$. □

3.2. Green function and regular minimizers

An argument analogous to the one used in the proof of Theorem 1.2 leads to the following theorem about regular minimizers ([11]).

Theorem 3.6. *Let $u \in H^2(\mathbb{R})$, $u \equiv 0$ in $\mathbb{R} \setminus (0, L)$ such that*

$$\begin{cases} u'''' = \sigma \text{ in } (0,L) \\ u(0) = u(L) = u'(0) = u'(L) = 0\,. \end{cases} \tag{3.4}$$

If

$$\|u''\|_{L^\infty(0,L)} \;\le\; \gamma \qquad (\textbf{stress regularity condition})\,, \tag{3.5}$$

then u is the unique minimizer of $\mathcal{F}$.

From now on any solution of (3.4) which is also a minimizer of $\mathcal{F}$ is called a *smooth minimizer* of $\mathcal{F}$.

Let $\mathcal{G}(x,y)$ be the Green function of the operator d^4/dx^4 in $(0,L)$, say

$$\begin{cases} \mathcal{G}_{xxxx}(\cdot\;, y) = \delta_y \;\text{ in } (0,L)\,, \\[2ex] \mathcal{G}(0,y) = \mathcal{G}_x(0,y) = \mathcal{G}(L,y) = \mathcal{G}_x(L,y) = 0\,. \end{cases} \tag{3.6}$$

Then the solution of (3.4) is given by

$$u(x) \;=\; \int_0^L \mathcal{G}(x,y)\,d\sigma(y)\,. \tag{3.7}$$

By setting $P_3(y) = L^{-3}\,(3L-2y)\,y^2$, $P_1(y) = L^{-1}y$, and

$$J_3(x,y) = \begin{cases} P_3(y) & \text{if } 0 \le y \le x \le L, \\ -P_3(L-y) & \text{if } 0 \le x < y \le L, \end{cases} \tag{3.8}$$

$$J_1(x,y) = \begin{cases} P_1(y) & \text{if } 0 \le y \le x \le L, \\ -P_1(L-y) & \text{if } 0 \le x < y \le L, \end{cases} \tag{3.9}$$

we have $J_3(x,\cdot) \in C([0,L] - \{x\})$, moreover $J_3(x,\cdot)$ is a bounded Borel function for every $x \in [0,L]$. Therefore

$$u'''(x) \;=\; \int_0^L J_3(x,y)\,d\sigma(y) \quad \text{for a.e. } x \in [0,L] \tag{3.10}$$

$$u''(x) \;=\; \int_0^L J_1(x,y)\left(\int_0^L J_3(x,\tau)\,d\sigma(\tau)\right)dy \tag{3.11}$$

for every $x \in [0, L]$ and direct calculations show that

$$u''(x) \ = \ \int_0^L K(x,y)\, d\sigma(y) \tag{3.12}$$

where

$$K(x,y) = \frac{1}{2}(2x-L)P_3(y) - \frac{1}{2L}y^2 + (y-x)^+. \tag{3.13}$$

By using (3.12), (3.13) and Hölder inequality we get the following theorem.

Theorem 3.7. *Let u be the unique solution of problem (3.4), then*

$$\|u''\|_{L^\infty} \ \leq \ \frac{4L}{27}\ |\sigma|_T. \tag{3.14}$$

Remark 3.8. By considering the special case $\sigma = \delta_{\frac{2L}{3}}$ we get

$$u''(x) = \frac{1}{2}(2x-L)P_3(\frac{2L}{3}) - \frac{2L}{9} + (\frac{2L}{3} - x)^+\,, \tag{3.15}$$

hence, $\sigma = \delta_{\frac{2L}{3}}$ entails the equality in (3.14):

$$\|u''\|_\infty = \frac{4L}{27}. \tag{3.16}$$

Therefore the constant $\frac{4L}{27}$ in (3.14) is the best possible.

A straightforward consequence of Theorems 3.7 and 3.6 is the following statement.

