Convergence of non-local finite element energies for fracture mechanics

LUCA LUSSARDI - MATTEO NEGRI

Dipartimento di Matematica - Università degli Studi di Pavia
Via Ferrata 1 - 27100 Pavia - Italy

e-mail: luca.lussardi@unipv.it - matteo.negri@unipv.it

Abstract. Usually *smeared crack* techniques are based on the following features: the fracture is represented by means of a band of finite elements and by a softening constitutive law of damage type. Often these methods are implemented with non-local operators which control the localization effects and reduce the mesh bias. We consider a non-local smeared crack energy defined for a finite element space on a structured grid. We characterize the limit energy as the mesh size h tends to zero and we establish a precise link between the discrete and continuum formulations of the fracture energies, showing the correct scaling and the explicit form of the mesh bias.

1 Introduction

It is well known that beyond the elastic limit most of the engineering materials exhibit dissipative phenomena, such as damage, plasticity and fracture, resulting with constitutive laws of softening type. Loss of coercivity (or ellipticity) is in some sense the main mathematical feature of these models. As a consequence, boundary value problems based on local formulations of the constitutive law are ill-posed in Sobolev spaces and their finite element simulations are usually affected by un-physical effects induced by the mesh.

In this work we take into account the class of elasto-brittle materials. In this case, after Griffith [1], the free energy has the form

$$\int_{\Omega\setminus K} W^e(\varepsilon(u))\,dx + \gamma\mathcal{H}^{n-1}(K)\,,$$

where $\Omega \subset \mathbf{R}^n$ is the reference configuration, u is the displacement field and K is the fracture; W^e is the density of linearized elastic energy, γ is the material toughness and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (roughly speaking, the length for $n = 2$ or the surface for $n = 3$). In particular we are interested in the finite element discretization of the free energy by means of the so-called *crack band* [2] and *smeared cracking* [3] methods, which are both based on a damage constitutive law. Our attention is focused on the behaviour of the discrete energies, introduced hereafter, as the mesh size $h \to 0$, with a special care for two crucial aspects: scaling and mesh bias. By contrast, for the moment we are not taking into account boundary value problems and crack tracking.

From the mathematical point of view a convenient framework for the study of fracture energies is provided by the spaces of functions with bounded variation and bounded deformation. For our purposes the natural choice is the space SBD (see §6 and [4] for details). In this contest the fracture set K is replaced by the set $J(u)$ of jump points of the displacement field $u : \Omega \to \mathbf{R}^n$; the Griffith energy is then written as

$$F(u) = \int_{\Omega \setminus J(u)} W^e(\boldsymbol{\varepsilon}(u))\, dx + \gamma \mathcal{H}^{n-1}(J(u))\,.$$

Now, let us start with a *crack band* approach [2]. Let $\boldsymbol{T}_h$ be a triangulation of $\Omega \subset \mathbf{R}^2$ and let V_h denote the space of P_1 elements on $\boldsymbol{T}_h$. Moreover, let $w : [0, +\infty) \to [0, 1]$ be a continuous, non-decreasing function such that $w(0) = 0$ and $\lim_{s \to +\infty} w(s) = 1$. Let $f(s)$ be such that $f'(s) = 1 - w(s)$. Then, for u_h in the space V_h, we consider the free energy

$$F_h(u_h) = \int_\Omega \frac{1}{h} f\big(hW^e(\boldsymbol{\varepsilon}_h)\big)\, dx = \sum_{T_h \in \boldsymbol{T}_h} |T_h| \frac{1}{h} f\big(hW^e(\boldsymbol{\varepsilon}_h)\big)\,.$$

Note that this energy is of isotropic damage type; indeed, for every element T_h, differentiation of the energy density gives

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}_h}\Big(\frac{1}{h} f\big(hW^e(\boldsymbol{\varepsilon}_h)\big)\Big) = f'\big(hW^e(\boldsymbol{\varepsilon}_h)\big)\boldsymbol{\sigma}_h = \big(1 - w(hW^e(\boldsymbol{\varepsilon}_h))\big)\boldsymbol{\sigma}_h\,.$$

Note that here the damage parameter w depends on the energy density and on the mesh size h. This scaling property, as already pointed out in similar cases in [2] and [5], is fundamental to recover to right order of the fracture energy as $h \to 0$. More precisely, it is shown in [6] that (as $h \to 0$) the discrete energies F_h converge (in the sense of Γ-convergence [7]) to a continuum energy of the form

$$F_\phi(u) = \int_{\Omega \setminus J(u)} W^e(\boldsymbol{\varepsilon}(u))\, dx + \gamma \int_{J(u)} \phi(\nu)\, d\mathcal{H}^{n-1}\,,$$

where ν is the normal vector to $J(u)$ and ϕ is an anisotropic norm accounting for the mesh bias. As a matter of fact, an explicit representation of ϕ is possible only when the triangulations are periodic in space and self similar with respect to h. Note that the isotropic energy F is recovered as a special case of F_ϕ when ϕ is the Euclidean norm, i.e. $\phi(\nu) = |\nu| = 1$.

Since the anisotropy effect of the *crack band* is quite strong, it was proposed in [3] a non-local *smeared cracking* approach which effectively reduces this numerical artifact. In the original form [3] the (local) strain was replaced by a non-local strain obtained by a weighted average on a finite volume. Later, this approach was employed in a similar way taking the average of other quantities, such as the damage parameter, the energy density or the stress. Referring to [8] for a comparative study, here we take into account a special form of the non-local operator, which is quite convenient for our analysis. Let $\boldsymbol{Q}_h$ be a square mesh in $\Omega \subset \mathbf{R}^2$ and let V_h be a space of Q_1 elements on $\boldsymbol{Q}_h$ (or of P_1 elements on a triangulation $\boldsymbol{T}_h$ with the same vertices as $\boldsymbol{Q}_h$). For every $Q_h \in \boldsymbol{Q}_h$, denoting by $\boldsymbol{x}$ its center, let

$$W^e(\boldsymbol{\varepsilon}_h) * \rho_h(\boldsymbol{x}) = \int_{P_h(\boldsymbol{x})} W^e(\boldsymbol{\varepsilon}_h(y))\rho_h(y - \boldsymbol{x})\, dy\,,$$

where ρ_h is a symmetric convolution kernel $\rho_h(x) = \rho(x/h)/h^2$ with compact support $P_h = hP$. Accordingly, the non-local discrete energy will be

$$F_h(u_h) = \sum_{Q_h \in \boldsymbol{Q}_h} |Q_h| \frac{1}{d_h} f\left(hW^e(\boldsymbol{\varepsilon}_h) * \rho_h(\boldsymbol{x})\right),$$

where d_h is the diameter of P_h. As in the case of the *crack band*, we prove that as $h \to 0$ the discrete non-local energies F_h Γ-converge [7] to a continuum, local energy of the form F_ϕ, where ϕ can be explicitly computed and depends only on the support P of the convolution operator.

Some remarks are worth. In principle, similar results holds true for more complex energies which may take into account the differences between traction, compression, shear and between loading and unloading regimes. By contrast, such models would be considerably more difficult from the mathematical point of view and possibly will be the subject for future investigations. We remark also that in our convergence result the diameter d_h of the non-local operator behaves like hd, where d is the diameter of P, and thus $d_h \to 0$ as $h \to 0$. Hence, d_h is not really a "characteristic length" of the material since it depends on the mesh size h. From this point of view, as it does not introduce any internal length in the continuum model, we may regard our approach also as a *non-local crack band* method. It seems that our convergence result supplies also a mathematical base for the so-called "1/3 *rule*": indeed, as shown in §3 the anisotropy of the mesh is considerably reduced when the support P_h contains at least a band of 3 square elements. Clearly, when larger supports P are employed the effect of the mesh is further reduced while the original isotropic model is recovered, in the limit as $h \to 0$, if the diameter d_h satisfies $d_h \to 0$ and $d_h/h \to +\infty$, i.e. when the mesh size is much smaller than the diameter d_h. As a matter of fact this case has just a theoretical relevance since the number of elements Q_h in the support P_h increases as $h \to 0$ and definitively tends to infinity.

Apart from the mechanical applications, it is right to mention that analogous results, based on non-local operators, are of common use in the approximation of free discontinuity problems [9]. Most of the results in this direction originated in the field of image processing from the non-local approximation of the Mumford-Shah functional [10]. A sort of *crack band* model has been considered in [11] combined with directional mesh adaption and in [12] for structured triangulations while a non-local approach has been employed in [13] discretizing [10] with convolution operators such that $d_h \to 0$ and $d_h/h \to +\infty$. To our knowledge, the truly non-local case where $P_h = hP$ was not treated before; it follows as a corollary of our main result. Finally, from a technical point of view we remark that our proof relies strongly on the use of the slicing technique which seems convenient for linearized elastic energy density (see also [14]). By contrast, in the case of the Mumford-Shah functional it is often possible to use a simpler two-dimensional construction, which is not easily extended to our case.

2 Constitutive assumptions and convergence results

2.1 Notation and statement of the convergence results

For $\boldsymbol{x} \in \mathbf{Z}^2$ let $Q(\boldsymbol{x})$ be the square with center $\boldsymbol{x}$ and edge length 1. For $r \geq \sqrt{2}/2$ let B_r be the ball with radius r centered in the origin and let P be the union of the squares $Q(\boldsymbol{x})$ such that $\boldsymbol{x} \in \mathbf{Z}^2$ and $Q(\boldsymbol{x}) \subset B_r$ (see Figure 2). We denote these squares by $Q(\boldsymbol{x}_k)$, for $k = 1, \dots, m = |P|/|Q|$, and the diameter of P by d. Finally, we say that ρ is a (discrete) convolution kernel on B_r if

$$\rho = \sum_{k=1}^{m} q_k \chi_{Q(\boldsymbol{x}_k)} \quad \text{where} \quad q_k > 0 \quad \text{and} \quad \sum_{k=1}^{m} q_k = 1\,.$$

Clearly, ρ is constant on the squares $Q(\boldsymbol{x}_k)$ (for $k = 1, \dots, m$) and $\int_P \rho = \int_{B_r} \rho = 1$.

For a sequence $h_n \searrow 0$ let us denote by $Q_n(\boldsymbol{x})$ the squares with center $\boldsymbol{x} \in h_n\mathbf{Z}^2$ and edge length h_n. Clearly, the vertices of $Q_n(\boldsymbol{x})$ belong to the lattice

$h_n[\mathbf{Z}^2 + (1/2, 1/2)]$. Moreover, let $P_n = h_n P$ be the union of the squares $Q_n(\boldsymbol{x})$ contained in $B_{h_n r}$. We write again $P_n = \cup_{k=1}^m Q_n(\boldsymbol{x}_k)$ for $\boldsymbol{x}_k \in h_n\mathbf{Z}^2$ and we denote the diameter of P_n by d_n. Then let $\rho_n(x) = \rho(x/h_n)/h_n^2$. Clearly we can write $\rho_n = \sum_{k=1}^m (q_k/h_n^2)\chi_{Q_n(\boldsymbol{x}_k)}$.

