REGULAR SUBMANIFOLDS, GRAPHS AND AREA FORMULA IN HEISENBERG GROUPS

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ABSTRACT. We describe intrinsically regular submanifolds in Heisenberg groups $\mathbb{H}^n$. Low dimensional and low codimensional submanifolds turn out to be of a very different nature. The first ones are Legendrian surfaces, while low codimensional ones are more general objects, possibly non Euclidean rectifiable. Nevertheless we prove that they are graphs in a natural group way, as well as that an area formula holds for the intrinsic Haudorff measure. Finally, they can be seen as Federer-Fleming currents given a natural complex of differential forms on $\mathbb{H}^n$.

CONTENTS

1. INTRODUCTION

Our aim is studying *intrinsically regular submanifolds* of the Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$. By that we mean submanifolds which, in the geometry

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of the Heisenberg group, have the same role as $\mathcal{C}^1$ submanifolds have inside Euclidean spaces. Here and in what follows, 'intrinsic' will denote properties defined only in terms of the group structure of $\mathbb{H}^n$, or, to be more precise, of its Lie algebra $\mathfrak{h}$.

We postpone complete definitions of $\mathbb{H}^n$ to the next section. Here we remind that $\mathbb{H}^n$, with group operation $\cdot$, is a (connected and simply connected) Lie group identified through exponential coordinates with $\mathbb{R}^{2n+1}$. If $\mathfrak{h}$ denotes the Lie algebra of all left invariant differential operators on $\mathbb{H}^n$, then $\mathfrak{h}$ admits the stratification $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$; $\mathfrak{h}_1$ is called horizontal layer. The horizontal layer defines, by left translation, the horizontal fiber bundle $H\mathbb{H}^n$. Since $H\mathbb{H}^n$ depends only on the stratification of $\mathfrak{h}$, we call 'intrinsic' any notion depending only on $H\mathbb{H}^n$. The stratification of $\mathfrak{h}$ induces, through the exponential map, a family of anisotropic dilations δ_λ for $\lambda > 0$. We refer to δ_λ as intrinsic dilations. A privileged role in the geometry of $\mathbb{H}^n$ is played by the so-called *horizontal curves*, these are curves tangent at any point to the fiber of $H\mathbb{H}^n$ at that point (if we think $\mathbb{H}^n$ as the configuration space of a non-holonomic mechanical system, horizontal curves describe admissible trajectories of the system).

Heisenberg groups provide the simplest non-trivial examples of nilpotent stratified, connected and simply connected Lie groups (Carnot groups in most of the recent literature).

Let us start with some comments about possible notions of regular submanifolds of a group.

It is barely worth to say that considering Euclidean regular submanifolds of $\mathbb{H}^n$, identified with the Euclidean space $\mathbb{R}^{2n+1}$, it is never a satisfactory choice and for many reasons. Indeed, Euclidean regular submanifolds need not to be group regular; this is absolutely obvious for low dimensional submanifolds: the 1-dimensional, group regular, objects are horizontal curves that are a small subclass of $\mathcal{C}^1$ lines, but, also a low codimensional Euclidean submanifold need not to be group regular due to the presence of the so called *characteristic points* where no intrinsic notion of tangent space to the surface exists (see [6], [18]). On the contrary in Carnot groups exist e.g. 1-codimensional surfaces, sometimes called $\mathbb{H}$-regular or $\mathbb{G}$-regular surfaces, that can be highly irregular as Euclidean objects but that enjoy very nice properties from the group point of view, so that it is very natural to think of them as 1-codimensional regular submanifolds of the group, (see [12], [11], [15]).

What do we mean by 'very nice properties'? The key words here are *intrinsic* and *regular*. We have already stated how *intrinsic* should be meant here. Now, the most natural requirements (and we believe non negotiable ones) to be made on a subset $S \subset \mathbb{H}^n$ to be considered as an intrinsic regular submanifold are

(i) S has, at each point, a tangent 'plane' and a normal 'plane' (or better a 'transverse plane');
(ii) tangent 'planes' depend continuously on the point;

the notion of 'plane' should be intrinsic to $\mathbb{H}^n$, i.e. depending only on the group structure and on the differential structure as given by the horizontal bundle. Since subgroups are the natural counterpart in groups of Euclidean planes through the origin, it seems accordingly natural to ask that

(iii) both the tangent 'plane' and the transversal 'plane' are subgroups (or better cosets of subgroups) of $\mathbb{H}^n$; $\mathbb{H}^n$ is the direct product of them (see later for a precise definition);
(iv) the tangent 'plane' to S in a point is the limit of group dilations of S centered in that point (see Definition 3.4).

Notice that the distinction between normal and transversal planes turns out to be a natural one. Indeed not necessarily at each point a natural normal subgroup exists, while always it does exist a (possibly not normal but) transversal subgroup. Moreover, the explicit requirement of the existence of both a tangent space and a transverse space is not pointless. Indeed there are subgroups in $\mathbb{H}^n$, as the T axis for example, without a (normal) complementary subgroup i.e a subgroup $\mathbb{G}$ of $\mathbb{H}^n$ such that $\mathbb{H}^n = \mathbb{G} \cdot T$. Finally, the transversal subgroups will appear naturally in the last section while dealing with natural differential forms and currents in $\mathbb{H}^n$.

We notice also that condition (iv) guarantees that the tangent plane has the natural geometric meaning of 'surface seen at infinite scale', the scale however being meant with respect to intrinsic dilations. This yields that – if S is both an Euclidean smooth manifold and a group regular manifold – the intrinsic tangent plane is usually different from the Euclidean one. On the other hand, as already pointed out, there are sets, 'bad' from the Euclidean point of view, that behave as regular sets with respect to group dilations.

Obviously, a natural check to be made in order to understand if requirements (i)–(iv) are reasonable ones is to see if they are met by the classes of regular submanifolds of $\mathbb{H}^n$ considered in the literature.

$\mathcal{C}^1$ *horizontal curves:* they are Euclidean $\mathcal{C}^1$ curves; their (Euclidean) tangent space in a point is a 1-dimensional affine subspace contained in the horizontal fiber through the point, hence it is also a coset of a 1-dimensional subgroup of $\mathbb{H}^n$. The normal space is the complementary subspace of the tangent space, and it is again a subgroup. Clearly both of them depend continuously on the point. It can also be shown (see Theorem 3.5) that the Euclidean tangent lines are also limits of group dilation of the curve, so that they are also tangent in the group sense.

Legendre submanifolds: they are n-dimensional, hence maximal dimensional, integral manifolds of the horizontal distribution (see [4]). The tangent spaces are n-dimensional affine subspaces of the horizontal fiber that are also cosets of subgroups of $\mathbb{H}^n$. The complementary affine subspaces are the normal subgroups. As before the tangent spaces are limit of intrinsic dilations of the surface (see Theorem 3.5).

1-*codimensional* $\mathbb{H}$-*regular surfaces:* (see [10], [11]) we recall that, locally, they are given as level sets of $\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ functions from $\mathbb{H}^n$ to $\mathbb{R}$ (see Definition 2.12), with P-differential of maximal rank (the notion of P-differential for maps between Carnot groups, introduced by Pansu in [23], provides the intrinsic notion we use systematically to be coherent with our purpose). It has been proved in ([11]) that

$\mathbb{H}$-regular surfaces have a natural normal space (i.e. the span of the *horizontal normal vector*) at each point, hence it is a coset of a 1-dimensional subgroup contained in the horizontal fibre; that the natural tangent space is a subgroup obtained as limit of intrinsic dilations of the surface; and finally, notwithstanding that these surfaces can be highly irregular as Euclidean surfaces, the intrinsic normal subgroup and the intrinsic tangent subgroup depend continuously on the point.

In conclusion, all the surfaces in these examples are *intrinsically regular submanifolds* in the sense that they satisfy requirements (i)–(iv). Notice that $\mathcal{C}^1$ horizontal curves have topological dimension 1, Legendre submanifolds have topological dimension n, and 1-codimensional $\mathbb{H}$-regular surfaces have topological dimension $2n$ (the systematic specification 'topological' is not pointless, because, as already noticed in [10], [11], other different dimensions play a role in the geometry of Carnot groups). Our aim is now to fill the picture, finding other classes of intrinsically regular submanifolds of arbitrary topological dimension.

Notice that, from the analytical point of view, horizontal curves and Legendre surfaces are given locally as images in $\mathbb{H}^n$ respectively of intervals $I \subset \mathbb{R}$ or of open sets in $\mathbb{R}^n$ through P-differentiable maps with injective differentials. On the contrary 1-codimensional $\mathbb{H}$-regular surfaces are given locally as level sets of P-differentiable functions with surjective differentials.

The first idea coming to the mind, and the one we take here, is to generalize both these approaches. Notice that, even if in the Euclidean setting they are fully equivalent, this is no longer true in Heisenberg groups. Thus, if $1 \leq k \leq n$,

from one side we look for k-dimensional integral surfaces of $H\mathbb{H}^n$ and, in Definition 3.1, we define them as images of continuously P-differentiable functions $\mathcal{V} \to \mathbb{H}^n$, $\mathcal{V}$ open in $\mathbb{R}^k$, with differentials of maximal rank, hence injective;

on the other side, in Definition 3.2, we look for k-codimensional surfaces as level sets of continuosly P-differentiable functions $\mathcal{U} \to \mathbb{R}^k$, $\mathcal{U}$ open in $\mathbb{H}^n$, with P-differential of maximal rank, hence surjective.

These two approaches are naturally different ones: indeed no nontrivial geometric object falls under the scope of both definitions. The reason of this, related with intrinsic properties of the geometry of $\mathbb{H}^n$, is simply that, for $k > n$, there is no k-dimensional subgroup of the horizontal fibre; hence surfaces having as a tangent space a subgroup of the horizontal fibre are limited to have dimension $\leq n$ and, dually, the ones with an horizontal normal space are limited to have codimension $\leq n$ (both phenomena depend on the fact that we can find at most n linearly independent and commuting elements of $\mathfrak{h}_1$).

We will call the first ones *low dimensional* (or *k-dimensional*) $\mathbb{H}$*-regular surfaces* and the second ones *low codimensional* (or *k-codimensional*) $\mathbb{H}$*-regular surfaces*. It is the object of part of this paper to prove that these so defined $\mathbb{H}$-regular surfaces enjoy properties (i)–(iv).

We recall the usual notions of Carnot-Carathéodory distance and Hausdorff measures $\mathbb{H}^n$. Once a scalar product is defined in $\mathfrak{h}$, each fiber of the horizontal bundle over a generic point p is consequently endowed with a scalar product $\langle\cdot,\cdot\rangle_p$. We denote also by $|\cdot|_p$ the associated norm. Thus, we can define the (sub-Riemannian) length of a horizontal curve $\gamma : [0,T] \to \mathbb{H}^n$ as $\int_0^T |\gamma'(t)|_{\gamma(t)}\ dt$. Given $p, q \in \mathbb{H}^n$, their *Carnot-Carathéodory distance* $d_c(p,q)$ is the minimal length of horizontal curves connecting p and q. This notion is equivalent to the definition given in the next section.

The Carnot-Carathéodory distance is not - strictly speaking - intrinsic in our sense, because it does not depend only on the horizontal bundle, but also on the scalar product we have chosen, that is somehow arbitrary. Nevertheless, we still refer to Carnot-Carathéodory distance and to related notions as intrinsic ones because different scalar products on the algebra yield equivalent Carnot-Carathéodory distances.

From Carnot-Carathéodory distance, one gets the notions of intrinsic Hausdorff measures $\mathcal{H}_c^s$ or $\mathcal{S}_c^s$, $s \geq 0$, and of intrinsic Hausdorff dimension. The s-dimensional Hausdorff measures $\mathcal{H}_c^s$ and $\mathcal{S}_c^s$ are obtained from d_c, following classical Carathéodory construction as in Federer's book [9], Section 2.10.2. The intrinsic metric (or Hausdorff) dimension $\dim_{\mathbb{H}}(\mathrm{S})$ of a set S is the number $\dim_{\mathbb{H}}(\mathrm{S}) \stackrel{\mathrm{def}}{=} \inf\{\mathrm{s} \geq 0\ :\ \mathcal{H}_{\mathrm{c}}^{\mathrm{s}}(\mathrm{S}) = 0\}$.

Let us come back to low dimensional and low codimensional $\mathbb{H}$-regular surfaces. These two families of surfaces contain very different objects. We give here a first brief sketch of their basic properties; some of them are well known while other ones are proved in this paper.

Proposition: *k-dimensional $\mathbb{H}$-regular surfaces are Euclidean submanifolds. For $k = 1$, they are horizontal curves. For $k = n$, they are Legendrian manifolds and for $k < n$ they are submanifolds of Legendrian manifolds. They have equal topological dimension, metric dimension and Euclidean dimension. Their intrinsic tangent k-planes coincide with their Euclidean tangent k-planes (both are cosets of subgroups of $\mathbb{H}^n$ contained in the horizontal fibre).*

Low codimensional $\mathbb{H}$-regular surfaces, on the contrary, can be very irregular objects from an Euclidean point of view. In general these surfaces are not Euclidean $\mathcal{C}^1$ submanifolds, not even locally (see [15] where it is constructed a 1-codimensional $\mathbb{H}$-regular surface in $\mathbb{H}^1 \equiv \mathbb{R}^3$ that is a fractal set with Euclidean dimension 2.5). Nevertheless we prove that

Proposition: *k-codimensional $\mathbb{H}$-regular surfaces have metric dimension $(2n+2-k)$, and topological dimension $(2n+1-k)$. They admit at each point an intrinsic tangent $(2n+1-k)$-plane and an intrinsic normal k-plane contained in the horizontal fibre. Both the tangent and the normal are (cosets of) subgroups of $\mathbb{H}^n$ and depend continuously on the point.*

Besides (i)–(iv), $\mathbb{H}$-regular surfaces also enjoy the following important properties (precise statements are Theorem 3.5, Theorem 3.27, and Theorem 4.1):

Theorem 1: *Any $\mathbb{H}$-regular surface is locally a graph, provided we define intrinsically the notion of graph in $\mathbb{H}^n$.*

Theorem 2: *Any $\mathbb{H}$-regular surface has locally finite intrinsic Hausdorff measure. Precisely: k-dimensional $\mathbb{H}$-regular surfaces have finite $\mathcal{H}_c^k$ measure; k-codimensional $\mathbb{H}$-regular surfaces have finite $\mathcal{H}_c^{2n+2-k}$ measure and* *the (equivalent) spherical Hausdorff measure $\mathcal{S}_\infty^{2n+2-k}$ can be even explicitly computed.*

Let us comment both these Theorems.

It is possible to introduce an intrinsic and very operative notion of graph inside $\mathbb{H}^n$ (or more generally in Carnot groups) as follows: observe that $\mathbb{H}^n$ is (in many different ways) a direct product of subgroups; that is there are couples of subgroups, let us call them $\mathbb{G}_{\mathfrak{w}}$ and $\mathbb{G}_{\mathfrak{v}}$, such that any $p \in \mathbb{H}^n$ can be written in a unique way as $p = p_{\mathfrak{w}} \cdot p_{\mathfrak{v}}$, with $p_{\mathfrak{w}} \in \mathbb{G}_{\mathfrak{w}}$ and $p_{\mathfrak{v}} \in \mathbb{G}_{\mathfrak{v}}$. Simply split the algebra $\mathfrak{h}$ as the direct sum, $\mathfrak{h} = \mathfrak{w} \oplus \mathfrak{v}$, of two subalgebras $\mathfrak{w}$ and $\mathfrak{v}$ and set $\mathbb{G}_{\mathfrak{w}} := \exp \mathfrak{w}$, $\mathbb{G}_{\mathfrak{v}} := \exp \mathfrak{v}$.

Hence $\mathbb{H}^n$ is foliated by the family $\mathcal{L}_{\mathfrak{v}}(\xi)$ of cosets of (say) $\mathbb{G}_{\mathfrak{v}}$, where $\mathcal{L}_{\mathfrak{v}}(\xi) := \xi \cdot \mathbb{G}_{\mathfrak{v}}$ for each $\xi \in \mathbb{G}_{\mathfrak{w}}$; the subgroup $\mathbb{G}_{\mathfrak{w}}$ is the 'space of parameters' of the foliation. Then we define

Definition: We say that a set $S \subset \mathbb{H}^n$ is a *graph along* $\mathbb{G}_{\mathfrak{v}}$ *(or along* $\mathfrak{v}$*)* if for each $\xi \in \mathbb{G}_{\mathfrak{w}}$, $S \cap \mathcal{L}_{\mathfrak{v}}(\xi)$ contains at most one point. Equivalently if there is a function $\varphi : E \subset \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ such that

$$S = \{\xi \cdot \varphi(\xi) : \xi \in E\}$$

and we say that *S is the graph of φ.*

An interesting special case arises when the subgroup $\mathbb{G}_{\mathfrak{w}}$ is a normal subgroup of $\mathbb{H}^n$ or, equivalently, the algebra $\mathfrak{w}$ is an ideal in $\mathfrak{h}$. Indeed, when $\mathbb{G}_{\mathfrak{w}}$ is a normal subgroup, graphs over $\mathbb{G}_{\mathfrak{w}}$ have some further useful properties (see Proposition 3.11) and we speak, in this case, of *regular graphs.*

Going back to Theorem 1, it is easy to chek that low dimensional $\mathbb{H}$-regular surfaces are graphs because they are Euclidean $\mathcal{C}^1$ submanifolds and because low dimensional intrinsic graphs in $\mathbb{H}^n$ turn out to be Euclidean graphs.

On the contrary low codimensional $\mathbb{H}$-regular surfaces need not to be graphs in the Euclidean sense. An easy example is shown in Example 3.10. One of the main result proved here (Theorem 3.27) states that any low codimensional $\mathbb{H}$-regular surface is, locally, a regular and orthogonal graph of a continuous function φ.

The proof follows from two results of independent interest. The first one (Theorem 3.12) is an Implicit Function Theorem that essentially states that if $f : \mathbb{H}^n \to \mathbb{R}^k$, $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$ is locally a bijective map from each leaf of a foliation as described above, than locally the level sets of f are intrinsic graphs with respect to that foliation.

The second result (Propositions 3.24 and 3.25) states that if S is a low codimensional $\mathbb{H}$-regular surface, then a foliation of $\mathbb{H}^n$ as required in the hypotheses of the Implicit Function Theorem in fact exists. Notice that this result is an algebraic one and that it has no counterpart in the Euclidean theory.

A more precise statement of Theorem 2 is

Theorem: *Let S be a k-codimensional $\mathbb{H}$-regular surface, $1 \le k \le n$. Then, by definition, there are an open $\mathcal{U} \subset \mathbb{H}^n$ and $f = (f_1, \dots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$ such that $S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\}$. We know that S is locally a regular graph, that is it is possible to choose two subalgebras $\mathfrak{v}$, $\mathfrak{w}$ with*

$\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{w}$*, a relatively open subset* $\mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}$ *and a continuous function* $\varphi : \mathcal{V} \to \mathbb{G}_{\mathfrak{v}}$ *such that*

$$S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\} = \{\Phi(\xi) \stackrel{\text{def}}{=} \xi \cdot \varphi(\xi), \quad \xi \in \mathcal{V}\}.$$

Choose left invariant horizontal vector fields $v_1, \ldots, v_k$ *such that* $\mathfrak{v} = \operatorname{span}\{v_1, \ldots, v_k\}$, $[v_i, v_j] = 0$ *for* $1 \le i < j \le k$, $|v_1 \wedge \cdots \wedge v_k| = 1$ *and*

$$\Delta(p) \stackrel{\text{def}}{=} \left| \det \left[v_i f_j(p) \right]_{1 \le i,j \le k} \right| \neq 0 \quad \text{for } p \in \mathcal{U}.$$

Then there is an explicit constant c*, such that*

$$\mathcal{S}_\infty^{2n+2-k} \llcorner (S \cap \mathcal{U}) = c\, \Phi_\sharp \left(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \right) \mathcal{L}^{2n+1-k} \llcorner \mathbb{G}_{\mathfrak{w}}.$$

Here, for $s \ge 0$, $\mathcal{S}_\infty^s$ denotes the s-dimensional spherical Hausdorff measures, equivalent with $\mathcal{S}_c^s$, associated with the left invariant distance d_∞ defined by $d_\infty(p, q) = d_\infty(q^{-1} \cdot p, 0)$, where, if $p = (p', p_{2n+1}) \in \mathbb{R}^{2n} \times \mathbb{R}^1 \equiv \mathbb{H}^n$, then $d_\infty(p, 0) = \max\{|p'|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\}$. For a measure μ, $\Phi_\sharp \mu$ is the image measure of μ ([21], Definition 1.17) and $\mathcal{L}^{2n+1-k}$ is the $(2n+1-k)$-dimensional Lebesgue measure.

As we pointed out repeatedly, low dimensional $\mathbb{H}$-regular surfaces are particular Euclidean $\mathcal{C}^1$ surfaces, whereas low codimensional $\mathbb{H}$-regular surfaces are 'more general' objects than Euclidean submanifolds, given that they can even be fractal sets. A further insight on this phenomenon is provided by Rumin's construction (see [26]) of a complex of differential forms in $\mathbb{H}^n$ that plays the role of the De Rham complex for Euclidean spaces.

We give here a brief sketch of Rumin's construction, that indeed holds in the more general setting of contact manifolds.

Let us denote by $\bigwedge^k \mathfrak{h}$ the vector space of the k-forms over $\mathfrak{h}$ and let $\theta := dx_{2n+1} + 2\sum_{j=1}^n (x_j dx_{n+j} - x_{n+j} dx_j) \in \bigwedge^1 \mathfrak{h}$ denote the contact form in $\mathbb{H}^n$. Then define $\mathcal{I}^k \subset \bigwedge^k \mathfrak{h}$ as the ideal generated by θ, i.e. $\mathcal{I}^k := \{\alpha \in \bigwedge^k \mathfrak{h} \,:\, \alpha = \theta \wedge \beta + d\theta \wedge \gamma\}$, and $\mathcal{J}^k \subset \bigwedge^k \mathfrak{h}$ as the set $\mathcal{J}^k := \{\alpha \in \bigwedge^k \mathbb{R}^{2n+1} : \theta \wedge \alpha = 0,\ d\theta \wedge \alpha = 0\}$.

Finally, for an open $\mathcal{U} \subset \mathbb{H}^n$, denote by $\mathcal{D}_{\mathbb{H}}^k(\mathcal{U})$ (*Heisenberg k-differential forms*) the space of smooth sections, compactly supported in $\mathcal{U}$, respectively of $\frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k}$, when $1 \le k \le n$ and of $\mathcal{J}^k$ when $n+1 \le k \le 2n+1$. These spaces are endowed with the natural topology induced by that of $\mathcal{D}^k(\mathcal{U})$. Then, Rumin proves that

Theorem(Rumin): *There is a linear second order differential operator* $D : \mathcal{D}_{\mathbb{H}}^n(\mathcal{U}) \to \mathcal{D}_{\mathbb{H}}^{n+1}(\mathcal{U})$ *such that the following sequence is locally exact and has the same cohomology as the De Rham complex on* $\mathcal{U}$*:*

$$0 \to \mathcal{C}_\infty^0(\mathcal{U}) \xrightarrow{d} \mathcal{D}_{\mathbb{H}}^1(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}_{\mathbb{H}}^n(\mathcal{U}) \xrightarrow{D}$$
$$\xrightarrow{D} \mathcal{D}_{\mathbb{H}}^{n+1}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}_{\mathbb{H}}^{2n+1}(\mathcal{U}) \to 0$$

where d *is the operator induced by the external differentiation from* $\mathcal{D}^k(\mathcal{U}) \to \mathcal{D}^{k+1}(\mathcal{U})$*, when* $k \neq n$*.*

Since we can think of surfaces as duals of forms, the picture turns out to be perfectly coherent: the objects of Rumin's complex in dimension $k \le n$ are quotient spaces of usual sets of k-forms, so that their duals are 'smaller' than the duals of usual k-forms, coherently with the fact that low dimensional surfaces are particular Euclidean $\mathcal{C}^1$ surfaces. On the other hand, the objects of Rumin's complex in dimension $k \geq n$ are subspaces of usual sets of k-forms, so that their duals are 'larger' than the duals of usual k-forms, coherently with the fact that low codimensional surfaces can be very singular sets from the Euclidean point of view. On the other hand, the second order operator D is related with the jump of the metric dimension when we pass from low dimensional to low codimensional $\mathbb{H}$-regular surfaces.