Theorem 3.9. *If*

$$|\sigma|_T \leq \ \frac{27\,\gamma}{4\,L} \qquad (\textbf{load regularity condition}) \tag{3.17}$$

then u is a smooth minimizer of $\mathcal{F}$ and is also the unique minimizer of $\mathcal{F}$.

Theorem 3.10. *Assume (3.2) holds true, $\sigma \geq 0$ (or $\sigma \leq 0$) and*

$$\sigma(x) \ = \ \sigma(L-x) \tag{3.18}$$

Then $\mathcal{F}$ has a unique and smooth minimizer which coincides with the solution of problem (3.4) .

Proof. Let u be the unique solution of (3.4). Then

$$J_3\left(\frac{L}{2} - \tau, y\right) = -J_3\left(\frac{L}{2} + \tau, L-y\right) \qquad \forall \tau \in [0, L/2]\,.$$

By taking into account (3.18), Green representation (3.10) yields for a.e. $\tau \in [0, \frac{L}{2}]$

$$\begin{gathered} u'''\left(\frac{L}{2}-\tau\right) = \int_0^L J_3\left(\frac{L}{2}-\tau, y\right)\, d\sigma(y) = \\ -\int_0^L J_3\left(\frac{L}{2}+\tau, L-y\right) d\nu(y) = -\int_0^L J_3\left(\frac{L}{2}+\tau, y\right) d\sigma(y) = \\ = -u'''\left(\frac{L}{2}+\tau\right). \end{gathered} \tag{3.19}$$

Hence u'' is convex (resp. concave) and symmetric with respect to $x = L/2$. Therefore $\|u''\|_{L^\infty} = \max\{|u''(L)|, |u''(\frac{L}{2})|\}$ and (3.12) entails

$$u''(L) = \int_0^L K(L,y)\, d\sigma(y) = \frac{1}{L^2}\int_0^L y^2(L-y)\, d\sigma(y) \tag{3.20}$$

$$u''\left(\frac{L}{2}\right) = \int_0^L K\left(\frac{L}{2}, y\right) d\sigma(y) = \int_0^L \left(-\frac{y^2}{2L} + \left(y - \frac{L}{2}\right)^+\right) d\sigma(y). \tag{3.21}$$

Hence

$$\|u''\|_{L^\infty} \le \frac{L}{8}|\sigma|_T \tag{3.22}$$

and recalling (3.2) and Theorem 3.6 the thesis follows. □

4. Existence and properties of creased minimizers

We have shown in Lemma 3.2 that a minimizer of $\mathcal{F}$ exists provided $|\sigma|_T < 8\gamma/L$, while in Theorem 3.9 we have proven that, whenever $|\sigma|_T \le \frac{27\gamma}{4L} < \frac{8\gamma}{L}$, this minimizer is smooth and coincides with the unique solution of (3.4).
We emphasize that the gap between the safe load condition (3.2) and the load regularity condition (3.15) is so small that, at a first glance one could think that no creased minimizer exists. Actually finding explicit examples of creased minimizers is a quite hard task, but the difficulty may be circumvented by exploiting the estimate (3.20).
Here we show an explicit example of of load ,whose total mass belongs to $(\frac{27\gamma}{4L}, \frac{8\gamma}{L})$, such that the corresponding minimizers of $\mathcal{F}$ are not solutions of (3.4), say they are not smooth minimizers.

4.1. An example of creased minimizer

We choose

$$\sigma = \frac{k\,\gamma}{L}\,\delta_{2L/3} \qquad \text{where} \quad \frac{27}{4} < k < 8\,. \tag{4.1}$$

Then

$$\frac{27\,\gamma}{4L} < |\sigma|_T = \frac{k\,\gamma}{L} < \frac{8\,\gamma}{L}$$

and the safe load condition is satisfied but condition (3.17) is violated.