Considering a structured mesh, we assume that the reference configuration is a rectangular domain Ω and that, for every $n \in \mathbf{N}$, Ω is the union of the squares $Q_n(\boldsymbol{x})$ with center $\boldsymbol{x} \in (\Omega \cap h_n\mathbf{Z}^2)$. We introduce for every $n \in \mathbf{N}$ a set of indexes I_n such that $\{\boldsymbol{x} \in (\Omega \cap h_n\mathbf{Z}^2)\} = \{x_i \,:\, i \in I_n\}$. Now, for $n \in \mathbf{N}$ let $\boldsymbol{T}_n$ be a (structured) triangulation of Ω with vertices on the lattice $h_n[\mathbf{Z}^2 + (1/2, 1/2)]$ (see Figure 1) and let $V_n = V_n(\Omega, \mathbf{R}^2)$ be the finite element space of continuous piecewise affine functions on $\boldsymbol{T}_n$ such that $\|u\|_\infty \leq k$, where k is a positive constant, arbitrarily large and fixed a priori. Similarly, we denote by V the set of deformations $u \in SBD^2(\Omega)$ with $\|u\|_\infty \leq k$. We remark that the constant k is not related to a physical constraints; indeed it is employed only for technical reasons related to symmetrized gradients and its value does not effect the convergence result.

Now, let us turn to the constitutive law. Let $w : [0, +\infty) \to [0, 1]$ be a non-decreasing continuous function such that $w(0) = 0$ and

$$\int_0^{+\infty} \big(1 - w(s)\big)\, ds < +\infty\,. \tag{1}$$

Note that the monotonicity of w and (1) imply that $\lim_{s\to+\infty} w(s) = 1$. Then, let $f : [0, +\infty) \to [0, +\infty)$ be such that $f(0) = 0$ and $f'(s) = 1 - w(s)$. Since $(1 - w)$ is non-negative and non-increasing, it turns out that f is non-decreasing and concave. Moreover

$$\begin{aligned} &\lim_{s\to 0^+} f(s)/s = 1 - w(0) = 1 \\ &\lim_{s\to+\infty} f(s) = f_\infty = \int_0^{+\infty} \big(1 - w(s)\big)\, ds < +\infty. \end{aligned} \tag{2}$$

A typical example is given by the exponentially increasing damage, i.e. by

$$w(s) = 1 - e^{-s} \quad \text{which gives} \quad f(s) = 1 - e^{-s}\,.$$

In some applications it is customary to consider also a damage parameter of the form

$$w(s) = \begin{cases} 0 & \text{for } s \leq s_e \\ (s - s_e)/(s_f - s_e) & \text{for } s_e < s < s_f \\ 1 & \text{for } s \geq s_f, \end{cases}$$

for $0 < s_e < s_f$, which gives

$$f(s) = \begin{cases} s & \text{for } s \le s_e \\ s - (s-s_e)^2/2(s_f - s_e) & \text{for } s_e < s < s_f \\ s_f - (s_f - s_e)/2 & \text{for } s \ge s_f, \end{cases}$$

and separates clearly the elastic ($s \le s_e$), damage ($s_e < s < s_f$) and fracture regimes ($s_f \le s$). Finally, denoting by $u : \Omega \to \mathbf{R}^2$ the displacement field and by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(u)$ its symmetrized gradient, let $W^e(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}\,\mathbf{C}\,\boldsymbol{\varepsilon}$ be the (density of) linearized elastic energy. Our main convergence result is summarized in the following Theorem.

Theorem 2.1 *Let f be a non-decreasing, concave function satisfying (2) and let ρ be a discrete convolution kernel. For $h_n \searrow 0$ let $f_n(t) = f(d_n t)/d_n$. Let $\{F_n\}$ be the sequence of discrete energies*

$$F_n(u) = \begin{cases} \sum_{i \in I_n} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u)) * \rho_n(\boldsymbol{x}_i)\Big) & \text{for } u \in V_n \\ +\infty & \text{for } u \in L^1 \setminus V_n. \end{cases} \tag{3}$$

Then F_n Γ-converges with respect to the strong topology of L^1 to a functional of the form

$$F(u) = \begin{cases} \int_\Omega W^e(\boldsymbol{\varepsilon}(u))\,dx + f_\infty \int_{J(u)} \phi(\nu)\,d\mathcal{H}^1 & \text{for } u \in V \\ +\infty & \text{for } u \in L^1 \setminus V. \end{cases} \tag{4}$$

The density ϕ which appears in the jump term is a norm in $\mathbf{R}^2$ which measures the mesh bias introduced by the approximation. It depends only on the support P of the convolution kernel ρ and it can be computed explicitly by the following formula (see §3 for further details)

$$\phi(\nu) = \max\left\{2\langle \nu, \xi\rangle/d \,:\, \text{for } \xi \in P\right\}. \tag{5}$$

First of all we show that our energy density is associated to a constitutive law of the form $\boldsymbol{\sigma} = (1-\bar{w}_n)\mathbf{C}\,\boldsymbol{\varepsilon}$ where $\bar{w}_n$ denotes a non-local damage parameter. For simplicity the effect of the boundary $\partial\Omega$ on the convolution operator is not taken into account. Since the functional is non-local we will compute the stress in terms of the differential $dF_n(u)$. For every (admissible) variation δu of the displacement u we have

$$\begin{aligned} \langle dF_n(u), \delta u\rangle &= \lim_{t\to 0} \frac{1}{t}\big(F_n(u + t\delta u) - F_n(u)\big) \\ &= \sum_{i\in I_n} |Q_n| \frac{1}{t}\left[f_n\Big(W^e(\boldsymbol{\varepsilon} + t\delta\boldsymbol{\varepsilon}) * \rho_n(\boldsymbol{x}_i)\Big) - f_n\Big(W^e(\boldsymbol{\varepsilon}) * \rho_n(\boldsymbol{x}_i)\Big)\right]. \end{aligned}$$

Then we can write

$$\begin{aligned}\langle dF_n(u), \delta u\rangle &= \sum_{i\in I_n} |Q_n| f_n'\Big(W^e(\boldsymbol{\varepsilon}) * \rho_n\,(\boldsymbol{x}_i)\Big)\Big(\boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon} * \rho_n\,(\boldsymbol{x}_i)\Big)\\ &= \sum_{i\in I_n} |Q_n| f_n'(\boldsymbol{x}_i)\Big(\boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon} * \rho_n\,(\boldsymbol{x}_i)\Big)\,,\end{aligned}$$

where $\boldsymbol{\sigma}^e = \partial W^e/\partial\boldsymbol{\varepsilon}$ is the elastic part of the stress and $f_n'(\boldsymbol{x}_i)$ stands for $f_n'\Big(W^e(\boldsymbol{\varepsilon}) * \rho_n\,(\boldsymbol{x}_i)\Big)$. Hence, writing explicitly the convolution operator,

$$\begin{aligned}\langle dF_n(u), \delta u\rangle &= \sum_{i\in I_n} |Q_n| f_n'(\boldsymbol{x}_i)\Big(\sum_{k=1}^{m}(q_k/h_n^2)\int_{Q(\boldsymbol{x}_i+\boldsymbol{x}_k)} \boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon}\, dy\Big)\\ &= \sum_{i\in I_n}\sum_{k=1}^{m}(q_k/h_n^2)|Q_n| f_n'(\boldsymbol{x}_i)\Big(\int_{Q(\boldsymbol{x}_i+\boldsymbol{x}_k)} \boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon}\, dy\Big)\,.\end{aligned}$$

Then, considering f_n' as piecewise constant on the square Q_n, the change of variable $\boldsymbol{x}_j = \boldsymbol{x}_i + \boldsymbol{x}_k$ gives

$$\begin{aligned}\langle dF_n(u), \delta u\rangle &= \sum_{j\in I_n}\Big(\sum_{k=1}^{m}(q_k/h_n^2)|Q_n| f_n'(\boldsymbol{x}_j - \boldsymbol{x}_k)\Big)\Big(\int_{Q(\boldsymbol{x}_j)} \boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon}\, dy\Big)\\ &= \sum_{j\in I_n}(f_n' * \rho_n)(\boldsymbol{x}_j)\int_{Q(\boldsymbol{x}_j)} \boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon}\, dy\\ &= \sum_{j\in I_n}\int_{Q(\boldsymbol{x}_j)} (f_n' * \rho_n)(\boldsymbol{x}_j)\, \boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon}\, dy\,.\end{aligned}$$

It follows that on each element contained in Q_n we can write $\boldsymbol{\sigma} = (f_n' * \rho_n)\,\boldsymbol{\sigma}^e$ while in terms of the damage parameter w we can write

$$\boldsymbol{\sigma} = (f_n' * \rho_n)\boldsymbol{\sigma}^e = (1 - w * \rho_n)\boldsymbol{\sigma}^e = (1 - \bar{w}_n)\boldsymbol{\sigma}^e\,, \tag{6}$$

where $\bar{w}_n = \bar{w}_n(\boldsymbol{x}_i) = w * \rho_n\,(\boldsymbol{x}_i)$.

