Time-like lorentzian minimal submanifolds as singular limits of nonlinear wave equations

Giovanni Bellettini * Matteo Novaga † Giandomenico Orlandi ‡

Abstract

We consider the sharp interface limit $\epsilon \to 0^+$ of the semilinear wave equation $\Box \mathbf{u} + \nabla W(\mathbf{u})/\epsilon^2 = 0$ in $\mathbb{R}^{1+n}$, where $\mathbf{u}$ takes values in $\mathbb{R}^k$, $k = 1, 2$, and W is a double-well potential if $k = 1$ and vanishes on the unit circle and is positive elsewhere if $k = 2$. For fixed $\epsilon > 0$ we find some special solutions, constructed around minimal surfaces in $\mathbb{R}^n$. In the general case, under some additional assumptions, we show that the solutions converge to a Radon measure supported on a time-like k-codimensional minimal submanifold of the Minkowski space-time. This result holds also after the appearence of singularities, and enforces the observation made by J. Neu that this semilinear equation can be regarded as an approximation of the Born-Infeld equation.

1 Introduction

In this paper we consider the following system of semilinear hyperbolic equations

$$\Box \mathbf{u} + \frac{1}{\epsilon^2} \nabla W(\mathbf{u}) = 0\,, \qquad (1)$$

for

$$\mathbf{u} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^k, \qquad n \geq 1,\ k = 1, 2,$$

where $\Box \mathbf{u} = \mathbf{u}_{tt} - \Delta \mathbf{u} = \partial_{x^0 x^0} \mathbf{u} - \partial_{x^i x^i} \mathbf{u}$ is the wave operator in $\mathbb{R}^{1+n}$ with coordinates $x^0 = t, x^1, \ldots, x^n$, $\epsilon > 0$ is a small parameter, and $W(\mathbf{u}) = \widetilde{W}(|\mathbf{u}|)$, where $\widetilde{W} : \mathbb{R} \to \mathbb{R}^+$ is a double-well potential. Equation (1) is a Lorentz invariant field equation, governing the dynamics of topological defects such as vortices [9]; it is also strictly related to time-like lorentzian minimal submanifolds of codimension k in Minkowski $(1+n)$-dimensional space-time [10]. We refer to [12] for a discussion on the existence of local and global solutions to (1). The elliptic/parabolic analog of (1) is called the Ginzburg-Landau equation, and has been recently investigated by many authors in connection with euclidean minimal surfaces and mean curvature flow in codimension k (see for instance [2] and references therein). Here we are interested in the asymptotic limit as $\epsilon \to 0^+$ of solutions $\mathbf{u}_\epsilon$ to (1). The case $k = 1$ will be referred to as the scalar case, since (1) reduces to a single equation, and solutions will be denoted by u_ϵ; note that in this case, the vacuum states ± 1 are stable solutions.

For $n = 3$ and $k = 1$, the asymptotic limit as $\epsilon \to 0^+$ of u_ϵ has been formally computed by Neu in [10], using suitable asymptotic expansions. The author shows that there are solutions which take the constant values ± 1 out of a transition layer of thickness ϵ, provided such a layer is suitably close to a one-codimensional time-like lorentzian minimal surface Σ (called kink). The one-codimensional time-like minimal surface equation can be described as follows: the points $(x^0, x^1, \cdots, x^n)$ on each time-slice $\Sigma(t) := \Sigma \cap \{x^0 = t\}$ of Σ must satisfy the equation

$$A = (1 - V^2)\kappa \qquad (2)$$

*Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy, and Laboratori Nazionali di Frascati dell'INFN, via E. Fermi, 40, 00044 Frascati (Roma), Italy, Giovanni.Bellettini@lnf.infn.it

†Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy, novaga@math.unipd.it

‡Dipartimento di Informatica, Università di Verona, Ca' Vignal 2, strada le Grazie 15, 37134 Verona, Italy, giandomenico.orlandi@univr.it

in normal direction, where A, V and κ are respectively the acceleration, the velocity and the euclidean mean curvature of $\Sigma(t)$ at $(x^0, x^1, \cdots, x^n)$. We refer to [3], [8], [5] for the analysis of various aspects of Eq. (2). Interestingly, Neu [10] showed also that, due to possible oscillations on a small scale on the initial interface, which are not dissipated in time, solutions to (1) may not converge to a solution of (2), as the oscillation scale tends to zero.