Rumin's theorem suggests to define, by duality, (Federer-Fleming) currents in $\mathbb{H}^n$, together with boundaries and masses.

Precisely, for $1 \leq k \leq 2n+1$, we call *Heisenberg current of dimension* k in $\mathcal{U}$, any continuous linear functional on $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$. If T is a k-dimensional Heisenberg current its *Heisenberg boundary* is the $(k-1)$-dimensional Heisenberg current $\partial_{\mathbb{H}} T$, defined by the identities $\partial_{\mathbb{H}} T(\alpha) = T(d\alpha)$ if $k \neq n+1$ and $\partial_{\mathbb{H}} T(\alpha) = T(D\alpha)$ if $k = n+1$. The mass $\mathbf{M}_{\mathcal{V}}(T)$, of T in an open $\mathcal{V}$, is given as one can imagine. Though, its definition requires a few algebraic preliminaries so that it will be given in full detail in Section 5.

As in the Euclidean setting, oriented $\mathbb{H}$-regular surfaces induce naturally, by integration, Heisenberg currents. The following Proposition sketches the mutual relationships among $\mathbb{H}$-regular surfaces, their intrinsic Hausdorff measures, Rumin complex and Heisenberg currents.

Proposition: *Assume $S \subset \mathcal{U}$ is a k-dimensional $\mathbb{H}$-regular surface oriented by a (horizontal) tangent k-vector field $t_{\mathbb{H}}$. Then the map*

$$\alpha \mapsto [\![S]\!]\,(\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | t_{\mathbb{H}} \rangle \, d\mathcal{S}^k_\infty$$

from $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$ to $\mathbb{R}$ is a k-dimensional Heisenberg current with locally finite mass. Precisely, if $\mathcal{V} \subset\subset \mathcal{U}$,

$$\mathbf{M}_{\mathcal{V}}([\![S]\!]) = \mathcal{S}^k_\infty(S \cap \mathcal{V}).$$

Analogously, assume S is a k-codimensional $\mathbb{H}$-regular surface oriented by a tangent $(2n+1-k)$-vector field $t_{\mathbb{H}}$, then the map

$$\alpha \mapsto [\![S]\!]\,(\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | t_{\mathbb{H}} \rangle \, d\mathcal{S}^{2n+2-k}_\infty$$

from $\mathcal{D}^{2n+1-k}_{\mathbb{H}}(\mathcal{U})$ to $\mathbb{R}$ is a $(2n+1-k)$-dimensional Heisenberg current with locally finite mass and there exists a geometric constant $c_{n,k} \in (0,1)$ such that, for any open $\mathcal{V} \subset\subset \mathcal{U}$

$$c_{n,k}\, \mathcal{S}^{2n+2-k}_\infty(S \cap \mathcal{V}) \leq \mathbf{M}_{\mathcal{V}}([\![S]\!]) \leq \mathcal{S}^{2n+2-k}_\infty(S \cap \mathcal{V}).$$

In Proposition 5.8 the last statement is made more precise, providing an explicit form of the mass of the current carried by a low codimensional $\mathbb{H}$-regular surface.

Finally, let us mention a few open problems that should be attacked starting from the results of the present paper.

- Can we extend our theory to arbitrary Carnot groups? In general Carnot groups the subject is well understood only for codimension

1 (see [11]), but seems open for general low codimensional surfaces. However notice that in a Carnot group, we may have strong restrictions to the possible dimensions of graphs (for instance, in the 3-step Engel's group only 1 dimensional and 1-codimensional graphs exist). Finally recall that, when a Carnot group is not a contact manifolds, Rumin's theory does not apply.

- A theory of rectifiable sets in any dimension and codimension should be developed, at least in $\mathbb{H}^n$.
- Is there a unifying approach for low dimensional and low codimensional $\mathbb{H}$-regular surfaces? May be looking at images of subgroups of $\mathbb{H}^n$ or, dually, at level sets of group valued functions defined on $\mathbb{H}^n$? (see the approach of Scott Pauls in [24])
- Characterize the functions such that their intrinsic graphs are $\mathbb{H}$-regular surfaces. As for hypersurfaces, we refer to [2].

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2. Multilinear Algebra and Miscellanea

2.1. **Notations.** For a general review on Heisenberg groups and their properties we refer to [27], [14] and to [29]. We limit ourselves to fix some notations.

$\mathbb{H}^n$ is the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p', p_{2n+1})$, with $p' \in \mathbb{R}^{2n}$ and $p_{2n+1} \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot q = (p' + q', p_{2n+1} + q_{2n+1} + 2\langle Jp', q'\rangle_{\mathbb{R}^{2n}})$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the $2n \times 2n$ symplectic matrix. We denote as $p^{-1} := (-p', -p_{2n+1})$ the inverse of p and as 0 the identity of $\mathbb{H}^n$.

For any fixed $q \in \mathbb{H}^n$ and for any $r > 0$ left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ and non isotropic dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are automorphisms of the group defined as

$$\tau_q(p) := q \cdot p \qquad \text{and as} \qquad \delta_r p := (rp', r^2 p_{2n+1}).$$

We denote as $\mathfrak{h}^n$ or, more frequently, as $\mathfrak{h}$ when the dimension n is intended, the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_i + 2(Jp')_i \partial_{2n+1}, \quad Y_i := \partial_{i+n} + 2(Jp')_{i+n} \partial_{2n+1}, \quad T := \partial_{2n+1}.$$

The only non-trivial commutation relations among them are $[X_j, Y_j] = -4T$, for $j = 1, \dots, n$. Sometimes we will shift notations putting

$$W_i := X_i, \quad W_{i+n} := Y_i, \quad W_{2n+1} := T, \quad \text{for } i = 1 \cdots, n.$$

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

Hence Heisenberg groups are a special instance of Carnot groups of step 2. A *Carnot group* $\mathbb{G}$ *of step* k is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a *step* k *stratification*, i.e. there exist linear subspaces $V_1, \cdots, V_k$ such that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k, \quad [V_1, V_j] = V_{j+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \text{ if } i > k.$$

An intrinsic distance on $\mathbb{H}^n$ is the *Carnot-Carathéodory* distance $d_c(\cdot,\cdot)$. To define it recall that an absolutely continuous curve $\gamma : [0,T] \to \mathbb{H}^n$ is a *subunit curve* with respect to $X_1, \dots, X_n,\ Y_1, \dots, Y_n$ if there are real measurable functions $c_1, \dots, c_{2n}$, defined in $[0,T]$, such that $\sum_j c_j^2(s) \le 1$ and $\dot\gamma(s) = \sum_{j=1}^{n} c_j(s) X_j(\gamma(s)) + c_{j+n}(s) Y_j(\gamma(s))$, for a.e. $s \in [0,T]$. Then, if $p, q \in \mathbb{H}^n$, the cc-distance (Carnot-Carathéodory distance) $d_c(p,q)$ is

$$d_c(p,q) \stackrel{\text{def}}{=} \inf\{T > 0 :\ \gamma \text{ is subunit},\ \gamma(0) = p,\ \gamma(T) = q\}.$$

The set of subunit curves joining p and q is not empty, by Chow's theorem, since the rank of the Lie algebra generated by $X_1, \dots, X_n,\ Y_1, \dots, Y_n$ is $2n+1$; hence d_c is a distance on $\mathbb{H}^n$ inducing the same topology as the standard Euclidean distance.

Several distances *equivalent* to d_c have been used in the literature. Later on, we shall use the following one, that can also be computed explicitly

$$d_\infty(p,q) = d_\infty(q^{-1} \cdot p, 0),$$

where, if $p = (p', p_{2n+1}) \in \mathbb{H}^n$, $d_\infty(p,0) := \max\{|p'|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\}$.

We shall denote respectively by $U_c(p,r)$ and by $B_c(p,r)$ the open and the closed ball associated with d_c and by $U_\infty(p,r)$ and $B_\infty(p,r)$ the open and closed balls associated with d_∞.

Both the cc-metric d_c and the metric d_∞ are well behaved with respect to left translations and dilations, that is

$$\begin{aligned} d_c(z \cdot x, z \cdot y) &= d_c(x,y) \quad , \quad d_c(\delta_\lambda x, \delta_\lambda y) = \lambda d_c(x,y) \\ d_\infty(z \cdot x, z \cdot y) &= d_\infty(x,y) \quad , \quad d_\infty(\delta_\lambda x, \delta_\lambda y) = \lambda d_\infty(x,y) \end{aligned} \tag{1}$$

for $x, y, z \in \mathbb{H}^n$ and $\lambda > 0$.

We recall that, because the topologies induced by d_c, d_∞ and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^n$ is $2n+1$. On the contrary the Hausdorff dimension of $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$, with respect to the cc-distance d_c or with respect to any other equivalent distance, is the integer $Q := 2n+2$ usually called the *homogeneous dimension* of $\mathbb{H}^n$ (see [22]).

For a nonnegative integer k, $\mathcal{L}^k$ denotes the k-dimensional Lebesgue measure. $\mathcal{L}^{2n+1}$ is the bi-invariant Haar measure of $\mathbb{H}^n$, hence, if $E \subset \mathbb{R}^{2n+1}$ is measurable, then $\mathcal{L}^{2n+1}(\tau_p(E)) = \mathcal{L}^{2n+1}(E)$ for all $p \in \mathbb{H}^n$. Moreover, if $\lambda > 0$ then $\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E)$. We explicitly observe that, $\forall p \in \mathbb{H}^n$ and $\forall r > 0$,

$$\mathcal{L}^{2n+1}(B_c(p,r)) = r^{2n+2}\mathcal{L}^{2n+1}(B_c(p,1)) = r^{2n+2}\mathcal{L}^{2n+1}(B_c(0,1)) \tag{2}$$

and as a consequence

$$\mathcal{L}^{2n+1}(\partial B_c(p,r)) = 0 \text{ and } \mathcal{L}^{2n+1}(B_c(p,r)) = \mathcal{L}^{2n+1}(U_c(p,r)). \tag{3}$$

Analogously for the d_∞ distance.

Related with the previously defined distances d_c and d_∞, different Hausdorff measures, obtained following Carathédory's construction as in [9] Section 2.10.2, are used in this paper. For $m \geq 0$, we denote by $\mathcal{H}_E^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$ and by $\mathcal{H}_c^m$ and $\mathcal{H}_\infty^m$ the m-dimensional Hausdorff measures in $\mathbb{H}^n$, obtained, respectively, from the distances d_c and d_∞. Analogously, $\mathcal{S}_E^m$, $\mathcal{S}_c^m$, and $\mathcal{S}_\infty^m$ denote the corresponding spherical Hausdorff measures. We have to be more precise about the constants appearing in the various definitions. Since explicit computations will be carried out only for the measures $\mathcal{S}_\infty^m$, with m a positive integer, we limit ourselves to this case. For each $A \subset \mathbb{H}^n$ and $\delta > 0$, $\mathcal{S}_\infty^m(A) := \lim_{\delta\to 0}\mathcal{S}_{\infty,\delta}^m(A)$, where

$$\mathcal{S}_{\infty,\delta}^m(A) = \inf\left\{\sum_i \zeta(B_\infty(p_i,r_i)) : \ A \subset \bigcup_i B_\infty(p_i,r_i) \text{ and } r_i \leq \delta\right\}$$

and the evaluation function ζ is

$$\zeta(B_\infty(p,r)) := \begin{cases} \omega_m r^m & \text{if } 1 < m \leq n, \\ 2\omega_{m-1} r^m & \text{if } m = n+1, \\ 2\omega_{m-2} r^m & \text{if } n+2 \leq m \end{cases} \tag{4}$$

where ω_m is m-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^m$. The motivation for this choice of the function ζ can be seen in the proof of Theorem 4.1.

Clearly, since d_c and d_∞ are equivalent distances, for each fixed $m > 0$, all the measures $\mathcal{H}_c^m$, $\mathcal{S}_c^m$, $\mathcal{H}_\infty^m$, and $\mathcal{S}_\infty^m$ are equivalent measures. We notice however that, due to the lack of an optimal isodiametric inequality in $\mathbb{H}^n$, it is not known if, in general, $\mathcal{H}_\infty^m(E) = \mathcal{S}_\infty^m(E)$ even for 'nice' subsets of $\mathbb{H}^n$ and for $m = Q$. Related to this point see the recent paper [25] by Severine Rigot. This is the reason why we state some of the theorems in this paper in terms of the measures $\mathcal{S}_\infty^m$ that are somehow more explicit.

Translation invariance and homogeneity under dilations of Hausdorff measures follow as usual from (1). More precisely we have

Proposition 2.1. *Let $A \subseteq \mathbb{H}^n$, $p \in \mathbb{H}^n$ and $m,r \in [0,\infty)$. Then*

$$\mathcal{S}_\infty^m(\tau_p A) = \mathcal{S}_\infty^m(A) \quad \textit{and} \quad \mathcal{S}_\infty^m(\delta_r A) = r^m \mathcal{S}_\infty^m(A).$$

The same holds for $\mathcal{S}_c^m$, $\mathcal{H}_\infty^m$ and $\mathcal{H}_c^m$.

Finally we recall the following geometric property of spheres, whose easy proof can be found in [12]

Proposition 2.2. *Let d be a translation invariant and 1-homogeneous distance in $\mathbb{H}^n$, that is d is such that $d(z\cdot x, z\cdot y) = d(x,y)$ and $d(\delta_\lambda(x), \delta_\lambda(y)) =$*

$\lambda d(x, y)$ *for* $x, y, z \in \mathbb{H}^n$ *and* $\lambda > 0$*, and denote by* U_d *or* B_d *the open or closed* d*-balls. Then*

$$\operatorname{diam}_d(B_d(x,r)) = \operatorname{diam}_d(U_d(x,r)) = 2r, \quad \textit{for } r > 0.$$

2.2. **Horizontal and integrable k-vectors and k-covectors.** We consider the vector spaces $\mathfrak{h} := \operatorname{span}\{X_1, \dots, Y_n, T\}$ and $\mathfrak{h}_1 := \operatorname{span}\{X_1, \dots, Y_n\}$, endowed with an inner product, indicated as $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and T orthonormal.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $X_1, \cdots, Y_n, T$, is the family of covectors $\{dx_1, \cdots, dx_{2n}, \theta\}$ where $\theta := dx_{2n+1} - 2\langle (Jx'), dx' \rangle_{\mathbb{R}^{2n}}$ is the *contact form* in $\mathbb{H}^n$. We indicate as $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $dx_1, \cdots, dx_{2n}, \theta$ an orthonormal basis. Sometimes it will be notationally convenient to put $\theta_1 := dx_1, \cdots, \theta_{2n} := dx_{2n}, \theta_{2n+1} := \theta$.

Following Federer (see [9] 1.3), the exterior algebras of $\mathfrak{h}$ and of $\bigwedge^1 \mathfrak{h}$ are the graded algebras indicated, as usual, as $\bigwedge_* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge_k \mathfrak{h}$ and $\bigwedge^* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge^k \mathfrak{h}$ where $\bigwedge_0 \mathfrak{h} = \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le k \le 2n+1$,

$$\bigwedge\nolimits_k \mathfrak{h} := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\},$$
$$\bigwedge\nolimits^k \mathfrak{h} := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\}.$$

The elements of $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$ are called *k-vectors* and *k-covectors*. As usual, the dual space $\bigwedge^1(\bigwedge_k \mathfrak{h})$ of $\bigwedge_k \mathfrak{h}$ can be naturally identified with $\bigwedge^k \mathfrak{h}$. The action of a k-covector φ on a k-vector v is denoted as $\langle \varphi | v \rangle$.

The *symplectic two form* $d\theta \in \bigwedge^2 \mathfrak{h}_1$ is $d\theta = 4 \sum_{i=1}^n dx_i \wedge dx_{i+n}$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{h}$ and to $\bigwedge^k \mathfrak{h}$ making the bases $W_{i_1} \wedge \cdots \wedge W_{i_k}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ orthonormal.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$. This way we obtain the algebras $\bigwedge_* \mathfrak{h}_1 = \bigoplus_{k=1}^{2n} \bigwedge_k \mathfrak{h}_1$ and $\bigwedge^* \mathfrak{h}_1 = \bigoplus_{k=1}^{2n} \bigwedge^k \mathfrak{h}_1$ whose elements are the *horizontal k-vectors* and *horizontal k-covectors*; here

$$\bigwedge\nolimits_k \mathfrak{h}_1 := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\}$$
$$\bigwedge\nolimits^k \mathfrak{h}_1 := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\}.$$

and clearly $\bigwedge_k \mathfrak{h}_1 \subset \bigwedge_k \mathfrak{h}$ for $1 \le k \le 2n$.

Definition 2.3. We define linear isomorphisms (see [9] 1.7.8)

$$* : \bigwedge\nolimits_k \mathfrak{h} \longleftrightarrow \bigwedge\nolimits_{2n+1-k} \mathfrak{h} \quad \text{and} \quad * : \bigwedge\nolimits^k \mathfrak{h} \longleftrightarrow \bigwedge\nolimits^{2n+1-k} \mathfrak{h},$$

for $1 \le k \le 2n$, putting, for $v = \sum_I v_I W_I$ and $\varphi = \sum_I \varphi_I \theta_I$,

$$*v := \sum\nolimits_I v_I (*W_I) \quad \text{and} \quad *\,\varphi := \sum\nolimits_I \varphi_I (*\theta_I)$$

where

$$*W_I := (-1)^{\sigma(I)} W_{I^*} \quad \text{and} \quad *\,\theta_I := (-1)^{\sigma(I)} \theta_{I^*}$$

with $I = \{i_1, \cdots, i_k\}$, $1 \le i_1 < \cdots < i_k \le 2n+1$, $W_I = W_{i_1} \wedge \cdots \wedge W_{i_k}$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$, $I^* = \{i_1^* < \cdots < i_{2n+1-k}^*\} = \{1, \cdots, 2n+1\} \setminus I$ and $\sigma(I)$ is the number of couples (i_h, i_ℓ^*) with $i_h > i_\ell^*$.

The following properties of the $*$ operator follow readily from the definition: $\forall v,w \in \bigwedge_k \mathfrak{h}$ and $\forall \varphi,\psi \in \bigwedge^k \mathfrak{h}$

$$
\begin{aligned}
&**v = (-1)^{k(2n+1-k)}v, \qquad **\varphi = (-1)^{k(2n+1-k)}\varphi,\\
&v\wedge *w = \langle v,w\rangle W_{\{1,\cdots,2n+1\}}, \qquad \varphi\wedge *\psi = \langle \varphi,\psi\rangle \theta_{\{1,\cdots,2n+1\}},\\
&\langle *\varphi | *v\rangle = \langle \varphi | v\rangle.
\end{aligned}
\tag{5}
$$

Notice that, if $v = v_1\wedge\cdots\wedge v_k$ is a simple k-vector, then $*v$ is a simple $(2n+1-k)$-vector. Moreover notice that

$$
\text{if } v \in \bigwedge\nolimits_k \mathfrak{h}_1, \text{ then } *v = \xi\wedge T, \text{ with } \xi \in \bigwedge\nolimits_{2n-k}\mathfrak{h}_1. \tag{6}
$$

If $v \in \bigwedge_k \mathfrak{h}$ we define $v^* \in \bigwedge^k \mathfrak{h}$ by the identity $\langle v^*|w\rangle := \langle v,w\rangle$, and analogously we define $\varphi^* \in \bigwedge_k \mathfrak{h}$ for $\varphi \in \bigwedge^k \mathfrak{h}$.

Remark 2.4. A simple non zero k-vector $v = v_1\wedge\cdots\wedge v_k \in \bigwedge_k \mathfrak{h}$ is naturally associated with a left invariant distribution of k-dimensional planes in $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$. In general, if $k>1$, this distribution is not integrable - by Frobenius Theorem - because not necessarily $[v_i,v_j] \in \operatorname{span}\{v_1,\cdots,v_k\}$. An easy example is provided by the 2-vector $X_1\wedge Y_1 \in \bigwedge_2 \mathfrak{h}_1$. Horizontal k-vectors that are also integrable (more precisely: k-vectors such that the associated distribution is integrable) will play an important role in the following. Notice that if $T \in \{v_1,\cdots,v_k\}$ then certainly (the distribution associated with) v is integrable. On the other hand, it is elementary to observe that $v \in \bigwedge_k \mathfrak{h}_1$ can be integrable only if $k \le n$. More explicit algebraic characterizations of k-vectors associated with integrable distributions are proved in Theorem 2.8.

We define the vector spaces ${}_H\bigwedge_k$ and ${}_H\bigwedge^k$ of *integrable k-vectors and k-covectors* as follows

Definition 2.5. We set ${}_H\bigwedge_0 = \mathbb{R}$ and, for $1\le k\le n$,

$$
\begin{aligned}
&{}_H\bigwedge\nolimits_k \stackrel{\text{def}}{=} \operatorname{span}\left\{v \in \bigwedge\nolimits_k \mathfrak{h}_1 : \quad v \text{ is simple and integrable}\right\},\\
&{}_H\bigwedge\nolimits_{2n+1-k} \stackrel{\text{def}}{=} *\left({}_H\bigwedge\nolimits_k\right).
\end{aligned}
$$

Integrable covectors are defined by duality: for $0\le k\le 2n+1$ we set

$$
{}_H\bigwedge\nolimits^k \stackrel{\text{def}}{=} \bigwedge\nolimits^1\left({}_H\bigwedge\nolimits_k\right) \simeq \left\{\varphi \in \bigwedge\nolimits^k \mathfrak{h} : \quad \varphi^* \in {}_H\bigwedge\nolimits_k\right\}.
$$

Notice that ${}_H\bigwedge_1 = \bigwedge_1 \mathfrak{h}_1 = \mathfrak{h}_1$. On the contrary, for $1<k\le n$, $0 \ne {}_H\bigwedge_k \subsetneq \bigwedge_k \mathfrak{h}_1$.

If $1\le k\le n$ and if $w \in {}_H\bigwedge_{2n+1-k}$ is a simple $(2n+1-k)$-vector, then one can choose $w_1,\dots,w_{2n+1-k}$ so that: $w = w_1\wedge\cdots\wedge w_{2n+1-k}$, $w_1\wedge\cdots\wedge w_{2n-k} \in \bigwedge_{2n-k}\mathfrak{h}_1$ and $w_{2n+1-k} = T$.

Recall now the definition of H-linear map (*horizontal linear map*) between Carnot groups (see [23] and also Chapter 3 of [19]). This notion plays the same central role that is played by linear maps between vector spaces.

Definition 2.6. Let $\mathbb{G}^1$ and $\mathbb{G}^2$ be Carnot groups with dilation automorphisms δ^1_λ and δ^2_λ. We say that $L : \mathbb{G}^1 \to \mathbb{G}^2$ is a H-linear map if L is a homogeneous Lie groups homomorphism, where homogeneous means that $\delta^2_\lambda(Lx) = L(\delta^1_\lambda x)$, for all $\lambda > 0$ and $x \in \mathbb{G}^1$.