2.2 Some corollaries

In the applications, e.g. [3], non-local operators are usually defined by means of convolution kernels supported on balls. Nonetheless, for the sake of generality and comparing with similar results [15], [16], it seems interesting to consider also the case of kernels supported on a convex, symmetric set. Thus, let C be a convex,

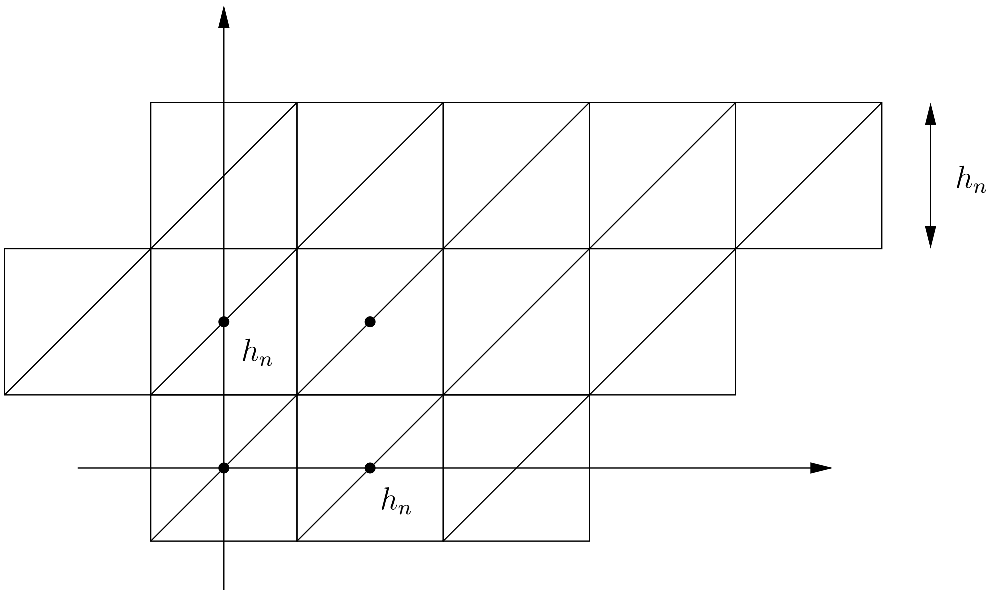

Figure 1: A structured grid on the lattice $h_n[\mathbf{Z}^2 + (1/2, 1/2)]$.

bounded set in $\mathbf{R}^2$ symmetric with respect to the x and y axes. Let P be the union of the squares $Q(\boldsymbol{x})$ contained in C and let d be the diameter of P. For $h_n \searrow 0$ let Q_n, P_n, d_n be defined as before. Then, following closely the proof of Theorem 2.1, it is easy to prove this similar result.

Corollary 2.2 *Let f be as in Theorem 2.1. Let ρ be a discrete kernel on a convex, bounded symmetric set C. For $h_n \searrow 0$ the sequence of the discrete energies F_n, defined as in (3), Γ-converges with respect to the strong topology of L^1 to a functional of the form (4). The density ϕ can be computed again by (5).*

Clearly, with minor changes in the proof, Theorem 2.1 and 2.2 hold true also in the framework of the space SBV. Denoting again by V_n the finite element space of piecewise affine functions on the mesh $\boldsymbol{T}_n$ we can state the following result.

Corollary 2.3 *With the notation and assumptions of Theorem 2.2 the sequence of the discrete energies*

$$F_n(u) = \begin{cases} \sum_{i \in I_n} |Q_n| f_n\Big(|\nabla u|^2 * \rho_n(\boldsymbol{x}_i)\Big) & \text{for } u \in V_n \\ +\infty & \text{for } u \in L^1 \setminus V_n \end{cases} \qquad (7)$$

Γ-converges with respect to the strong topology of L^1 to a functional of the form

$$F(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, dx + f_\infty \int_{S(u)} \phi(\nu)\, d\mathcal{H}^1 & \text{for } u \in SBV \\ +\infty & \text{for } u \in L^1 \setminus SBV. \end{cases} \tag{8}$$

The density ϕ is given again by (5).

Finally we remark that, with a small change in Lemma 5.1, all the previous convergence results holds true also in the case of Q_1 elements on a square mesh $\boldsymbol{Q}_n$.

2.3 A further relationship between damage and fracture

As we have seen, Theorem 2.1 gives a fracture energy as the limit of a rescaled damage energy in a discrete setting. This relationship is strengthen in the continuum framework by considering the scale effect of damage. Let $\rho : B_r \to (0, +\infty)$ be a symmetric convolution kernel with support B_r. We consider a non-local operator given by

$$W^e(\boldsymbol{\varepsilon}) * \rho\,(x) = \int_{B_r(x)} W^e(\boldsymbol{\varepsilon}(y))\, \rho(y - x)\, dy\,.$$

Under these assumptions, we will consider a free energy of the form

$$\mathcal{F}(u) = \int_\Omega f\Big(W^e(\boldsymbol{\varepsilon}) * \rho\,(x)\Big)\, dx\,, \tag{9}$$

where, for simplicity, the effect of the boundary $\partial\Omega$ on the convolution operator is not taken into account. As before, this energy density is associated to a constitutive law of the form $\boldsymbol{\sigma} = (1 - \bar{w})\mathbf{C}\, \boldsymbol{\varepsilon}$ where $\bar{w}$. Considering a variation δu, the Gateaux differential $\partial\mathcal{F}(u)$ is given by

$$\begin{aligned} \langle \partial\mathcal{F}(u), \delta u\rangle \; &= \lim_{t\to 0} \frac{1}{t}\big(\mathcal{F}(u + t\delta u) - \mathcal{F}(u)\big) \\ &= \frac{1}{t}\int_\Omega f\Big(W^e(\boldsymbol{\varepsilon} + t\delta\boldsymbol{\varepsilon}) * \rho\,(x)\Big) - f\Big(W^e(\boldsymbol{\varepsilon}) * \rho\,(x)\Big)\, dx\,. \end{aligned}$$

Then, assuming all the quantities to be sufficiently regular, we can write

$$\begin{aligned} \langle \partial\mathcal{F}(u), \delta u\rangle \; &= \int_\Omega f'\Big(W^e(\boldsymbol{\varepsilon}) * \rho\,(x)\Big)\Big(\boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon} * \rho\,(x)\Big)\, dx \\ &= \int_\Omega f'(x)\Big(\boldsymbol{\sigma}^e \cdot \delta\boldsymbol{\varepsilon} * \rho\,(x)\Big)\, dx\,, \end{aligned}$$

where $f'(x)$ stands for $f'\Big(W^e(\boldsymbol{\varepsilon}) * \rho\,(x)\Big)$. Again, writing explicitly the convolution operator,

$$\begin{aligned}
\langle \partial\mathcal{F}(u), \delta u\rangle &= \int_\Omega f'(x)\Big(\int_\Omega \boldsymbol{\sigma}^e(y)\cdot\delta\boldsymbol{\varepsilon}(y)\rho(x-y)\,dy\Big)\,dx \\
&= \int_\Omega \boldsymbol{\sigma}^e(y)\cdot\delta\boldsymbol{\varepsilon}(y)\Big(\int_\Omega f'(x)\rho(x-y)\,dx\Big)\,dy \\
&= \int_\Omega (f' * \rho)\,\boldsymbol{\sigma}^e\cdot\delta\boldsymbol{\varepsilon}\,dy\,.
\end{aligned}$$

It follows that $\boldsymbol{\sigma} = (f' * \rho)\,\boldsymbol{\sigma}^e$ and with the damage parameter w we get

$$\boldsymbol{\sigma} = (f' * \rho)\boldsymbol{\sigma}^e = (1 - w * \rho)\boldsymbol{\sigma}^e = (1-\bar{w})\boldsymbol{\sigma}^e\,,$$

where

$$\bar{w}(x) = w * \rho\,(x) = \int_\Omega w(y)\rho(x-y)\,dy = \int_\Omega w\Big(W^e(\boldsymbol{\varepsilon}) * \rho\,(y)\Big)\rho(x-y)\,dy\,.$$

Now, let us take into account the non-linear scale effects of this energy: let the domain be $\lambda\Omega$, for $\lambda > 1$, and consider the deformation $u_\lambda : \lambda\Omega \to \mathbf{R}^2$ defined by $u_\lambda(x') = u(x'/\lambda)$. The energy (9) is then given by

$$\mathcal{F}(u_\lambda) = \int_{\lambda\Omega} f\Big(W^e(\boldsymbol{\varepsilon}_\lambda) * \rho\,(x')\Big)\,dx'\,.$$

Being $\nabla u_\lambda(y') = \nabla u(y'/\lambda)/\lambda$ we have $W^e(\boldsymbol{\varepsilon}_\lambda(y')) = W^e(\boldsymbol{\varepsilon}(y'/\lambda))/\lambda^2$ and then can write

$$\begin{aligned}
\mathcal{F}(u_\lambda) &= \int_{\lambda\Omega} f\Big(\int_{B_r(x')} W^e(\boldsymbol{\varepsilon}_\lambda(y'))\,\rho(y'-x')\,dy'\Big)\,dx' \\
&= \int_{\lambda\Omega} f\Big(\int_{B_r(x')} \frac{1}{\lambda^2}W^e(\boldsymbol{\varepsilon}(y'/\lambda))\,\rho(y'-x')\,dy'\Big)\,dx'\,.
\end{aligned}$$

By the change of variables $x' = \lambda x$ and $y' = \lambda y$ we get

$$\begin{aligned}
\mathcal{F}(u_\lambda) &= \int_\Omega f\Big(\frac{1}{\lambda^2}\int_{B_r(\lambda x)} W^e(\boldsymbol{\varepsilon}(y'/\lambda))\,\rho(y'-\lambda x)\,dy'\Big)\lambda^2\,dx \\
&= \int_\Omega \lambda^2 f\Big(\frac{1}{\lambda^2}\int_{B_{(r/\lambda)}(x)} W^e(\boldsymbol{\varepsilon}(y))\,\rho(\lambda y-\lambda x)\lambda^2\,dy\Big)\,dx \\
&= \int_\Omega \lambda^2 f\Big(\frac{1}{\lambda^2}\int_{B_{(r/\lambda)}(x)} W^e(\boldsymbol{\varepsilon}(y))\,\rho_\lambda(y-x)\,dy\Big)\,dx\,,
\end{aligned}$$

where $\rho_\lambda(z) = \lambda^2\rho(\lambda z)$. In particular, note that the radius of the non-local operator is now $r_\lambda = r/\lambda$.

Denoting by $f_\lambda(t) = \lambda^2 f(t/\lambda^2)$ we can say that at the scale λ the energy $\mathcal{F}$ is equivalent to the rescaled energy $\mathcal{F}_\lambda$ in the reference domain, i.e.

$$\mathcal{F}(u_\lambda) = \mathcal{F}_\lambda(u) = \int_\Omega f_\lambda\Big(W^e(\boldsymbol{\varepsilon}) * \rho_\lambda(x)\Big)\, dx\,.$$

It is well known that for large domains, i.e. for $\lambda \gg 1$, quasi-brittle materials behaves like brittle materials; indeed, as $\lambda \to +\infty$, the rescaled (non-local) energies $\mathcal{F}_\lambda$ converge [16] to the Griffith (local) energy

$$\mathcal{G}(u) = \int_{\Omega\setminus J(u)} W^e(\boldsymbol{\varepsilon})\, dx + f_\infty \mathcal{H}^1(J(u))\,,$$

which describes an elasto-brittle behaviour.