In the first part of the present paper we compute some explicit selfsimilar solutions of (1). In particular, on the basis of the results of [11] we show that, given any euclidean nondegenerate minimal hypersurface M in $\mathbb{R}^n$, there exists a solution to (1) traveling around M (see Propositions 2.2 and 2.4).

In the second part of the paper we adapt to the hyperbolic setting the parabolic strategy followed in [1]. Given a solution $\mathbf{u}_\epsilon$ to (1) let

$$\ell_\epsilon(\mathbf{u}_\epsilon) := c_k(\epsilon)\left(\frac{-|\mathbf{u}_{\epsilon t}|^2 + |\nabla \mathbf{u}_\epsilon|^2}{2} + \frac{W(\mathbf{u}_\epsilon)}{\epsilon^2}\right)$$

be the rescaled lagrangian integrand, where

$$c_k(\epsilon) := \begin{cases} \epsilon & \text{if } k = 1, \\ \frac{\epsilon}{|\log \epsilon|} & \text{if } k = 2. \end{cases}$$

In our main result (Theorem 3.3) we show that, under some technical assumptions, $\ell_\epsilon(\mathbf{u}_\epsilon)$ concentrates on a k-codimensional set Γ, as $\epsilon \to 0^+$. Moreover, Γ is a time-like lorentzian minimal submanifold whenever it is smooth. In order to prove this result we suitably extend the notion of rectifiable varifold to the lorentzian setting, and prove that the stress-energy tensor of the solutions of (1) converges to a stationary lorentzian varifold, as $\epsilon \to 0^+$. Finally, we conclude the paper by discussing the validity of our assumptions in relation to the example of Neu [10].

1.1 Notation

Throughout the paper bold letters will refer to the case $k = 2$. The greek indices $\alpha, \beta, \gamma, \delta$ run from 0 to n, while the roman indices i, j run from 1 to n; we adopt the Einstein summation convention over repeated indices.

We let $\eta^{-1} = \mathrm{diag}(-1, 1, \ldots, 1)$ be the inverse Minkowski metric tensor with contravariant components $\eta^{\alpha\beta}$; $\eta_{\alpha\beta}$ are the covariant components of the Minkowski metric tensor η.

Given $\xi = (\xi_0, \hat{\xi}) \in \mathbb{R} \times \mathbb{R}^n$ we set $|\hat{\xi}|^2 := \eta^{ij}\hat{\xi}_i\hat{\xi}_j$,

$$\langle \xi, \xi \rangle_m := -(\xi_0)^2 + |\hat{\xi}|^2 = \eta^{\alpha\beta}\xi_\alpha\xi_\beta,$$

and if $\langle \xi, \xi \rangle_m \neq 0$ we set $|\xi|_m := \frac{\langle \xi, \xi \rangle_m}{|\langle \xi, \xi \rangle_m|^{1/2}}$. We say that ξ is space-like (resp. time-like) if $\langle \xi, \xi \rangle_m > 0$ (resp. $\langle \xi, \xi \rangle_m < 0$). Given a $(2,0)$-tensor A, we set $\mathrm{tr}_m(A) := \eta_{\alpha\beta}A^{\beta\alpha}$, while $\mathrm{tr}(A)$ is the euclidean trace of A. We say that A is space-like (resp. time-like) if $A\xi$ is space-like (resp. time-like) for all $\xi \in \mathbb{R} \times \mathbb{R}^n \setminus \{(0,0)\}$.

∇ (resp. $\overline{\nabla}$) indicates the euclidean gradient in $\mathbb{R}^n$ (resp. in $\mathbb{R}^{1+n}$); for a smooth function $g : \mathbb{R}^{1+n} \to \mathbb{R}$ we set $\nabla_m g := (-g_t, \nabla g) = \eta^{\alpha\beta}\frac{\partial g}{\partial x^\beta} = \eta^{-1}\overline{\nabla} g$.

$\mathcal{H}^h$ denotes the h-dimensional euclidean area (i.e. the Hausdorff measure) either in $\mathbb{R}^n$ or in $\mathbb{R}^{1+n}$ for $h \in \{0, \ldots, n\}$; $\llcorner$ is the symbol of restriction of measures and $\rightharpoonup$ denotes the weak* convergence of Radon measures. If μ is a measure absolutely continuous with respect to ν, we denote by $d\mu/\mathrm{d}\nu$ the Radon-Nikodym derivative of μ with respect to ν.