Let u the solution of (3.4): we want to show that if in addition

$$0 \,<\, \beta \,<\, \frac{L}{4}\,(\ddot{u}(L)-\gamma)^2$$

then there exists $w \in SBH(\mathbb{R}),\ w \equiv 0$ outside $(0,L)$ such that $\mathcal{F}(w) < \mathcal{F}(u)$.
We first observe that by (4.1) and (3.20) we get

$$u''(L) = \frac{1}{L^2}\int_0^L \tau^2(L-\tau)\,d\sigma(\tau) = \frac{4k\gamma}{27} > \gamma. \tag{4.2}$$

Let now v be solution of

$$\begin{cases} v'''' \;=\; 0 \qquad \text{in } (0,L) \\ v(0)=v'(0)=v(L)=0, \qquad\qquad v''(L)=\gamma-u''(L), \end{cases}$$

explicitly

$$v(x) = \frac{\gamma-u''(L)}{4L}\,x^2(x-L)\,, \qquad\qquad v'(L)=\frac{L}{4}(\gamma-u''(L)).$$

Then $w=u+v$ is a solution of

$$\begin{cases} w'''' \;=\; f \qquad (0,L) \\ w(0)=w'(0)=w(L)=0; \quad \ddot{w}(L)=\gamma \end{cases}$$

If we define $w \equiv 0$ in $\mathbb{R}\setminus(0,L)$ then $S_{\dot{w}}=\{L\}$, $[\dot{w}](L)=\dfrac{L}{4}(u''(L)-\gamma)>0$, $\ddot{w}(L)=\gamma=\gamma\,\mathrm{sign}([w](L))$. Hence all Euler equations in Theorem 3.3 are fulfilled. Then by taking into account $u(0)=u(L)=\dot{u}(0)=\dot{u}(L)=0=v(0)=\dot{v}(0)=v(L)$ and $\dddot{v}\equiv$ constant, the compliance identity (Lemma 3.5) gives:

$$\mathcal{F}(w) = \beta - \frac{1}{2}\int_0^L |\ddot{w}|^2 = \beta - \frac{1}{2}\int_0^L |\ddot{u}|^2 - \int_0^L \ddot{u}\,\ddot{v} - \frac{1}{2}\int_0^L |\ddot{v}|^2 =$$

$$= \beta - \frac{1}{2}\int_0^L |\ddot{u}|^2 - (\dot{u}\,\ddot{v})\big|_0^L + \int_0^L \dot{u}\,\dddot{v} - \frac{1}{2}(\dot{v}\,\ddot{v})\big|_0^L + \frac{1}{2}\int_0^L \dot{v}\,\dddot{v} =$$

$$= \beta - \frac{1}{2}\int_0^L |\ddot{u}|^2 - \frac{1}{2}\dot{v}(L)\,\ddot{v}(L) =$$

$$= -\frac{1}{2}\int_0^L |\ddot{u}|^2 + \beta - \frac{L}{4}(\ddot{u}(L)-\gamma)^2 < -\frac{1}{2}\int_0^L |\ddot{u}|^2 = \mathcal{F}(u).$$

4.2. Structure of creased minimizers

We conclude our analysis by showing that the number of creases can be estimated independently of data β, γ, L, σ.

Theorem 4.1. *If* $v \in \mathrm{argmin}\,\mathcal{F}$ *then* $\sharp(S_{\dot{v}}) \le 2$.

Proof. Assume by contradiction that $w \in \operatorname{argmin} \mathcal{F}$ and $x_1 < x_2 < x_3$ are three distinct points in $S_{\dot w}$. Then we can modify w by eliminating one of them and reducing at the same time the energy. Set

$$\lambda = \frac{x_2 - x_1}{x_3 - x_1}$$

$$\varepsilon_0 = \min \left\{ |[\dot w](x_1)| \, , \, \frac{1-\lambda}{\lambda} \, |[\dot w](x_3)| \right\} \tag{4.3}$$