3 Anisotropy

Let P be the polygon defined in the previous section. Note that, just by definition, P inherits the axial symmetries of $\mathbf{Z}^2$. Let $co(P)$ be the convex envelope of P; then we define the anisotropy function $\phi : \mathbf{R}^2 \to [0, +\infty)$ as

$$\phi(v) = \max\{2\langle v, \xi\rangle/d \,:\, \text{for } \xi \in P\} = \max\{2\langle v, \xi\rangle/d \,:\, \text{for } \xi \in co(P)\}\,. \quad (10)$$

Moreover, let ξ_i for $i = 1, \dots, m$ be the vectors of the vertices of $co(P)$; being $co(P)$ a convex polygon, we get also $\phi(v) = \max\{2\langle v, \xi_i\rangle/d \,:\, \text{for } i = 1, \dots, m\}$. In particular, ϕ is the dual of a norm with unit ball $2co(P)/d$. Given P, the unit (anisotropic) ball $\{\phi(v) \le 1\}$ can be computed graphically in a very easy way (see Figure 2): by (5) it is sufficient to draw the lines $\langle v, 2\xi_i/d\rangle = 1$ and then to take the intersection of the half planes $\langle v, 2\xi_i/d\rangle \le 1$.

Now, let us see how the anisotropy ϕ is related to the smeared representation of the fracture. Let J denote a line segment in $\mathbf{R}^2$ with unit normal ν. For $h > 0$ let $Q_h = hQ$ and $P_h = hP$ be the rescaling of Q and P in the lattice $h\mathbf{Z}^2$. We denote by $d_h = hd$ the diameter of P_h. Since it is not restrictive, we assume that $J \cap h\mathbf{Z}^2 = \emptyset$ and let $\boldsymbol{x}_i$ for $i = 1, \dots, M$ be the points of $h\mathbf{Z}^2$ such that $Q_h(\boldsymbol{x}_i) \cap J \ne \emptyset$. For convenience, we define a further set J_h as the piecewise linear curve joining the points $\boldsymbol{x}_i$ (for $i = 1, \dots, M$) with horizontal and vertical line segments of length h (see Figure 3). Note that J_h is in some sense the representation of J in the discrete geometry of the lattice $h\mathbf{Z}^2$ and that it does not depend on the support P_h. Now, for every point $\boldsymbol{x}_i$ (for $i = 1, \dots, M$) we consider the polygon $P_h(\boldsymbol{x}_i) = P_h + \boldsymbol{x}_i$. The smeared representation $\mathcal{J}_h$ of the "crack" J is then given by the union of

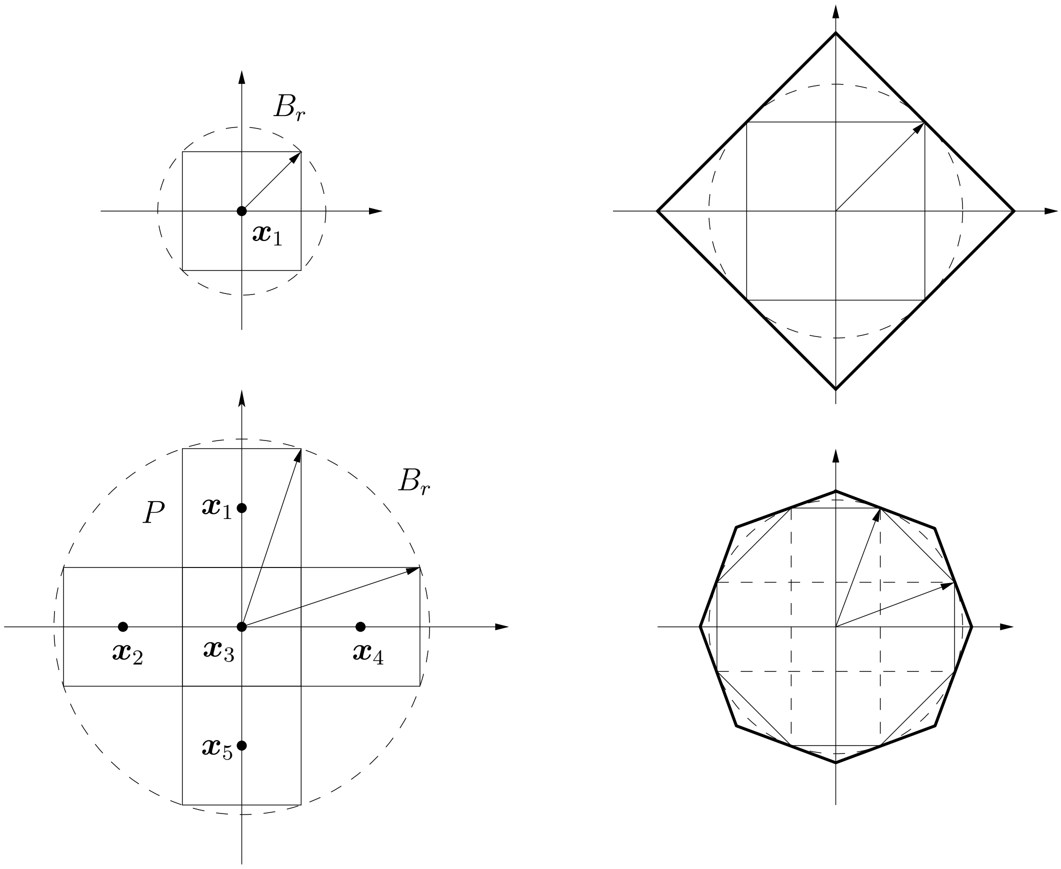

Figure 2: Left: balls B_r for $r = \sqrt{2}/2$ and $r = \sqrt{10}/2$ (dashed) and polygons P (solid). Right: anisotropic unit circle $\{\phi(v) = 1\}$ (bold), unit circle $|v| = 1$ (dashed) and polygon $2co(P)/d$ (solid).

the sets $P_h(\boldsymbol{x}_i)$ and is a closed neighbourhood of J. The following proposition establish a precise link between the smeared crack $\mathcal{J}_h$ and the mesh bias (10) with the diameters d_h playing the role of the characteristic lengths.

Proposition 3.1 *Under the previous assumptions we have*

$$\lim_{h \to 0} |\mathcal{J}_h|/d_h = \mathcal{H}^1(J)\phi(\nu).$$

Proof. Given ν with $|\nu| = 1$ let $\xi \in co(P)$ such that $\phi(\nu) = 2\langle \nu, \xi \rangle/d$. Moreover let $J_h^\xi = \{J_h + t\xi \text{ for } |t| \le h\}$. We will prove that

$$|\mathcal{J}_h| = |J_h^\xi| + O(h^2)\,. \tag{11}$$

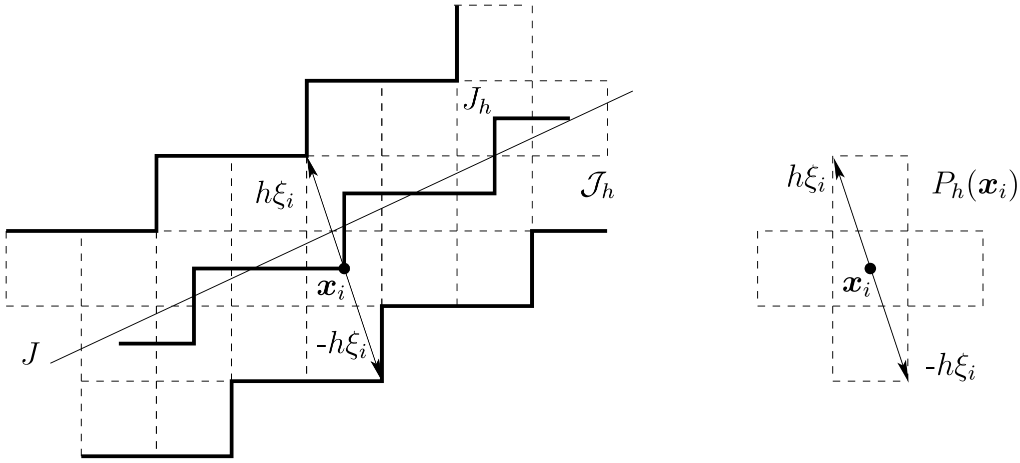

Figure 3: J and J_h.

From this estimate follows $\lim_{h\to 0} |\mathcal{J}_h|/d_h = \mathcal{H}^1(J)\phi(\nu)$. Indeed, apart from the endpoints, whose contribution is negligible, the sections of J_h^ξ in directions ξ have length $2h|\xi|$; hence by Fubini's Theorem we can write that

$$
\begin{aligned}
|\mathcal{J}_h|/d_h &= |J_h^\xi|/d_h + O(h) = |\{J + t\xi \text{ for } |t| \le h\}|/d_h + O(h) \\
&= 2|J|\langle \nu, \xi\rangle h/d_h + O(h) = |J|\phi(\nu) + O(h)\,,
\end{aligned}
$$

which gives the required limit.

Considering that the contribution near the endpoints is of order $O(h^2)$, to show (11) it is sufficient to prove that $\mathcal{J}_h = J_h^\xi$ whenever J is a (infinite) line with normal ν.

Step 1: $J_h^\xi \subset \mathcal{J}_h$. First of all observe that by construction the sections of P_h parallel to $\hat{e}_1 = (1,0)$ and $\hat{e}_2 = (0,1)$ are convex, and that the same holds for $P_h(\boldsymbol{x}_i) \cup P_h(\boldsymbol{x}_{i+1})$ for $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ consecutive points of J_h. Let $y \in J_h^\xi$; then by definition $y = x + t\xi$ with $x \in J_h$ and $|t| \le h$. Note that x belongs to a segment $[\boldsymbol{x}_i, \boldsymbol{x}_{i+1}]$ parallel to $\hat{e}_i$ (for $i = 1$ or $i = 2$) where $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ are consecutive points of J_h. Then $x = \lambda\boldsymbol{x}_i + (1-\lambda)\boldsymbol{x}_{i+1}$ for some $\lambda \in [0,1]$, and thus $y = \lambda(\boldsymbol{x}_i + t\xi) + (1-\lambda)(\boldsymbol{x}_{i+1} + t\xi)$ for $(\boldsymbol{x}_i + t\xi) \in P_h(\boldsymbol{x}_i)$ and $(\boldsymbol{x}_{i+1} + t\xi) \in P_h(\boldsymbol{x}_{i+1})$. From the convexity property of $P_h(\boldsymbol{x}_i) \cup P_h(\boldsymbol{x}_{i+1})$ along the directions $\hat{e}_i$ it follows that

$$
y \in P_h(\boldsymbol{x}_i) \cup P_h(\boldsymbol{x}_{i+1}) \subset \mathcal{J}_h.
$$

Step 2: $J_h^\xi \supset \mathcal{J}_h$. For every vertex $\boldsymbol{x}_i \in J_h$ let J^o denote the line parallel to J through $\boldsymbol{x}_i$; first of all we prove that $P_h(\boldsymbol{x}_i) \subset \{J^o + t\xi \text{ for } |t| \le h\}$ (see Figure 4).