We recall that a smooth k-codimensional submanifold M of $\mathbb{R}^n$ without boundary is said minimal if M has vanishing mean curvature. A minimal submanifold $M \subset \mathbb{R}^n$ is said nondegenerate if the second variation of its $(n-k)$-dimensional area, represented by the associated Jacobi operator, is injective.

2 Selfsimilar solutions

Unless otherwise specified, in what follows we take $W(\mathbf{u}) = \frac{1}{4}(1 - |\mathbf{u}|^2)^2$ if $n \leq 4$, and if $n > 4$ we suppose W to be a function of $|\mathbf{u}|$ with the proper growth at infinity in order problem (1) to be well-posed [12].

We let

$$e_\epsilon(\mathbf{u}_\epsilon) := c_k(\epsilon)\left(\frac{|\mathbf{u}_{\epsilon t}|^2 + |\nabla \mathbf{u}_\epsilon|^2}{2} + \frac{W(\mathbf{u}_\epsilon)}{\epsilon^2}\right)$$

be the rescaled energy integrand of a solution $\mathbf{u}_\epsilon$ of (1). By $|\mathbf{u}_{\epsilon t}|^2$ (resp. $|\nabla \mathbf{u}_\epsilon|^2$) we mean

the square euclidean norm of $\mathbf{u}_{\epsilon t} \in \mathbb{R}^k$ (resp. of $\nabla \mathbf{u}_\epsilon$, i.e., the sum of the squares of the elements of the matrix $\nabla \mathbf{u}_\epsilon$).
We notice that the following quantity is conserved for any $t \geq 0$:

$$\int_{\mathbb{R}^n} e_\epsilon(\mathbf{u}_\epsilon(t,x))\ dx = \int_{\mathbb{R}^n} e_\epsilon(\mathbf{u}_\epsilon(0,x))\ dx, \tag{3}$$

assuming the proper growth conditions on the right hand side.

2.1 Traveling waves

Let $k = 1, 2$. We construct solutions of (1), which are traveling waves along a prescribed direction $\nu \in \mathbb{R}^n$, $|\nu| = 1$. Up to a rotation of $\mathbb{R}^n$, we can assume $\nu = (0, \ldots, 0, 1)$. Letting $x = (y, z) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, we look for traveling wave solutions of (1) of the form

$$\mathbf{u}_\epsilon(t,x) = \mathbf{v}(y, z - vt), \tag{4}$$

for some $v \in (-1,1)$ and a suitable map $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^k$. Then, (1) becomes

$$-\Delta_y \mathbf{v} - (1 - v^2)\mathbf{v}_{zz} + \frac{1}{\epsilon^2}\nabla W(\mathbf{v}) = 0, \tag{5}$$

where Δ_y is the Laplacian in $\mathbb{R}^{n-1}$ with respect to the $y = (y^1, \ldots, y^{n-1})$-coordinates. Let

$$\mathbf{f}(y,z) := \mathbf{v}(y, \sqrt{1-v^2}z). \tag{6}$$

Then $\mathbf{f}$ satisfies the elliptic Ginzburg-Landau system

$$-\Delta \mathbf{f} + \frac{1}{\epsilon^2}\nabla W(\mathbf{f}) = 0. \tag{7}$$

Hence traveling wave solutions of (1), with $v \in (-1, 1)$, correspond to solutions of the elliptic system (7).
We now recall the following result, which follows from [11].

Theorem 2.1. *For any smooth nondegenerate minimal submanifold $M \subset \mathbb{R}^n$ of codimension 1, there exist solutions f_ϵ of (7) such that*

$$\epsilon \left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2} \right) \rightharpoonup \sigma\, \mathcal{H}^{n-1} \llcorner M$$

as $\epsilon \to 0^+$, where $\sigma = \sigma(W, n)$ is a positive constant independent of M.

As a consequence our first result is the existence of traveling waves close to any nondegenerate minimal hypersurface of $\mathbb{R}^n$.