For every $0 < \varepsilon \le \varepsilon_0$ we define a function $w_\varepsilon \in SBH(\mathbb{R})$ such that $w_\varepsilon(0_-) = 0$ and

$$w'_\varepsilon = \dot w_\varepsilon \;=\; \dot w - \varepsilon \; \mathrm{sign}([\dot w](x_1)) \, \chi_{[x_1, x_2)} + \frac{\lambda \, \varepsilon \; \mathrm{sign}([\dot w](x_1))}{1-\lambda} \, \chi_{[x_2, x_3]}, \tag{4.4}$$

hence w_ε is different from w only outside $[x_1, x_3]$, moreover

$$\int_0^L \dot w_\varepsilon = \int_0^L \dot w = 0, \qquad \mathrm{spt}\, w \subset [0, L],$$

$$[\dot w_\varepsilon](x_2) \;=\; \frac{\varepsilon}{1-\lambda} \, \mathrm{sign}[\dot w](x_1) + [\dot w](x_2).$$

At first we assume $\varepsilon_0 = |[\dot w](x_1)|$. Then

$$[\dot w_{\varepsilon_0}](x_1) = 0,$$

and either $[\dot w_{\varepsilon_0}](x_2) \, [\dot w](x_2) \ge 0$ or $[\dot w_{\varepsilon_0}](x_2) \, [\dot w](x_2) < 0$.
In the first case w_{ε_0} fulfills all the Euler equations in Theorem 3.3 (since the sign of all survived jumps are preserved by (4.3)) and $\ddot w_{\varepsilon_0} = \ddot w$, $\sharp(S_{\dot w_{\varepsilon_0}}) \le 2 < 3 = \sharp(S_{\dot w})$. Hence $\mathcal{F}(w_{\varepsilon_0}) < \mathcal{F}(w)$ by the compliance identity.
In the second case since $[\dot w_\varepsilon](x_2)]$ and $[\dot w](x_2)$ has the same (resp. opposite) sign for $\varepsilon = 0$ (respectively $\varepsilon = \varepsilon_0$) we can choose $\overline{\varepsilon} \in (0, \varepsilon_0]$ such that

$$[\dot w_{\overline{\varepsilon}}](x_2) \;=\; \frac{\overline{\varepsilon}}{1-\lambda} \, \mathrm{sign}[\dot w](x_1) + [\dot w](x_2) = 0$$

then, by (4.3) $w_{\overline{\varepsilon}}$ fulfills all conditions in Theorem 3.3 and $\ddot w_{\varepsilon_0} = \ddot w$, $\sharp(S_{\dot w_{\varepsilon_0}}) \le 2 < 3 = \sharp(S_{\dot w})$. Hence $\mathcal{F}(w_{\varepsilon_0}) < \mathcal{F}(w)$ by the compliance identity.
Eventually, still assuming (4.3), we examine the case

$$\varepsilon_0 = \frac{1-\lambda}{\lambda} \, |[w](x_3)|$$

and we define a function $\omega_\varepsilon \in SBH(\mathbb{R})$ such that $\omega_\varepsilon(0_-) = 0$ and

$$\omega'_\varepsilon \;=\; \dot\omega_\varepsilon \;=\; \dot w \;-\; \varepsilon \; \mathrm{sign}([\dot w](x_3)) \, \chi_{[x_1, x_2]} \;+\; \frac{\lambda \, \varepsilon \; \mathrm{sign}([\dot w](x_3))}{1-\lambda} \; \chi_{[x_2, x_3]} \, . \tag{4.5}$$

Then

$$\int_0^L \dot\omega_\varepsilon = \int_0^L \dot w = 0, \qquad \mathrm{spt}\, w \subset [0, L],$$

$$[\dot\omega_\varepsilon](x_2) \;=\; \frac{\varepsilon}{1-\lambda} \, \mathrm{sign}[\dot w](x_3) + [\dot w](x_2),$$

and

$$[\dot{\omega}_{\varepsilon_0}](x_3) = 0\,.$$

so we can proceed as above by interchanging x_1 and x_3. □

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