STABILITY AND MEMORY EFFECTS IN A HOMOGENIZED MODEL GOVERNING THE ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES

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Abstract. We present a macroscopic model of electrical conduction in biological tissues. This model is derived via a homogenization limit by a microscopic formulation, based on Maxwell's equations, taking into account the periodic geometry of the microstructure. We also study the asymptotic behaviour of the model for large times. Our results imply that periodic boundary data lead to an asymptotically periodic solution. The model is relevant in applications like electric impedance tomography.

1. Introduction

In this paper we deal with a model of electrical conduction in composite media and, specifically, biological tissues. The classical governing equation is

$$-\operatorname{div}(\kappa\nabla u_t + \sigma\nabla u) = 0\,, \tag{1-1}$$

which is derived from the Maxwell equations in the quasi-stationary approximation (see e.g., [31]). Here, u is the electrical potential and κ, σ are the permittivity and the conductivity of the material, respectively. The geometry of the composite media we have in mind is a periodic array of the unit cell depicted in Figure 1. More precisely, we look at a phase E_1^η, modelling the cell cytosol, coated by a shell $\mathit{\Gamma}^\eta$, modelling the cell membrane, included in a phase E_2^η, modelling the extracellular fluid ([20]). In particular, the permittivity κ (respectively, the conductivity σ) in E_1^η and E_2^η is lower (respectively, greater) than in $\mathit{\Gamma}^\eta$. The diameter of the cell is of the order of tens of micrometers, while the width of the membrane is of the order of ten nanometers. This suggests that the thin shell $\mathit{\Gamma}^\eta$ could be preferably modelled as a two dimensional interface $\mathit{\Gamma}$, in order to get a simpler model, and, possibly, a better understanding of the effect of the geometric features of the microscopic structure. This simpler model can be obtained from equation (1-1) via a concentration-of-capacity procedure [5], leading to Problem (2-1)–(2-6), below. In particular, equation (2-3) takes into account the conductive/capacitive behaviour of the concentrated membrane. As shown in (2-3), the electric potential jumps across the interface $\mathit{\Gamma}$, and its jump satisfies a dynamical condition (roughly speaking, in the form of a hyperbolic differential equation on the interface itself).

Our model is designed to investigate the response of biological tissues to the injection of electrical currents in the radiofrequency range, that is the Maxwell–Wagner interfacial polarization effect [12], [20], at higher frequencies than those considered in [1, 3, 4, 5, 6]. This effect is relevant in clinical applications like electric impedance tomography and body composition [14, 16].

Problem (2-1)–(2-6) contains a small parameter ε, coinciding with the period of the microstructure. The typical structure of the periodic array we have in mind is given in Figure 2. Applications

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Key words and phrases. Homogenization; electrical conduction in biological tissues; dynamical boundary conditions; asymptotic decay.

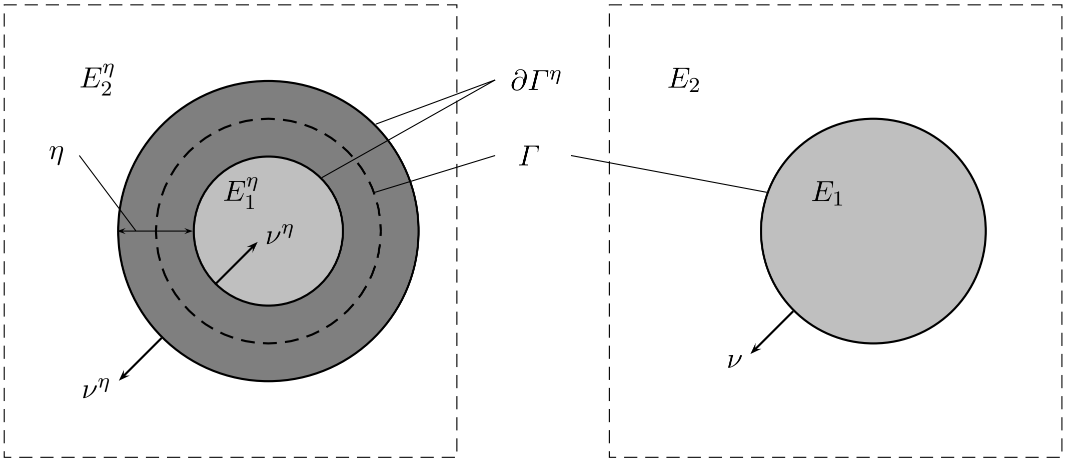

FIGURE 1. The periodic cell Y. *Left*: before concentration; Γ^η is the dark gray region, and $E^\eta = E_1^\eta \cup E_2^\eta$ is the union of the light gray and white regions. *Right*: after concentration; Γ^η shrinks to Γ as $\eta \to 0$.

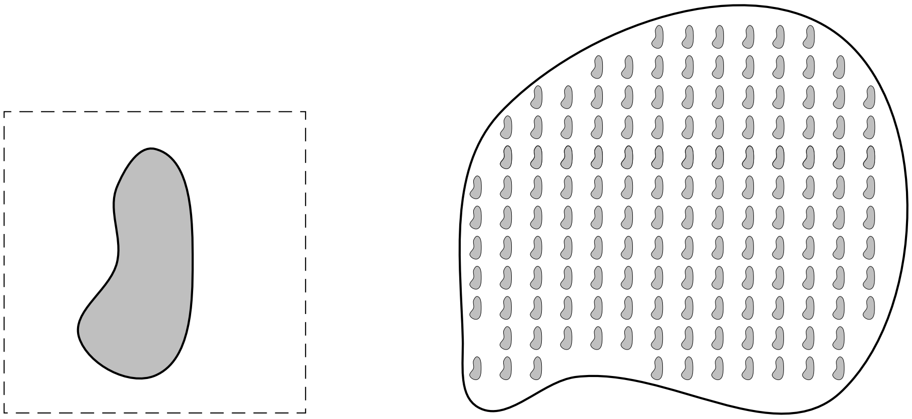

FIGURE 2. *Left*: an example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \Gamma$ in $\boldsymbol{R}^2$. Here E_1 is the light gray region and Γ is its boundary. The remaining part of Y (the white region) is E_2. *rRight*: the corresponding domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Here Ω_1^ε is the light gray region and Γ^ε is its boundary. The remaining part of Ω (the white region) is Ω_2^ε.