Proposition 2.2. *Let $k = 1$. Let $M \subset \mathbb{R}^n$ be a smooth nondegenerate minimal submanifold of codimension 1 without boundary, and let $v \in (-1, 1)$. Define*

$$\Sigma := \Big\{ \Big(t, y, \sqrt{1-v^2}z + vt\Big) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : (y,z) \in M \Big\}.$$

Then there exist traveling wave solutions $u_\epsilon : \mathbb{R}^{1+n} \to \mathbb{R}$ of (1) of the form (4) such that

$$\ell_\epsilon(u_\epsilon) \rightharpoonup \sigma\, \mathcal{H}^n \llcorner \Sigma \tag{8}$$

as $\epsilon \to 0^+$.

Proof. Set $\gamma := (1 - v^2)^{-1/2}$. If f_ϵ are as in Theorem 2.1, we define $u_\epsilon(t,x) := f_\epsilon(y, \gamma(z - vt))$. Then $\ell_\epsilon(u_\epsilon) = \epsilon \left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2} \right)$, hence if $\varphi \in C_c(\mathbb{R}^{1+n})$,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \ell_\epsilon(u_\epsilon)\varphi\ dxdt \\ &= \int_0^T \int_{\mathbb{R}^n} \epsilon \left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2} \right) \varphi\ dxdt \end{aligned} \tag{9}$$

where the integrand on the right hand side is evaluated at $(y, \gamma(z - vt))$. Therefore, making the change of variables $t' = t$, $y' = y$, $z' = \gamma(z - vt)$, and setting $x' = (y', z')$, we have that (9) equals

$$\begin{aligned} &\gamma^{-1} \int_0^T \int_{\mathbb{R}^n} \epsilon \left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2} \right) \varphi\ dx'dt' \\ &\to \sigma\gamma^{-1} \int_0^T \int_M \varphi\ d\mathcal{H}^{n-1}dt' \\ &= \sigma \int_\Sigma \varphi\ d\mathcal{H}^n, \end{aligned}$$

and (8) is proved. $\square$

Remark 2.3. Σ is a time-like lorentzian minimal hypersurface. Indeed, let $d : \mathbb{R}^n \to \mathbb{R}$ be the signed distance function from M, negative in the interior of M, so that $M = \{(y,z) \in \mathbb{R}^n : d(y,z) = 0\}$, $|\nabla d|^2 = 1$ in a neighbourhood of M, and $\Delta d = 0$ on M. Define $g : \mathbb{R}^{1+n} \to \mathbb{R}$ as $g(t,x) := d(y, \gamma(z-vt))$, $x = (y,z)$. Observe that $\Sigma = \{g = 0\}$, so that the Minkowskian mean curvature of Σ is given by the euclidean divergence in $\mathbb{R}^{1+n}$ of $\nabla_m g/|\nabla_m g|_m$, namely by

$$\Big(\frac{-g_t}{\sqrt{-(g_t)^2 + |\nabla g|^2}}\Big)_t + \Big(\frac{g_{x^i}}{\sqrt{-(g_t)^2 + |\nabla g|^2}}\Big)_{x^i}$$

evaluated on Σ. The equality $|\nabla d|^2 = 1$ implies $\sqrt{-(g_t)^2 + |\nabla g|^2} = 1$ in a neighbourhood of Σ. Therefore we only have to check that

$$-g_{tt} + g_{x^i x^i} = 0 \qquad \text{on } \Sigma, \tag{10}$$

which is verified because $-g_{tt} + g_{x^i x^i}$ on Σ coincides with Δd on M.

Note that $\ell_\epsilon(u_\epsilon)$ concentrates on Σ in the limit $\epsilon \to 0^+$; the same happens for $e_\epsilon(u_\epsilon)$, since $e_\epsilon(u_\epsilon)$, and $\ell_\epsilon(u_\epsilon)$ in Proposition 2.2 are mutually absolutely continuous.

2.2 Rotating waves

In this section we let $W(\mathbf{u}) = (1 - |\mathbf{u}|^2)^2/4$, $\widetilde{W} : \mathbb{R} \to \mathbb{R}$ be defined as $\widetilde{W}(s) := (1 - s^2)^2/4$, and let $k = 2$; we identify the target space $\mathbb{R}^2$ with the complex plane. We look for solutions of (1) of the form

$$\mathbf{u}_\epsilon(t,x) = \rho(x) e^{i\omega t}, \qquad \rho : \mathbb{R}^n \to \mathbb{R}, \tag{11}$$

for some $\omega \in \mathbb{R}$. Substituting (11) into (1), we get that ρ must satisfy

$$-\Delta\rho - \omega^2 \rho + \frac{1}{\epsilon^2}\widetilde{W}'(\rho) = 0. \tag{12}$$