In this paper we deal only with H-linear maps from $\mathbb{R}^k \to \mathbb{H}^n$ and, viceversa, from $\mathbb{H}^n \to \mathbb{R}^k$. H-linear maps are closely related with integrable k-vectors, precisely, for $1 \le k \le n$, there is a one to one correspondence between injective H-linear maps $\mathbb{R}^k \to \mathbb{H}^n$ and integrable simple k-vectors.

The following Proposition, characterizing H-linear maps $\mathbb{R}^k \to \mathbb{H}^n$, is a special instance of a more general statement proved in [19].

Proposition 2.7. *Let $k \ge 1$ and $L : \mathbb{R}^k \to \mathbb{H}^n$ be H-linear. Then there is a $2n \times k$ matrix A with $A^T JA = 0$, such that*

$$Lx = (Ax, 0), \qquad \forall x \in \mathbb{R}^k.$$

Moreover, L can be injective only if $1 \le k \le n$.

Proof. First notice that $L(\mathbb{R}^k) \subset \{p \in \mathbb{H}^n : p_{2n+1} = 0\}$. Indeed, for all $x \in \mathbb{R}^k$: $2(Lx)_{2n+1} = (Lx \cdot Lx)_{2n+1} = (L(2x))_{2n+1} = (\delta_2 Lx)_{2n+1} = 4(Lx)_{2n+1}$. Here we used the notations $\lambda p = (\lambda p', \lambda p_{2n+1})$ for $\lambda \in \mathbb{R}$ while $\delta_\lambda p = (\lambda p', \lambda^2 p_{2n+1})$. Moreover L is linear as a map $\mathbb{R}^k \to \mathbb{R}^{2n}$, hence $Lx = (Ax, 0)$ for some matrix A. Finally, for all $x, y \in \mathbb{R}^k$, $0 = (L(x + y))_{2n+1} = (Lx \cdot Ly)_{2n+1} = 2\langle JAx, Ay\rangle_{\mathbb{R}^{2n}}$ that yields $A^T JA = 0$. □

Theorem 2.8. *Assume $2 \le k \le n$ and $v = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k \mathfrak{h}_1$, $v \neq 0$. Then the following four statements are equivalent*

(1) $v \in {}_H\bigwedge_k$;
(2) $[v_i, v_j] = 0$ *for* $1 \le i, j, \le k$;
(3) $\langle \gamma \wedge d\theta | v\rangle = 0$ *for all* $\gamma \in \bigwedge^{k-2} \mathfrak{h}$;
(4) *there is an injective H-linear map $L : \mathbb{R}^k \to \mathbb{H}^n$ such that $Le_1 \wedge \cdots \wedge Le_k = v$; L can be explicitly defined as $Lx = \delta_{x_1} v_1 \cdot \delta_{x_2} v_2 \cdot \ldots \cdot \delta_{x_k} v_k$.*

Notice that for $k = 1$ statements (1) to (4) are either meaningless or trivially equivalent.

Proof.
$(1 \Rightarrow 2)$: because $[v_i, v_j]$ is always a multiple of T and $v_i, v_j \in \mathfrak{h}_1$, the necessity of (2) for the integrability of the distribution associated with v is just Frobenius theorem;
$(2 \Rightarrow 1)$: follows from Frobenius theorem;
$(2 \Leftrightarrow 3)$: a direct computation yields $[v_i, v_j] = \langle d\theta | v_i \wedge v_j\rangle = 4\langle Jv_i, v_j\rangle_{\mathbb{R}^{2n}}$. If $v = v_1, \ldots, v_k \in \mathfrak{h}_1$ and if $\gamma \in \bigwedge^{k-2} \mathfrak{h}_1$ then

$$\langle \gamma \wedge d\theta | v\rangle = \sum_\pi \sigma(\pi) \langle \gamma | v_{\pi(1)} \wedge \cdots \wedge v_{\pi(k-2)}\rangle \langle d\theta | v_{\pi(k-1)} \wedge v_{\pi(k)}\rangle$$

where the sum is extended to all the permutations π of $\{1, \ldots, k\}$ and $\sigma(\pi)$ is ± 1 accordingly with the parity of the permutation π; hence, $\forall \gamma \in \bigwedge^{k-2} \mathfrak{h}_1$, $\langle \gamma \wedge d\theta | v_1 \wedge \cdots \wedge v_k\rangle = 0$, is equivalent with $[v_i, v_j] = \langle d\theta | v_i \wedge v_j\rangle = 0$ for $1 \le i < j \le k$;
$(3 \Leftrightarrow 4)$: let $v_j = \sum_{i=1}^{2n} v_{i,j} W_i \in \mathfrak{h}_1$. Put $\tilde{v_j} := (v_{1,j}, \cdots, v_{2n,j}) \in \mathbb{R}^{2n}$. Then if A is the $2n \times k$ matrix $A = [v_{i,j}] = \left[\begin{array}{c|c|c} \tilde{v}_1 & \cdots & \tilde{v}_k \end{array}\right]$, then $A^T JA = [\langle J\tilde{v_i}, \tilde{v_j}\rangle_{\mathbb{R}^{2n}}]_{1 \le i < j \le k}$; so that recalling Proposition 2.7 the required equivalence follows. □

We want to show now that the spaces of integrable covectors are isomorphic with the spaces defined by Rumin in [26]. Indeed Rumin's paper largely inspired the present one. We begin recalling Rumin's approach: first define $\mathcal{I}^*$ and $\mathcal{J}^* \subset \bigwedge^* \mathfrak{h}$, where $\mathcal{I}^*$ is the graded ideal generated by θ, that is $\mathcal{I}^* := \{\beta \wedge \theta + \gamma \wedge d\theta : \beta, \gamma \in \bigwedge^* \mathfrak{h}\}$ and $\mathcal{J}^*$ is the annihilator of $\mathcal{I}^*$, that is $\mathcal{J}^* := \{\alpha \in \bigwedge^* \mathfrak{h} : \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0\}$. Both $\mathcal{I}^*$ and $\mathcal{J}^*$ are graded, indeed $\mathcal{I}^* = \oplus_{k=1}^{2n+1} \mathcal{I}^k$ and $\mathcal{J}^* = \oplus_{k=1}^{2n+1} \mathcal{J}^k$, where $\mathcal{I}^k, \mathcal{J}^k \subset \bigwedge^k \mathfrak{h}$ and

$$\mathcal{I}^k = \{\beta \wedge \theta + \gamma \wedge d\theta : \beta \in \bigwedge^{k-1} \mathfrak{h}, \gamma \in \bigwedge^{k-2} \mathfrak{h}\}$$

$$\mathcal{J}^k = \{\alpha \in \bigwedge^k \mathfrak{h} : \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0\}.$$

Observe that, from a well known Lemma in symplectic geometry, for $1 \leq k \leq n$: $\mathcal{I}^{2n+1-k} = \bigwedge^{2n+1-k} \mathfrak{h}$ and $\mathcal{J}^k = 0$.

The following identities, or natural isomorphisms, hold

Theorem 2.9. *For $1 \leq k \leq n$,*

$$_H\bigwedge_k = \ker \mathcal{I}^k \quad \textit{and} \quad _H\bigwedge_{2n+1-k} \simeq \frac{\bigwedge_{2n+1-k} \mathfrak{h}}{\ker \mathcal{J}^{2n+1-k}}, \tag{7}$$

$$_H\bigwedge^k \simeq \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} \quad \textit{and} \quad _H\bigwedge^{2n+1-k} = \mathcal{J}^{2n+1-k}, \tag{8}$$

where $\ker \mathcal{I}^k = \{v \in \bigwedge_k \mathfrak{h} : \langle \varphi | v \rangle = 0 \quad \forall \varphi \in \mathcal{I}^k\}$ *and* $\ker \mathcal{J}^{2n+1-k}$ *is analogously defined.*

Proof. To prove the first equality in (7) notice that, if $v \in \bigwedge_k \mathfrak{h}$, the condition $\langle \beta \wedge \theta | v \rangle = 0$ for all $\beta \in \bigwedge^{k-1} \mathfrak{h}$ implies $v \in \bigwedge_k \mathfrak{h}_1$, hence we get $\ker \mathcal{I}^k = \{v \in \bigwedge_k \mathfrak{h}_1 : \langle \gamma \wedge d\theta | v \rangle = 0 \quad \forall \gamma \in \bigwedge^{k-2} \mathfrak{h}\}$, and we conclude by the equivalence of (1) and (3) in Theorem 2.8.

To prove the second one in (7) recall that, by Definition 2.5, $_H\bigwedge_{2n+1-k} = *_H\bigwedge_k = * \ker \mathcal{I}^k$. Moreover $\ker \mathcal{I}^k = \{v \in \bigwedge_k \mathfrak{h} : \langle \varphi^*, v \rangle = 0 \quad \forall \varphi \in \mathcal{I}^k\}$ where $\varphi^* \in \bigwedge_k \mathfrak{h}$ is such that $\langle \varphi | v \rangle = \langle \varphi^*, v \rangle$, $\forall v \in \bigwedge_k \mathfrak{h}$. Hence

$$* \left(\ker \mathcal{I}^k \right) = \{v \in \bigwedge_{2n+1-k} \mathfrak{h} : \langle *\varphi^*, v \rangle = 0 \quad \forall \varphi \in \mathcal{I}^k\}. \tag{9}$$

Now notice that

$$\varphi \in \mathcal{I}^k \iff *\varphi^* \in \ker \mathcal{J}^{2n+1-k} \tag{10}$$

indeed, $*\varphi^* \in \ker \mathcal{J}^{2n+1-k} \iff \langle \psi | *\varphi^* \rangle = 0$, $\forall \psi \in \mathcal{J}^{2n+1-k} \iff \langle *\psi | \varphi^* \rangle = 0$, $\forall \psi \in \mathcal{J}^{2n+1-k}$; hence $*\varphi^* \in \ker \mathcal{J}^{2n+1-k} \iff \langle \alpha | \varphi^* \rangle = 0$, $\forall \alpha \in *(\mathcal{J}^{2n+1-k}) = \left(\mathcal{I}^k\right)^\perp$, $\iff \langle \alpha, \varphi \rangle = 0$, $\forall \alpha \in \left(\mathcal{I}^k\right)^\perp \iff \varphi \in \mathcal{I}^k$.

Finally, from (9) and (10) it follows

$$\begin{aligned} * \left(_H\bigwedge_k \right) &\equiv * \left(\ker \mathcal{I}^k \right) \\ &= \{v \in \bigwedge_{2n+1-k} \mathfrak{h} : \langle \psi, v \rangle = 0 \quad \forall \psi \in \ker \mathcal{J}^{2n+1-k}\} \\ &= \left(\ker \mathcal{J}^{2n+1-k} \right)^\perp \simeq \frac{\bigwedge_{2n+1-k} \mathfrak{h}}{\ker \mathcal{J}^{2n+1-k}}. \end{aligned}$$

This concludes the proof of the second part of (7).

To prove (8), recall that by Definition 2.5, for $1 \le k \le 2n+1$, ${}_H\bigwedge^k := \bigwedge^1 \left({}_H\bigwedge_k\right)$. Now, given that for any two finite dimensional vector spaces V and W with V subspace of W, it holds that $\bigwedge^1 \left(\frac{W}{V}\right) \simeq \ker(V)$ and $\bigwedge^1 V \simeq \frac{\bigwedge^1 W}{\ker(V)}$, we have, for $k = 1, \cdots, n$,

$$\bigwedge^1 (\ker \mathcal{I}^k) \simeq \frac{\bigwedge^1 \bigwedge_k \mathfrak{h}}{\ker(\ker \mathcal{I}^k)} \simeq \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k},$$

and, for $k = n+1, \cdots, 2n+1$,

$$\bigwedge^1 \left(\frac{\bigwedge_k \mathfrak{h}}{\ker \mathcal{J}^k} \right) \simeq \ker(\ker \mathcal{J}^k) = \mathcal{J}^k.$$

$\square$

Finally we observe that our previous algebraic construction yields canonically several bundles over $\mathbb{H}^n$. These are the bundles of k-vectors and k-covectors, still indicated as $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$, the bundles $\bigwedge_k \mathfrak{h}_1$ and $\bigwedge^k \mathfrak{h}_1$ of the horizontal k-vectors and k-covectors and the bundles ${}_H\bigwedge_k$ and ${}_H\bigwedge^k$ of the integrable k-vectors and k-covectors. The fiber of $\bigwedge_k \mathfrak{h}$ over $p \in \mathbb{H}^n$ is denoted by $\bigwedge_{k,p} \mathfrak{h}$ and analogously for the other ones.

It is customary to call *horizontal bundle* $H\mathbb{H}^n$ the bundle generated by $X_1, \dots, X_n, Y_1, \dots, Y_n$, or, with our previous notations, $H\mathbb{H}^n := \bigwedge_1 \mathfrak{h}_1$.

The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge_k \mathfrak{h}$ and on $\bigwedge^k \mathfrak{h}$ induces an inner product on each fiber of the previous bundles.

2.3. Calculus on $\mathbb{H}^n$.

Definition 2.10 (Pansu [23]). Let $(\mathbb{G}^1, \cdot)$ and $(\mathbb{G}^2, \cdot)$ be Carnot groups with dilation automorphisms δ^1_λ and δ^2_λ. Let $\mathcal{U}$ be an open subset of $\mathbb{G}^1$, and $f : \mathcal{U} \to \mathbb{G}^2$. We say that f is P-differentiable at $p_0 \in \mathcal{U}$ if there is a (unique) H-linear map $d_{\mathbb{H}} f_{p_0} : \mathbb{G}^1 \to \mathbb{G}^2$ such that

$$d_{\mathbb{H}} f_{p_0}(p) := \lim_{\lambda \to 0} \delta^2_{1/\lambda} \left(f(p_0)^{-1} \cdot f(p_0 \cdot \delta^1_\lambda p) \right)$$

uniformly for p in compact subsets of $\mathcal{U}$.

In the sequel, we shall deal only with the cases $\mathbb{G}^1 = \mathbb{R}^k$, $\mathbb{G}^2 = \mathbb{H}^n$, and $\mathbb{G}^1 = \mathbb{H}^n$, $\mathbb{G}^2 = \mathbb{R}^k$. The structure of the differential map in the first case has been already described in Proposition 2.7. In the second case, because of the commutativity of the target space, the differential can be thought as the k-uple of the P-differentials of the components of f. Again, the differential can be written in the form $d_H f_{p_0}(p) = A_{p_0} p'$, where A_{p_0} is a $(k \times 2n)$–matrix (see e.g. [12], Proposition 2.5). Thus, if $k = 1$, $d_H f_{p_0}$ can be identified with an element of $\bigwedge^1 \mathfrak{h}_1$.

Definition 2.11. If $f : \mathcal{U} \subset \mathbb{H}^n \to \mathbb{R}$ is differentiable at p, then the *horizontal gradient* of f at p is defined as

$$\nabla_H f(p) := d_H f(p)^* \in \bigwedge\nolimits_1 \mathfrak{h}_1$$

or equivalently as

$$\nabla_H f(p) = \sum_{j=1}^n (X_j f(p)) X_j + (Y_j f(p)) Y_j.$$

Definition 2.12. In the sequel, we shall use the following notations for function spaces. If $\mathcal{U} \subset \mathbb{H}^n$ and $\mathcal{V} \subset \mathbb{R}^k$ are open subsets, we denote

- $\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})$ is the vector space of continuous functions $f : \mathcal{U} \to \mathbb{R}$ such that also the P-differential $d_H f$ is continuous. $[\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$ is the set of k-uples $f = \{f_1, \cdots, f_k\}$ such that each $f_i \in \mathcal{C}^1_{\mathbb{H}}(\mathcal{U})$, for $1 \le i \le k$.
- $\mathcal{C}^1(\mathcal{V}; \mathbb{H}^n)$ is the vector space of continuous functions $f : \mathcal{V} \to \mathbb{H}^n$ such that the P-differential $d_H f(p)$ depends continuously on $p \in \mathcal{V}$.
- $\mathrm{Lip}(\mathbb{R}^k; \mathbb{H}^n)$, $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k; \mathbb{H}^n)$, $\mathrm{Lip}(\mathbb{H}^n; \mathbb{R}^k)$, $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{H}^n; \mathbb{R}^k)$ are the vector spaces of Lipshitz continuous (locally Lipshitz continuous) functions, where the metric used in the definition are the cc-metric of the corresponding spaces.

3. Regular Surfaces and Regular Graphs

3.1. **Regular submanifolds in $\mathbb{H}^n$.** Here we give the definition of $\mathbb{H}$-regular surfaces in the spirit illustrated in the introduction. We distinguish low dimensional from low codimensional surfaces, the first ones being images of open subset of Euclidean spaces while the second ones are level sets of intrinsically regular functions.

Definition 3.1. Let $1 \le k \le n$. A subset $S \subset \mathbb{H}^n$ is a *k-dimensional $\mathbb{H}$-regular surface* (or a $\mathcal{C}^1_{\mathbb{H}}$ surface of dimension k) if for any $p \in S$ there are open sets $\mathcal{U} \subset \mathbb{H}^n$, $\mathcal{V} \subset \mathbb{R}^k$ and a function $\varphi : \mathcal{V} \to \mathcal{U}$ such that $p \in \mathcal{U}$, φ is injective, φ is continuously P-differentiable with $d_H\phi$ injective, and

$$S \cap \mathcal{U} = \varphi(\mathcal{V}).$$

Definition 3.2. Let $1 \le k \le n$. A subset $S \subset \mathbb{H}^n$ is a *k-codimensional $\mathbb{H}$-regular surface* (or a $\mathcal{C}^1_{\mathbb{H}}$ surface of codimension k or a $\mathcal{C}^1_{\mathbb{H}}$ surface of topological dimension $(2n+1-k)$) if for any $p \in S$ there are an open set $\mathcal{U} \subset \mathbb{H}^n$ and a function $f : \mathcal{U} \to \mathbb{R}^k$ such that $p \in \mathcal{U}$, $f = (f_1, \dots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, $\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k \neq 0$ in $\mathcal{U}$ (equivalently, $d_H f$ is onto) and

$$S \cap \mathcal{U} = \{q \in \mathcal{U} : f(q) = 0\}.$$

Remark 3.3. For $k = 1$, Definition 3.1 gives back the notion of horizontal, continuously differentiable, curve. On the other hand, Definition 3.1 cannot be extended to the case $k > n$. Indeed, for $k > n$, as proved in [1] (see also [19]), the set of maps φ satisfying the assumptions of Definition 3.1 is empty. Even more, they show that $\mathbb{H}^n$ is *purely k–unrectifiable*, i.e., if $k > n$, for any $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k, \mathbb{H}^n)$ we have $\mathcal{H}^k_c(f(A)) = 0$ for any $A \subset \mathbb{R}^k$.

In turn Definition 3.2, for $k = 1$, gives the notion of $\mathbb{H}$–regular hypersurface introduced in [10] and [11]. Definition 3.2 – unlike the previous one – could be formally extended to $k > n$, but we restrict ourselves to $1 \le k \le n$ because only in this situation it is possible to prove (see below) that a $\mathcal{C}^1_{\mathbb{H}}$ surface of codimension k is locally a graph in a consistent suitable sense.

As we said in the introduction, the surfaces of these two families are very different from one another. The first ones are particular Euclidean C^1 submanifolds, precisely for $k = n$ they are Legendrian submanifolds ([5]), on the contrary the second ones may be very irregular from an Euclidean point of view (see [15]). We will prove that both k-dimensional and k-codimensional $\mathbb{H}$-regular surfaces are intrinsic regular surfaces as defined in the introduction. We begin recalling the definition of *Heisenberg tangent cone* to a set A in a point p

Definition 3.4. Let $A \subset \mathbb{H}^n$. The *intrinsic (Heisenberg) tangent cone* to A in 0 is the set

$$\mathrm{Tan}_{\mathbb{H}}(A,0) \stackrel{\mathrm{def}}{=} \left\{ x = \lim_{h\to+\infty} \delta_{r_h} x_h \in \mathbb{H}^n, \text{with } r_h \to +\infty \text{ and } x_h \in A \right\}$$

and the cone in a point p is given as $\mathrm{Tan}_{\mathbb{H}}(A,p) \stackrel{\mathrm{def}}{=} \tau_p \mathrm{Tan}_{\mathbb{H}}(\tau_{-p}A,0)$.

We prove, in Theorem 3.5, that a k-dimensional $\mathbb{H}$-regular surface S has an intrinsic tangent cone $\mathrm{Tan}_{\mathbb{H}}(S,p)$ at each point p and that $\mathrm{Tan}_{\mathbb{H}}(S,p)$ is a k-plane, precisely, the Euclidean tangent plane $\mathrm{Tan}(S,p)$ to S in p. Notice that this statement is far from being evident, because $\mathrm{Tan}_{\mathbb{H}}(S,p)$ is the limit of S under intrinsic dilations δ_λ, while $\mathrm{Tan}(S,p)$ is the limit under Euclidean dilations.

If $S = \{p : f(p) = 0\}$ is a k-codimensional $\mathbb{H}$-regular surfaces, in Theorem 3.29 we prove that the Heisenberg tangent cone $\mathrm{Tan}_{\mathbb{H}}(S,p)$ is always a $(2n+1-k)$-plane and that it is the translated in p of the kernel of the differential $d_H f_p$. On the contrary, as we observed before, an Euclidean $(2n+1-k)$-tangent plane to S may never exist.

On the other side, not necessarily a k-dimensional, smooth, Euclidean submanifold of $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$ belongs to any of these families: clearly it does not for $1 \le k \le n$ because of the necessary condition of being tangent to $H\mathbb{H}^n$, but also for $n < k$ because of the possible presence of the so-called characteristic points.

The following theorem provides a description of the class of the k-dimensional $\mathbb{H}$-regular surfaces.

Theorem 3.5. *If S is a k-dimensional $\mathbb{H}$-regular surface, $1 \le k \le n$, then*

(1) *S is an Euclidean k-dimensional submanifold of $\mathbb{R}^{2n+1}$ of class $\mathcal{C}^1$.*
(2) *The Euclidean tangent bundle $\mathrm{Tan}S$ is a subbundle of $\bigwedge_k \mathfrak{h}_1$ and*

$$\mathrm{Tan}(S,p) = \mathrm{Tan}_{\mathbb{H}}(S,p)$$

for any point $p \in S$.
(3) *$\mathcal{S}^k_\infty \llcorner S$ is comparable with $\mathcal{H}^k_E \llcorner S$.*

Proof. Let $\mathcal{V} \subset \mathbb{R}^k$, $\mathcal{U} \subset \mathbb{H}^n$ be open sets such that $\phi : \mathcal{V} \to \mathcal{U}$, $\phi \in \mathcal{C}^1(\mathcal{V};\mathbb{H}^n)$, ϕ injective, $d_H\phi$ injective and $S \cap \mathcal{U} = \phi(\mathcal{V})$. Assume $p = \phi(x) \in S \cap \mathcal{U}$ and $x \in \mathcal{V}$. To prove (1) it is enough to show that the Euclidean differential $d\phi_x$ exists for every $x \in \mathcal{V}$, depends continuously on x and that $d\phi_x$ is injective. Notice that $\phi \in \mathcal{C}^1(\mathbb{R}^k;\mathbb{H}^n)$ yields that $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k;\mathbb{H}^n)$ and this in turn implies that $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k;\mathbb{R}^{2n+1})$. Hence ϕ is Euclidean differentiable a.e. in $\mathcal{V}$. Let $x_0 \in \mathcal{V}$ be such that both $d\phi_{x_0}$ and $d_H\phi_{x_0}$ exist.