deal with measurements of the electric potential at the macroscopic (body) scale: this suggests to investigate the homogenization limit of Problem (2-1)–(2-6) when we let $\varepsilon \to 0$. Extensive surveys on this topic are, e.g., in [9, 10, 11, 13, 15, 24, 28, 32, 33, 34]. It turns out that the partial differential equation obtained in the limit is nonstandard (see (3-39) below). Indeed, it is an equation exhibiting memory effects, i.e., it contains explicitly the history of the unknown, hence it is markedly different from the Laplace equation presently used as a standard in the bioelectrical impedance literature [14].

Our model can be compared to some papers where homogenization theory is applied to linear stationary elliptic problems involving imperfect interfaces, arising in fields like elasticity [27] or heat conduction [29]. See also papers [17, 34], where hyperbolic problems with interfaces are considered in the framework of elastodynamics and electrodynamics.

In view of the applications, it is also of interest to study the evolution in time of the homogenized potential (see Section 2C). In particular, it is of interest to show that a time-harmonic boundary data elicits a time-harmonic solution for large times. In this regard, reasoning as in [6], it is enough to prove that the solution u_0 of (3-39) exponentially decays to zero as time increases, provided that a zero Dirichlet boundary condition is assigned (see Theorem 2.1 and Corollary 2.2).

From a mathematical point of view, the asymptotic behaviour of evolutive equations with memory is a classical problem [19, 35, 18, 26], currently drawing much interest in the literature [21, 25, 22, 30, 8]. We note that the exponential decay of the memory kernel, in general, does not imply the existence of bounded solutions, as shown by a counterexample presented in Section 5 (see, also, [19, 18]).

We finally note that our methods could be easily applied to study the homogenization problem and the time-asymptotic behaviour of Kelvin-Voigt viscoelastic composites with coated inclusions.

2. Position of the problem and main results

We look at the homogenization limit ($\varepsilon \searrow 0$) of the following problem for $u_\varepsilon(x,t)$:

$$
\begin{aligned}
-\operatorname{div}(\kappa\nabla u_{\varepsilon t} + \sigma\nabla u_\varepsilon) &= 0\,, && \text{in } (\varOmega_1^\varepsilon \cup \varOmega_2^\varepsilon) \times (0,+\infty); && \text{(2-1)}\\
[(\kappa\nabla u_{\varepsilon t} + \sigma\nabla u_\varepsilon)\cdot\nu] &= 0\,, && \text{on } \varGamma^\varepsilon \times (0,+\infty); && \text{(2-2)}\\
\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + \frac{\beta}{\varepsilon}[u_\varepsilon] &= ((\kappa\nabla u_{\varepsilon t} + \sigma\nabla u_\varepsilon)\cdot\nu)^{(2)}\,, && \text{on } \varGamma^\varepsilon \times (0,+\infty); && \text{(2-3)}\\
u_\varepsilon(x,t) &= 0\,, && \text{on } \partial\varOmega \times (0,+\infty); && \text{(2-4)}\\
\nabla u_\varepsilon(x,0) &= \boldsymbol{G}_\varepsilon(x)\,, && \text{in } \varOmega_1^\varepsilon \cup \varOmega_2^\varepsilon; && \text{(2-5)}\\
[u_\varepsilon](x,0) &= S_\varepsilon(x)\,, && \text{on } \varGamma^\varepsilon. && \text{(2-6)}
\end{aligned}
$$

The operators div and ∇ act with respect to the space variable x; $\varOmega = \varOmega_1^\varepsilon \cup \varOmega_2^\varepsilon \cup \varGamma^\varepsilon$, where $\varOmega_1^\varepsilon$ and $\varOmega_2^\varepsilon$ are two disjoint open subsets of $\varOmega$, and $\varGamma^\varepsilon = \partial\varOmega_1^\varepsilon \cap \varOmega = \partial\varOmega_2^\varepsilon \cap \varOmega$; ν is the normal unit vector pointing into $\varOmega_2^\varepsilon$; the typical geometry we have in mind is depicted in Figure 2. We refer to Section 2A for a precise definition of the structure of $\varOmega_1^\varepsilon$, $\varOmega_2^\varepsilon$, $\varGamma^\varepsilon$.

Moreover, we assume that:

$$
\begin{aligned}
\alpha > 0\,; \qquad \beta \geq 0\,; \qquad &\kappa = \kappa_1 > 0\,,\ \sigma = \sigma_1 > 0 \quad \text{in } \varOmega_1^\varepsilon;\\
&\kappa = \kappa_2 > 0\,,\ \sigma = \sigma_2 > 0 \quad \text{in } \varOmega_2^\varepsilon;
\end{aligned}
\qquad \text{(2-7)}
$$

where κ_1, κ_2, σ_1, σ_2, α and β are constants. From a physical point of view, $\varGamma^\varepsilon$ represents the cell membranes, having capacitance α/ε and conductance β/ε per unit area, whereas $\varOmega_1^\varepsilon$ (resp., $\varOmega_2^\varepsilon$) is the intracellular (resp., extracellular) space, having permittivity κ_1 (resp., κ_2) and conductivity σ_1 (resp., σ_2).