This scalar equation can be rewritten as

$$-\Delta\rho + \frac{1}{\epsilon^2}\widetilde{W}_\epsilon'(\rho) = 0, \tag{13}$$

where

$$\begin{aligned}\widetilde{W}_\epsilon(\rho) &:= \frac{(1 + \epsilon^2\omega^2 - \rho^2)^2}{4} \\ &= (1 + \epsilon^2\omega^2)^2\, \widetilde{W}\left(\frac{\rho}{\sqrt{1+\epsilon^2\omega^2}}\right).\end{aligned}$$

Therefore (13) reduces to (7) with $k = 1$ and W replaced by $\widetilde{W}$, after the change of variables

$$f(x) = \frac{\rho\left(\frac{x}{\sqrt{1+\epsilon^2\omega^2}}\right)}{\sqrt{1+\epsilon^2\omega^2}},$$

and we can still apply Theorem 2.1. In particular, we get the following

Proposition 2.4. *Let $M \subset \mathbb{R}^n$ be a smooth nondegenerate minimal submanifold of codimension 1, and let $\omega \in \mathbb{R}$. Define*

$$\Sigma := \mathbb{R} \times M.$$

Then there exist solutions $\mathbf{u}_\epsilon : \mathbb{R}^{1+n} \to \mathbb{R}^2$ of (1) of the form (11) such that

$$\epsilon\left(\frac{-|\mathbf{u}_{\epsilon t}|^2 + |\nabla \mathbf{u}_\epsilon|^2}{2}\right) + \frac{W(\mathbf{u}_\epsilon)}{\epsilon} \rightharpoonup \sigma \mathcal{H}^n \llcorner \Sigma$$

as $\epsilon \to 0^+$.

Proof. If φ is a test function, we have

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} \frac{\epsilon}{c_2(\epsilon)} e_\epsilon(\mathbf{u}_\epsilon)\varphi\, dxdt \\ &= \int_0^T \int_{\mathbb{R}^n} \Big[\epsilon \frac{|\nabla\rho|^2}{2} \\ &\quad + \frac{1}{\epsilon}\left(\widetilde{W}(\rho) + \epsilon^2 \frac{\rho^2\omega^2}{2}\right)\Big]\varphi\, dxdt \\ &\to \sigma \int_0^T \int_M \varphi\, d\mathcal{H}^{n-1} dt.\end{aligned}$$

□

Note that in Proposition 2.4 $\frac{\epsilon}{c_2(\epsilon)} e_\epsilon(\mathbf{u}_\epsilon)$ concentrates on the lorentzian minimal submanifold Σ of codimension 1, even if $k = 2$.

3 Convergence as $\epsilon \to 0^+$

We are interested in passing to the limit in (1), as $\epsilon \to 0^+$. As already mentioned in the introduction, a formal limit has been performed in [10]. Rigorous asymptotic results for well prepared initial data have been recently announced in [7].

From now on we shall assume that

(A1) there exists a constant $C > 0$ such that

$$\sup_{\epsilon \in (0,1)} \int_{\mathbb{R}^n} e_\epsilon(\mathbf{u}_\epsilon(0,x))\, dx \le C.$$

3.1 Assumptions on ℓ and e

Under assumption (A1), using (3) it follows that the measures $e_\epsilon(\mathbf{u}_\epsilon)\ dtdx$ converge, up to a (not relabelled) subsequence; namely

$$e_\epsilon(\mathbf{u}_\epsilon)\ dtdx \rightharpoonup e,$$

where e is a measure in $\mathbb{R}^{1+n}$. Since $|\ell_\epsilon(\mathbf{u}_\epsilon)|$ and $c_k(\epsilon)W(\mathbf{u}_\epsilon)/\epsilon^2$ are both bounded by $e_\epsilon(\mathbf{u}_\epsilon)$, they converge, up to a subsequence, to two measures ℓ and w respectively,

$$\ell_\epsilon(\mathbf{u}_\epsilon)\ dtdx \rightharpoonup \ell, \tag{14}$$
$$c_k(\epsilon)W(\mathbf{u}_\epsilon)/\epsilon^2\ dtdx \rightharpoonup w, \tag{15}$$

and ℓ and w are absolutely continuous with respect to e, with density less than or equal to 1. In the following, we shall also assume that