By a translation argument, we can suppose $\boldsymbol{x}_i = 0$; then

$$J^o = \{v \in \mathbf{R}^2 : \langle v, \nu \rangle = 0\}, \qquad J^o + h\xi = \{v \in \mathbf{R}^2 : \langle v, \nu \rangle = \langle h\xi, \nu \rangle\}.$$

We introduce also the half-space

$$[J^o + h\xi]^- := \{v \in \mathbf{R}^2 : \langle v, \nu \rangle \leq \langle h\xi, \nu \rangle\}.$$

Observe that by the choice of ξ we have $\max_{w \in P_h} \langle w, \nu \rangle = \langle h\xi, \nu \rangle$; then $\langle w, \nu \rangle \leq \langle h\xi, \nu \rangle$ for every $w \in P_h$. This implies that $P_h \subset [J^o + h\xi]^-$. By the same argument $P_h \subset [J^o - h\xi]^+$, where

$$[J^o - h\xi]^+ := \{v \in \mathbf{R}^2 : \langle v, \nu \rangle \geq -\langle h\xi, \nu \rangle\}.$$

Then

$$P_h \subset [J^o + h\xi]^- \cap [J^o - h\xi]^+ = \{J^o + t\xi \text{ for } |t| \leq h\}.$$

This implies that $P_h \subset J_h^\xi$; indeed, suppose by contradiction that

$$(J_h - h\xi) \cap \operatorname{int}(P_h) \neq \emptyset\,,$$

then (see Figure 4), being $J_h - h\xi$ a piecewise linear curve with vertices on the lattice $h\mathbf{Z}^2 - h\xi$, it follows that an entire line segment of $J_h - h\xi$ is contained in P_h. We denote it by $[x_1', x_2']$. Hence, there are two squares Q_1 and Q_2 (represented with dashed lines in Figure 4) contained in P_h and such that $Q_1 \cap Q_2 = [x_1', x_2']$. Since $J^o - h\xi$ is not contained in P_h in the worst case it will pass through a vertex of $Q_1 \cup Q_2$. Then, with the notation of Figure 4 by the definition of J_h it follows that $J - h\xi$ intersects the interior of Q_1. At the same time, the definition of J^o implies that $J - h\xi$ intersects the interior of Q_2 since by translation both J^o and J intersect $Q_h(\boldsymbol{x}_i)$ (see again Figure 4). This is clearly a contradiction. ■

4 The Γ-limsup inequality

To obtain the estimate from above for the Γ-limsup, we take into account the density results [17] and [18]; thus we can limit ourselves to considering functions u which are $W^{k,\infty}(\Omega \setminus J(u), \mathbf{R}^2)$ (for k arbitrarily large), where $J(u)$ is the disjoint union of segments J_j, for $j = 1, \ldots, M$. Moreover, without loss of generality, we can suppose that $J(u)$ does not intersect the vertices of $\boldsymbol{T}_n$.

Let u_n be the Lagrange interpolation of u in V_n. Let us denote by J_n the set $\{i \in I_n : Q_n(\boldsymbol{x}_i) \cap J(u) \neq \emptyset\}$ and by $J_{n,j}$ the set $\{i \in I_n : Q_n(\boldsymbol{x}_i) \cap J_j \neq \emptyset\}$ for $j = 1, \ldots, M$. Using the notation of the previous section let

$$\mathcal{J}_n = \bigcup_{i \in J_n} P_n(\boldsymbol{x}_i) \quad \text{and} \quad \mathcal{J}_{n,j} = \bigcup_{i \in J_{n,j}} P_n(\boldsymbol{x}_i).$$

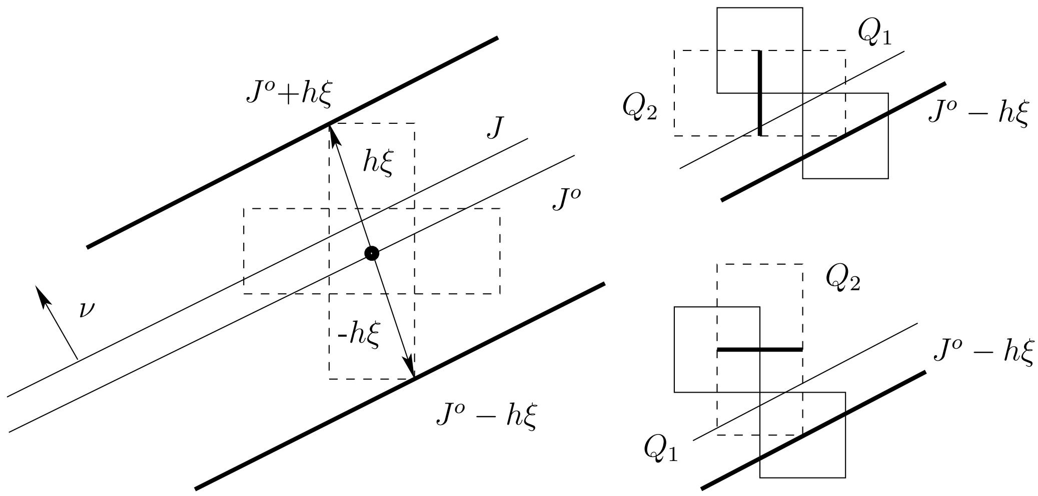

Figure 4: Representation of some sets used in the proof of Proposition 3.1.

Define $I''_n = \{i \in I_n : Q_n(\boldsymbol{x}_i) \subset \mathcal{J}_n\}$ ans $I'_n = I_n \setminus I''_n$; then we can write

$$\begin{aligned}\limsup_{n\to+\infty} F_n(u_n) \leq &\limsup_{n\to+\infty} \sum_{i\in I'_n} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) \\ &+ \limsup_{n\to+\infty} \sum_{i\in I''_n} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i)\Big).\end{aligned} \tag{12}$$

We will show that

$$\limsup_{n\to+\infty} \sum_{i\in I'_n} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) \leq \int_\Omega W^e(\boldsymbol{\varepsilon}(u))\, dx \tag{13}$$

and

$$\limsup_{n\to+\infty} \sum_{i\in I''_n} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) \leq f_\infty \int_{J(u)} \phi(\nu)\, d\mathcal{H}^1. \tag{14}$$

From (12), (13) and (14) it follows that $\limsup_{n\to+\infty} F_n(u_n) \leq F(u)$.

Before proving (13) and (14) it is convenient to write the non-local operator in different ways: denoting by $P_n(\boldsymbol{x}_i)$ the set $P_n + \boldsymbol{x}_i$ we get

$$\begin{aligned}
W^e(\boldsymbol{\varepsilon}(u)) * \rho_n(\boldsymbol{x}_i) &= \int_{P_n} W^e(\boldsymbol{\varepsilon}(u))\rho_n(x-\boldsymbol{x}_i)\,dx \\
&= \sum_{k=1}^m (q_k/h_n^2) \int_{Q_n(\boldsymbol{x}_k+\boldsymbol{x}_i)} W^e(\boldsymbol{\varepsilon}(u))\,dx \\
&= \sum_{k=1}^m q_k \fint_{Q_n(\boldsymbol{x}_k+\boldsymbol{x}_i)} W^e(\boldsymbol{\varepsilon}(u))\,dx\,.
\end{aligned}$$

First consider the sum on I_n'; since f is concave, by condition (2) $f(t) \le t$ and then $f_n(t) \le t$. Thus we can write, recalling the definition of the convolution kernel,

$$\begin{aligned}
\sum_{i\in I_n'} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) &\le \sum_{i\in I_n'} |Q_n|\, W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i) \\
&= \sum_{i\in I_n'} |Q_n| \sum_{k=1}^m (q_k/h_n^2) \int_{Q_n(\boldsymbol{x}_k+\boldsymbol{x}_i)} W^e(\boldsymbol{\varepsilon}(u_n))\,dx \\
&= \sum_{k=1}^m q_k \sum_{i\in I_n'} \int_{Q_n(\boldsymbol{x}_k+\boldsymbol{x}_i)} W^e(\boldsymbol{\varepsilon}(u_n))\,dx.
\end{aligned}$$

Observe that given k for every $i \in I_n'$ we have $Q_n(\boldsymbol{x}_k + \boldsymbol{x}_i) = Q_n(\boldsymbol{x}_j)$ for some $j \in I_n \setminus J_n$; moreover, for every such j there exists at most one index $i \in I_n'$ such that $Q_n(\boldsymbol{x}_k + \boldsymbol{x}_i) = Q_n(\boldsymbol{x}_j)$. Then, as $\sum_{k=1}^m q_k = 1$, we have

$$\begin{aligned}
\sum_{i\in I_n'} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) &\le \sum_{k=1}^m q_k \sum_{j\notin J_n} \int_{Q_n(\boldsymbol{x}_j)} W^e(\boldsymbol{\varepsilon}(u_n))\,dx \\
&= \sum_{j\notin J_n} \int_{Q_n(\boldsymbol{x}_j)} W^e(\boldsymbol{\varepsilon}(u_n))\,dx.
\end{aligned}$$

Now we show that

$$\lim_{n\to+\infty} \sum_{j\notin J_n} \int_{Q_n(\boldsymbol{x}_j)} W^e(\boldsymbol{\varepsilon}(u_n))\,dx = \int_\Omega W^e(\boldsymbol{\varepsilon}(u))\,dx.$$

Let $\Omega_n = \bigcup_{j\notin J_n} Q_n(\boldsymbol{x}_j)$; then

$$\begin{aligned}
&\left| \sum_{j\notin J_n} \int_{Q_n(\boldsymbol{x}_j)} W^e(\boldsymbol{\varepsilon}(u_n))\,dx - \int_\Omega W^e(\boldsymbol{\varepsilon}(u))\,dx \right| \le \\
&\qquad \le \sum_{j\notin J_n} \int_{Q_n(\boldsymbol{x}_j)} \left| W^e(\boldsymbol{\varepsilon}(u_n)) - W^e(\boldsymbol{\varepsilon}(u)) \right|\,dx + \int_{\Omega\setminus\Omega_n} W^e(\boldsymbol{\varepsilon}(u))\,dx.
\end{aligned}$$