Since u_ε is not in general continuous across $\varGamma^\varepsilon$ we have set

$$
u_\varepsilon^{(2)} := \text{ trace of } u_{\varepsilon|\varOmega_2^\varepsilon} \text{ on } \varGamma^\varepsilon\,, \quad u_\varepsilon^{(1)} := \text{ trace of } u_{\varepsilon|\varOmega_1^\varepsilon} \text{ on } \varGamma^\varepsilon\,, \quad \text{and} \quad [u_\varepsilon] := u_\varepsilon^{(2)} - u_\varepsilon^{(1)}\,.
$$

A similar convention is employed for the current flux density across the membrane $(\kappa\nabla u_{\varepsilon t}+\sigma\nabla u_\varepsilon)\cdot\nu$.

We assume that the restrictions of $\boldsymbol{G}_\varepsilon$ to Ω_1^ε and Ω_2^ε are gradients of scalar fields and $\boldsymbol{G}_\varepsilon$ strongly converges in L^2. Moreover, we assume that $S_\varepsilon\in H^1(\Omega)$ and $S_\varepsilon/\varepsilon$ strongly converges in L^2. These assumptions are introduced in order to rule out the appearance of an initial layer (see [7]). Further assumptions on $\boldsymbol{G}_\varepsilon$ and S_ε are introduced in Section 2B.

2A. **Geometry.** Following [3], we introduce a periodic open subset E of $\boldsymbol{R}^N$, so that $E+z=E$ for all $z\in\boldsymbol{Z}^N$. For all $\varepsilon>0$ we define $\Omega_1^\varepsilon=\Omega\cap\varepsilon E$, $\Omega_2^\varepsilon=\Omega\setminus\overline{\varepsilon E}$, $\Gamma^\varepsilon=\Omega\cap\partial(\varepsilon E)$. We assume that Ω, E have regular boundary, say of class C^∞ for the sake of simplicity. We also employ the notation $Y=(0,1)^N$, and $E_1=E\cap Y$, $E_2=Y\setminus\overline{E}$, $\Gamma=\partial E\cap\overline{Y}$. We stipulate that E_1 is a connected smooth subset of Y such that $\operatorname{dist}(\overline{E_1},\partial Y)>0$. Some generalizations may be possible, but we do not dwell on this point here. Finally, we assume that $\operatorname{dist}(\Gamma^\varepsilon,\partial\Omega)>\gamma\varepsilon$ for some constant $\gamma>0$ independent of ε, by dropping the inclusions contained in the cells $\varepsilon(Y+z)$, $z\in\boldsymbol{Z}^N$ which intersect $\partial\Omega$ (see Figure 2). For later usage, we introduce the set:

$$\boldsymbol{Z}^N_\varepsilon:=\{z\in\boldsymbol{Z}^N : \varepsilon(Y+z)\subseteq\Omega\}\,. \tag{2-8}$$

2B. **Energy estimate.** Multiply (2-1) by u_ε and integrate by parts. Using (2-2)–(2-6), we arrive, for all $t>0$, to the energy estimate

$$\begin{aligned}\int_\Omega\frac{\kappa}{2}|\nabla u_\varepsilon(x,t)|^2\,\mathrm{d}x+\int_0^t\!\!\int_\Omega\sigma|\nabla u_\varepsilon(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau+\frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon(x,t)]^2\,\mathrm{d}\sigma\\+\frac{\beta}{\varepsilon}\int_0^t\!\!\int_{\Gamma^\varepsilon}[u_\varepsilon(x,\tau)]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau=\int_\Omega\frac{\kappa}{2}|\boldsymbol{G}_\varepsilon(x)|^2\,\mathrm{d}x+\frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}S^2_\varepsilon(x)\,\mathrm{d}\sigma\,.\end{aligned} \tag{2-9}$$

We assume that

$$\int_\Omega\frac{\kappa}{2}|\boldsymbol{G}_\varepsilon(x)|^2\,\mathrm{d}x+\frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}S^2_\varepsilon(x)\,\mathrm{d}\sigma<\gamma\,, \tag{2-10}$$

for a constant γ independent of ε. In fact (2-9), coupled with the Poincaré's inequality (Lemma 4.1), is a main tool in the rigorous proof of convergence of u_ε to its limit. In particular, up to a subsequence, u_ε weakly converges in $L^2(\Omega\times(0,\overline{T}))$ as $\varepsilon\to0$ to a limit u_0, for every $\overline{T}>0$. The equation satisfied by u_0 will be formally derived via a homogenization procedure in Section 3.

2C. **Exponential decay.**

Theorem 2.1. *Let $\Omega_1^\varepsilon,\Omega_2^\varepsilon,\Gamma^\varepsilon$ be as before. Assume that* (2-7) *holds and the initial data $\boldsymbol{G}_\varepsilon$ are gradients of scalar fields and together with S_ε satisfy* (2-10). *Let u_ε be the solution of* (2-1)–(2-6). *Then*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C(\varepsilon+\mathrm{e}^{-\lambda t})\quad a.e.\ in\ (0,+\infty), \tag{2-11}$$

where C and λ are positive constants independent of ε. Moreover, if $\beta>0$, or else if S_ε has null mean average over each connected component of Γ^ε, it follows that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C\,\mathrm{e}^{-\lambda t}\quad a.e.\ in\ (0,+\infty). \tag{2-12}$$

This result easily yields the following exponential time-decay estimate for the limit u_0 under homogeneous Dirichlet boundary data:

Corollary 2.2. *Under the assumptions of Theorem 2.1, if $u_\varepsilon\to u_0$ weakly in $L^2(\Omega\times(0,\overline{T}))$ for every $\overline{T}>0$, then*

$$\|u_0(\cdot,t)\|_{L^2(\Omega)}\le C\,\mathrm{e}^{-\lambda t}\quad a.e.\ in\ (0,+\infty). \tag{2-13}$$

3. Formal homogenization

We summarize here, to establish the notation, some well known asymptotic expansions needed in the two-scale method (see, e.g., [11], [34]). Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$u_\varepsilon = u_\varepsilon(x, y, t) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \dots . \tag{3-1}$$

Note that u_0, u_1, u_2 are periodic in y, and u_1, u_2 are assumed to have zero integral average over Y. Recalling that

$$\operatorname{div} = \frac{1}{\varepsilon} \operatorname{div}_y + \operatorname{div}_x , \qquad \nabla = \frac{1}{\varepsilon}\nabla_y + \nabla_x , \tag{3-2}$$

we compute, e.g.,

$$\nabla u_\varepsilon = \frac{1}{\varepsilon}\nabla_y u_0 + \big(\nabla_x u_0 + \nabla_y u_1\big) + \varepsilon\big(\nabla_y u_2 + \nabla_x u_1\big) + \dots . \tag{3-3}$$

We also stipulate

$$\boldsymbol{G}_\varepsilon = \boldsymbol{G}_\varepsilon(x, y) = \boldsymbol{G}_0(x, y) + \varepsilon \boldsymbol{G}_1(x, y) + \varepsilon^2 \boldsymbol{G}_2(x, y) + \dots ; \tag{3-4}$$

$$S_\varepsilon = S_\varepsilon(x, y) = S_0(x, y) + \varepsilon S_1(x, y) + \varepsilon^2 S_2(x, y) + \dots , \tag{3-5}$$

where the restrictions of $\boldsymbol{G}_0(x, \cdot)$, $\boldsymbol{G}_1(x, \cdot)$, … to E_1 and E_2 are the gradient of scalar fields. According to equation (2-10), recalling that $|\varGamma^\varepsilon|_{N-1} \sim 1/\varepsilon$, we assume $S_0 \equiv 0$ in (3-5); moreover, according to the assumption on the strong convergence of $\boldsymbol{G}_\varepsilon$ and $S_\varepsilon/\varepsilon$, the functions $\boldsymbol{G}_0(x, y)$ and $S_1(x, y)$ do not depend on y, i.e. $\boldsymbol{G}_0(x, y) = \boldsymbol{G}_0(x)$ and $S_1(x, y) = S_1(x)$.

For the sake of brevity, we introduce the operator:

$$\mathcal{D} := \kappa\frac{\partial}{\partial t} + \sigma . \tag{3-6}$$

Applying (3-2)–(3-3) to Problem (2-1)–(2-6), one readily obtains by matching corresponding powers of ε, that u_0 solves,

$$\begin{aligned}
-\mathcal{D}\,\Delta_y\, u_0 &= 0 , && \text{in } (E_1 \cup E_2) \times (0, +\infty); && (3\text{-}7)\\
[\mathcal{D}\nabla_y u_0 \cdot \nu] &= 0 , && \text{on } \varGamma \times (0, +\infty); && (3\text{-}8)\\
\alpha\frac{\partial[u_0]}{\partial t} + \beta[u_0] &= (\mathcal{D}\nabla_y u_0 \cdot \nu)^{(2)} , && \text{on } \varGamma \times (0, +\infty). && (3\text{-}9)\\
\nabla_y u_{0|t=0} &= 0 , && \text{on } E_1 \cup E_2; && (3\text{-}10)\\
[u_0]_{|t=0} &= 0 , && \text{on } \varGamma. && (3\text{-}11)
\end{aligned}$$

Reasoning as in Section 2B we obtain an energy estimate for (3-7)–(3-11), which implies that $[u_0] = 0$ for all times, and

$$u_0 = u_0(x, t) .$$

Next we find for u_1:

$$\begin{aligned}
-\mathcal{D}\,\Delta_y\, u_1 &= 0 , && \text{in } (E_1 \cup E_2) \times (0, +\infty); && (3\text{-}12)\\
[\mathcal{D}(\nabla_y u_1 + \nabla_x u_0) \cdot \nu] &= 0 , && \text{on } \varGamma \times (0, +\infty); && (3\text{-}13)\\
\alpha\frac{\partial[u_1]}{\partial t} + \beta[u_1] &= (\mathcal{D}(\nabla_y u_1 + \nabla_x u_0) \cdot \nu)^{(2)} , && \text{on } \varGamma \times (0, +\infty). && (3\text{-}14)\\
\nabla_y u_{1|t=0} + \nabla_x u_{0|t=0} &= \boldsymbol{G}_0 , && \text{on } E_1 \cup E_2; && (3\text{-}15)\\
[u_1]_{|t=0} &= S_1 , && \text{on } \varGamma. && (3\text{-}16)
\end{aligned}$$

Since both u_0 and $\boldsymbol{G}_0$ do not depend on y, equation (3-15) implies $\nabla_y u_{1|t=0} = 0$ on $E_1 \cup E_2$.