(A2) e is absolutely continuous with respect to ℓ.

and, as in [1], that

(A3) there exists an absolute constant $c > 0$ such that

$$c \leq \lim_{\rho \to 0^+} \frac{\ell(B_\rho(t,x))}{\omega_{n+1-k}\rho^{n+1-k}} < +\infty$$

for ℓ-almost every (t,x), where $B_\rho(t,x)$ denotes the euclidean ball of radius ρ centered at (t,x) and $\omega_{n+1-k} := \mathcal{H}^{n+1-k}(B_1(0,0))$. From Preiss' Theorem [4] it follows that the support of the measures e and ℓ,

$$\Gamma := \mathrm{spt}(e) = \mathrm{spt}(\ell), \tag{16}$$

is a rectifiable set of dimension $n+1-k$, and

$$\ell \geq c\,\mathcal{H}^{n+1-k} \llcorner \Gamma$$

in the sense of measures.

3.2 Lorentzian rectifiable varifolds

A matrix P represents a lorentzian orthogonal projection on a time-like subspace of codimension k of $\mathbb{R}^{1+n}$ if there exists a Lorentz transformation L such that

$$L^{-1}\,P\,L = \begin{cases} \mathrm{diag}(1,0,1,\ldots,1) & \text{if } k=1, \\ \mathrm{diag}(1,0,0,1,\ldots,1) & \text{if } k=2. \end{cases}$$

The pair of Radon measures $V = (\mu_V, \delta_P)$ is a rectifiable time-like lorentzian varifold of codimension k if $\mathrm{spt}(\mu_V) \subset \mathbb{R}^{1+n}$ is an $(n+1-k)$-rectifiable set whose tangent space is time-like $\mathcal{H}^{n+1-k}$-almost everywhere, and δ_P is the Dirac delta concentrated on P, where P is the lorentzian orthogonal projection onto the tangent space to $\mathrm{spt}(\mu_V)$.

Definition 3.1. *We say that the rectifiable lorentzian varifold $V = (\mu_V, \delta_P)$ is stationary if*

$$\int_{\mathbb{R}^{1+n}} \mathrm{tr}\left(\eta^{-1} P\ \overline{\nabla}\mathbf{X}\right)\, d\mu_V = 0 \tag{17}$$

for all $\mathbf{X} \in (C^1_c(\mathbb{R}^{1+n}))^{n+1}$.

Notice that (17) is equivalent to require that the generalized varifold $(\mu_V, \delta_{\eta^{-1}P})$ is stationary in the sense of [1, Def. 3.4].

Remark 3.2. When $\mathrm{spt}(\mu_V)$ is smooth, a direct computation [13] shows that condition (17) implies that $\mathrm{spt}(\mu_V)$ is a time-like minimal submanifold of codimension k, and $\mu_V = \theta\mathcal{H}^{n+1-k} \llcorner \Gamma$, for some constant $\theta > 0$.

3.3 The stress-energy tensor

We let

$$\begin{aligned} T^{\alpha\beta}_\epsilon(\mathbf{u}) \quad := \quad & -c_k(\epsilon)\eta^{\alpha\gamma}\partial_{x^\gamma}\mathbf{u} \cdot \eta^{\beta\delta}\partial_{x^\delta}\mathbf{u} \\ & +\ell_\epsilon(\mathbf{u})\,\eta^{\alpha\beta} \end{aligned}$$

be the contravariant components of the symmetric stress-energy tensor, where $\cdot$ is the euclidean scalar product in $\mathbb{R}^k$. Notice that

$$|T^{\alpha\beta}_\epsilon(\mathbf{u})| \leq e_\epsilon(\mathbf{u}), \tag{18}$$

for any $\alpha, \beta \in \{0,\ldots,n\}$. A direct computation shows that a solution $\mathbf{u}_\epsilon$ of (1) satisfies

$$\partial_{x^\beta} T^{\alpha\beta}_\epsilon(\mathbf{u}_\epsilon) = 0. \tag{19}$$

As a consequence, for all $X \in C^1_c(\mathbb{R}^{1+n})$ we have

$$\int_{\mathbb{R}^{1+n}} T^{\alpha\beta}_\epsilon(\mathbf{u}_\epsilon)\,\partial_{x^\beta} X\ dtdx = 0. \tag{20}$$