By Proposition 2.7, there exist a $2n \times k$ matrix A_{x_0} with $A_{x_0}^T J A_{x_0} = 0$, such that

$$d_H\phi_{x_0}(\xi) = (A_{x_0}\xi, 0),$$

for all $\xi \in \mathbb{R}^k$. By the very definition of P-differential, it is easy to see that the rows of A_{x_0} are just the first $2n$ rows of the (Euclidean) Jacobian matrix of $\phi = (\phi_1, \dots, \phi_{2n+1})$ in x_0. Because $d_H\phi_x$ is continuous and everywhere defined in $\mathcal{V}$, it follows that $\nabla\phi_j(x)$, $1 \le j \le 2n$, exist in $\mathcal{V}$ and is continuous in x.

Because the last component of $d_H\phi_x$ is zero, once more by the definition of P-differentiability, it follows

$$\nabla\phi_{2n+1}(x) = 2\sum_{j=1}^{n} \big(\phi_{j+n}(x)\,\nabla\phi_j(x) - \phi_j(x)\,\nabla\phi_{j+n}(x)\big), \tag{11}$$

for all $x \in \mathcal{V}$. This implies that $\nabla\phi_{2n+1}(x)$ is a continuous function and eventually that ϕ is continuously differentiable.

Because the rank of A_x equals k for any x, since $d_H\phi$ is 1–1, also the Jacobian matrix of ϕ is a $(2n+1)\times k$ matrix with rank k and the proof of (1) is completed.

Let us now prove $\mathrm{Tan}(S,p) = \mathrm{Tan}_{\mathbb{H}}(S,p)$ for any point $p \in S$.

First observe that, if $x \in \mathcal{V}$ and $p = \phi(x)$, an explicit computation, using (11), gives

$$p + d\phi_x(h) = p \cdot d_H\phi_x(h), \quad \text{for any } h \in \mathbb{R}^k.$$

Because $\mathrm{Tan}(S,p) = p + d\phi_x(\mathbb{R}^k)$, to achieve point (2) of the thesis, it is enough to show that

$$\mathrm{Tan}_{\mathbb{H}}(S,p) = p \cdot d_H\phi_x(\mathbb{R}^k). \tag{12}$$

Without loss of generality, we can assume $p = 0 = \phi(0)$, so that we have to prove that $d_H\phi_0(\mathbb{R}^k) = \mathrm{Tan}_{\mathbb{H}}(S,0)$.

Let $\xi = d_H\phi_0(h)$ be given. Consider the points $p_n = \phi(\frac{1}{n}h)$ that belong to S for $n \in \mathbb{N}$ sufficiently large. By definition of P-differential

$$\delta_n(p_n) \to d_H\phi_0(h) = \xi \quad \text{as } n \to \infty,$$

so that $\xi \in \mathrm{Tan}_{\mathbb{H}}(S,0)$ and $d_H\phi_0(\mathbb{R}^k) \subset \mathrm{Tan}_{\mathbb{H}}(S,0)$.

To prove the reverse inclusion, let $\xi \in \mathrm{Tan}_{\mathbb{H}}(S,0)$ be of the form $\xi = \lim_{h\to+\infty} \delta_{r_h}p_h$ with $r_h \to +\infty$ and $p_h \in S$. Since $r_h d_c(p_h,0) = d_c(\delta_{r_h}p_h,0) \to d_c(\xi,0)$, necessarily $p_h \to 0$ as $h \to \infty$. Thus, by local inverse function theorem, we can assume without loss of generality that $p_h = \phi(z_h)$, with $z_h \in \mathbb{R}^k$, $z_h \to 0$ as $h \to \infty$. Notice now that there exist $c > 0$ and $\rho > 0$ such that

$$|z|_{\mathbb{R}^k} \le c\, d_c(\phi(z),0), \qquad \text{provided } |z|_{\mathbb{R}^k} \le \rho. \tag{13}$$

Indeed, suppose by contradiction the statement is false: then there exists a sequence of points $w_h \in \mathbb{R}^k$ such that $w_h \to 0$ and

$$d_c(\phi(w_h),0)/|w_h|_{\mathbb{R}^k} \to 0 \quad \text{as} \quad h \to \infty.$$

Without loss of generality, we may assume $w_h/|w_h|_{\mathbb{R}^k} \to w$ as $h \to \infty$, with $|w| = 1$. Then, by definition of P-differential, because the converge is required to be uniform with respect to the direction, we have

$$\begin{aligned} 0 &= \lim_{h\to\infty} \frac{d_c(\phi(w_h),0)}{|w_h|_{\mathbb{R}^k}} = \lim_{h\to\infty} d_c(\delta_{1/|w_h|_{\mathbb{R}^k}}(\phi(|w_h|_{\mathbb{R}^k}\frac{w_h}{|w_h|_{\mathbb{R}^k}})),0) \\ &= d_c(d_H\phi(0)w,0), \end{aligned}$$

that yields $w = 0$ because of the injectivity of $d_H\phi_0$ and hence a contradiction. Thus, we can apply (13) with $z = z_h$ for h sufficiently large, and we get $r_h|z_h| \le c\, r_h d_c(p_h,0) = c\, d_c(\delta_{r_h}p_h,0) \le C$, for $h \in \mathbb{N}$, and therefore we can assume $r_h z_h \to z_0$ as $h \to \infty$. Finally, once more by definition of P-differential, we get that $\xi \in d_H\phi_0(\mathbb{R}^k)$, because $\xi = \lim_{h\to+\infty} \delta_{r_h} p_h = \lim_{h\to+\infty} \delta_{r_h} \phi(\frac{1}{r_h} r_h z_h) = d_H\phi_0(z_0)$, achieving the proof of (2).

The proof of (3) follows from the following area formula (once more a special instance of a more general formula in Carnot groups: see Theorem 4.3.4 in [19] or the paper [17])

$$\int_{\mathcal{V}} J_k(d_H\phi_x)\, dx = H^k_\infty(S \cap \mathcal{U})$$

where

$$J_k(d_H\phi_x) = \frac{\mathcal{H}^k_\infty(d_H\phi_x(B(x,1))}{\mathcal{L}^{2n+1}(B(x,1))}.$$

Because $d_H\phi_x(B(x,1)) \subset H\mathbb{H}^n_0$, it follows that $\mathcal{H}^k_\infty(d_H\phi_x(B(x,1))$ is proportional to $\mathcal{H}^k_E(d_H\phi_x(B(x,1)) = \mathcal{L}^k(d_H\phi_x(B(x,1))$. Indeed group translations and Euclidean translations restricted to $d_H\phi_x(\mathbb{R}^k)$ coincide as well as group and Euclidean dilations. Hence $\mathcal{H}^k_\infty$ and $\mathcal{H}^k_E$ restricted to $d_H\phi_x(\mathbb{R}^k)$ have the same invariances so that are proportional (see [21]). This concludes the proof. $\square$

3.2. **Foliations and graphs in a Lie group** $\mathbb{G}$. The Heisenberg group $\mathbb{H}^n$ and also any other Carnot group $\mathbb{G}$ is a product of subgroups in many different ways. Hence it makes sense in a natural way to speak of subsets that are graphs inside $\mathbb{G}$. The following definition seems to share with the usual Euclidean notion many good features.

Assume that the algebra $\mathfrak{g}$ of $\mathbb{G}$ is the direct sum of two subalgebras $\mathfrak{w}$ and $\mathfrak{v}$, that is

$$\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}.$$

Set now $\mathbb{G}_{\mathfrak{w}} := \exp \mathfrak{w}$, and $\mathbb{G}_{\mathfrak{v}} := \exp \mathfrak{v}$. We denote *system of coordinate planes* (i.e. left laterals) of $\mathbb{G}$ the double family $\mathcal{L}_{\mathfrak{v}}$ and $\mathcal{L}_{\mathfrak{w}}$ defined as

$$\mathcal{L}_{\mathfrak{v}}(p) := p \cdot \mathbb{G}_{\mathfrak{v}}, \quad \forall p \in \mathbb{G}_{\mathfrak{w}} \quad \text{and} \quad \mathcal{L}_{\mathfrak{w}}(q) := q \cdot \mathbb{G}_{\mathfrak{w}}, \quad \forall q \in \mathbb{G}_{\mathfrak{v}}.$$

Observe that each $x \in \mathbb{G}$ belongs exactly to one leaf in $\mathcal{L}_{\mathfrak{v}}$ and to one in $\mathcal{L}_{\mathfrak{w}}$. Observe also that the leaves in $\mathcal{L}_{\mathfrak{v}}$ (or in $\mathcal{L}_{\mathfrak{w}}$) are invariant by translations, that is $x \in \mathcal{L}_{\mathfrak{v}}(p) \implies \tau_x\mathcal{L}_{\mathfrak{v}}(p) = \mathcal{L}_{\mathfrak{v}}(p)$. Then

Definition 3.6. We say that a set $S \subset \mathbb{G}$ is a *graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$ *(or along* $\mathfrak{v}$*)* if, for each $\xi \in \mathbb{G}_{\mathfrak{w}}$, $S \cap \mathcal{L}_{\mathfrak{v}}(\xi)$ contains at most one point. Equivalently if there is a function $\varphi : E \subset \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ such that

$$S = \{\xi \cdot \varphi(\xi) : \xi \in E\}$$

and we say that S *is the graph of* φ. The set $\mathbb{G}_{\mathfrak{w}}$ will be mentioned as the space of the parameters of the graph.

If we assume that $v_1, \cdots, v_k$ and $w_1, \cdots, w_{2n+1-k}$ are bases respectively of $\mathfrak{v}$ and $\mathfrak{w}$, then φ can be univocally associated with a k-uple $(\varphi_1, \cdots, \varphi_k)$:

$\tilde{E} \subset \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$, that makes the following diagram commutative

$$\begin{array}{ccc} \mathbb{G}_{\mathfrak{w}} & \xrightarrow{\varphi} & \mathbb{G}_{\mathfrak{v}} \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathbb{R}^{2n+1-k} & \xrightarrow[(\varphi_1,\cdots,\varphi_k)]{} & \mathbb{R}^k \end{array}$$

that is

$$\varphi\left(\exp\left(\sum_{l=1}^{2n+1-k} \xi_l w_l\right)\right) = \exp\left(\sum_{l=1}^{k} \varphi_l(\xi_1, \cdots, \xi_{d-k}) v_l\right)$$

when $\exp\left(\sum_{l=1}^{2n+1-k} \xi_l w_l\right) \in E$. Notice that one can always assume that $|v_1 \wedge \cdots \wedge v_k| = |w_1 \wedge \cdots \wedge w_{2n+1-k}| = 1$. Finally, we can restate once more Definition 3.6 in the following way: S is a graph (over $\mathbb{G}_{\mathfrak{w}}$ along $\mathbb{G}_{\mathfrak{v}}$) if

$$S = \left\{\xi \cdot \exp\left(\sum_{l=1}^{k} \varphi_l(\xi_1, \cdots, \xi_{d-k}) v_l\right), \quad \xi \in E\right\}, \tag{14}$$

where $\xi := \exp\left(\sum_{l=1}^{d-k} \xi_l w_l\right)$.

When $\mathfrak{w}$ is an ideal in $\mathfrak{g}$, and not simply a subalgebra, the graphs enjoy further useful properties. Hence we define

Definition 3.7. [Regular Graph] Assume $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}$, where $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras and $\mathfrak{w}$ is also an ideal. We say that $S \subset \mathbb{G}$ is a *regular graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$ if for each $p \in \mathbb{G}_{\mathfrak{w}}$, $S \cap \mathcal{L}_{\mathfrak{v}}(p)$ contains at most one point.

Remark 3.8. When $\mathbb{G} \equiv \mathbb{H}^n$, if $\mathfrak{h} = \mathfrak{w} \oplus \mathfrak{v}$ and $\mathfrak{w}$, $\mathfrak{v}$ are subalgebras, then the larger one of the two algebras is necessarily an ideal, that is, in $\mathbb{H}^n$ graphs of codimension strictly smaller than $n+1$ are necessarily regular graphs. We are indebted with Adam Korányi for this remark and for the following elegant proof ([16]).

Assume that $\dim \mathfrak{w} \geq n+1$, then there are two cases

(1) $\mathfrak{w}$ is not abelian. Then it contains some non zero bracket, hence it contains T, hence it contains $\mathfrak{h}_2$ so that $\mathfrak{w}$ is an ideal.
(2) $\mathfrak{w}$ is abelian. Consider the bilinear form B on $\mathfrak{h}$ defined by

$$B(X,Y)T := [X,Y].$$

Observe that B restricted to $\mathfrak{h}_1$ is simplectic. Because B is invariant under the projection $P : \mathfrak{h} \to \mathfrak{h}_1$, then $P\mathfrak{w}$ is an isotropic subspace of $\mathfrak{h}_1$, hence $\dim \mathfrak{w} \leq n$. Clearly $\mathfrak{w}$ is a subspace of $P\mathfrak{w} + \mathfrak{h}_2$. Then

$$n+1 \leq \dim \mathfrak{w} \leq \dim(P\mathfrak{w} + \mathfrak{h}_2) = \dim(P\mathfrak{w}) + 1 \leq n+1.$$

Hence $\dim \mathfrak{w} = \dim(P\mathfrak{w} + \mathfrak{h}_2)$ so that $\mathfrak{w} = P\mathfrak{w} + \mathfrak{h}_2$ and, consequently, $\mathfrak{w}$ contain $\mathfrak{h}_2$ so that it is an ideal.

When $\mathbb{G} \equiv \mathbb{H}^n$, a special instance of Definition 3.6, corresponding to the notion of orthogonal graphs in Euclidean spaces is available. It is somehow simpler to work with and can be given as follows

Definition 3.9. [Orthogonal Graph] Suppose $\mathbb{G} \equiv \mathbb{H}^n$, with our previous notations, if $(w_1, \cdots, w_{2n+1-k})$ and $(v_1, \cdots, v_k)$ are basis respectively of $\mathfrak{w}$ and of $\mathfrak{v}$, if $|v_1 \wedge \cdots \wedge v_k| = |w_1 \wedge \cdots \wedge w_{2n+1-k}| = 1$ and if

$$w_1 \wedge \cdots \wedge w_{2n+1-k} = *(v_1 \wedge \cdots \wedge v_k)$$

we refer to S as an *orthogonal graph along* $\mathfrak{v}$.

As usual properties of the function φ are attributed to the graph of φ; in particular we say that the graph of φ is continuous exactly when the map $\varphi : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ is continuous.

We stress here that these intrinsic notions of graphs, adapted to the geometry of the group, are not a pointless generalization. From one side, the fact that a surface is locally a graph is, as usual, a powerful tool; here the fact that $\mathbb{H}$-regular surfaces are locally intrinsic graphs is a key tool in studying their local structure (see sections 3.5 and 4). On the other side, one could not have used the usual Euclidean notion. Indeed, as the following example shows, *$\mathbb{H}$-regular surfaces (of low codimension), in general, are not graphs in the usual Euclidean sense, while they are always, locally, graphs in the intrinsic Heisenberg sense.*

Example 3.10. See Figure 1 and Figure 2. In $\mathbb{H}^1$, with the notations of Definition 3.6, let $\mathfrak{v} = \mathrm{span}\{X\}$ and $\mathfrak{w} = \mathrm{span}\{Y, T\}$. Then $\mathbb{G}_{\mathfrak{v}} = \{(x, 0, 0) : x \in \mathbb{R}\}$ and $\mathbb{G}_{\mathfrak{w}} = \{(0, \eta, \tau) : \eta, \tau \in \mathbb{R}\}$. Then, fix $1/2 < \alpha < 1$, and take $\varphi : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ as $\varphi(0, \eta, \tau) = (|\tau|^\alpha, 0, 0)$. Define S as the graph of φ, precisely

$$S = \{\xi \cdot \varphi(\xi) : \xi \in \mathbb{G}_{\mathfrak{w}}\} = \{(|\tau|^\alpha, \eta, \tau + 2\eta|\tau|^\alpha) : \eta, \tau \in \mathbb{R}\}.$$

A non trivial theorem, proved in [2], states that if φ is sufficiently regular then its Heisenberg graph is a $\mathbb{H}$-regular surface. Our φ satisfies the hypotheses of that theorem hence S is a $\mathbb{H}$-regular surface. But, as one can easily check, S is not an Euclidean graph in any neighborhood of the origin.

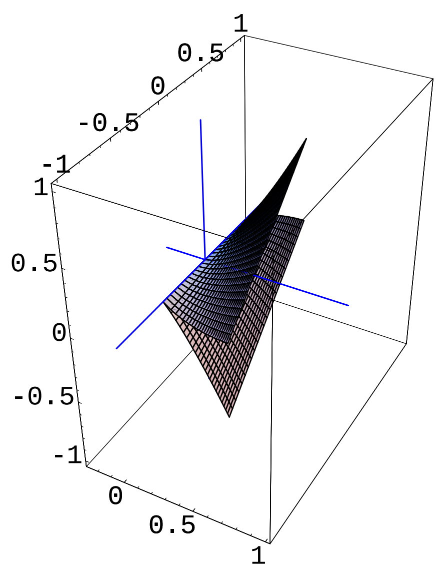

FIGURE 1. The surface $S \subset \mathbb{H}^1$ of Example 3.10 when $\alpha = 2/3$

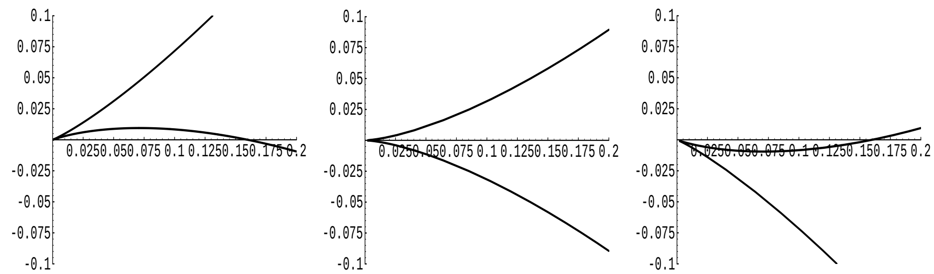

FIGURE 2. Sections of S for $x = .2$, $x = 0$ and $x = -.2$

Notice that one could have defined intrinsic graphs in more general ways. For example, one can drop the assumption that $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras asking only that they are linear subspaces such that $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}$. Everything said up to now about graphs is true in this more general setting, but for the fact that the coordinate planes in $\mathcal{L}_{\mathfrak{v}}$ and $\mathcal{L}_{\mathfrak{w}}$ are not anymore cosets of $\mathbb{G}$. This more general setting has been taken by many authors, for example when sets (graphs) as $\{(x_1, x_2, f(x_1, x_2))\} \subset \mathbb{H}^1$ are studied. In our notation this amounts to the choice of $\mathfrak{v} = \text{span}\{T\}$ and $\mathfrak{w} = \text{span}\{X_1, Y_1\}$. Here clearly $\mathfrak{w}$ is not a subalgebra and $\exp \mathfrak{w}$ is not a group.

On the other hand, intrinsic graphs, as in Definition 3.6, enjoy some nice properties that are not anymore true admitting more general definitions. For example, if $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras the intrinsic Hausdorff dimensions of the coordinate planes add up correctly to the total homogeneous dimension of $\mathbb{H}^n$. This may be false in more general settings. Think again to $\mathbb{H}^1$ with, as before, $\mathfrak{v} = \text{span}\{T\}$ and $\mathfrak{w} = \text{span}\{X_1, Y_1\}$; then $dim(\exp \mathfrak{v}) = 2$, $dim(\exp \mathfrak{w}) = 3$ (at least in a generic non characteristic point) while $dim(\mathbb{H}^1) = 4$.

Moreover, if $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras and if S is an intrinsic graph over $\mathbb{G}_{\mathfrak{w}}$ then also any left translation of S along $\mathbb{G}_{\mathfrak{w}}$ is an intrinsic graph. Precisely, if $p \in \mathbb{G}_{\mathfrak{w}}$ then $\tau_p S = \{p \cdot \xi \cdot \varphi(\xi) : \ \xi \in E\} = \{\eta \cdot \varphi \circ \tau_{-p}(\eta) : \ \eta \in \tau_p E\}$. That is, as it happens with Euclidean graphs, if S is the graph of φ then $\tau_p S$ is the graph of $\varphi \circ \tau_{-p}$.

If, in addition, S is a regular graph in the sense of Definition 3.7, it is possible to write explicitly how S behaves under a generic translation. This is the content of next Proposition.

Proposition 3.11. *Assume that S is a regular graph, as in Definition 3.7, over $\mathbb{G}_{\mathfrak{w}}$ along $\mathbb{G}_{\mathfrak{v}}$, that is $S = \{\Phi(\xi) := \xi \cdot \varphi(\xi) \ : \xi \in E\}$ and let $q \in \mathbb{G}$, $q = q_{\mathfrak{w}} \cdot q_{\mathfrak{v}}$ with $q_{\mathfrak{w}} \in \mathbb{G}_{\mathfrak{w}}$ and $q_{\mathfrak{v}} \in \mathbb{G}_{\mathfrak{v}}$. Then the translated set $\tau_q S$ is again a regular graph over $\mathbb{G}_{\mathfrak{w}}$ along $\mathbb{G}_{\mathfrak{v}}$, precisely*

$$\tau_q S = \{\Phi_q(\eta) := \eta \cdot \varphi_q(\eta) : \ \eta \in E' := q \cdot E \cdot (q_{\mathfrak{v}})^{-1} \subset \mathbb{G}_{\mathfrak{w}}\},$$

where $\varphi_q : E' \to \mathbb{G}_{\mathfrak{v}}$ is defined as $\varphi_q(\eta) := q_{\mathfrak{v}} \cdot \varphi(q^{-1} \cdot \eta \cdot q_{\mathfrak{v}})$. In addition $\Phi_q = \tau_{q^{-1}} \circ \Phi \circ \sigma_{q^{-1}}$, where $\sigma_p : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{w}}$ is defined by $\sigma_p(\eta) = p \cdot \eta \cdot p_{\mathfrak{v}}^{-1}$.

Proof. Because $\mathbb{G}_{\mathfrak{w}}$ is a normal subgroup of $\mathbb{G}$ then $E' = q_{\mathfrak{w}} \cdot q_{\mathfrak{v}} \cdot E \cdot q_{\mathfrak{v}}^{-1} \subset \mathbb{G}_{\mathfrak{w}}$. Given this, the proof is an elementary computation. $\square$

3.3. **Implicit Function Theorem.** In the first part of this section we prove a preliminary version of the Implicit Function Theorem, precisely we prove it under the assumption of the existence of $v \in {}_H\bigwedge_k$ that is 'transverse' to the surface. In the next section we will prove that for any k-codimensional, $\mathbb{H}$-regular surface the previous assumption holds true. In the second part of the section we provide a number of results related with the regularity of the implicitly defined functions. The argument in the following proof was suggested by an argument in [8], about codimension 1 surfaces in nilpotent groups.

We assume that $1 \le k \le n$ and that $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$, $v \neq 0$. That is $v_1, \cdots, v_k$ are linearly independent, left invariant vector fields in $\mathfrak{h}_1$ satisfying

$$[v_i, v_j] = 0, \quad \text{for all } 1 \le i, j \le k. \tag{15}$$

By definition, $*v \in {}_H\bigwedge_{2n+1-k}$ and we can assume $*v = w_1 \wedge \cdots \wedge w_{2n+1-k}$, with $w_1, \cdots, w_{2n-k} \in \mathfrak{h}_1$ and $w_{2n+1-k} = T$. We set $\mathfrak{v} := \operatorname{span}\{v_1, \cdots, v_k\}$ and $\mathfrak{w} := \operatorname{span}\{w_1, \cdots, w_{2n+1-k}\}$. Notice that both are subalgebras, $\mathfrak{w}$ is also an ideal and that $\mathfrak{w} \oplus \mathfrak{v} = \mathfrak{h}$.