In order to represent u_1 in a suitable way, let $\boldsymbol{g} \in L^2(E_1 \cup E_2)$ and $s \in L^2(\Gamma)$ be assigned, such that the restrictions of $\boldsymbol{g}$ to E_1 and E_2 are gradients of scalar fields, and consider the problem

$$-\mathcal{D}\,\Delta_y\, v = 0\,, \qquad \text{in } (E_1 \cup E_2) \times (0,+\infty); \tag{3-17}$$

$$[\mathcal{D}\nabla_y v \cdot \nu] = 0\,, \qquad \text{on } \Gamma \times (0,+\infty); \tag{3-18}$$

$$\alpha \frac{\partial [v]}{\partial t} + \beta [v] = (\mathcal{D}\nabla_y v \cdot \nu)^{(2)}\,, \quad \text{on } \Gamma \times (0,+\infty). \tag{3-19}$$

$$\nabla_y v_{|t=0} = \boldsymbol{g}\,, \qquad \text{on } E_1 \cup E_2; \tag{3-20}$$

$$[v]_{|t=0} = s\,, \qquad \text{on } \Gamma. \tag{3-21}$$

where v is a periodic function in Y, such that $\int_Y v(y,t)\,\mathrm{d}y = 0$. Define the transform $\mathcal{T}$ by

$$\mathcal{T}(\boldsymbol{g}, s)(y,t) = v(y,t)\,, \qquad y \in Y\,, t > 0\,.$$

Then, introduce the cell functions $\boldsymbol{\chi}^0 : Y \to \boldsymbol{R}^N$ and $\boldsymbol{\chi}^1 : Y \times (0,+\infty) \to \boldsymbol{R}^N$, whose components χ^0_h and $\chi^1_h(\cdot,t)$, $h = 1, \dots, N$, are required to be periodic functions with vanishing integral average over Y for $t \geq 0$. The function χ^0_h of the components of $\boldsymbol{\chi}^0$ satisfies

$$-\kappa\,\Delta_y\,\chi^0_h = 0\,, \qquad \text{in } E_1 \cup E_2; \tag{3-22}$$

$$[\kappa(\nabla_y \chi^0_h - \boldsymbol{e}_h) \cdot \nu] = 0\,, \qquad \text{on } \Gamma; \tag{3-23}$$

$$\alpha[\chi^0_h] = (\kappa(\nabla_y \chi^0_h - \boldsymbol{e}_h) \cdot \nu)^{(2)}\,, \qquad \text{on } \Gamma. \tag{3-24}$$

The initial value $\chi^1_h(\cdot,0)$ of the components of $\boldsymbol{\chi}^1$ satisfies

$$-\kappa\,\Delta_y\,\chi^1_h(\cdot,0) - \sigma\,\Delta_y\,\chi^0_h = 0\,, \qquad \text{in } E_1 \cup E_2; \tag{3-25}$$

$$[(\kappa\nabla_y \chi^1_h(\cdot,0) + \sigma(\nabla_y \chi^0_h - \boldsymbol{e}_h)) \cdot \nu] = 0\,, \qquad \text{on } \Gamma; \tag{3-26}$$

$$((\kappa\nabla_y \chi^1_h(\cdot,0) + \sigma(\nabla_y \chi^0_h - \boldsymbol{e}_h)) \cdot \nu)^{(2)} = \alpha[\chi^1_h(\cdot,0)] + \beta[\chi^0_h]\,, \qquad \text{on } \Gamma. \tag{3-27}$$

Finally, χ^1_h is defined for $t > 0$ by

$$\chi^1_h = \mathcal{T}\left(\nabla_y \chi^1_h(\cdot,0), [\chi^1_h(\cdot,0)]\right)\,. \tag{3-28}$$

Straightforward calculations show that u_1 may be written in the form

$$\begin{aligned} u_1(x,y,t) = {} & -\boldsymbol{\chi}^0(y) \cdot \nabla_x u_0(x,t) - \int_0^t \boldsymbol{\chi}^1(y,t-\tau) \cdot \nabla_x u_0(x,\tau)\,\mathrm{d}\tau \\ & + \mathcal{T}\big(\nabla_y(\boldsymbol{\chi}^0 \cdot \boldsymbol{G}_0(x)), S_1(x) + [\boldsymbol{\chi}^0] \cdot \boldsymbol{G}_0(x)\big)(y,t)\,, \end{aligned} \tag{3-29}$$

so that

$$\begin{aligned} \mathcal{D}u_1(x,y,t) = {} & -\kappa\boldsymbol{\chi}^0(y) \cdot \nabla_x u_{0t}(x,t) - (\kappa\boldsymbol{\chi}^1(y,0) + \sigma\boldsymbol{\chi}^0(y)) \cdot \nabla_x u_0(x,t) \\ & - \int_0^t (\mathcal{D}\boldsymbol{\chi}^1)(y,t-\tau) \cdot \nabla_x u_0(x,\tau)\,\mathrm{d}\tau \\ & + \mathcal{D}\mathcal{T}\big(\nabla_y(\boldsymbol{\chi}^0 \cdot \boldsymbol{G}_0(x)), S_1(x) + [\boldsymbol{\chi}^0] \cdot \boldsymbol{G}_0(x)\big)(y,t)\,. \end{aligned} \tag{3-30}$$

Next we find for u_2:

$$-\mathcal{D}\left(\Delta_y\, u_2 + 2\frac{\partial^2 u_1}{\partial x_j \partial y_j} + \Delta_x\, u_0\right) = 0\,, \qquad \text{in } (E_1 \cup E_2) \times (0, +\infty); \tag{3-31}$$

$$[\mathcal{D}(\nabla_y u_2 + \nabla_x u_1) \cdot \nu] = 0\,, \qquad \text{on } \mathit{\Gamma} \times (0, +\infty); \tag{3-32}$$

$$(\mathcal{D}(\nabla_y u_2 + \nabla_x u_1) \cdot \nu)^{(2)} = \alpha \frac{\partial [u_2]}{\partial t} + \beta [u_2]\,, \quad \text{on } \mathit{\Gamma} \times (0, +\infty). \tag{3-33}$$

$$\nabla_y u_{2|t=0} + \nabla_x u_{1|t=0} = \boldsymbol{G}_1\,, \qquad \text{on } E_1 \cup E_2; \tag{3-34}$$

$$[u_2]_{|t=0} = S_2\,, \qquad \text{on } \mathit{\Gamma}. \tag{3-35}$$

Let us find the solvability conditions for this problem. Integrating by parts the partial differential equations (3-31) solved by u_2, both in E_1 and in E_2, adding the two contributions, and using (3-32), we get

$$\left[\int_{E_1} + \int_{E_2}\right] \mathcal{D}\Big\{\, \Delta_x\, u_0(x, t) + 2\frac{\partial^2 u_1}{\partial x_j \partial y_j} \Big\} \,\mathrm{d}y = -\int_{\mathit{\Gamma}} [\mathcal{D}\nabla_x u_1 \cdot \nu] \,\mathrm{d}\sigma\,. \tag{3-36}$$