Since $|\Omega \setminus \Omega_n| \to 0$ and $W^e(\varepsilon(u)) \in L^1(\Omega)$ the second integral goes to 0. For every $j \notin J_n$ we have the following pointwise estimate, which descends from the definition of W^e:

$$\begin{aligned} |W^e(\varepsilon(u_n)) - W^e(\varepsilon(u))| &= \frac{1}{2} |(\varepsilon(u) - \varepsilon(u_n))\mathbf{C}(\varepsilon(u) + \varepsilon(u_n))| \\ &\leq c|\varepsilon(u) - \varepsilon(u_n)||\varepsilon(u) + \varepsilon(u_n)| \\ &\leq 2c|\varepsilon(u) - \varepsilon(u_n)| \, ||\nabla u||_{L^\infty(Q_n(\boldsymbol{x}_j), \mathbf{R}^2)} \\ &\leq c'|\nabla u_n - \nabla u|, \end{aligned}$$

where $c' > 0$ does not depends on j and n. On every square $Q_n(\boldsymbol{x}_j)$ for $j \notin J_n$ by [19] Theorem 3.1.6 we have, for every $p \in [1, +\infty]$,

$$||u_n - u||_{W^{1,1}} \leq c|Q_n|^{1-1/p} h_n |u|_{W^{2,p}} \,,$$

where $|u|$ is the seminorm of the second derivative of u. Thus for every $j \notin J_n$

$$\begin{aligned} \int_{Q_n(\boldsymbol{x}_j)} |W^e(\varepsilon(u_n)) - W^e(\varepsilon(u))| dx &\leq c' \int_{Q_n(\boldsymbol{x}_i)} |\nabla u_n - \nabla u| dx \\ &\leq c''|Q_n|^{1-1/p} h_n |u|_{W^{2,p}}, \end{aligned}$$

and then

$$\begin{aligned} \sum_{j \notin J_n} \int_{Q_n(\boldsymbol{x}_j)} |W^e(\varepsilon(u_n)) dx - W^e(\varepsilon(u))| \, dx &\leq \sum_{j \in I'_n} c''|Q_n|^{1-1/p} h_n |u|_{W^{2,p}} \\ &= c''|u|_{W^{2,p}} h_n |Q_n|^{-1/p} \sum_{j \in I'_n} |Q_n| \\ &\leq c''|u|_{W^{2,p}} h_n^{1-2/p} |\Omega| \,, \end{aligned}$$

which tends to 0 for $p > 2$ and this conclude the proof of (13).

For the proof of (14) we observe that as

$$f_n\Big(W^e(\varepsilon(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) \leq \frac{f_\infty}{d_n},$$

we can write

$$\limsup_{n \to +\infty} \sum_{i \in I''_n} |Q_n| f_n\Big(W^e(\varepsilon(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) \leq f_\infty \sum_{j=1}^{M} \limsup_{n \to +\infty} \sum_{i \in I''_{n,j}} \frac{|Q_n|}{d_n},$$

where $I''_{n,j} = \{i \in I''_n : Q_n(\boldsymbol{x}_i) \subset \mathcal{J}^j_n\}$. By Proposition 3.1 we get (14), since

$$\begin{aligned}\limsup_{n\to+\infty} \sum_{i\in I''_n} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n)) * \rho_n(\boldsymbol{x}_i)\Big) &\le f_\infty \sum_{j=1}^{M} \limsup_{n\to+\infty} \frac{|\mathcal{J}_{n,j}|}{d_n} \\ &\le f_\infty \sum_{j=1}^{M} \phi(\nu)\mathcal{H}^1(J_j) \\ &= f_\infty \int_{J(u)} \phi(\nu)\, d\mathcal{H}^1,\end{aligned}$$

and this concludes the proof. ■

5 The Γ-liminf inequality

For every open subset A of Ω let $I'_n = \{i \in I_n : P_n(\boldsymbol{x}_i) \subset A\}$. We define the localized functionals

$$F_n(u,A) = \begin{cases} |Q_n| \displaystyle\sum_{i\in I'_n} f_n\Big(W^e(\boldsymbol{\varepsilon}(u)) * \rho_n(\boldsymbol{x}_i)\Big) & \textit{for } u \in V_n \\ +\infty & \textit{for } u \in L^1 \setminus V_n, \end{cases} \tag{15}$$

and

$$F(u,A) = \begin{cases} \displaystyle\int_A W^e(\boldsymbol{\varepsilon}(u))dx + f_\infty \int_{J(u)\cap A} \phi(\nu) d\mathcal{H}^1 & \textit{for } u \in V \\ +\infty & \textit{for } u \in L^1 \setminus V. \end{cases}$$

Let $u_n \to u$ in L^1, with $u_n \in V_n$ and $\liminf_{n\to+\infty} F_n(u_n, A) < +\infty$; we will show that

$$\begin{aligned} \liminf_{n\to+\infty} F_n(u_n,A) &\ge \int_A W^e(\boldsymbol{\varepsilon}(u))dx \\ \liminf_{n\to+\infty} F_n(u_n,A) &\ge f_\infty \int_{J(u)\cap A} \phi(\nu) d\mathcal{H}^1. \end{aligned} \tag{16}$$

By Proposition 6.2, since the function

$$\mu(\cdot) = \Gamma\text{-}\liminf_{n\to+\infty} F_n(u_n,\cdot) = \inf\{\liminf_{n\to+\infty} F_n(u_n,\cdot) : u_n \to u\}$$

is superadditive on disjoint open subsets of Ω and since the densities $W^e(\boldsymbol{\varepsilon}(u))$ and $\phi(\nu)$ are supported on the disjoint Borel sets $\Omega \setminus J(u)$ and $J(u)$ we obtain

$$\Gamma\text{-}\liminf_{n\to+\infty} F_n(u_n,A) \ge \int_A W^e(\boldsymbol{\varepsilon}(u))dx + f_\infty \int_{J(u)\cap A} \phi(\nu) d\mathcal{H}^1 = F(u)\,.$$

5.1 Estimate for the bulk energy and compactness

Let $\delta > 0$ small and $A^\delta = \{x \in A : d(x, \partial A) > \delta\}$. For n sufficiently large A^δ is covered by the union of the sets $P_n(\boldsymbol{x}_i)$ for $i \in I'_n$. From (15) and by Jensen's inequality we have

$$\begin{aligned} F_n(u_n, A) &\geq \sum_{i \in I'_n} |Q_n| f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) * \rho_n(\boldsymbol{x}_i) \\ &= \sum_{k=1}^m \sum_{i \in I'_n} |Q_n| \frac{q_k}{h_n^2} \int_{Q_n(\boldsymbol{x}_k + \boldsymbol{x}_i)} f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) dx \\ &= \sum_{k=1}^m q_k \sum_{i \in I'_n} \int_{Q_n(\boldsymbol{x}_k + \boldsymbol{x}_i)} f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) dx. \end{aligned}$$

Observe that for every $k = 1, \ldots, m$ and for n sufficiently large A^δ is covered by the union of the squares $Q_n(\boldsymbol{x}_k + \boldsymbol{x}_i)$ for $i \in I'_n$; thus

$$\sum_{i \in I'_n} \int_{Q_n(\boldsymbol{x}_k + \boldsymbol{x}_i)} f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) dx \geq \int_{A^\delta} f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) dx.$$

Then, recalling that $\sum_{k=1}^m q_k = 1$, we have

$$\begin{aligned} F_n(u, A) &\geq \sum_{k=1}^m q_k \sum_{i \in I'_n} \int_{Q_n(\boldsymbol{x}_k + \boldsymbol{x}_i)} f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) dx \\ &\geq \sum_{k=1}^m q_k \int_{A^\delta} f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) dx \\ &= \int_{A^\delta} f_n\Big(W^e(\boldsymbol{\varepsilon}(u_n))\Big) dx. \end{aligned} \tag{17}$$

Arguing for instance as in §5.1 in [6] we get

$$\liminf_{n \to +\infty} F_n(u_n, A) \geq \int_A W^e(\boldsymbol{\varepsilon}(u)) dx\,,$$

which gives the lower estimate for the bulk energy. Moreover, by (17) and §5.1 in [6], if $u_n \in V_n$ and $F_n(u_n) < +\infty$ then u_n is precompact with respect to the strong topology of L^1.

5.2 Estimate for the fracture energy

We want to show that

$$\liminf_{n \to +\infty} F_n(u_n, A) \geq f_\infty \int_{J(u) \cap A} \phi(\nu) d\mathcal{H}^1.$$

Since $W^e(\boldsymbol{\varepsilon}(u)) \geq \mu|\boldsymbol{\varepsilon}(u)|^2$, by the monotonicity of f we get

$$F_n(u_n, A) \geq \sum_{i \in I'_n} |Q_n| f_n\Big(\mu|\boldsymbol{\varepsilon}(u_n)|^2 * \rho_n(\boldsymbol{x}_i)\Big). \tag{18}$$

The idea of the proof is a reduction from $P_n(\boldsymbol{x}_i)$ to a finite family of symmetric rectangles $R_n^\xi(\boldsymbol{x}_i)$ contained in $co(P_n(\boldsymbol{x}_i))$; then the supremum of the family of the anisotropy functions ψ_ξ, associated to every $R_n^\xi(\boldsymbol{x}_i)$, will be ϕ. Hence, as a preliminary case, we consider our discrete functional to be defined on auxiliary rectangular meshes.

5.2.1 A preliminary case

Let us consider the lattice $2a\mathbf{Z} \times 2b\mathbf{Z}$ and the rectangle $R = (-a, a) \times (-b, b)$, with $a, b > 0$. Considering the mesh with vertices on the lattice $(2a\mathbf{Z} \times 2b\mathbf{Z}) + (a, b)$ let $\boldsymbol{T}$ be the triangulation obtained dividing every rectangle R in two triangles T as in Figure 5.

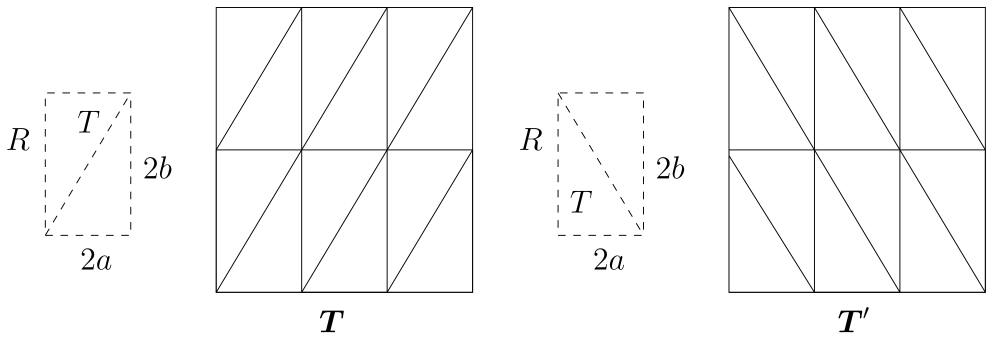

Figure 5: The triangulations $\boldsymbol{T}$ and $\boldsymbol{T}'$.