With these notations we can state the following rather straightforward version of the classical implicit function theorem.

Proposition 3.12 (Implicit Function Theorem). *Let $\mathcal{U} \subset \mathbb{H}^n$ be an open set, $p^0 \in \mathcal{U}$, $p^0 = p^0_{\mathfrak{w}} \cdot p^0_{\mathfrak{v}}$ with $p^0_{\mathfrak{w}} \in \mathbb{G}_{\mathfrak{w}}$ and $p^0_{\mathfrak{v}} \in \mathbb{G}_{\mathfrak{v}}$. We assume that $1 \le k \le n$ and that $f = (f_1, \cdots, f_k) : \mathcal{U} \to \mathbb{R}^k$ is a continuous function such that $f(p^0) = 0$, $v_j f_i$ are continuous functions in $\mathcal{U}$ for $1 \le i, j \le k$ and*

$$\Delta \overset{\text{def}}{=} \left| \det \left(\left[v_i f_j(p^0) \right]_{1 \le i,j \le k} \right) \right| > 0 \tag{16}$$

Finally define $S := \{p \in \mathcal{U} : f(p) = 0\}$.

Then there are an open set $\mathcal{U}' \subset \mathcal{U}$, with $p^0 \in \mathcal{U}'$, such that $S \cap \mathcal{U}'$ is a $(2n+1-k)$-dimensional continuous graph over $\mathbb{G}_{\mathfrak{w}}$ along $\mathfrak{v}$, that is, there is a relatively open $\mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}$, $p^0_{\mathfrak{w}} \in \mathcal{V}$ and a function $\varphi : \mathcal{V} \to \mathbb{G}_{\mathfrak{v}}$, with $\varphi(p^0_{\mathfrak{w}}) = p^0_{\mathfrak{v}}$, such that

$$S \cap \mathcal{U}' = \{\Phi(\xi) \overset{\text{def}}{=} \xi \cdot \varphi(\xi), \quad \xi \in \mathcal{V}\} \tag{17}$$

Proof. Let $d := 2n+1$. Consider the one to one map $\psi : \mathbb{R}^{d-k} \times \mathbb{R}^k \to \mathbb{H}^n \simeq \mathbb{R}^d$, defined as

$$\psi(x_1, \cdots, x_{d-k}, y_1, \cdots, y_k) \overset{\text{def}}{=} \exp \sum_{l=1}^{d-k} x_l w_l \cdot \exp \sum_{l=1}^{k} y_l v_l. \tag{18}$$

Observe that ψ as a map $\mathbb{R}^d \to \mathbb{R}^d$ is a global diffeomorphism. Moreover by definition, $\psi(\mathbb{R}^{d-k} \times \{0\}) = \mathbb{G}_{\mathfrak{w}}$ and $\psi(\{0\} \times \mathbb{R}^k) = \mathbb{G}_{\mathfrak{v}}$. We define $\psi_{\mathfrak{w}} : \mathbb{R}^{d-k} \to \mathbb{G}_{\mathfrak{w}}$ as $\psi_{\mathfrak{w}}(x_1, \cdots, x_{d-k}) := \psi(x_1, \cdots, , x_{d-k}, 0)$ and $\psi_{\mathfrak{v}} : \mathbb{R}^k \to \mathbb{G}_{\mathfrak{v}}$ analogously. Let $(x^0_1, \cdots, y^0_k) = \psi_{\mathfrak{v}}^{-1}(p^0)$. Define the map $g : \mathbb{R}^{d-k} \times \mathbb{R}^k \to \mathbb{R}^k$ as $g = f \circ \psi$, that is

$$g(x_1, \cdots, y_k) = f \left(\exp \sum_{l=1}^{d-k} x_l w_l \cdot \exp \sum_{l=1}^{k} y_l v_l \right),$$

so that the following diagram is commutative

$$\begin{array}{lll} \mathbb{H}^n & \supset \mathcal{U} & \stackrel{f}{\rightarrow} \quad \mathbb{R}^k \\ \psi \uparrow & & \nearrow_{g} \\ \mathbb{R}^d & \supset \psi^{-1}(\mathcal{U}) & \end{array}$$

Clearly g is continuous in the open set $\psi^{-1}(\mathcal{U}) \subset \mathbb{R}^d$, $(x_1^0, \cdots, y_k^0) \in \psi^{-1}(\mathcal{U})$ and $g(x_1^0, \cdots, y_k^0) = 0$. The derivatives $\dfrac{\partial g_i}{\partial y_j}$ exist, are continuous in $\psi^{-1}(\mathcal{U})$ and $\frac{\partial g_i}{\partial y_j}(x_1, \cdots, y_k) = (v_j f_i)(\phi(x_1, \cdots, y_k))$. Hence, assumption (16) reads as

$$\det\left(\left[\frac{\partial g_i}{\partial y_j}(x_1^0, \cdots, y_k^0)\right]_{1 \le i,j \le k}\right) \neq 0 \tag{19}$$

Then classical Implicit Function Theorem applied to g yields that there are an open $\tilde{\mathcal{U}} \subset \psi^{-1}(\mathcal{U})$, such that $(x_1^0, \cdots, y_k^0) \in \tilde{\mathcal{U}}$, an open $\tilde{\mathcal{V}} \subset \mathbb{R}^{d-k}$ with $(x_1^0, \cdots, x_{d-k}^0) \in \tilde{\mathcal{V}}$ and a continuous $\mathbb{R}^k$ valued function $\tilde{\varphi} = (\tilde{\varphi}_1, \cdots, \tilde{\varphi}_k) : \tilde{\mathcal{V}} \to \mathbb{R}^k$, such that

$$\begin{aligned} \tilde{S} &:= \{(x_1, \cdots, y_k) \in \tilde{\mathcal{U}} : g(x_1, \cdots, y_k) = 0\} \\ &= \{g(x_1, \cdots, x_{d-k}, \tilde{\varphi}(x_1, \cdots, x_{d-k})) : \quad (x_1, \cdots, x_{d-k}) \in \tilde{\mathcal{V}}\}. \end{aligned}$$

Finally, assertion (17) follows with $\mathcal{U}' = \psi(\tilde{\mathcal{U}})$, $\mathcal{V} = \psi(\tilde{\mathcal{V}} \times \{0\})$ and

$$\varphi \stackrel{\text{def}}{=} \psi_{\mathfrak{v}} \circ \tilde{\varphi} \circ \psi_{\mathfrak{w}}^{-1} : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}.$$

$\square$

The regularity of the implicitly defined functions φ and Φ is a more delicate issue. One can address both the problems of Euclidean and of intrinsic regularity.

Example 3.13. Let $f : \mathbb{H}^1 \to \mathbb{R}$ be defined as $f(x) = x_1 - 1$. Then $S = \{x \in \mathbb{H}^1 : x_1 = 1\}$ is 1-codimensional $\mathbb{H}$-regular surface. The function φ is constant: $\varphi(\xi_1, \xi_2) = (1, 0, 0)$ while Φ – even if it is $\mathcal{C}^\infty$ in Euclidean sense from $\mathbb{R}^2 \to \mathbb{R}^3$ – is not Lipschitz as a map from $\mathbb{G}_{\mathfrak{v}} \to \mathbb{G}_{\mathfrak{w}}$.

More generally, if the defining function f is Euclidean regular – say $\mathcal{C}^\infty$ – then both φ and Φ are Euclidean $\mathcal{C}^\infty$ and, consequently, $\varphi \in \mathcal{C}^\infty(\mathbb{H}^n, \mathbb{R}^k)$. Here the fact that φ is $\mathbb{R}^k$ valued plays a key role, indeed, as the previous example shows, in general $\Phi \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{H}^n, \mathbb{H}^n)$.

If we do not assume Euclidean regularity on f, in general the implicitly defined functions φ and Φ do not have any Euclidean regularity.

Example 3.14. Let $f : \mathbb{H}^1 \to \mathbb{R}$ be defined as

$$f(x) = x_1 - \sqrt{x_1^4 + x_2^4 - x_3^\alpha}$$

with $1 < \alpha < 2$. Then S is a 1-codimensional $\mathbb{H}$-regular surface. In this case φ, as a map from $\mathbb{R}^2$ to $\mathbb{R}$, is not Euclidean Lipschitz continuous in 0.

Notice that a much more dramatic example, in this line, is exibited in [15] where the corresponding function φ is non differentiable almost everywhere.

In the Euclidean setting, a $\mathcal{C}^1$-surface is locally the graph of a $\mathcal{C}^1$-function and viceversa. In $\mathbb{H}^n$ the characterization of those functions φ whose graphs are $\mathbb{H}$-regular surfaces is a hard problem, surprisingly somehow connected with the regularity of solutions of non linear diffusion equations. This problem is addressed in a forthcoming paper by Ambrosio, Serra Cassano and Vittone ([2]). In particular, as it is shown in that paper, in general it is false that φ is a Lipschitz function from $\mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$. Nevertheless, it is true that, if $\varphi(p) = 0$, then $|\varphi(q)| \le c\ d_c(p,q)$, (see Corollary 3.17). This fact is a key point in our proof of the existence of the tangent plane to any k-codimensional regular surface.

Proposition 3.15. *Given the same hypotheses and notations of Proposition 3.12 we assume also that there are $\alpha \in (0,1]$ and $c_\alpha > 0$, such that*

$$|f(\xi_2 \cdot \varphi(\xi_1))|_{\mathbb{R}^k} \le c_\alpha (d_c(\xi_1, \xi_2))^\alpha \tag{20}$$

for fixed $\xi_1, \xi_2 \in \mathcal{V}$ with $\xi_2 \cdot \varphi(\xi_1) \in \mathcal{U}'$. Then there is $c > 0$ such that

$$d_c(\varphi(\xi_1), \varphi(\xi_2)) \le c(d_c(\xi_1, \xi_2))^\alpha. \tag{21}$$

Proof. First observe that (19) yields that there is $r > 0$ such that the map

$$y_1, \cdots, y_k \mapsto f(\xi \cdot \phi_{\mathfrak{v}}(y_1, \cdots, y_k)),$$

from $\mathbb{R}^k$ to $\mathbb{R}^k$, is invertible in $\psi^{-1}\left(B(p^0, r) \cap \mathcal{U}'\right)$, for each fixed $\xi \in \mathbb{G}_{\mathfrak{w}}$, when ξ close to $p^0_{\mathfrak{w}}$. Moreover the inverse map is bounded, that is there is $c_1 > 0$ such that

$$|\psi^{-1}(\eta_2) - \psi^{-1}(\eta_1)|_{\mathbb{R}^k} \le c_1 |f(\xi \cdot \eta_2) - f(\xi \cdot \eta_1)|_{\mathbb{R}^k}$$

when η_1 and η_2 are sufficiently close to $p^0_{\mathfrak{v}}$. Observe also that assumption (15) yields that the map $\psi_{\mathfrak{v}} : \mathbb{R}^k \to \mathbb{G}_{\mathfrak{v}}$ is globally bilipschitz.

Hence there is an open $\mathcal{V}' \subset \mathbb{G}_{\mathfrak{w}}$, with $p^0_{\mathfrak{w}} \equiv \psi_{\mathfrak{w}}(x^0_1, \cdots, x^0_{d-k}) \in \mathcal{V}'$, and a costant $c_2 > 0$ such that for $\xi_1, \xi_2 \in \mathcal{V}'$, we have

$$\begin{aligned} |f(\xi_1 \cdot \varphi(\xi_2))|_{\mathbb{R}^k} &= |f(\xi_1 \cdot \varphi(\xi_2)) - f(\xi_1 \cdot \varphi(\xi_1))|_{\mathbb{R}^k} \\ &\ge c_2\ d_c(\varphi(\xi_1), \varphi(\xi_2)). \end{aligned}$$

On the other side, from assumption (20) we get

$$\begin{aligned} |f(\xi_1 \cdot \varphi(\xi_2))|_{\mathbb{R}^k} &= |f(\xi_1 \cdot \varphi(\xi_2)) - f(\xi_2 \cdot \varphi(\xi_2))|_{\mathbb{R}^k} \\ &\le c_\alpha (d_c(\xi_1, \xi_2))^\alpha. \end{aligned}$$

Hence we get (21). $\square$

Remark 3.16. Hypothesis (20) is not an easy one to verify. A special instance of it, that we will use later, is the following: if $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, then $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{U}; \mathbb{R}^k)$ hence there is $L = L(\mathcal{V}) > 0$ such that for $\xi_1, \xi_2 \in \mathcal{V}$,

$$\begin{aligned} |f(\xi_2 \cdot \varphi(\xi_1))|_{\mathbb{R}^k} &= |f(\xi_2 \cdot \varphi(\xi_1)) - f(\xi_1 \cdot \varphi(\xi_1))|_{\mathbb{R}^k} \\ &\le L d_c(\xi_2 \cdot \varphi(\xi_1), \xi_1 \cdot \varphi(\xi_1)). \end{aligned}$$

Now if

$$d_c(\xi_2 \cdot \varphi(\xi_1), \xi_1 \cdot \varphi(\xi_1)) = d_c(\xi_2, \xi_1) \tag{22}$$

then (20) holds with $\alpha = 1$. Notice that (22) trivially holds when $\varphi(\xi_1) = 0$.

Corollary 3.17. *Given the same assumptions and notations of Theorem 3.12, assume also that* $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{U}, \mathbb{R}^k)$*. Then, for any relatively compact* $\mathcal{V}' \subset \mathcal{V}$*, there is a positive constant* c *such that the implicitly defined function* φ *satisfies*

$$d_c\left(\varphi(\xi_1), \varphi(\xi_0)\right) \le c\, d_c\left(\xi_1 \cdot \varphi(\xi_0), \xi_0 \cdot \varphi(\xi_0)\right) \tag{23}$$

for all $\xi_0, \xi_1 \in \mathcal{V}'$*. Moreover if* $\varphi(\xi_0) = 0$*, that is if* $\xi_0 \in S \cap \mathcal{U} \cap \mathbb{G}_{\mathfrak{w}}$*, then (23) becomes*

$$|\varphi(\xi)| \le c\, d_c(\xi, \xi_0), \qquad \forall \xi \in \mathcal{V}'. \tag{24}$$

Proof. If $p = \xi_0 \cdot \varphi(\xi_0) \in S$ then, working as in Proposition 3.11 – here we use that $\mathbb{G}_{\mathfrak{w}}$ is a normal subgroup of $\mathbb{H}^n$ because $\mathfrak{w}$ is an ideal in $\mathfrak{h}$ – we get $\tau_{p^{-1}} S = \{\eta \cdot \varphi_{p^{-1}}(\eta) \ : \eta \in E'\}$, where $\varphi_{p^{-1}}(\eta) := \varphi(\xi_0)^{-1} \cdot \varphi\left(p \cdot \eta \cdot \varphi(\xi_0)^{-1}\right)$. Now $\varphi_{p^{-1}}(0) = 0$ hence, keeping in mind the preceding Remark, from Theorem 3.12 we get $|\varphi_{p^{-1}}(\xi)| \le c\,|\xi|$, for all $\xi \in \mathbb{G}_{\mathfrak{w}} \cap \mathcal{V}'$, that is $\left|\varphi(\xi_0)^{-1} \cdot \varphi\left(p \cdot \xi \cdot \varphi(\xi_0)^{-1}\right)\right| = d_c\left(\varphi\left(p \cdot \xi \cdot \varphi(\xi_0)^{-1}\right), \varphi(\xi_0)\right) \le c\,|\xi|$. Putting now $\xi_1 := p \cdot \xi \cdot \varphi(\xi_0)^{-1}$ we get (23) and (24). $\square$

Coherently with our purpose, previous results were stated in an intrinsic form, that is in coordinate free formulation. Later on we need also identities written ‘in coordinates ’. To this end we define a function $\tilde{\Phi}$ that is nothing but the function Φ seen in exponential coordinates.

Definition 3.18. Using the notations in the proof of Proposition 3.12 we define $\tilde{\Phi} : \tilde{\mathcal{V}} \to \mathbb{R}^{2n+1}$ by the commutative diagram

$$\begin{tikzcd} \mathbb{G}_{\mathfrak{w}} \arrow[r, "\Phi"] & \mathbb{H}^n \\ \mathbb{R}^{2n+1-k} \arrow[u, "\psi_{\mathfrak{w}}"] \arrow[r, "\tilde{\Phi}"'] & \mathbb{R}^d \arrow[u, "\psi"'] \end{tikzcd}$$

Hence, if $x = (x_1, \dots, x_{d-k})$,

$$\tilde{\Phi}(x) = \left(x_1, \dots, x_{d-k}, \varphi_1(x_1, \dots, x_{d-k}), \dots, \varphi_k(x_1, \dots, x_{d-k})\right),$$

where $\varphi_1, \dots \varphi_k$ have been defined in (14).

We evaluate here the Jacobian of the map ψ defined in (18).

Proposition 3.19. *Let* $1 \le k \le n$*, with the same notations of Proposition 3.12, we assume* $v = v_1 \wedge \dots \wedge v_k \in {}_H\bigwedge_k$ *and* $w = w_1 \wedge \dots \wedge w_{2n+1-k} \in {}_H\bigwedge_{2n+1-k}$*, with* $w_1 \wedge \dots \wedge w_{2n-k} \in \bigwedge_{2n-k} \mathfrak{h}_1$ *and* $w_{2n+1-k} = T$*. Then*

$$|\det J_\psi| = |w_1 \wedge \dots \wedge w_{2n+1-k} \wedge v_1 \wedge \dots \wedge v_k| \tag{25}$$

and

$$|\det J_{\psi_{\mathfrak{w}}}| = |w_1 \wedge \dots \wedge w_{2n+1-k}|\,. \tag{26}$$

Hence in particular if we choose $w = *v$ *and* $|v| = 1$ *we have*

$$|\det J_\psi| = 1. \tag{27}$$

Proof. Let $d = 2n+1$, then, for $\ell = 1, \dots, d-k$,

$$\begin{aligned}\psi(\xi, \eta) &:= \exp(\sum_{j=1}^{d-k} \xi_j w_j) \cdot \exp(\sum_{j=1}^{k} \eta_j v_j) \\ &= \exp(\sum\nolimits_j \eta_j v_j) \cdot \exp(\sum\nolimits_{j \neq \ell} \xi_j w_j) \cdot \exp(\xi_\ell w_\ell) + \alpha_\ell T,\end{aligned}$$

α_ℓ depend on all the variables ξ and η but not on ξ_{d-k}. Hence, because v_j and w_j are invariant by translations, we have

$$\begin{aligned}&\frac{\partial \psi}{\partial \xi_\ell} = w_\ell + \frac{\partial \alpha_\ell}{\partial \xi_\ell} T, \quad \text{ for } \ell \neq d-k, \text{ and } \quad \frac{\partial \psi}{\partial \xi_{d-k}} = T; \\ &\frac{\partial \psi}{\partial \eta_j} = v_j, \quad \text{ for } j = 1, \dots, k.\end{aligned}$$

Hence

$$\begin{aligned}|\det J_\psi| = \left| \frac{\partial \psi}{\partial \xi_1} \wedge \cdots \wedge \frac{\partial \psi}{\partial \eta_k} \right| &= \left| \left(w_1 + \frac{\partial \alpha_1}{\partial \xi_1} T \right) \wedge \cdots \wedge T \wedge v_1 \wedge \cdots \wedge v_k \right| \\ &= |w_1 \wedge \cdots \wedge w_{d-k} \wedge v_1 \wedge \cdots \wedge v_k| \\ &= |\langle *(w_1 \wedge \cdots \wedge w_{d-k}), v_1 \wedge \cdots \wedge v_k \rangle| \\ \text{and if } w = *v, \text{ with } |v| = 1, & \\ &= |v|^2 = 1.\end{aligned}$$

The proof of (26) follows analogously. $\square$

The following result is well-known.

Lemma 3.20. *Let $\xi = \xi_1 \wedge \cdots \wedge \xi_k, \eta = \eta_1 \wedge \cdots \wedge \eta_k \in \bigwedge_k \mathfrak{h}$ be simple k-vectors in $\mathbb{R}^{2n+1}$. Then*

$$\langle \xi, \eta \rangle_{\bigwedge_k \mathbb{R}^{2n+1}} = \det \left[\langle \xi_i, \eta_j \rangle_{\mathbb{R}^{2n+1}} \right]_{i,j=1,\dots,k}. \tag{28}$$

Lemma 3.21. *Let $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in \bigwedge_k \mathfrak{h}$ and $\eta = \eta_1 \wedge \cdots \wedge \eta_{d-k} \in \bigwedge_{d-k} \mathfrak{h}$ be simple. If*

$$\langle \xi_i, \eta_j \rangle_{\mathbb{R}^{2n+1}} = 0 \quad \text{for} \quad i = 1, \dots, k, \quad j = 1, \dots, d-k, \tag{29}$$

then ξ and $*\eta$ are linearly dependent, where here the $*$ operator is the Hodge operator associated with the Euclidean scalar product in $\mathbb{R}^{2n+1}$.

Proof. Put $d := 2n+1$. Since $\langle \cdot, \cdot \rangle_{\bigwedge_k \mathbb{R}^d}$ is a positive definite scalar product in $\bigwedge_k \mathfrak{h}$, we need only to show that

$$|\langle \xi, *\eta \rangle_{\bigwedge_k \mathbb{R}^d}| = (\langle \xi, \xi \rangle_{\bigwedge_k \mathbb{R}^d})^{1/2} (\langle *\eta, *\eta \rangle_{\bigwedge_k \mathbb{R}^d})^{1/2}.$$

First notice that, by definition, $\langle *\eta, *\eta \rangle_{\bigwedge_k \mathbb{R}^d} \, dp_1 \wedge \cdots \wedge dp_{2n+1} = (-1)^{k(d-k)} (*\eta) \wedge \eta = \eta \wedge *\eta = \langle \eta, \eta \rangle_{\bigwedge_{d-k} \mathbb{R}^d} \, dp_1 \wedge \cdots \wedge dp_{2n+1}$, so that we have to show that

$$|\langle \xi, *\eta \rangle_{\bigwedge_k \mathbb{R}^d}| = (\langle \xi, \xi \rangle_{\bigwedge_k \mathbb{R}^d})^{1/2} (\langle \eta, \eta \rangle_{\bigwedge_{d-k} \mathbb{R}^d})^{1/2}.$$

Denote now by $C = [c_{i,j}]_{i,j=1,\dots,d}$ the $(d \times d)$-matrix with rows ordinately given by $\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_{d-k}$, i.e., if $\xi_i = (\xi_{i1}, \dots, \xi_{id})$ and $\eta_i = (\eta_{i1}, \dots, \eta_{id})$, then

$$c_{i,j} = \begin{cases} \xi_{ij} & \text{if} \quad i = 1, \dots, k \ , j = 1, \dots, d; \\ \eta_{ij} & \text{if} \quad i = k+1, \dots, d \ , j = 1, \dots, d. \end{cases}$$

Keeping in mind (29) and (28), we have

$$
\begin{aligned}
(\det C)^2 &= \det C\,{}^tC \\
&= \det \begin{bmatrix} \langle \xi_1,\xi_1\rangle_{\mathbb{R}^d} & \cdots & \langle \xi_1,\xi_k\rangle_{\mathbb{R}^d} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \langle \xi_k,\xi_1\rangle_{\mathbb{R}^d} & \cdots & \langle \xi_k,\xi_k\rangle_{\mathbb{R}^d} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \langle \eta_1,\eta_1\rangle_{\mathbb{R}^d} & \cdots & \langle \eta_1,\eta_{d-k}\rangle_{\mathbb{R}^d} \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \langle \eta_{d-k},\eta_1\rangle_{\mathbb{R}^d} & \cdots & \langle \eta_{d-k},\eta_{d-k}\rangle_{\mathbb{R}^d} \end{bmatrix} \\
&= \langle \xi,\xi\rangle_{\bigwedge_k \mathbb{R}^d} \langle \eta,\eta\rangle_{\bigwedge_{d-k} \mathbb{R}^d},
\end{aligned}
$$

and the lemma is proved. □

Proposition 3.22. *With the notation of the Implicit Function Theorem (see Propostion 3.12) and of Definition 3.18, suppose now that f is continuously differentiable in the Euclidean sense. Then the implicitly defined function $\Phi\circ\psi_{\mathfrak{w}}$ – that is continuosly differentiable by usual Euclidean Implicit Function Theorem – satisfies the identity*

$$
\begin{aligned}
&|\nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p)|_{\bigwedge_k \mathbb{R}^d} \\
&\qquad = \frac{\Delta}{|\det J_\psi|} \left| \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi) \right|_{\bigwedge_k \mathbb{R}^d}
\end{aligned}
$$

where $\xi = (\Phi\circ\psi_{\mathfrak{w}})^{-1}(p)$ and $\Delta = \Delta(\Phi\circ\psi_{\mathfrak{w}}(\xi))$.