Since $|T^{\alpha\beta}_\epsilon(\mathbf{u}_\epsilon)|$ is bounded by $e_\epsilon(\mathbf{u}_\epsilon)$, for any $\alpha, \beta \in \{0,\ldots,n\}$ there exists a measure $T^{\alpha\beta}$ such that

$$T^{\alpha\beta}_\epsilon(\mathbf{u}_\epsilon) dtdx \rightharpoonup T^{\alpha\beta} \tag{21}$$

as $\epsilon \to 0^+$. We denote by T the measure-valued tensor with components $T^{\alpha\beta}$. Note that $T^{\alpha\beta} = T^{\beta\alpha}$ and $\mathrm{spt}(T) = \Gamma$.
From (18) it follows that $T^{\alpha\beta}$ on the right hand side of (21) is absolutely continuous with respect to e; in turn, by (A2), e is absolutely continuous with respect to ℓ, and therefore $T^{\alpha\beta}$ is absolutely continuous with respect to ℓ. We denote by $\widetilde{T}^{\alpha\beta}$ the density of the measure $T^{\alpha\beta}$ with respect to the measure ℓ, i.e.,

$$\widetilde{T}^{\alpha\beta} := \frac{dT^{\alpha\beta}}{d\ell}, \tag{22}$$

and by $\widetilde{T}$ the tensor with components $\widetilde{T}^{\alpha\beta}$.
In addition to (A3), we shall also assume that

(A4) for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$ the difference $\widetilde{T}(x) - \eta^{-1}$ is space-like.

Recalling the expression of $T_\epsilon(\mathbf{u}_\epsilon) - \ell_\epsilon(\mathbf{u}_\epsilon)\eta^{-1}$, we observe that (A4) is reminiscent to require that $\overline{\nabla}\mathbf{u}_\epsilon$ becomes space-like near Γ in the limit $\epsilon \to 0^+$, and that Γ is time-like $\mathcal{H}^{n+1-k}$-almost everywhere. This is for instance consistent with the explicit solution corresponding to a singular pulsating spherical kink considered in [10] (see also [7]).
Finally, in the case $k = 1$, we shall suppose

(A5) $\dfrac{dw}{dl} = \dfrac{1}{2}$.

This assumption corresponds to the so-called *equipartition of energy* (see [6] for a similar condition in the parabolic case).

3.4 Main result

We are now in a position to prove the main result of the paper.

Theorem 3.3. *Let ℓ, w and $\widetilde{T}$ be defined as in* (14), (15) *and* (22) *respectively. The following two statements hold.*

(i) Let $k = 1$. Assume (A1)-(A5). Then $(\ell, \delta_{\eta\widetilde{T}})$ is a stationary lorentzian rectifiable varifold of codimension one.

(ii) Let $k = 2$. Assume (A1)-(A4). Then $(\ell, \delta_{\eta\widetilde{T}})$ is a stationary lorentzian rectifiable varifold of codimension two.

Therefore, in the two cases, the set Γ defined in (16) *is a time-like minimal submanifold of codimension k in the regions where it is smooth.*

Proof. Passing to the limit in the linear condition (20) we have

$$\int_{\mathbb{R}^{1+n}} \partial_{x^\beta} X \, dT^{\alpha\beta} = 0, \tag{23}$$

for any $\alpha \in \{0, \ldots, n\}$. Note that (23) is (the generic component of) the stationarity condition for the lorentzian varifold $(\ell, \delta_{\eta\widetilde{T}})$. Therefore, it is enough to prove that for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$ the matrix $\eta\widetilde{T}(x)$ is the lorentzian orthogonal projection onto the tangent space to Γ at x.
By a rescaling argument around $\mathcal{H}^{n+1-k}$-almost every point $x \in \Gamma$ as in [1, eq. (3.6)], from (23) we obtain

$$\widetilde{T}(x) \int_{\mathbb{R}^{1+n}} \overline{\nabla}\phi \, d\nu = 0, \tag{24}$$

for all test functions ϕ supported in the euclidean unit ball of $\mathbb{R}^{1+n}$, and for all ν in the tangent space to ℓ at x. As in [1, Lemma 3.9], from (24) it follows that for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$

$$\text{at least } k \text{ eigenvalues of } \eta\widetilde{T}(x) \text{ are zero}. \tag{25}$$