For $h_n \searrow 0$ let $T_n = h_n T$ and let V_n be the space of finite elements on $\boldsymbol{T}_n = h_n\boldsymbol{T}$. Let A be an open subset of $\mathbf{R}^2$ and $A_n = \{T \text{ of } \boldsymbol{T} \colon T \subseteq A\}$. In this section we take into account the functional

$$G_n(u_n, A) = \begin{cases} \displaystyle\sum_{T \in A_n} |T_n| \frac{1}{h_n} f\Big(ch_n|\boldsymbol{\varepsilon}(u_n)|^2\Big) & \textit{for } u_n \in V_n \\ +\infty & \textit{for } u_n \in L^1 \setminus V_n, \end{cases}$$

where $c > 0$ is a constant. Following the proof in §5.2 of [6] we obtain

$$\liminf_{n \to +\infty} G_n(u_n, A) \geq f_\infty \int_{J(u) \cap A} \psi(\nu)\, d\mathcal{H}^1, \tag{19}$$

where $\psi(\nu) = \sup_{i=1,2,3}\{2|\langle \nu, \eta_i\rangle|\}$ with $\eta_1 = (a,0)$, $\eta_2 = (a,b)$, $\eta_3 = (0,b)$ (see Figure 6). Let us consider the symmetric triangulation $\boldsymbol{T}'$ as in Figure 5. With obvious modifications in the notation, it is clear that, by symmetry,

$$\liminf_{n\to+\infty} G_n(u_n, A) \geq f_\infty \int_{J(u)\cap A} \psi'(\nu)\, d\mathcal{H}^1, \tag{20}$$

where $\psi'(\nu) = \sup_{i=1,2,3}\{2|\langle \nu, \eta_i\rangle|\}$ with $\eta_1 = (a,0)$, $\eta_2 = (-a,b)$, $\eta_3 = (0,b)$.

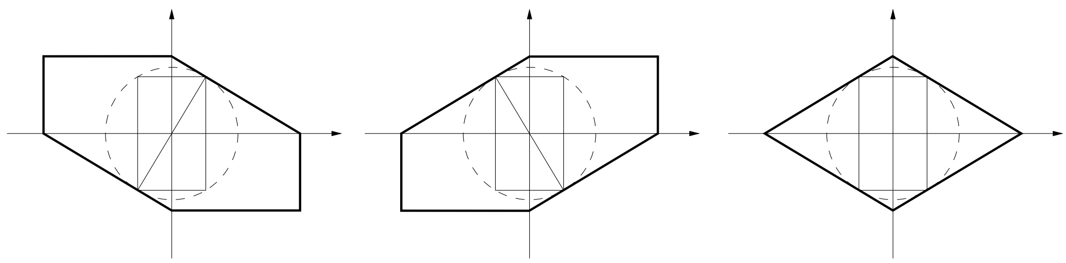

Figure 6: The rectangle $R = (-a,a)\times(-b,b)$; the circle $\{|v| = \sqrt{a^2+b^2}\}$; the level curves (from left to right) $\{\psi(v) = \sqrt{a^2+b^2}\}$, $\{\psi'(v) = \sqrt{a^2+b^2}\}$ and $\{(\psi\vee\psi')(v) = \sqrt{a^2+b^2}\}$.

5.2.2 The general case

Let $\xi = (a,b)$ be a vertex of $co(P)$; by symmetry it is not restrictive to consider only the case $a, b > 0$. Let $R^\xi = (-a,a)\times(-b,b)$. For $h_n \searrow 0$ let $R^\xi_n = h_n R^\xi$. Recalling that $\rho = \sum_{k=1}^m q_k \chi_{Q(\boldsymbol{x}_k)}$, $q_k > 0$, $\sum_{k=1}^m q_k = 1$, let $q = \min\{q_k : k = 1,\dots,m\} > 0$; since $P_n(\boldsymbol{x}_i) \supset R^\xi_n(\boldsymbol{x}_i)$ we have

$$\begin{aligned}|\boldsymbol{\varepsilon}(u_n)|^2 * \rho_n(\boldsymbol{x}_i) &\geq \frac{q}{h_n^2}\int_{R^\xi_n(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(u_n)|^2 dx\\ &= q\frac{|R^\xi_n|}{|Q_n|} ⨍_{R^\xi_n(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(u_n)|^2 dx.\end{aligned}$$

Since f_n is non-decreasing from (18) we obtain

$$F_n(u_n, A) \geq \sum_{i\in I'_n} |Q_n| f_n\left(c ⨍_{R^\xi_n(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(u_n)|^2 dx\right),$$

with $c = \mu q |R^\xi_n|/|Q_n|$ a positive constant independent of n. Let $R^\xi = \bigcup_{j=1}^{m'} Q(\boldsymbol{x}_j)$, with $\boldsymbol{x}_j \in \mathbf{Z}^2$; then $\mathbf{Z}^2$ can be written as the disjoint union of the sets $2(a\mathbf{Z}\times b\mathbf{Z})+$

$\boldsymbol{x}_j$, for $j = 1, \ldots, m'$, and similarly $I'_n = \bigcup_{j=1}^{m'} I'_{j,n}$, where $I'_{j,n} = \{i \in I'_n : \boldsymbol{x}_i = h_n \boldsymbol{x}_j + 2h_n(a\mathbf{Z} \times b\mathbf{Z})\}$. Hence we have

$$\begin{aligned} F_n(u_n, A) &\geq \sum_{i \in I'_n} |Q_n| f_n \left(c \fint_{R_n^\xi(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(u_n)|^2 dx \right) \\ &\geq \sum_{i \in I'_n} \frac{|Q_n|}{|R_n^\xi|} |R_n^\xi| f_n \left(c \fint_{R_n^\xi(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(u_n)|^2 dx \right) \\ &= \sum_{j=1}^{m'} \frac{1}{m'} \sum_{i \in I'_{j,n}} |R_n^\xi| f_n \left(c \fint_{R_n^\xi(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(u_n)|^2 dx \right). \end{aligned}$$

Fix $j = 1, \ldots, m'$ and consider the functional

$$F_{j,n}(u_n, A) = \sum_{i \in I'_{j,n}} |R_n^\xi| f_n \left(c \fint_{R_n^\xi(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(u_n)|^2 dx \right).$$

For every $j = 1, \ldots, m'$ consider the auxiliary triangulation $\boldsymbol{T}_j$ with triangles oriented as in Figure 7 and with vertices on the lattice $2(a\mathbf{Z} \times b\mathbf{Z}) + \boldsymbol{x}_j + \xi$. Denote by

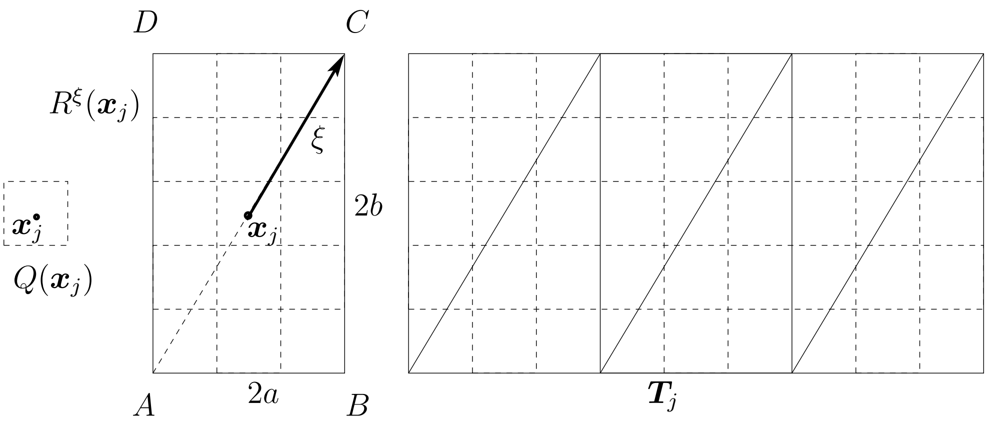

Figure 7: The triangulation $\boldsymbol{T}_j$.

$W_{j,n}$ the space of finite elements on $\boldsymbol{T}_{j,n} = h_n \boldsymbol{T}_j$. For $u \in V_n$ let $w \in W_{j,n}$ such that $w(x) = u(x)$ for every x vertex of $\boldsymbol{T}_{j,n}$ (the field w re-interpolates the displacement field u on the triangulation $\boldsymbol{T}_{j,n}$). In general the elastic energy density $W^e(\boldsymbol{\varepsilon}(u))$ could be greater than $W^e(\boldsymbol{\varepsilon}(w))$; however we have the following control.

Lemma 5.1 *There exists $c > 0$, independent of n, such that for all $u \in V_n$*

$$\int_{R_n^\xi} |\boldsymbol{\varepsilon}(u)|^2 dx \geq c \int_{R_n^\xi} |\boldsymbol{\varepsilon}(w)|^2 dx.$$

Proof. A simple rescaling argument show that the constant c does not depend on n. Thus we can assume $h_n = 1$ and we will drop the subscript n.

By contradiction we assume that for all $m \in \mathbf{N}$ there exists $u^m \in V$ with

$$\int_{R^\xi} |\boldsymbol{\varepsilon}(u^m)|^2 dx \leq \frac{1}{m} \int_{R^\xi} |\boldsymbol{\varepsilon}(w^m)|^2 dx.$$

By normalization we can suppose $\int_{R^\xi} |\boldsymbol{\varepsilon}(w^m)|^2 dx = 1$, and then $\int_{R^\xi} |\boldsymbol{\varepsilon}(u^m)|^2 dx \leq \frac{1}{m}$; thus, being u^m piecewise affine, $\boldsymbol{\varepsilon}(u^m) \to 0$ uniformly in R^ξ. Let A, B, C, D be the vertices of R^ξ, as in Figure 7. By translation we can suppose $u^m(B) = 0$ for all $m \in \mathbf{N}$; let $u^m(A) = (\alpha_1^m, \alpha_2^m)$, $u^m(C) = (\gamma_1^m, \gamma_2^m)$. Then, on ABC, we have

$$Dw^m = \begin{pmatrix} -\alpha_1^m/2a & \gamma_1^m/2b \\ \\ -\alpha_2^m/2a & \gamma_2^m/2b \end{pmatrix}$$

from which

$$\boldsymbol{\varepsilon}(w^m) = \begin{pmatrix} -\dfrac{\alpha_1^m}{2a} & \dfrac{1}{4}\left(\dfrac{\gamma_1^m}{b} - \dfrac{\alpha_2^m}{a}\right) \\ \\ \dfrac{1}{4}\left(\dfrac{\gamma_1^m}{b} - \dfrac{\alpha_2^m}{a}\right) & \dfrac{\gamma_2^m}{2b} \end{pmatrix}.$$