Thus we obtain

$$\left(\kappa_0 \frac{\partial}{\partial t} + \sigma_0\right) \Delta_x\, u_0 = 2\int_{\mathit{\Gamma}} [\mathcal{D}\nabla_x u_1 \cdot \nu] \,\mathrm{d}\sigma - \int_{\mathit{\Gamma}} [\mathcal{D}\nabla_x u_1 \cdot \nu] \,\mathrm{d}\sigma = \int_{\mathit{\Gamma}} [\mathcal{D}\nabla_x u_1 \cdot \nu] \,\mathrm{d}\sigma\,, \tag{3-37}$$

where

$$\kappa_0 = \kappa_1 |E_1| + \kappa_2 |E_2|\,; \quad \sigma_0 = \sigma_1 |E_1| + \sigma_2 |E_2|\,. \tag{3-38}$$

Then we substitute the representation (3-29) into equation (3-37) and, after simple algebra, obtain the homogenized equation for u_0 in $\mathit{\Omega} \times (0, +\infty)$ as

$$-\operatorname{div}\left(K\nabla_x u_{0t} + A\nabla_x u_0 + \int_0^t B(t - \tau)\nabla_x u_0(\cdot, \tau) \,\mathrm{d}\tau - \mathcal{F}\right) = 0\,, \tag{3-39}$$

where the matrices K, A, $B(t)$ and the vector $\mathcal{F}(x, t)$ are defined as follows:

$$K = \kappa_0 I + \int_{\mathit{\Gamma}} \nu \otimes [\kappa \boldsymbol{\chi}^0(y)] \,\mathrm{d}\sigma\,, \qquad A = \sigma_0 I + \int_{\mathit{\Gamma}} \nu \otimes [\kappa \boldsymbol{\chi}^1(y, 0) + \sigma \boldsymbol{\chi}^0(y)] \,\mathrm{d}\sigma\,,$$

$$B(t) = \int_{\mathit{\Gamma}} \nu \otimes [(\mathcal{D}\boldsymbol{\chi}^1)(y, t)] \,\mathrm{d}\sigma\,,$$

$$\mathcal{F}(x, t) = \int_{\mathit{\Gamma}} [\mathcal{D}\mathcal{T}\big(\nabla_y(\boldsymbol{\chi}^0 \cdot \boldsymbol{G}_0(x)), S_1(x) + [\boldsymbol{\chi}^0] \cdot \boldsymbol{G}_0(x)\big)(y, t)]\nu \,\mathrm{d}\sigma\,. \tag{3-40}$$

Equation (3-39) is complemented with the initial condition

$$\nabla_x u_{0|t=0} = \boldsymbol{G}_0\,, \qquad \text{on } \mathit{\Omega}. \tag{3-41}$$

Finally, integrating in time equation (3-39), changing the order in the double integral thus appearing and using (3-41), we obtain also the following formulation

$$-\operatorname{div}\left(K\nabla_x u_0 + \int_0^t \Big(A + \int_0^{t-s} B(\tau) \,\mathrm{d}\tau\Big)\nabla_x u_0(\cdot, s) \,\mathrm{d}s - K\boldsymbol{G}_0 - \int_0^t \mathcal{F}(\cdot, \tau) \,\mathrm{d}\tau\right) = 0\,, \tag{3-42}$$

which shows that the homogenized equation has exactly the form of an equation with memory of the type derived in [1, 3] and studied in [2].

4. TIME-EXPONENTIAL ASYMPTOTIC DECAY: PROOF OF THEOREM 2.1

The case $\beta>0$ is quite simple. We introduce the space

$$H^1_\varepsilon(\Omega):=\{v\in L^2(\Omega)\ :\ v_{|\Omega^\varepsilon_i}\in H^1(\Omega^\varepsilon_i),\ i=1,2;\ v=0\text{ on }\partial\Omega\}\,. \tag{4-1}$$

It turns out that, for all $v\in H^1_\varepsilon(\Omega)$,

$$\int_\Omega \sigma|\nabla v|^2\,\mathrm{d}x+\frac{\beta}{\varepsilon}\int_{\Gamma^\varepsilon}[v]^2\,\mathrm{d}\sigma\geq\lambda\left(\int_\Omega\frac{\kappa}{2}|\nabla v|^2\,\mathrm{d}x+\frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[v]^2\,\mathrm{d}\sigma\right), \tag{4-2}$$

for $\lambda=\min\{2\sigma_1/\kappa_1,2\sigma_2/\kappa_2,2\beta/\alpha\}$. Taking $v=u_\varepsilon(\cdot,t)$ in the previous estimate and using equations (2-9), (2-10) and the differential version of Gronwall's Lemma, we obtain:

$$\int_\Omega\frac{\kappa}{2}|\nabla u_\varepsilon(\cdot,t)|^2\,\mathrm{d}x+\frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon(\cdot,t)]^2\,\mathrm{d}\sigma\leq\gamma\,\mathrm{e}^{-\lambda t}\,,\quad\text{a.e. in }(0,+\infty), \tag{4-3}$$

and (2-12) follows from Poincaré's inequality (Lemma 4.1).