These eigenvalues correspond to the directions in the normal space to Γ at x.
From the equalities

$$\mathrm{tr}_m(\eta^{\alpha\gamma}\partial_{x^\gamma}\mathbf{u}_\epsilon \cdot \eta^{\beta\delta}\partial_{x^\delta}\mathbf{u}_\epsilon) = -|\mathbf{u}_{\epsilon t}|^2 + |\nabla\mathbf{u}_\epsilon|^2$$

and

$$c_k(\epsilon)(|\mathbf{u}_{\epsilon t}|^2 - |\nabla\mathbf{u}_\epsilon|^2) = 2\frac{c_k(\epsilon)W(\mathbf{u}_\epsilon)}{\epsilon^2} - 2\ell_\epsilon(\mathbf{u}_\epsilon),$$

we obtain

$$\mathrm{tr}_m(T_\epsilon(\mathbf{u}_\epsilon)) = 2\frac{c_k(\epsilon)W(\mathbf{u}_\epsilon)}{\epsilon^2} + (n-1)\ell_\epsilon(\mathbf{u}_\epsilon). \tag{26}$$

Passing to the limit as $\epsilon \to 0^+$ we get

$$\mathrm{tr}_m(T) = 2w + (n-1)\ell \tag{27}$$

in the sense of measures. Considering the density with respect to ℓ we get

$$\mathrm{tr}_m(\widetilde{T}) = 2\frac{dw}{d\ell} + (n-1). \tag{28}$$

Thanks to assumption (A4), for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$ the tensor $\widetilde{T}(x) - \eta^{-1}$, hence also $\eta\widetilde{T}(x) - \mathrm{Id}$, is space-like. Therefore, letting $\lambda_0, \lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\eta\widetilde{T}(x)$, there exists a Lorentz transformation $L(x)$ such that

$$\begin{aligned} & L^{-1}(x)(\eta\widetilde{T}(x) - \mathrm{Id})L(x) \\ = \;& L^{-1}(x)\,\eta\widetilde{T}(x)\,L(x) - \mathrm{Id} \\ = \;& \mathrm{diag}(0, \lambda_1 - 1, \ldots, \lambda_n - 1). \end{aligned}$$

In particular

$$\lambda_0 = 1.$$

Passing to the limit in the expression of $T_\epsilon(\mathbf{u}_\epsilon) - \ell_\epsilon(\mathbf{u}_\epsilon)\eta^{-1}$ as $\epsilon \to 0^+$, we get that $\widetilde{T} - \eta^{-1} = \eta^{-1}(\eta\widetilde{T} - \mathrm{Id})$ is negative semidefinite (in the euclidean sense), which implies

$$\lambda_i \leq 1 \qquad \forall i \in \{1, \ldots, n\}. \tag{29}$$

From (29) and (25) we then obtain

$$\mathrm{tr}_m(\widetilde{T}(x)) = \sum_{i=0}^{n} \lambda_i \leq n - k + 1\,. \tag{30}$$

Note that equality in (29) holds if and only if $\eta\widetilde{T}(x)$ is a lorentzian orthogonal projection on a time-like subspace of codimension k. Consequently, our aim is now to prove equality in (30).

Case (i): $k = 1$. Using (A5), (28) becomes $\mathrm{tr}_m(\widetilde{T}(x)) = n$.

Case (ii): $k = 2$. From (28) and (30) it follows

$$\frac{dw}{d\ell} \leq 0.$$

Since w is a positive measure, as well as ℓ (thanks to assumption (A3)), we deduce

$$\frac{\mathrm{d}w}{\mathrm{d}\ell} = 0.$$

Therefore, (28) becomes $\mathrm{tr}_m(\widetilde{T}(x)) = n - 1$. □

Remark 3.4. In [10] Neu shows by a formal asymptotic argument (made rigorous by Jerrard as announced in [7]) that the thesis of Theorem 3.3 holds, when $k = 1$, for well-prepared initial data and before the appearence of singularities. However, the small ripples example of Neu [10] suggests that the thesis of case (i) of Theorem 3.3 may not hold without assuming (A4)-(A5). Therefore, we expect that (A4)-(A5) are not necessarily satisfied for not well-prepared initial data.

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