Proof. Let $d = 2n+1$. Since $f_i(\Phi\circ\psi_{\mathfrak{w}}(\xi)) \equiv 0$ for $\xi \in \tilde{\mathcal{V}}$, we have

$$\langle \nabla f_i(\Phi\circ\psi_{\mathfrak{w}}), \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_j} \rangle_{\mathbb{R}^d} \equiv 0$$

for $i = 1,\dots,k$ and $j = 1,\cdots,d-k$. By Lemma 3.21, this implies that, for $\xi \in \tilde{\mathcal{V}}$,

$$\nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p) = \lambda(p) \; * \left(\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi) \right),$$

where, as in Lemma 3.21 and through all this proof, $*$ denotes the Hodge operator with respect to the Euclidean scalar product.

To evaluate $\lambda(p)$, from the above identity, setting $dV := dp_1 \wedge \cdots \wedge dp_d$ and $v = v_1 \wedge \cdots \wedge v_k$, we get

$$
\begin{aligned}
&\langle v, \nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p)\rangle_{\bigwedge_k \mathbb{R}^d} \; dV \\
&= \lambda(p) \left\langle v, * \left(\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_1} \wedge \cdots \wedge \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_{d-k}} \right)(\xi) \right\rangle_{\bigwedge_k \mathbb{R}^d} dV \\
&= \lambda(p) \left(v_1 \wedge \cdots \wedge v_k \wedge \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi) \right).
\end{aligned} \tag{30}
$$

By Lemma 3.20, we can also write

$$
\begin{aligned}
&\left| \langle v, \nabla f_1(p) \wedge \cdots \wedge \nabla f_k(p)\rangle_{\bigwedge_k \mathbb{R}^d} \right| = \left| \det \left[\langle v_i, \nabla f_j\rangle_{\mathbb{R}^d} \right]_{i,j=1,\dots,k} \right| \\
&= \left| \det \left[v_i f_j \right]_{i,j=1,\dots,k} \right| = \Delta.
\end{aligned} \tag{31}
$$

By Definition 3.18, $v_\ell(\Phi\circ\psi_{\mathfrak{w}}(\xi)) = v_\ell(\psi\circ\tilde{\Phi}(\xi)) = J_\psi(\tilde{\Phi}(\xi))e_{d-k+\ell}$, for $\ell = 1,\dots,k$. Indeed, for any point $(x,y) = (x_1,\dots,x_{d-k},y_1,\dots,y_k)$, we can always write $\psi(x,y) = \exp \sum_{j=1}^{d-k} x_j w_j \cdot \exp\left(\sum_{i\neq \ell} y_i v_i + y_\ell v_\ell\right) = \exp(y_\ell v_\ell)\cdot$ $(\exp \sum_{j=1}^{d-k} x_j w_j \cdot \exp \sum_{i\neq\ell} y_i v_i)$, so that $\dfrac{\partial \psi}{\partial y_\ell} = v_\ell(\psi(x,y))$. Analogously $\dfrac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial \xi_j}(\xi) = J_\psi(\tilde\Phi(\xi))\dfrac{\partial\tilde\Phi}{\partial\xi_j}(\xi)$, for $j = 1,\dots,d-k$. Hence

$$
\begin{aligned}
&v_1\wedge\cdots\wedge v_k \wedge \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\\
&= \det J_\psi(\tilde\Phi(\xi))\cdot\left(e_{d-k+1}\wedge\cdots\wedge e_d\wedge\frac{\partial\tilde\Phi}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial\tilde\Phi}{\partial\xi_{d-k}}(\xi)\right).
\end{aligned} \tag{32}
$$

On the other hand, by construction, $\frac{\partial\tilde\Phi}{\partial\xi_j}(\xi) = \sum_{\ell=1}^k \frac{\partial\phi_\ell}{\partial\xi_j}(\xi)e_{d-k+\ell} + e_j$. Using this and keeping into account Proposition 3.19, (32) becomes

$$
\begin{aligned}
&v_1\wedge\cdots\wedge v_k \wedge \frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\\
&= \det J_\psi\ \left(e_{d-k+1}\wedge\cdots\wedge e_d\wedge e_1\wedge\cdots\wedge e_{d-k}\right)\\
&= \varepsilon_I \det J_\psi\ dV,
\end{aligned} \tag{33}
$$

where ε_I is 1 or -1 according to the parity of the permutation $(d-k+1,\dots,d,1,\dots,d-k)$. Thus, combining (30), (31), and (33), we get $\Delta = |\lambda|\,|\det J_\psi|$, and, consequently,

$$
\begin{aligned}
&|\nabla f_1(p)\wedge\cdots\wedge\nabla f_k(p)|\\
&= \frac{|\Delta|}{|\det J_\psi|}\left| *\left(\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\right)\right|\\
&= \frac{|\Delta|}{|\det J_\psi|}\left|\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial(\Phi\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}}(\xi)\right|.
\end{aligned} \tag{34}
$$

□

3.4. **Regular Surfaces locally are graphs.** In this section we prove that k-codimensional $\mathbb{H}$-regular surfaces are, locally, graphs in the Heisenberg sense. That is we have to show that assumptions of Proposition 3.12 hold true. In particular, if we assume, accordingly with the notations of Proposition 3.12, that the surface S is locally defined by the equation $S = \{p\in\mathcal{U} :\ f(p) = 0\}$, we have to check that if $\nabla_H f_1\wedge\cdots\wedge\nabla_H f_k \neq 0$, then there exist k, linearly independent, horizontal vectors $v_1,\dots,v_k$ such that

$$
[v_i, v_j] = 0, \text{ for } 1\le i,j\le k, \tag{35}
$$

$$
\Delta \overset{\text{def}}{=} \left|\det\left([v_i f_j]_{1\le i,j\le k}\right)\right| > 0. \tag{36}
$$

Notice that this problem does not appear when $k=1$; indeed if $\nabla_H f\neq 0$ then there is at least one $i\in\{1,\dots,2n\}$ with $W_i f\neq 0$ and we can take $v_1 = W_i$.

When $k>1$, condition $\nabla_H f_1\wedge\cdots\wedge\nabla_H f_k\neq 0$ yields the existence of k vectors in $X_1,\dots,Y_n$ such that (36) holds but not necessarily (35). For instance consider the following example

Example 3.23. Let $f = (f_1,f_2) : \mathbb{H}^2\to\mathbb{R}^2$ be defined as

$$
f(p_1,\dots,p_5) = (p_1,p_3).
$$

Then S is the 2-codimensional plane $S = \{p_1 = p_3 = 0\}$. Writing explicitly the 2×4 matrix associated with $d_H f$, we see that all 2×2 minors vanish but for

$$\begin{bmatrix} X_1 f_1, & Y_1 f_1 \\ X_1 f_2, & Y_1 f_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{37}$$

Clearly, the choice $v_1 = X_1$ and $v_2 = Y_1$ satisfy (36) but not (35). Hence we cannot foliate $\mathbb{H}^2$ using integral surfaces of v_1 and v_2, by Frobenius Theorem. Nevertheless an adapted foliation, satisfying both (35) and (36), exists: indeed it is enough to take

$$v_1 \stackrel{\text{def}}{=} X_1 + X_2, \qquad v_2 \stackrel{\text{def}}{=} Y_1 - Y_2. \tag{38}$$

Clearly this is a typical non Euclidean phenomenon.

In the following part of this section we prove that the procedure in (38) can be generalized. First we start showing how to generalize it to the case of 2-codimensional surfaces in $\mathbb{H}^n$, $n \geq 2$.

Proposition 3.24. *For $n > 1$, let $f = (f_1, f_2) : \mathbb{H}^n \to \mathbb{R}^2$, $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^2$. Assume there is $p^0 \in \mathbb{H}^n$ such that*

$$\begin{bmatrix} X_1 f_1(p^0), & \cdots, & Y_n f_1(p^0) \\ X_1 f_2(p^0), & \cdots, & Y_n f_2(p^0) \end{bmatrix} \neq 0.$$

Then there are an open $\mathcal{U} \ni p^0$ and a simple, integrable $v = v_1 \wedge v_2 \in {}_H\bigwedge_2$ such that, for $p \in \mathcal{U}$,

$$\det \begin{bmatrix} v_1 f_1(p), & v_2 f_1(p) \\ v_1 f_2(p), & v_2 f_2(p) \end{bmatrix} \neq 0.$$

Proof. We adopt the notation $W_1 := X_1, \cdots, W_{2n} := Y_n$. Then we assume, without loss of generality that $W_1 f \neq 0$.

If there is (i, j) with $i < j$ and $(i, j) \neq (i, i+n)$ such that $(W_i \wedge W_j) f \neq 0$ then the proposition is proved with $v = W_i \wedge W_j$.
On the contrary assume that

$$(W_i \wedge W_j) f = 0 \qquad \forall (i, j) \text{ with } i < j \text{ and } (i, j) \neq (i, i+n). \tag{39}$$

From (39) with $i = 1$ we get that $\forall h, l \neq n+1$, both $W_h f$ and $W_l f$ are a multiple of $W_1 f$, since $W_1 f \neq 0$ we get that $W_h f$ and $W_l f$ are linerly dependent, so that

$$(W_h \wedge W_l) f = 0 \qquad \forall (h, l) \text{ with } 1 < h < l \leq 2n; h, l \neq 1+n. \tag{40}$$

From (39) and (40) we have that only $(W_1 \wedge W_{n+1}) f \neq 0$. Hence if we choose $v = (W_1 + W_2) \wedge (W_{1+n} - W_{2+n})$, then v is simple, $v \in {}_H\bigwedge_2$ (because $v \in \bigwedge_2 \mathfrak{h}_1$ and $\langle d\theta | v \rangle = 0$) and

$$v(f) = (W_1 \wedge W_{1+n}) f \neq 0.$$

$\square$

We deal with the general situation of a k-codimensional surface in $\mathbb{H}^n$ in the following Proposition. We keep using the notation $W_1 := X_1, \cdots, W_{2n} := Y_n$.

Proposition 3.25. *For $2 < k \leq n$, let $f = (f_1, \dots, f_k) : \mathbb{H}^n \to \mathbb{R}^k$, $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$. If there is $p^0 \in \mathbb{H}^n$, such that*

$$\operatorname{rank}\left[W_i f_j(p^0)\right]_{1 \leq i \leq 2n, 1 \leq j \leq k} = k,$$

then there are an open $\mathcal{U} \ni p^0$ and a simple, integrable k-vector $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$ such that, for all $p \in \mathcal{U}$,

$$\det\left[v_i f_j(p)\right]_{1 \leq i,j \leq k} \neq 0.$$

Proof. During the proof of the present theorem we use the following notations: a h-uple $I = (i_1, \dots, i_h)$ is said to have *degree* j, a non negative integer, if I contains exactly j different couples of the form $i_l, i_l + n$ or, equivalently, if there are exactly j (different) elements $i_{l_1}, \dots, i_{l_j} \in I$ such that also $i_{l_1} + n, \dots, i_{l_j} + n \in I$. Clearly $j \leq [h/2]$. Notice also that, to avoid trivialities, we always assume that $i_1 \neq \cdots \neq i_h$.
We also write W_J for $W_{j_1} \wedge \cdots \wedge W_{j_h}$ for any h-uple $J = (j_1, \dots, j_h)$.

Let us now come to the proof.

If there is a k-uple I^* with degree 0 such that $W_{I^*} f \neq 0$ then the proposition is proved with

$$v = W_{I^*}.$$

If this is not true, recalling that $\operatorname{rank}(Wf(p)) = k$, we have that there is j^*, $0 < j^* \leq [k/2]$, such that

$$W_I f = 0 \qquad \text{for any } k\text{-uple } I \text{ with degree} < j^* \tag{41}$$

while there is at least one k-uple I^* with degree j^*,

$$I^* = (J^*, i_1, \cdots, i_{j^*}, i_1 + n, \cdots, i_{j^*} + n),$$

such that

$$W_{I^*} f \neq 0. \tag{42}$$

Notice that the degree of $J^* = 0$. Clearly, J^* can even be empty.

Now let us choose indices $h_1, \cdots, h_{j^*}$ such that

- $1 \leq h_1, \cdots, h_{j^*} \leq n$;
- $h_1, \cdots, h_{j^*}, h_1 + n, \cdots, h_{j^*} + n \notin I^*$.

It is easy to convince oneself that such a choice of $h_1, \cdots, h_{j^*}$ is always possible.

The simple vector v in the statement of this proposition is defined as

$$\begin{aligned} v \overset{\text{def}}{=} & W_{J^*} \wedge (W_{i_1} + W_{h_1}) \wedge (W_{i_1+n} - W_{h_1+n}) \wedge \cdots \\ & \wedge \left(W_{i_{j^*}} + W_{h_{j^*}}\right) \wedge \left(W_{i_{j^*}+n} - W_{h_{j^*}+n}\right). \end{aligned} \tag{43}$$

Clearly, $v \in \bigwedge_k \mathfrak{h}_1$ but also

$$v \in {}_H\bigwedge_k.$$

Indeed, from Theorem 2.8 we know that any $v = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k \mathfrak{h}_1$ belongs also to ${}_H\bigwedge_k$ if and only if $\langle d\theta | v_i \wedge v_j \rangle = 0$, for all $1 \leq i, j \leq n$. In this case, it is enough to observe that for $1 \leq l \leq j^*$

$$\langle d\theta | (W_{i_l} + W_{h_l}) \wedge (W_{i_l+n} - W_{h_l+n}) \rangle = 0.$$

The proposition then follows from the following

Claim:

$$v(f) = W_{I^*} f \neq 0.$$

Proof of Claim: Write

$$v(f) = \left(\tilde{W} \wedge \left(W_{i_{j^*}} + W_{h_{j^*}}\right) \wedge \left(W_{i_{j^*}+n} - W_{h_{j^*}+n}\right)\right)(f) \tag{44}$$

where

$$\begin{aligned}\tilde{W} = W_{J^*} \wedge (W_{i_1} + W_{h_1}) \wedge (W_{i_1+n} - W_{h_1+n}) \wedge \cdots \\ \wedge \left(W_{i_{j^*-1}} + W_{h_{j^*-1}}\right) \wedge \left(W_{i_{j^*-1}+n} - W_{h_{j^*-1}+n}\right).\end{aligned}$$

The first step in the proof is to show that

$$v(f) = \left(\tilde{W} \wedge W_{i_{j^*}} \wedge W_{i_{j^*}+n}\right)(f). \tag{45}$$

Indeed

$$\begin{aligned}v(f) = &\left(\tilde{W} \wedge W_{i_{j^*}} \wedge W_{i_{j^*}+n}\right)(f) - \left(\tilde{W} \wedge W_{h_{j^*}} \wedge W_{h_{j^*}+n}\right)(f) \\ &+ \left(\tilde{W} \wedge W_{h_{j^*}} \wedge W_{i_{j^*}+n}\right)(f) - \left(\tilde{W} \wedge W_{i_{j^*}} \wedge W_{h_{j^*}+n}\right)(f).\end{aligned}$$

It is easy to see that the last two terms vanish. This follows from (41) observing that both of them are sums of terms as $W_I f$ where the degrees of the various k-uples I are all $< j^*$. It is more complicated to show that also

$$\left(\tilde{W} \wedge W_{h_{j^*}} \wedge W_{h_{j^*}+n}\right)(f) = 0$$

This can be done observing that $(\tilde{W} \wedge W_{h_{j^*}} \wedge W_{h_{j^*}+n})(f)$ is the sum of terms as $(W_{(J,h_{j^*},h_{j^*}+n)})(f)$ where J is a $(k-2)$-uple of degree $< j^*$, then remembering (41) and (42), all these terms vanish as proved in the following lemma.

Lemma 3.26. *Assume that $J = (j_1, \ldots, j_{k-2})$ is a $(k-2)$-uple with* degree$(J) \leq j^* - 1$*; assume also that i, h are two fixed integers such that $1 \leq i \neq h \leq n$ and that $i, n+i, h, n+h \notin J$. If, for any k-uple I with* degree$(I) < j^*$,

$$W_{(J,i,i+n)} f \neq 0 \text{ and } W_I f = 0$$

then

$$(W_J \wedge W_h \wedge W_{h+n}) f = 0. \tag{46}$$

Proof. Observe that both the k-uples (i, J, h) and $(i, J, h+n)$ have degree $\leq j^* - 1$. Hence, from the hypothesis of the Lemma,

$$(W_J \wedge W_i \wedge W_h) f = 0 \quad \text{and } (W_J \wedge W_i \wedge W_{h+n}) f = 0.$$

Thus there are k-uples of real numbers $(\alpha_{1,i}, \alpha_{1,h}, \alpha_{1,j_1}, \ldots, \alpha_{1,j_{k-2}},)$ and $(\alpha_{2,i}, \alpha_{2,h+n}, \alpha_{2,j_1}, \ldots, \alpha_{2,j_{k-2}},)$, both non vanishing, such that

$$\begin{aligned}\alpha_{1,i} W_i f + \alpha_{1,h} W_h f + \sum_{l=1}^{k-2} \alpha_{1,j_l} W_{j_l} f = 0 \\ \alpha_{2,i} W_i f + \alpha_{2,h+n} W_{h+n} f + \sum_{l=1}^{k-2} \alpha_{2,j_l} W_{j_l} f = 0.\end{aligned} \tag{47}$$

If $\alpha_{1,i} = 0$ then $W_h f$ and $W_{j_l} f$ are linearly dependent, hence (46) follows. If $\alpha_{2,i} = 0$ we get the same conclusion. If, on the contrary, both $\alpha_{1,i} \neq 0$ and $\alpha_{2,i} \neq 0$ then (47) yields

$$\beta_{1,h} W_h f + \sum_{l=1}^{k-2} \beta_{1,j_l} W_{j_l} f = \beta_{2,h+n} W_{h+n} f + \sum_{l=1}^{k-2} \beta_{1,j_l} W_{j_l} f, \tag{48}$$

where $\beta_{t,s} = \alpha_{t,s}/\alpha_{t,1}$. Hence $W_h f$, $W_{h+n} f$ and $W_{j_l} f$ are linearly dependent and once more (46) follows. Notice that the assumption $W_{(J,i,i+n)} f \neq 0$ yields $W_i f \neq 0$ and this one in turn is used exactly here to be sure that the $\beta_{t,s}$ are not all 0. □

To conclude the proof we consider one by one all the other indexes $i_1, \cdots, i_{j^*-1}$.
Precisely, starting from (45), we can write

$$\begin{aligned} v(f) = &\Big(W_{J^*} \wedge (W_{i_1} + W_{h_1}) \wedge (W_{i_1+n} - W_{h_1+n}) \wedge \cdots \\ &\wedge \left(W_{i_{j^*-1}} + W_{h_{j^*-1}}\right) \wedge \left(W_{i_{j^*-1}+n} - W_{h_{j^*-1}+n}\right) \\ &\wedge W_{i_{j^*}} \wedge W_{i_{j^*}+n}\Big)(f) \\ = &\left(\tilde{W} \wedge \left(W_{i_{j^*-1}} + W_{h_{j^*-1}}\right) \wedge \left(W_{i_{j^*-1}+n} - W_{h_{j^*-1}+n}\right)\right)(f) \end{aligned}$$

where the new $\tilde{W}$ is

$$\begin{aligned} \tilde{W} \stackrel{\text{def}}{=} &(\pm) W_{J^*} \wedge (W_{i_1} + W_{h_1}) \wedge (W_{i_1+n} - W_{h_1+n}) \wedge \cdots \\ &\wedge \left(W_{i_{j^*-2}} + W_{h_{j^*-2}}\right) \wedge \left(W_{i_{j^*-2}+n} - W_{h_{j^*-2}+n}\right) \\ &\wedge W_{i_{j^*}} \wedge W_{i_{j^*}+n}. \end{aligned}$$

We use the same argument as before to get that

$$\begin{aligned} v(f) &= \left(\tilde{W} \wedge W_{i_{j^*-1}} \wedge W_{i_{j^*}+n-1}\right)(f) \\ &= \Big(W_{J^*} \wedge (W_{i_1} + W_{h_1}) \wedge (W_{i_1+n} - W_{h_1+n}) \wedge \cdots \\ &\quad \wedge \left(W_{i_{j^*-2}} + W_{h_{j^*-2}}\right) \wedge \left(W_{i_{j^*-2}+n} - W_{h_{j^*-2}+n}\right) \\ &\quad \wedge W_{i_{j^*-1}} \wedge W_{i_{j^*-1}+n} \wedge W_{i_{j^*}} \wedge W_{i_{j^*}+n}\Big)(f). \end{aligned} \tag{49}$$

Now it is clear how to proceed to exhaust all the remaining indexes.
This concludes the proof of the Claim and of the Proposition. □

Theorem 3.27. *Let $S \subset \mathbb{H}^n$ be a k-codimensional $\mathbb{H}$-regular surface, $1 \leq k \leq n$. Then S is locally a regular graph, that is, for each $p \in S$ it is possible to choose an open subset $\mathcal{U} \subset \mathbb{H}^n$, with $p \in \mathcal{U}$, a simple k-vector $v \in {}_H\bigwedge_k$, a simple $(2n+1-k)$-vector w and a function $\varphi : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ such that*

$$S \cap \mathcal{U} = \{\xi \cdot \varphi(\xi) : \ \xi \in \mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}\}.$$

Moreover it is possible to choose v and w such that $|v| = |w| = 1$.

Proof. The statement follows combining Propositions 3.12, 3.24 and 3.25. □

3.5. **The tangent group to a $\mathbb{H}$-regular surface (low codimension).**

Definition 3.28. Let $S = \{x : f(x) = 0\}$ be a k-codimensional $\mathbb{H}$-regular surface in $\mathbb{H}^n$ (with $1 \le k \le n$). The *tangent group* to S in p, indicated as $T^g_{\mathbb{H}}S(p)$, is the subgroup of $\mathbb{H}^n$ defined as

$$T^g_{\mathbb{H}}S(p) \stackrel{\text{def}}{=} \{x \in \mathbb{H}^n : d_H f_p(x) = 0\}.$$

The *group normal* (or *horizontal normal*) $n_{\mathbb{H}}(p) \in \bigwedge_{k,p} \mathfrak{h}_1$ is defined by

$$n_{\mathbb{H}}(p) \stackrel{\text{def}}{=} \frac{\nabla_H f_1(p) \wedge \cdots \wedge \nabla_H f_k(p)}{|\nabla_H f_1(p) \wedge \cdots \wedge \nabla_H f_k(p)|}.$$

The $(2n+1-k)$-vector $t_{\mathbb{H}}(p) \in \bigwedge_{2n+1-k,p} \mathfrak{h}$ defined as

$$t_{\mathbb{H}}(p) \stackrel{\text{def}}{=} * n_{\mathbb{H}}(p)$$

will be said to be the *group tangent* to S in p.