Now we consider the case $\beta=0$. We introduce the space $\widetilde{H}^{1/2}(\Gamma^\varepsilon)\subset H^{1/2}(\Gamma^\varepsilon)$ of the functions which have null average over each connected component of Γ^ε, i.e. on $\varepsilon(\Gamma+z)$, for each z belonging to the set $\boldsymbol{Z}^N_\varepsilon$ defined in (2-8). We decompose the initial datum $S_\varepsilon(x)$ in (2-6) as $S_\varepsilon(x)=\overline{S}_\varepsilon(x)+\widetilde{S}_\varepsilon(x)$, where

$$\begin{aligned}&\overline{S}_\varepsilon(x)=\fint_{\varepsilon(\Gamma+z)}S_\varepsilon\,\mathrm{d}\sigma=:C_{\varepsilon z}\qquad\text{on each }\varepsilon(\Gamma+z),\ z\in\boldsymbol{Z}^N_\varepsilon\,;\\&\widetilde{S}_\varepsilon(x)\in\widetilde{H}^{1/2}(\Gamma^\varepsilon)\,,\end{aligned} \tag{4-4}$$

and the initial datum $\boldsymbol{G}_\varepsilon(x)$ in (2-5) as $\boldsymbol{G}_\varepsilon(x)=\overline{\boldsymbol{G}}_\varepsilon(x)+\widetilde{\boldsymbol{G}}_\varepsilon(x)$, where $\overline{\boldsymbol{G}}_\varepsilon(x)=0$ and $\widetilde{\boldsymbol{G}}_\varepsilon(x)=\boldsymbol{G}_\varepsilon(x)$. Accordingly, the solution u_ε of Problem (2-1)–(2-6) is decomposed as $\overline{u}_\varepsilon+\widetilde{u}_\varepsilon$. Clearly,

$$\overline{u}_\varepsilon(x,t)=\begin{cases}0 & \text{for }(x,t)\in\Omega^\varepsilon_2\times(0,+\infty)\,,\\ -C_{\varepsilon z} & \text{for }(x,t)\in(\varepsilon(E_1+z))\times(0,+\infty),\ z\in\boldsymbol{Z}^N_\varepsilon\,.\end{cases} \tag{4-5}$$

Using the previous equation, we compute:

$$\int_\Omega|\overline{u}_\varepsilon|^2\,\mathrm{d}x=\sum_{z\in\boldsymbol{Z}^N_\varepsilon}\int_{\varepsilon(E_1+z)}|\overline{u}_\varepsilon|^2\,\mathrm{d}x=\varepsilon^N|E_1|\sum_{z\in\boldsymbol{Z}^N_\varepsilon}\left|\fint_{\varepsilon(\Gamma+z)}S_\varepsilon\,\mathrm{d}\sigma\right|^2\,. \tag{4-6}$$

On the other hand, by Hölder's inequality, we estimate:

$$\sum_{z\in\boldsymbol{Z}^N_\varepsilon}\left|\fint_{\varepsilon(\Gamma+z)}S_\varepsilon\,\mathrm{d}\sigma\right|^2\leq\frac{\gamma}{\varepsilon^{N-1}}\int_{\Gamma^\varepsilon}S^2_\varepsilon\,\mathrm{d}\sigma\,. \tag{4-7}$$

Hence, as a consequence of (2-10), it follows that

$$\|\overline{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\leq C\varepsilon\,, \tag{4-8}$$

where C is a constant independent of ε.

In order to obtain an estimate for $\widetilde{u}_\varepsilon$, we introduce the space

$$\widetilde{H}^1_\varepsilon(\Omega):=\{v\in H^1_\varepsilon(\Omega)\ :[v]\in\widetilde{H}^{1/2}(\Gamma^\varepsilon)\}\,, \tag{4-9}$$

and, using Lemma 4.2 and Remark 4.3 below, we compute, for every $v \in \widetilde{H}^1_\varepsilon(\Omega)$:

$$\int_\Omega \sigma |\nabla v|^2 \,\mathrm{d}x \geq \sum_{z \in \boldsymbol{Z}^N_\varepsilon} \int_{\varepsilon(Y+z)} \sigma |\nabla v|^2 \,\mathrm{d}x \geq \frac{\alpha \widetilde{\lambda}}{\varepsilon} \sum_{z \in \boldsymbol{Z}^N_\varepsilon} \int_{\varepsilon(\Gamma+z)} [v]^2 \,\mathrm{d}\sigma = \frac{\alpha \widetilde{\lambda}}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \,, \tag{4-10}$$

where $\widetilde{\lambda}$ is defined in (4-15) and is independent of ε. Hence,

$$\int_\Omega \sigma |\nabla v|^2 \,\mathrm{d}x \geq \lambda \left(\int_\Omega \frac{\kappa}{2} |\nabla v|^2 \,\mathrm{d}x + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \right), \tag{4-11}$$

for $\lambda = (\max\{\kappa_1/(2\sigma_1), \kappa_2/(2\sigma_2)\} + 1/(2\widetilde{\lambda}))^{-1}$.

On the other hand, reasoning as in Section 2B and using (4-4) and (2-10), we get that $\widetilde{u}_\varepsilon$ satisfies the following energy estimate:

$$\int_\Omega \frac{\kappa}{2} |\nabla \widetilde{u}_\varepsilon(x,t)|^2 \,\mathrm{d}x + \int_0^t \int_\Omega \sigma |\nabla \widetilde{u}_\varepsilon(x,\tau)|^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [\widetilde{u}_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma < \gamma \,. \tag{4-12}$$

Hence, by using (4-11) written for $\widetilde{u}_\varepsilon(\cdot,t)$ and the differential version of Gronwall's Lemma, we obtain:

$$\int_\Omega \frac{\kappa}{2} |\nabla \widetilde{u}_\varepsilon(\cdot,t)|^2 \,\mathrm{d}x + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [\widetilde{u}_\varepsilon(\cdot,t)]^2 \,\mathrm{d}\sigma \leq \gamma \,\mathrm{e}^{-\lambda t} \,, \quad \text{a.e. in } (0,+\infty), \tag{4-13}$$

and (2-11) follows from Poincaré's inequality (Lemma 4.1) and (4-8).

Lemma 4.1. *Poincaré's inequality.* [23, 3]. *Let v belong to the space $H^1_\varepsilon(\Omega)$ introduced in equation* (4-1)*. Then,*

$$\int_\Omega v^2 \,\mathrm{d}x \leq C \left\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \varepsilon^{-1} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \right\}. \tag{4-14}$$

Here C depends only on Ω and E.

Lemma 4.2. [6]. *Set $\widetilde{H}^1(Y) := \{v \in L^2(Y) \;:\; v_{|E_i} \in H^1(E_i),\ i=1,2,\ [v] \in \widetilde{H}^{1/2}(\Gamma)\}$, where $\widetilde{H}^{1/2}(\Gamma)$ is comprised by the functions of $H^{1/2}(\Gamma)$ with null integral average. Then,*

$$\widetilde{\lambda} := \min_{v \in \widetilde{H}^1(Y),\ [v] \not\equiv 0} \frac{\displaystyle\int_Y \sigma |\nabla v|^2 \,\mathrm{d}y}{\alpha \displaystyle\int_\Gamma [v]^2 \,\mathrm{d}\sigma} > 0 \,. \tag{4-15}$$

Remark 4.3. [6]. *The change of variables $y = x/\varepsilon$ applied to equation* (4-15) *yields:*

$$\min_{v \in \widetilde{H}^1(\varepsilon Y),\ [v] \not\equiv 0} \frac{\displaystyle\int_{\varepsilon Y} \sigma |\nabla v|^2 \,\mathrm{d}x}{\dfrac{\alpha}{\varepsilon} \displaystyle\int_{\varepsilon \Gamma} [v]^2 \,\mathrm{d}\sigma} = \widetilde{\lambda} > 0 \,, \tag{4-16}$$

where $\widetilde{H}^1(\varepsilon Y) := \{v \in L^2(\varepsilon Y) \;:\; v_{|\varepsilon E_i} \in H^1(\varepsilon E_i),\ i=1,2,\ [v] \in \widetilde{H}^{1/2}(\varepsilon\Gamma)\}$, $\widetilde{H}^{1/2}(\varepsilon\Gamma)$ is comprised by the functions of $H^{1/2}(\varepsilon\Gamma)$ with null integral average, and $\widetilde{\lambda}$ is the positive constant introduced in Lemma 4.2.

5. A COUNTEREXAMPLE

As pointed out in the Introduction, the structure of equation (3-39) is not enough to imply the exponential decay of the solution to zero or its boundedness, even if exponentially decaying memory kernel and source are considered. Indeed, let $\Omega = (-1,1)$, $\mu > 0$, $a > 0$, $b \in \boldsymbol{R}$, and $f(x), h(x)$ be smooth functions. Consider the problem

$$\begin{cases} -\Big(u_{0xt} + a u_{0x} + b \displaystyle\int_0^t \mathrm{e}^{-\mu(t-\tau)} u_{0x}(x,\tau)\,\mathrm{d}\tau + f(x)\mathrm{e}^{-\mu t} \Big)_x = 0\,, \\ u_0(\pm 1, 0) = 0\,, \\ u_{0x} = h(x)\,. \end{cases} \tag{5-1}$$

Multiplying the previous equation by $\mathrm{e}^{\mu t}$, we obtain

$$u_{0xxt}\mathrm{e}^{\mu t} + a u_{0xx}\mathrm{e}^{\mu t} + b \int_0^t \mathrm{e}^{\mu\tau} u_{0xx}(x,\tau)\,\mathrm{d}\tau = f'(x)\,. \tag{5-2}$$

Setting $v(x,t) = u_{0xx}\mathrm{e}^{\mu t}$ and differentiating with respect to t, equation (5-2) can be rewritten as

$$v_{tt} + (a-\mu)v_t + bv = 0\,,$$

which must be complemented with the initial conditions

$$\begin{cases} v(x,0) = h'(x)\,, \\ v_t(x,0) = f'(x) + (\mu - a)h'(x)\,. \end{cases}$$

This last equation has an explicit solution of the form, if $(\mu-a)^2 - 4b > 0$,

$$v(x,t) = C_1(x)\exp\big(\frac{\mu - a + \sqrt{(\mu-a)^2 - 4b}}{2}t\big) + C_2(x)\exp\big(\frac{\mu - a - \sqrt{(\mu-a)^2 - 4b}}{2}t\big)\,,$$

where $C_1(x)$ and $C_2(x)$ are easily determined by using the initial conditions, thus implying that

$$u_{0xx}(x,t) = C_1(x)\exp\big(\frac{-\mu - a + \sqrt{(\mu-a)^2 - 4b}}{2}t\big) + C_2(x)\exp\big(\frac{-\mu - a - \sqrt{(\mu-a)^2 - 4b}}{2}t\big)\,.$$

Hence, u_0 can be obtained by integrating twice with respect to x and using the previous mentioned boundary conditions.

Note that, in general, if b is negative and $-b > \mu a$, the first exponential tends to infinity as $t \to +\infty$. Taking into account that, except for a very particular choice of the initial data, C_1 is different from zero, we have that solutions to Problem (5-1) do not exponentially decay in time, in general.

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