Notice that the group tangent vector is never horizontal. It can always be written in the form $t_{\mathbb{H}}(p) = \xi \wedge T$, where $\xi \in \bigwedge_{2n-k,p} \mathfrak{h}_1$ (Proposition 5). Moreover, if $t_{\mathbb{H}}(p) = v_1 \wedge \cdots \wedge v_{2n+1-k}$, then $T^g_{\mathbb{H}}S(p) = \exp(\mathrm{span}\{v_1, \dots, v_{2n+1-k}\})$.

As in the Euclidean setting, a $\mathbb{H}$*–orientation of* S will be identified with a continuous horizontal group vector field, or, equivalently, with a continuous group tangent vector field. If they exist, then S is said to be $\mathbb{H}$–orientable.

Finally notice that the definitions of $t_{\mathbb{H}}$ and of $n_{\mathbb{H}}$ are good ones. Indeed, as proved in the following Proposition, the notions of Heisenberg tangent group and consequently of horizontal normal to S do not depend on the defining function f.

Proposition 3.29. *If S is a k-codimensional $\mathbb{H}$-regular surface (with $1 \le k \le n$) and $p \in S$, then*

$$\mathrm{Tan}_{\mathbb{H}}(S, p) = \tau_p T^g_{\mathbb{H}}S(p). \tag{50}$$

Proof: Since $T^g_{\mathbb{H}}(\tau_{-p}S)(0) = T^g_{\mathbb{H}}S(p)$ it is enough to prove that if $0 \in S$ then

$$\mathrm{Tan}_{\mathbb{H}}(S, 0) = T^g_{\mathbb{H}}S(0).$$

With the same notations as used in Proposition 3.12, fix $r_0 > 0$ such that $B(0, r_0) \subset \mathcal{U}'$ and $S \cap B(0, r_0) = \{x \in B(0, r_0) : f(x) = 0\} = \{\Phi(\xi) := \xi \cdot \varphi(\xi) : \xi \in \mathcal{V}\}$. For $r \ge 1$, define $S_r := \delta_r S$ and $f_r := r f \circ \delta_{1/r}$. Clearly,

$$\begin{aligned} S_r \cap B(0, r r_0) &= \left\{x : \delta_{1/r} x \in S \cap B(0, r_0)\right\} = \{x \in B(0, r r_0) : f_r(x) = 0\} \\ &= \{\delta_r \Phi(\xi) : \quad \xi \in \mathcal{V}\} = \{\Phi_r(\xi) : \quad \xi \in \delta_r \mathcal{V}\}. \end{aligned}$$

Where we have defined $\Phi_r := \delta_r \circ \Phi \circ \delta_{1/r}$. Notice also that $f_r \in \mathcal{C}^1_{\mathbb{H}}(B(0, r r_0))$ and for any left invariant, horizontal vector field W and for all $x \in B(0, r r_0)$ $W f_r(x) = W f(\delta_{1/r} x)$.

Define now $f_\infty : \mathbb{H}^n \to \mathbb{R}^k$ as $f_{\infty,i}(x) = \langle \nabla_H f_i(0), \pi x \rangle_0$, for $i = 1, \cdots, k$. Observe that, because $f \in \mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)$, $f_r \to f_\infty$ as $r \to +\infty$ uniformly on each compact subset of $\mathbb{H}^n$ and that, by definition of tangent group,

$$T^g_{\mathbb{H}}S(0) = \{x : f_\infty(x) = 0\} = \{\Phi_\infty(\xi) : \quad \xi \in \mathbb{G}_{\mathfrak{w}}\},$$

where $\Phi_\infty : \mathbb{G}_{\mathfrak{w}} \to T^g_{\mathbb{H}} S(0)$ is implicitly defined by $f_\infty(\Phi_\infty(\xi)) = 0$, but can be also explicitly written solving the equation $f_\infty(\xi \cdot \exp(\sum_{l=1}^k \lambda_l V_l)) = 0$ with respect to ξ.

We want to prove that, for each $\xi \in \mathbb{G}_{\mathfrak{w}}$,

$$\Phi_r(\xi) \to \Phi_\infty(\xi) \quad \text{as } r \to +\infty. \tag{51}$$

First observe that, for each fixed ξ, $r \mapsto \Phi_r(\xi)$ is bounded for $r \to +\infty$. Indeed, from the Lipschitz continuity of φ (see(21)) it follows $|\Phi(\xi)|_c = |\xi \cdot \varphi(\xi)|_c \le |\xi|_c + |\varphi(\xi)|_c \le (1+c)|\xi|_c$, where c is the constant in Corollary 3.17. Hence

$$|\Phi_r(\xi)|_c = |(\delta_r \circ \Phi \circ \delta_{1/r})(\xi)|_c = r|\Phi(\delta_{1/r}\xi)|_c \le r(1+c)|\delta_{1/r}\xi|_c = (1+c)|\xi|_c.$$

Hence, for each fixed ξ, the limit class of $\Phi_r(\xi)$ as $r \to +\infty$, is not empty. Moreover, if $\Phi_{r_h}(\xi) \to l(\xi)$ as $r_h \to +\infty$, because $f_r \to f_\infty$ as $r \to +\infty$ uniformly on compact subsets, it follows that $l(\xi) = \Phi_\infty(\xi)$, and we have proved (51).

Since for r large $\Phi_r(\xi) \in S_r$, from (51) it follows

$$T^g_{\mathbb{H}} S(0) \subset \mathrm{Tan}_{\mathbb{H}}(S, 0).$$

To prove the opposite inequality, assume $p_h \in S_{r_h}$ and $p_h \to p$ as $r_h \to +\infty$. For $h \ge h_0$, $p_h \in S_{r_h} \cap B(0, r_h r_0)$, hence $p_h = \Phi_{r_h}(\xi_h)$ with $\xi_h \in \mathbb{G}_{\mathfrak{w}}$. But, $0 = f_{r_h}(\Phi_{r_h}(\xi_h)) \to f_\infty(p)$, hence $f_\infty(p) = 0$ and $p \in T^g_{\mathbb{H}} S(0)$.

$\square$

4. Surface measures and their representation (low codimension)

Theorem 4.1. *Let S be a k-codimensional $\mathbb{H}$-regular surface, $1 \le k \le n$. By Theorem 3.27 and with the notations therein, we know that S is locally a normal graph, that is we can assume that there are an open subset $\mathcal{U} \subset \mathbb{H}^n$, a function $f = (f_1, \dots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, a simple k-vector $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$, with $|v| = 1$, a simple $(2n+1-k)$-vector $w \stackrel{\text{def}}{=} *v \in {}_H\bigwedge_{2n+1-k}$, a relatively open $\mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}$ and a continuous function $\varphi : \mathcal{V} \to \mathbb{G}_{\mathfrak{v}}$ such that $S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\} = \{\Phi(\xi) \stackrel{\text{def}}{=} \xi \cdot \varphi(\xi), \quad \xi \in \mathcal{V}\}$. Now, if we put*

$$\Delta(p) \stackrel{\text{def}}{=} \left| \det\left[v_i f_j(p)\right]_{1 \le i,j \le k} \right| \neq 0 \qquad \text{for } p \in \mathcal{U},$$

then

$$\mathcal{S}^{Q-k}_\infty \llcorner (S \cap \mathcal{U}) = \Phi_\sharp \left(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \right) \mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}}. \tag{52}$$

Here, for a measure μ, $\Phi_\sharp \mu$ is the image measure of μ ([21], Definition 1.17). Notice also that, since $\mathbb{G}_{\mathfrak{w}}$ is a linear space, $\mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}} = \mathcal{L}^{2n+1-k} \llcorner \mathbb{G}_{\mathfrak{w}}$, the $(2n+1-k)$-dimensional Lebesgue measure.

Remark 4.2. If we assume simply that $S \cap \mathcal{U}$ is a regular graph (and not an orthogonal graph) then formula (52) takes the following more general form

$$\begin{aligned}
&\mathcal{S}_\infty^{Q-k} \llcorner (S \cap \mathcal{U}) \\
&= \frac{|\det J_\psi|}{|\det J_{\psi_{\mathfrak{w}}}|}\, \Phi_\sharp \left(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \right) \mathcal{H}_E^{2n+1-k} \llcorner \mathbb{G}_{\mathfrak{w}}
\end{aligned}$$

and recalling the computations in Proposition 3.19,

$$= \frac{|v \wedge w|}{|w|}\, \Phi_\sharp \left(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \right) \mathcal{H}_E^{2n+1-k} \llcorner \mathbb{G}_{\mathfrak{w}}.$$

Proof. Let $d = 2n+1$. We need the following Differentiation Theorem whose proof can be found in Federer's book (see [9], Theorems 2.10.17 and 2.10.18).

Theorem 4.3. *[Differentiation theorem] Let μ be a regular measure and ζ the valuation function defined in (4) and used in the definition of the measure $\mathcal{S}_\infty^{Q-k}$. If*

$$\lim_{r \to 0} \frac{\mu(B_\infty(x,r))}{(\zeta(B_\infty(x,r)))^{Q-k}} = \mathfrak{s}(x), \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{H}^n, \tag{53}$$

then

$$\mu = \mathfrak{s}(x) \mathcal{S}_\infty^{Q-k}. \tag{54}$$

We are going to apply the Theorem 4.3 to the measure $\mu = \mu_S$ defined as

$$\mu_S(\mathcal{O}) \stackrel{\text{def}}{=} \int_{\Phi^{-1}(\mathcal{O})} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}}$$

for any Borel set $\mathcal{O} \subset \mathbb{H}^n$. By Theorem 4.3, identity (52) follows once proved that

$$\begin{aligned}
\lim_{r \to 0} \frac{1}{r^{Q-k}} \int_{\Phi^{-1}(B_\infty(p,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \, d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
= 2\omega_{Q-k-2}.
\end{aligned} \tag{55}$$

Hence we shall prove (55).

Step 1. Without loss of generality, in (55) we can assume $p = 0$. Indeed, using the fact that, with the notations of Proposition 3.11, the Jacobian of the map $\eta \to \sigma_p(\eta) \stackrel{\text{def}}{=} p \cdot \eta \cdot p_{\mathfrak{v}}^{-1}$ from $\mathbb{G}_{\mathfrak{w}} \simeq \mathbb{R}^{d-k}$ to itself is identically 1, we have

$$\begin{aligned}
&\int_{\Phi^{-1}(B_\infty(p,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \; d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&= \int_{\sigma_p^{-1}\circ\Phi^{-1}(B_\infty(p,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \circ \sigma_p \; d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&= \int_{(\Phi_{p^{-1}})^{-1}(B_\infty(p,r))} \frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \tau_{p^{-1}} \circ \Phi_{p^{-1}} \; d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&= \int_{(\Phi_{p^{-1}})^{-1}(B_\infty(p,r))} \frac{|\nabla_H (f_1\circ\tau_p) \wedge \cdots \wedge \nabla_H (f_k\circ\tau_p)|}{\Delta_p} \circ \Phi_{p^{-1}} \; d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}},
\end{aligned}$$

where $\Delta_p := \left|\det\left([v_i(f_j\circ\tau_p)]_{1\le i,j\le k}\right)\right|$. Remember that $\tau_{p^{-1}}(S) = \{x : (f\circ\tau_p)(x) = 0\}$. Hence, the limit in (55) equals the same limit when we replace S by $\tau_{p^{-1}}(S)$ and accordingly p with 0. This concludes the proof of Step 1.

Set now, for $\rho > 0$, $f_{1/\rho} \stackrel{\text{def}}{=} \frac{1}{\rho} f\circ\delta_\rho$ and $\Phi_{1/\rho} \stackrel{\text{def}}{=} \delta_{1/\rho}\circ\Phi\circ\delta_\rho$. Notice that for the dilated set $\delta_{1/\rho}S$ we have $\delta_{1/\rho}S = \{x \in \delta_{1/\rho}\mathcal{U} \;:\; f_{1/\rho}(x) = 0\} = \{\Phi_{1/\rho}(\xi) \;:\; \xi \in \delta_{1/\rho}\mathcal{V}\}$. Then defining, analogously to μ_S,

$$\begin{aligned}
&\mu_{(\delta_{1/\rho}S)}(B_\infty(0,r)) \\
&\stackrel{\text{def}}{=} \int_{\Phi_{1/\rho}^{-1}(B_\infty(0,r))} \frac{|\nabla_H f_{1/\rho,1} \wedge \cdots \wedge \nabla_H f_{1/\rho,k}|}{\Delta_{1/\rho}} \circ \Phi_{1/\rho} \; d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}},
\end{aligned}$$

we have

Step 2. If $\rho > 0$, then

$$\frac{\mu_S(B_\infty(0,\rho))}{\rho^{Q-k}} = \mu_{(\delta_{1/\rho}S)}(B_\infty(0,1)). \tag{56}$$

Given the homogeneity of the horizontal vector fields with respect to group dilations, (56) follows by the change of variables $x' = \delta_\rho(x)$. Indeed, the Jacobian of this tranformation from $\mathbb{G}_{\mathfrak{w}}$ to itself is equal to ρ^{k-Q}, since $T \in \mathfrak{w}$, and $\Phi^{-1}(B_\infty(0,\rho)) = \delta_\rho\big(\Phi_{1/\rho}^{-1}(B_\infty(0,1)\big)$.

Step 3. We can prove that

$$\begin{aligned}
&\lim_{\rho\to 0} \mu_{(\delta_{1/\rho}S)}(B_\infty(0,1)) \\
&= \int_{\Phi_\infty^{-1}(B_\infty(0,1))} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \circ \Phi_\infty \; d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}} \\
&= \int_{\Phi_\infty^{-1}(B_\infty(0,1))} \frac{|\nabla_H f_1(0) \wedge \cdots \wedge \nabla_H f_k(0)|}{\Delta_\infty} \circ \Phi_\infty \; d\mathcal{H}_E^{d-k} \llcorner \mathbb{G}_{\mathfrak{w}},
\end{aligned} \tag{57}$$

where, as in Proposition 3.29, $\Phi_\infty : \mathbb{G}_{\mathfrak{w}} \to T^g_{\mathbb{H}}S(0) = \mathrm{Tan}_{\mathbb{H}}(S,0)$ is implicitly defined by the equation $f_\infty(\Phi_\infty(\xi)) = 0$, $f_{\infty,i}(x) = d_H f_{i0}(x)$, for $i = 1, \cdots, k$, and Δ_∞ is defined accordingly.

Indeed, let $\psi_{1,\varepsilon}$ and $\psi_{2,\varepsilon}$ be nonnegative Lipschitz continuous functions supported, respectively, in an ε-neighborhood of $B_\infty(0,1)$ and in $B_\infty(0,1)$ and such that $\psi_{1,\varepsilon} \equiv 1$ on $B_\infty(0,1)$ and $\psi_{2,\varepsilon} \equiv 1$ on $B_\infty(0,1-\varepsilon)$. Then

$$\begin{aligned}
&\int\limits_{\mathbb{G}_{\mathfrak{w}}} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty}\, \psi_{2,\varepsilon} \circ \Phi_\infty \, d\mathcal{H}_E^{d-k} \\
&\le \liminf_{\rho\to 0}\ \mu_{(\delta_{1/\rho}S)}(B_\infty(0,1)) \le \limsup_{\rho\to 0}\ \mu_{(\delta_{1/\rho}S)}(B_\infty(0,1)) \\
&\le \int\limits_{\mathbb{G}_{\mathfrak{w}}} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty}\, \psi_{1,\varepsilon} \circ \Phi_\infty \, d\mathcal{H}_E^{d-k},
\end{aligned} \tag{58}$$

thanks to the uniform convergence of $f_{1/\rho} \to f_\infty$ and of $\Phi_{1/\rho} \to \Phi_\infty$. Letting now $\varepsilon \to 0$, we get eventually (57). This concludes the proof of Step 3.

The function $f_\infty = d_H f_0$ is an H-linear map, hence, as a map from $\mathbb{R}^d \to \mathbb{R}^k$, it does not depend on the variable p_{2n+1}. It follows that

$$|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}| = |\nabla f_{\infty,1} \wedge \cdots \wedge \nabla f_{\infty,k}|_{\bigwedge_k \mathbb{R}^d}.$$

Remember that the first norm in the preceding inequality is the norm induced in $\bigwedge_k \mathfrak{h}_1$ by the norm in $\bigwedge_1 \mathfrak{h}_1$. Moreover notice that f_∞ is Euclidean smooth, so that we can apply Proposition 3.22. Starting from (57), with $U = B_\infty(0,1)$, we get

$$\begin{aligned}
&\int\limits_{\Phi_\infty^{-1}(U)} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \circ \Phi_\infty \, d\mathcal{H}_E^{d-k} \llcorner\, G_{\mathfrak{w}} \\
&= \int\limits_{(\Phi_\infty\circ\psi_{\mathfrak{w}})^{-1}(U)} \frac{|\nabla_H f_{\infty,1} \wedge \cdots \wedge \nabla_H f_{\infty,k}|}{\Delta_\infty} \circ (\Phi_\infty \circ \psi_{\mathfrak{w}})\, |\det J_{\psi_{\mathfrak{w}}}|\, d\mathcal{H}_E^{d-k} \\
&= \int\limits_{(\Phi_\infty\circ\psi_{\mathfrak{w}})^{-1}(U)} \frac{|\nabla f_{\infty,1} \wedge \cdots \wedge \nabla f_{\infty,k}|_{\bigwedge_k \mathbb{R}^d}}{\Delta_\infty} \circ (\Phi_\infty \circ \psi_{\mathfrak{w}})\, |\det J_{\psi_{\mathfrak{w}}}|\, d\mathcal{H}_E^{d-k}
\end{aligned}$$

and using Proposition 3.22

$$\begin{aligned}
&= \int\limits_{(\Phi_\infty\circ\psi_{\mathfrak{w}})^{-1}(U)} \left| \frac{\partial(\Phi_\infty\circ\psi_{\mathfrak{w}})}{\partial\xi_1} \wedge \cdots \wedge \frac{\partial(\Phi_\infty\circ\psi_{\mathfrak{w}})}{\partial\xi_{d-k}} \right|_{\bigwedge_k \mathbb{R}^d} \frac{|\det J_{\psi_{\mathfrak{w}}}|}{|\det J_\psi|}\, d\mathcal{H}_E^{d-k}, \\
&= \frac{|\det J_{\psi_{\mathfrak{w}}}|}{|\det J_\psi|}\, \mathcal{H}_E^{d-k}(\mathrm{Tan}_{\mathbb{H}}(S,0) \cap B_\infty(0,1)) = \frac{|w|}{|v \wedge w|} 2\omega_{Q-2},
\end{aligned}$$

from Proposition 3.19. □

As in [11], Corollay 3.7 we can prove the following Corollary.

Corollary 4.4. *If S is k-codimensional $\mathbb{H}$-regular surface with $1 \le k \le n$, then the Hausdorff dimension of S with respect to the cc–distance d_c, or any other metric comparable with it, is $Q-k$.*

Recall that regular surfaces in general are not Euclidean regular. In fact, as we already stressed, recently Kirchheim and Serra Cassano provided an example of a 1-codimensional $\mathbb{H}$-regular surface S in $\mathbb{H}^1$ that has Euclidean

Hausdorff dimension 2.5 and hence it is not a 2-dimensional Euclidean rectifiable set. Thus, the topological dimension of S equals 2, its Euclidean Hausdorff dimension equals 2.5 and its intrinsic Hausdorff dimension equals 3.

Nevertheless, if it happens that S is a k-codimensional Euclidean $\mathcal{C}^1$ submanifold of $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$, $1 \le k \le n$, then the surface measure $\mathcal{H}_E^{2n+1-k} \llcorner S$ is locally finite and its relation with the spherical Hausdorff measure $\mathcal{S}_\infty^{Q-k} \llcorner S$ takes a particularly simple form. This is the content of Theorem 4.6. In codimension 1, the formula has been proved by the authors in [10], and, with the $\mathbb{H}$-perimeter taking place of the Hausdorff measure, by Capogna, Danielli and Garofalo in [7].

Lemma 4.5. *Let S be an $\mathbb{H}$-regular surface of codimension k and suppose, in addition, that S is also an Euclidean $\mathcal{C}^1$-manifolds. With the notations of Theorem 4.1, we have*

$$\begin{aligned} &\mathcal{S}_\infty^{2n+2-k} \llcorner S \\ &= \left(\sum_{1 \le i_1 < \cdots < i_k \le 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n \rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \right)^{1/2} \mathcal{H}_E^{2n+1-k} \llcorner S, \end{aligned} \tag{59}$$

where

$$n = n_1 \wedge \cdots \wedge n_k = \frac{\nabla f_1 \wedge \cdots \wedge \nabla f_k}{|\nabla f_1 \wedge \cdots \wedge \nabla f_k|_{\bigwedge_k \mathbb{R}^{2n+1}}} = \frac{\nabla f}{|\nabla f|_{\bigwedge_k \mathbb{R}^{2n+1}}}$$

is a continuous Euclidean unit normal k-vector field and $W_1 = X_1, \ldots, W_{2n} = Y_n$.

Proof. Denote by $\Theta : \bigwedge_1 \mathfrak{h}_1 \to \mathbb{R}^{2n}$ the map that associates with an horizontal vector its canonical coordinates with respect to the orthonormal basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. Clearly, Θ is a vector space isomorphism and an isometry. We still denote by Θ the induced operator acting from $\bigwedge_k \mathfrak{h}_1$ to $\bigwedge_k \mathbb{R}^{2n}$. We have, for $1 \le j \le k$, $\Theta(\nabla_H f_j) = (W \bullet \nabla f_j)$ where we have set

$$(W \bullet \nabla f) \stackrel{\text{def}}{=} \left(\langle X_1, \nabla f \rangle_{\mathbb{R}^{2n+1}}, \ldots, \langle Y_n, \nabla f \rangle_{\mathbb{R}^{2n+1}} \right) \in \bigwedge\nolimits_1 \mathbb{R}^{2n}.$$

Notice that, thanks to the assumed Euclidean regularity of f, the local parametrization Φ of S is continuously differentiable in the Euclidean sense.

Hence

$$
\begin{aligned}
&|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|_{\bigwedge_k \mathfrak{h}_1} \\
&= |(W \bullet \nabla f_1) \wedge \cdots \wedge (W \bullet \nabla f_k)|_{\bigwedge_k \mathbb{R}^{2n}} \\
&= \left(\sum_{1 \le i_1 < \cdots < i_k \le 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, \nabla f_1 \wedge \cdots \wedge \nabla f_k \rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \right)^{1/2} \\
&= \left(\sum_{1 \le i_1 < \cdots < i_k \le 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n \rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \right)^{1/2} \quad |\nabla f|_{\bigwedge_k \mathbb{R}^{2n+1}}
\end{aligned}
$$

and by (34), it follows

$$
\begin{aligned}
&= \left(\sum_{1 \le i_1 < \cdots < i_k \le 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n \rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \right)^{1/2} \\
&\qquad\qquad \frac{\Delta}{|\det J_\psi|} \left| \frac{\partial (\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_1}(\xi) \wedge \cdots \wedge \frac{\partial (\Phi \circ \psi_{\mathfrak{w}})}{\partial \xi_{d-k}}(\xi) \right| .
\end{aligned}
$$

Replacing in (52) we obtain eventually (59). □

Strictly speaking, an Euclidean regular surface S may be not $\mathbb{H}$-regular. Indeed, even if S is locally the zero set of a function $f \in [\mathcal{C}^1(\mathbb{R}^{2n+1})]^k \subset [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$ with non-vanishing Euclidean gradient, nevertheless the non-degeneracy condition $\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k \neq 0$ may fail to hold at some points. As in [18], a point p of an Euclidean $\mathcal{C}^1$ submanifold S is said to be a *characteristic point* of S if $\mathrm{Tan}(S,p) \subset H\mathbb{H}^n_p$ and, consequently, the non-degeneracy condition fails. We denote by $C(S)$ the set of these points.

When $k = 1$, it is known that $C(S)$ is *small* inside S. There are many results in this line, under various regularity hypotheses on the surfaces and using different surface measures (Euclidean versus intrinsic) to estimate the smallness. Balogh (see [6]) was the first one to prove that, in the Heisenberg groups, the intrinsic $(Q-1)$-Hausdorff measure of the characteristic set of an Euclidean $\mathcal{C}^1$ surface vanishes. Recently, Magnani ([18], 2.16) extended this result to Euclidean $\mathcal{C}^1$-submanifold of arbitrary codimension in general Carnot groups. Precisely, in the setting of the Heisenberg group, we have

$$
\mathcal{S}^{Q-k}_\infty(C(S)) = 0 \tag{60}
$$

if S is an Euclidean $\mathcal{C}^1$-submanifold of codimension k, $1 \le k \le n$ in $\mathbb{H}^n$. Since a $\mathcal{C}^1$-submanifold S in $\mathbb{H}^n$ can be written as $S = C(S) \cup (S \setminus C(S))$ and $S \setminus C(S)$ is a $\mathbb{H}$-regular surface, then, by Lemma 4.5, we have

Theorem 4.6. *If S is an Euclidean $\mathcal{C}^1$-submanifold of codimension k, $1 \le k \le n$ in $\mathbb{H}^n$, then*

$$
\begin{aligned}
&\mathcal{S}_\infty^{2n+2-k} \llcorner S \\
&= \left(\sum_{1\le i_1<\cdots<i_k\le 2n} \langle W_{i_1} \wedge \cdots \wedge W_{i_k}, n \rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \right)^{1/2} \mathcal{H}_E^{2n+1-k} \llcorner S \qquad (61)\\
&= \left(\sum_{1\le i_1<\cdots<i_k\le 2n} \left(\det \left[\langle W_{i_\ell}, n_j \rangle_{\mathbb{R}^{2n+1}} \right]_{\ell,j=1,\dots,k} \right)^2 \right)^{1/2} \mathcal{H}_E^{2n+1-k} \llcorner S,
\end{aligned}
$$

where $n = n_1 \wedge \cdots \wedge n_k$ is a continuous Euclidean unit normal k-vector field and $W = (X_1, \dots, Y_n)$.

5. Appendix: Federer-Fleming Currents

We give here a natural definition of (Federer-Fleming) currents with respect to an intrinsic complex of differential forms on $\mathbb{H}^n$ and we also see that $\mathbb{H}$-regular surfaces can be naturally identified with currents defined in this way.

Let $\mathcal{U}$ be an open subset of $\mathbb{H}^n$ and let $\mathcal{D}^*(\mathcal{U}) = \mathcal{D}^0(\mathcal{U}) \oplus \cdots \oplus \mathcal{D}^{2n+1}(\mathcal{U})$ be the graded algebra of $\mathcal{C}^\infty$ differential forms on $\mathbb{R}^{2n+1}$ with compact support in $\mathcal{U}$.

Definition 5.1. Following Rumin [26] we denote by $\mathcal{D}_{\mathbb{H}}^k(\mathcal{U})$ (*Heisenberg k-differential forms*) the space of compactly supported smooth sections respectively of ${}_H\bigwedge_k \equiv \frac{\bigwedge_k \mathfrak{h}}{\mathcal{I}^k}$, when $1 \le k \le n$ and of ${}_H\bigwedge_k \equiv \mathcal{J}^k$ when $n+1 \le k \le 2n+1$. These spaces are endowed with the natural topology induced by that of $\mathcal{D}^k(\mathcal{U})$. We denote by $\mathcal{D}_{\mathbb{H}}^*(\mathcal{U}) = \mathcal{D}_{\mathbb{H}}^0(\mathcal{U}) \oplus \cdots \oplus \mathcal{D}_{\mathbb{H}}^{2n+1}(\mathcal{U})$ the graded algebra of all Heisenberg differential forms with compact support, where $\mathcal{D}_{\mathbb{H}}^0(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U})$.

The following Theorem is proved in [26].

Theorem 5.2 (Rumin)**.** *There is a linear second order differential operator $D : \mathcal{D}_{\mathbb{H}}^n(\mathcal{U}) \to \mathcal{D}_{\mathbb{H}}^{n+1}(\mathcal{U})$ such that the following sequence has the same cohomology as the De Rham complex on $\mathcal{U}$:*

$$
\begin{aligned}
0 \to \mathcal{D}_{\mathbb{H}}^0(\mathcal{U}) \xrightarrow{d} \mathcal{D}_{\mathbb{H}}^1(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}_{\mathbb{H}}^n(\mathcal{U}) \xrightarrow{D} \\
\xrightarrow{D} \mathcal{D}_{\mathbb{H}}^{n+1}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}_{\mathbb{H}}^{2n+1}(\mathcal{U}) \to 0
\end{aligned}
$$

where d is the operator induced by the external differentiation from $\mathcal{D}^k(\mathcal{U}) \to \mathcal{D}^{k+1}(\mathcal{U})$, with $k \ne n$.

Definition 5.3. We call Heisenberg k-current, $1 \le k \le 2n+1$ any continuous linear functional on $\mathcal{D}_{\mathbb{H}}^k(\mathcal{U})$ and we denote by $\mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ the set of all Heisenberg k-currents.

Proposition 5.4. *If $1 \le k \le n$, any $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ can be identified with an element $\tilde{T}$ of $\mathcal{D}_k(\mathcal{U})$, the space of all Euclidean k-currents by setting*

$$\tilde{T}(\omega) \stackrel{\text{def}}{=} T([\omega])$$

for any $\omega \in \mathcal{D}^k(\mathcal{U})$.

On the other hand, if $S \in \mathcal{D}_k(\mathcal{U})$ is such that $S(\alpha \wedge \theta) = 0$ for any $\alpha \in \mathcal{D}^{k-1}(\mathcal{U})$ and $S(\beta \wedge \theta) = 0$ for any $\beta \in \mathcal{D}^{k-2}(\mathcal{U})$ if $k \geq 2$, then S induces a Heisenberg k-current $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ by the identity

$$T([\omega]) \stackrel{\text{def}}{=} S(\omega)$$

for all $[\omega] \in \mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$. Obviously, with our previous notations, $\tilde{T} = S$.

Definition 5.5. Let T be k-dimensional $\mathbb{H}$-current in an open set $\mathcal{U} \subset \mathbb{H}^n$, then the *mass* $\mathbf{M}_{\mathcal{V}}(T)$ of T in $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ open, is

$$\mathbf{M}_{\mathcal{V}}(T) \stackrel{\text{def}}{=} \sup\{T(\alpha) :\ \alpha \in \mathcal{D}^k_{\mathbb{H}}(\mathcal{V}), |\alpha| \leq 1\}$$

Remark 5.6. In the last few years a very general theory of currents in metric spaces was developed by Ambrosio and Kirkhheim in [3]. As pointed by the same authors in [1], this approach, when particularized to Heisenberg groups with Carnot–Carathéodory distance, is not satisfactory. Indeed, they prove the non existence of rectifiable $2n+1-k$-currents (in their sense) in $\mathbb{H}^n$ when $k < n$. This depends, once more, on the non existence of Lipschitz injective maps from $\mathbb{R}^{2n+1-k}$ to $\mathbb{H}^n$ when $k < n$.

On the contrary there are plenty of Heisenberg $(2n+1-k)$-currents given as integration on $\mathbb{H}$-regular surfaces of codimension $k < n$, as we shall see below (see Proposition 5.8). These Heisenberg currents carried by $\mathbb{H}$-regular surfaces play a major role in applications since most naturally they will be the building blocks of Heisenberg rectifiable currents (whose theory has to be developed).

On the other hand, Ambrosio and Kirkhheim (see Theorem 4.5 in [3]) proved that rectifiable metric k-currents in $\mathbb{H}^n$, when $k \leq n$, are carried by k-dimensional rectifiable sets of $\mathbb{H}^n$. These sets are, up to negligeable subsets, countable unions of Lipschitz images of Borel sets in $\mathbb{R}^k$. Since our k-dimensional $\mathbb{H}$-regular surfaces, with $k \leq n$, are intrinsically $\mathcal{C}^1$ images of open sets in $\mathbb{R}^k$, it turns out again that our Heisenberg currents given by integration on $\mathbb{H}$-regular surfaces of dimension $k \leq n$ play the role of building blocks for a theory of Heisenberg rectifiable currents of low dimension.

Remark 5.7. For 1-codimensional currents – as already pointed out by Magnani – the perimeter measure (see e.g. [10]) can be seen as the mass of the boundary of a suitable $(2n+1)$-dimensional $\mathbb{H}$-current. Indeed, if $F = (F_1, \dots, F_{2n})$ is an horizontal vector field in an open subset of $\mathbb{H}^n$, and if we identify F with the form $\sum_{j=1}^{2n} F_j dx_j \in \bigwedge^1 \mathfrak{h}_1$, then we have

$$\mathrm{div}_{\mathbb{H}}\ F = *d(*F),$$

where $*$ denotes here the Hodge operator defined in Definition 2.3. Thus, we can argue e.g. as in [28], Remark 27.7.

As in the Euclidean setting, we notice that Heisenberg currents are generalizations of Heisenberg regular submanifolds, in the sense that any oriented $\mathbb{H}$-regular surface induces, by integration, in a natural way a k-dimensional Heisenberg current.

Proposition 5.8. *Let $S \subset \mathcal{U}$ be a $\mathbb{H}$-regular surface as in Definitions 3.1 and 3.2. Assume S is oriented by a group tangent k-vector field $t_{\mathbb{H}}$. Then, if S is k-dimensional, $1 \le k \le n$, the map*

$$\alpha \to [\![S]\!]\,(\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | t_{\mathbb{H}} \rangle \, d\mathcal{S}^k_\infty$$

from $\mathcal{D}^k_{\mathbb{H}}$ to $\mathbb{R}$ is a Heisenberg k-current with locally finite mass. Precisely, if $\mathcal{V} \subset\subset \mathcal{U}$,

$$\mathbf{M}_{\mathcal{V}}([\![S]\!]) = \mathcal{S}^k_\infty(S \cap \mathcal{V}).$$

Analogously, if S is k-codimensional, $1 \le k \le n$, the map

$$\alpha \to [\![S]\!]\,(\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | \omega \rangle \, d\mathcal{S}^{Q-k}_\infty$$

from $\mathcal{D}^{2n+1-k}_{\mathbb{H}}$ to $\mathbb{R}$ is a Heisenberg $(2n+1-k)$-current with locally finite mass. Precisely, if $\mathcal{V} \subset\subset \mathcal{U}$,

$$\mathbf{M}_{\mathcal{V}}([\![S]\!]) = \int_{S\cap\mathcal{V}} |\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}}(t_{\mathbb{H}})| \, d\mathcal{S}^{Q-k}_\infty, \tag{62}$$

where $\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}} : \bigwedge_{2n+1-k} \mathfrak{h}_1 \to {}_H\bigwedge_{2n+1-k}$ is the orthogonal projection with respect to the Riemannian scalar product defined in Section 2.2.

Corollary 5.9. *There exists a geometric constant $c_{n,k} \in (0,1)$ such that for any k-codimensional $\mathbb{H}$-regular surface S, $1 \le k \le n$, we have*

$$c_{n,k}\, \mathcal{S}^{Q-k}_\infty(S \cap \mathcal{V}) \le \mathbf{M}_{\mathcal{V}}([\![S]\!]) \le \mathcal{S}^{Q-k}_\infty(S \cap \mathcal{V}),$$

for every Borel set $\mathcal{V}$.

Proof. By (62), it is enough to show that

$$c_{n,k} \stackrel{\text{def}}{=} \inf\{|\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}}(v)| \ : \ v \in \bigwedge\nolimits_{2n+1-k} \mathfrak{h}, v \text{ simple}, |v| = 1\} > 0.$$

Indeed, by Propositions 3.24 and 3.25, $|\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}}(v)| > 0$ for all $v \in \bigwedge_{2n+1-k} \mathfrak{h}$,$v$ simple, $|v| = 1$. Then the assertion follows by the compactness of the set of simple vectors of unit norm. □

Example 5.10. We stress that the mass of the current carried by a k-codimensional $\mathbb{H}$-regular surface S can be different (though equivalent) from its $\mathcal{S}^{Q-k}_\infty$-measure. Clearly, by (62) this does not happen when $t_{\mathbb{H}} \in {}_H\bigwedge_{2n+1-k}$ on S. On the other hand, if for instance we consider the surface S of Example 3.23, then a direct computation shows that, taking $t_{\mathbb{H}} = W_2 \wedge W_4 \wedge T$, we obtain

$$\mathbf{M}_{\mathcal{V}}([\![S]\!]) = \frac{1}{\sqrt{2}}\, \mathcal{S}^{Q-k}_\infty(S \cap \mathcal{V}).$$

Thanks to Rumin's result, the operators d and D act in the complex as external differentiation does in De Rham complex, and we can give the following (obvious) definition.

Definition 5.11. Let T be a Heisenberg k-current in an open set $\mathcal{U} \subset \mathbb{H}^n$ with $1 \le k \le d$. Then we define the Heisenberg $(k-1)$-current $\partial_{\mathbb{H}} T$, the *Heisenberg boundary of* T, by the identity

$$\partial_{\mathbb{H}} T(\alpha) = T(d\alpha) \quad \text{if } k \neq n+1$$

and

$$\partial_{\mathbb{H}} T(\alpha) = T(D\alpha) \quad \text{if } k = n+1.$$

The following trivial statement says that – also when boundaries are concerned – low dimension $\mathbb{H}$-currents are but particular Euclidean currents.

Proposition 5.12. *If $1 \le k \le n$, the Heisenberg boundary $\partial_{\mathbb{H}} T$ of $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ can be identified as in Proposition 5.4 with the Euclidean $(k-1)$-current $\partial \tilde{T}$.*

Proof. Let us notice first that $\partial\tilde{T}(\alpha \wedge \theta) = 0$ for any $\alpha \in \mathcal{D}^{k-1}(\mathcal{U})$ and $\partial\tilde{T}(\beta \wedge \theta) = 0$ for any $\beta \in \mathcal{D}^{k-2}(\mathcal{U})$ if $k \ge 2$. Indeed (e.g.) $\partial\tilde{T}(\alpha \wedge \theta) = \tilde{T}(d\alpha \wedge \theta) + \tilde{T}(\alpha \wedge d\theta) = T([d\alpha \wedge \theta]) + T([\alpha \wedge d\theta]) = T([0]) + T([0]) = 0$. Thus, $\partial\tilde{T}$ induces a $(k-1)$-dimensional $\mathbb{H}$-current T'. On the other hand, for any $[\omega] \in \mathcal{D}_{\mathbb{H}}^{k-1}$, we have $T'([\omega]) = \partial\tilde{T}(\omega) = \tilde{T}(d\omega) = T([d\omega]) = T(d[\omega]) = \partial_{\mathbb{H}} T([\omega])$, so that $T' = \partial_{\mathbb{H}} T$. $\square$

When $k \ge n+1$, the structure of the boundary of a current is much more difficult to describe, even in the simplest situation of a current carried by a low codimensional $\mathbb{H}$-regular surface. As an example, consider the case $n = 1$, and let S be a 1-codimensional $\mathbb{H}$-regular (hyper)surface. We want to state here something similar to Stokes formula that yields that the boundary of a 2-dimensional current in $\mathbb{R}^3$ carried by a sufficiently regular portion of a 2-dimensional Euclidean differentiable manifold (a 2-dimensional oriented Euclidean differentiable manifold with boundary) is carried by the boundary itself, endowed with a suitable induced orientation.

First of all, we cannot think in general of a portion of $\mathbb{H}$-regular hypersurface – whatever regularity we assume for the boundary – as a differentiable manifold with boundary, since, as we pointed out repeatedly, $\mathbb{H}$-regular surfaces may be very "bad" from the Euclidean point of view ([15]). On the other hand, even when dealing with (Euclidean) smooth hypersurfaces with boundary, the mass of the boundary of the associated current may be not locally finite, unless the topological boundary is a horizontal curve.

Let us start by illustrating the last phenomenon: if $[\omega] \in \mathcal{D}_{\mathbb{H}}^1(\mathbb{H}^1)$ we can alway choose ω to be its horizontal representative $\omega = \omega_1 dp_1 + \omega_2 dp_2$. In this case, accordingly with Rumin's theorem ([26]), the operator D has the form

$$D[\omega] = d(\omega + \tilde{\omega}\theta),$$

where $\tilde{\omega} \in \mathcal{C}^\infty(\mathbb{H}^1)$, is chosen in order to have $d(\omega + \tilde{\omega}\theta) \in \mathcal{D}_{\mathbb{H}}^2(\mathbb{H}^1)$, i.e. such that $d(\omega + \tilde{\omega}\theta) \wedge \theta = 0$. An esplicit computation (see also [13], Section 6) shows that

$$\tilde{\omega} = \frac{1}{4}(W_2\omega_1 - W_1\omega_2).$$

Consider now the 2-dimensional $\mathbb{H}$-current $[\![S]\!]$ carried by the hypersurface $S = \{p_1 = 0,\ p_2 > 0\}$ oriented by $W_2 \wedge T$. Let t_0 be the boundary of S, i.e. $t_0 = \{p_1 = p_2 = 0\}$. If $[\omega] \in \mathcal{D}_{\mathbb{H}}^1(\mathbb{H}^1)$, with $\omega = \omega_1 dp_1 + \omega_2 dp_2$ as above, by definition and by Stokes theorem (keeping also in mind that

$\mathcal{S}^3_\infty \llcorner S = \mathcal{H}^2_E \llcorner S$, by (52)), we have

$$\begin{aligned}\partial_{\mathbb{H}} [\![S]\!] ([\omega]) &\stackrel{\text{def}}{=} \int_S \langle D([\omega]) | W_2 \wedge T\rangle d\mathcal{H}^2_E = \int_S \langle d(\omega + \tilde\omega\theta) | W_2 \wedge T\rangle d\mathcal{H}^2_E \\ &= \int_{t_0} \langle \omega + \tilde\omega\theta | T\rangle d\mathcal{H}^1_E = \frac{1}{4}\int_{t_0} (\partial_2\omega_1 - \partial_1\omega_2)\, d\mathcal{H}^1_E.\end{aligned}$$

Clearly, the above quantity can be made arbitrary large still keeping $|[\omega]| \le 1$. This shows that $\partial_{\mathbb{H}} [\![S]\!]$, though being a well defined current in our sense, has no locally finite mass.

An analysis of the example above shows quickly that the reason making the boundary of the current carried by S not being of finite mass relies precisely on the fact that the operator D is a second order differential operator because of the derivatives of ω hidden in $\tilde\omega$. These derivatives remain in the integration after applying Stokes theorem. Thus, we can expect the boundary of the current carried by a smooth 2-dimensional Euclidean manifold S with boundary ∂S to have finite mass if (and only if) ∂S is horizontal, since in this case $\langle \tilde\omega\theta | t_{\mathbb{H}}\rangle \equiv 0$, and no derivatives are left after applying Stokes theorem.

In fact, this is coherent with our definition of $\mathbb{H}$-regular surface, providing a further evidence for it: the boundary of an hypersurface in $\mathbb{H}^1$ according with our definition has finite mass only if the boundary is a 1-dimensional surface again in our sense (i.e. an horizontal curve). We stress that, if $n > 1$, this phenomenon is typical of n-codimensional $\mathbb{H}$-regular surfaces, since the surface itself and its boundary belong to the two different classes of $\mathbb{H}$-surfaces: the surface is a low-codimensional, whereas the the boundary is low-dimensional. This is clearly strictly connected with the fact that the derivation in Rumin's complex is a second order operator only when we pass from dimension n to dimension $n+1$. For instance, if we consider in $\mathbb{H}^2$ the 1-codimensional $\mathbb{H}$-regular surface $S = \{p_1 = 0,\ p_2 > 0\}$ oriented by $W_2 \wedge W_3 \wedge W_4 \wedge T$, again classical Stokes theorem yields that now $\partial [\![S]\!]$ is carried by the 2-codimensional $\mathbb{H}$-regular surface $\{p_1 = p_2 = 0\}$ oriented by $W_3 \wedge W_4 \wedge T$, despite of the analogy with the preceding example. This because in $\mathbb{H}^2$ both $\{p_1 = 0,\ p_2 > 0\}$ and $\{p_1 = p_2 = 0\}$ are low-codimensional $\mathbb{H}$-regular surfaces.

If $k < n$, the above example can be easily generalized to that of a continuously differentiable $(2n+1-k)$-manifold with boundary $S \subset \mathbb{H}^n$ that locally has the form $\{f_1(p) = \dots = f_k(p) = 0,\ f_{k+1}(p) \ge 0\}$, with (for sake of simplicity)

$$\det[W_i f_j]_{1\le i,j\le k} \ne 0 \quad \text{and} \quad \det[W_i f_j]_{1\le i,j\le k+1} \ne 0.$$

So far, we have dealt with Euclidean regular hypersurfaces in $\mathbb{H}^1$. If we want a more intrinsic result – still for 1-codimensional hypersurfaces in $\mathbb{H}^1$ – we have to deal with pieces of 1-codimensional $\mathbb{H}$-regular hypersurfaces that are sufficiently regular. The following result can be derived from Theorem 5.4 in [13].

Theorem 5.13. *Let $S \subset \mathbb{H}^1$ be a $\mathbb{H}$-regular $\mathbb{H}$-oriented hypersurface, and let $V \subset S$ be the closure of a relatively open subset V_0 of S. We assume that V is a topological 2-manifold with boundary ∂V that is a finite union of* *disjoint simple closed $\mathbf{C}^1$-piecewise horizontal curves. Then $\partial \llbracket V_0 \rrbracket$ is carried by ∂V and has finite mass.*

We stress that in the above theorem we do not assume any further non-intrinsic regularity on the surface, since we require V is but a *topological* 2-*manifold with boundary*. Indeed, any $\mathbb{H}$-regular hypersurface is a topological 2-manifold.

The proof of Theorem 5.4 in [13] relies on a somehow delicate approximation procedure. The same procedure is much easier when both the surface and its boundary belong to the class of low-codimensional $\mathbb{H}$-regular surfaces, and thus our previous remark concerning $(2n+1-k)$-currents carried by continuously differentiable $(2n+1-k)$-manifolds with boundary that locally take the form $\{f_1(p) = \cdots = f_k(p) = 0,\ f_{k+1}(p) \geq 0\}$ can be extended to the case when $f_1, \dots, f_{k+1}$ are $\mathcal{C}^1_{\mathbb{H}}$-functions. Indeed, let J_ε be an usual Friedrichs' mollifier; if we put $f_{i,\varepsilon} = f_i * J_\varepsilon$ for $i = 1, \dots, k+1$, then $f_{i,\varepsilon} \to f$ and $W_k f_{i,\varepsilon} \to f$ as $\varepsilon \to 0$ uniformly on compact sets for $k = 1, \dots, 2n$, as proved in [10], Theorem 6.5, Step 1.

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