Considering the triangle ABC and the edge AB, since w^m is affine on AB and interpolates u^m on the vertices of R^ξ, we have (using the notation of §6.1)

$$|D^{\hat{e}_1} w_{\hat{e}_1}^m|^2 \leq \fint_{AB} |D^{\hat{e}_1} u_{\hat{e}_1}^m|^2 \, dx = \fint_{AB} |\langle \boldsymbol{\varepsilon}(u^m)\hat{e}_1, \hat{e}_1\rangle|^2 \, dx \leq \fint_{AB} |\boldsymbol{\varepsilon}(u^m)|^2 \, dx.$$

Moreover, as $\boldsymbol{\varepsilon}(u^m)$ is constant on triangles of $\boldsymbol{T}$, we get

$$\fint_{AB} |\boldsymbol{\varepsilon}(u^m)|^2 \, dx \leq c \fint_{R^\xi} |\boldsymbol{\varepsilon}(u^m)|^2 \, dx\,,$$

with $c > 0$ independent of m. Then $|D^{\hat{e}_1} w_{\hat{e}_1}^m| \to 0$. The same holds for $\hat{e}_2$ and for $\zeta = 2a\hat{e}_1 + 2b\hat{e}_2$ (considering the edges BC and AC respectively), i.e. $|D^{\hat{e}_2} w_{\hat{e}_2}^m| \to 0$ and $|D^\zeta w_\zeta^m| \to 0$. Thus we obtain $\alpha_1^m, \gamma_2^m \to 0$ and

$$D^\zeta w_\zeta^m = (-2a\alpha_1^m + 2b\gamma_2^m) + ba\left(\frac{\gamma_1^m}{b} - \frac{\alpha_2^m}{a}\right) \to 0,$$

from which $\gamma_1^m/b - \alpha_2^m/a \to 0$. Then $|\boldsymbol{\varepsilon}(w^m)| \to 0$ on ABC. Similarly one shows $|\boldsymbol{\varepsilon}(w^m)| \to 0$ on ACD; this is a contradiction, since $\int_{R^\xi} |\boldsymbol{\varepsilon}(w^m)|^2 dx = 1$. ■

Remark 5.2 *The proof of previous Lemma can be obtained in other ways; for instance applying inequality 2.47 in [20] we know that for all* u^m *there exists* $S(u^m)$*, where* $S(x)$ *is an affine function* $Ax+d$ *with* A *skew symmetric* $n\times n$*-matrix, such that*

$$\int_{R^\xi} |u^m - S(u^m)|^2 dx \le c(R) \int_{R^\xi} |\boldsymbol{\varepsilon}(u^m)|^2 dx.$$

Assume that $\int_{R^\xi} |\boldsymbol{\varepsilon}(w^m)|^2 dx = 1$*; arguing by contradiction as in the previous proof, if* $|\boldsymbol{\varepsilon}(u^m)| \to 0$ *then* u^m *and* w^m *converge to an affine function* $S = Ax+d$ *with* A *skew symmetric. This would imply* $\int_{R^\xi} |\boldsymbol{\varepsilon}(w^m)|^2 dx \to 0$.

By Lemma 5.1 we have

$$F_{j,n}(u_n, A) \ge \sum_{i\in I'_{j,n}} |R_n^\xi| f_n \left(c' \fint_{R_n^\xi(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(w_{j,n})|^2 dx \right),$$

for some $c' > 0$ independent of n, where $w_{j,n} \in W_{j,n}$ is the re-interpolated of u_n in the triangulation $\boldsymbol{T}_{j,n}$. We have

$$\begin{aligned}\fint_{R_n^\xi(\boldsymbol{x}_i)} |\boldsymbol{\varepsilon}(w_{j,n})|^2 dx &= \frac{1}{|R_n^\xi|}|T_n^\ell||\boldsymbol{\varepsilon}(w_{j,n}^\ell)|^2 + \frac{1}{|R_n^\xi|}|T_n^r||\boldsymbol{\varepsilon}(w_{j,n}^r)|^2 \\ &= \frac{|\boldsymbol{\varepsilon}(w_{j,n}^\ell)|^2 + |\boldsymbol{\varepsilon}(w_{j,n}^r)|^2}{2},\end{aligned}$$

where $w_{j,n}^\ell$ and $w_{j,n}^r$ are the restrictions of $w_{j,n}$ on, respectively, T_n^ℓ and T_n^r (the left and right-triangle that divide R_n^ξ); then by concavity of f_n

$$f_n\left(c' \frac{|\boldsymbol{\varepsilon}(w_{j,n}^\ell)|^2 + |\boldsymbol{\varepsilon}(w_{j,n}^r)|^2}{2} \right) \ge \frac{1}{2} f_n\Big(c'|\boldsymbol{\varepsilon}(w_{j,n}^\ell)|^2\Big) + \frac{1}{2} f_n\Big(c'|\boldsymbol{\varepsilon}(w_{j,n}^r)|^2\Big).$$

Now, we want to apply the preliminary case; to do this, let $\delta > 0$ small and $A^\delta = \{x \in A : d(x, \partial A) > \delta\}$; then for n sufficiently large

$$F_{j,n}(u_n, A) \ge \sum_{T_n \subset A} |T_n| f_n\Big(c'|\boldsymbol{\varepsilon}(w_{j,n})|^2\Big) \ge \sum_{T_n\subset A} |T_n| \frac{1}{h_n d} f\Big(h_n c' |\boldsymbol{\varepsilon}(w_{j,n})|^2\Big).$$

For n sufficiently large the union of the triangles $T \subset A$ contains A^δ. Thus by (19) we get

$$\liminf_{n\to+\infty} F_{j,n}(u_n, A) \ge f_\infty \int_{J(u)\cap A^\delta} \psi_\xi(\nu) d\mathcal{H}^1,$$

with $\psi_\xi(\nu) = \sup_{i=1,2,3}\{(2/d)|\langle \nu, \eta_i\rangle|\}$ with $\eta_1 = (a,0)$, $\eta_2 = (a,b) = \xi$ and $\eta_3 = (0,b)$. Then

$$\liminf_{n\to+\infty} F_n(u_n, A) \geq \frac{f_\infty}{m'} \sum_{j=1}^{m'} \int_{J(u)\cap A^\delta} \psi_\xi(\nu) d\mathcal{H}^1 = f_\infty \int_{J(u)\cap A^\delta} \psi_\xi(\nu) d\mathcal{H}^1.$$

By taking the supremum for $\delta \to 0$ we get

$$\liminf_{n\to+\infty} F_n(u_n, A) \geq f_\infty \int_{J(u)\cap A} \psi_\xi(\nu) d\mathcal{H}^1. \tag{21}$$

Now, let us consider the triangulation $\boldsymbol{T}'_j$ with elements oriented as in Figure 8 and vertices on the lattice $2(a\mathbf{Z} \times b\mathbf{Z}) + \boldsymbol{x}_j + \xi$. Following the proof above it is

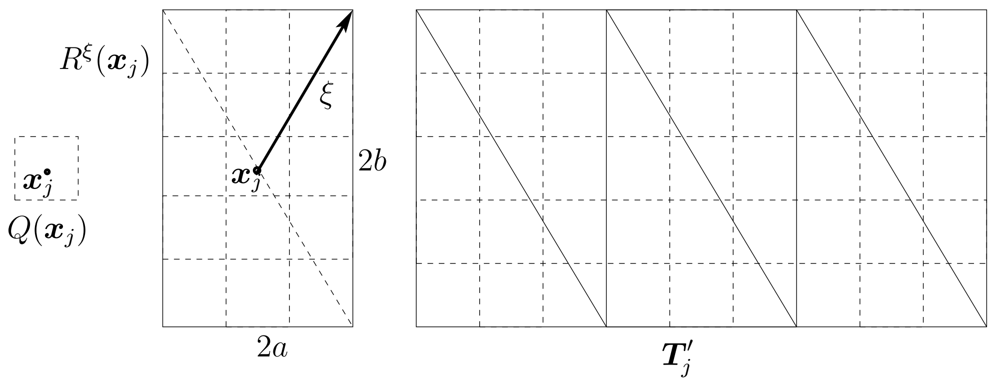

Figure 8: The triangulation $\boldsymbol{T}'_j$.

easy to see that

$$\liminf_{n\to+\infty} F_n(u_n, A) \geq f_\infty \int_{J(u)\cap A} \psi'_\xi(\nu) d\mathcal{H}^1\,, \tag{22}$$

where ψ'_ξ is given as in (20), i.e. $\psi'_\xi(\nu) = \sup_{i=1,2,3}\{(2/d)|\langle \nu, \eta_i\rangle|\}$ with $\eta_1 = (a,0)$, $\eta_2 = (-a,b)$ and $\eta_3 = (0,b)$. From the definition of ψ_ξ and ψ'_ξ it is easy to check that

$$\phi_\xi(\nu) := (\psi_\xi \vee \psi'_\xi)(\nu) = (2/d)\Big(|\langle \nu, \xi\rangle| \vee |\langle \nu, \xi'\rangle|\Big)\,,$$

where ξ' is the symmetric vector with respect to the horizontal (or vertical) axis. By (10) it follows that $\phi(\nu) = \sup_\xi \phi_\xi(\nu)$ where the supremum is taken over the vectors $\xi \in co(P)$. Then setting $\lambda = \mathcal{H}^1 \llcorner J(u)$, by (21), (22) and by Proposition 6.2 we conclude that

$$\liminf_{n\to+\infty} F_n(u_n, A) \geq f_\infty \int_{J(u)\cap A} \phi(\nu) d\mathcal{H}^1 ,$$

which gives the required estimate.

6 Technical results

6.1 Special functions of bounded deformation

Let Ω be an open, bounded and Lipschitz subset of $\mathbf{R}^2$; a function $u \in L^1(\Omega, \mathbf{R}^2)$ is called a *special function of bounded deformation* if the symmetric part of its distributional derivative

$$Eu = \frac{1}{2}\Big(Du + Du^T\Big)$$

is a finite measure on Ω which can be written as $Eu = \boldsymbol{\varepsilon}(u)\mathcal{L}^n + (u^+ - u^-) \odot \nu_u \mathcal{H}^1 \llcorner J(u)$. We denote by $SBD(\Omega)$ the space of special functions with bounded deformation; moreover, we denote by $SBD^p(\Omega)$, for $p > 1$, the space of all SBD-functions with $\boldsymbol{\varepsilon}(u) \in L^p(\Omega, \mathbf{R}^{2\times 2})$ and $\mathcal{H}^1(J(u)) < +\infty$.

For $\zeta \in S^1$ let $u_\zeta = \langle u, \zeta\rangle$. If $u \in SBD(\Omega)$ then $D^\zeta u_\zeta = \langle Du_\zeta, \zeta\rangle$ is a Radon measure and $D^\zeta u_\zeta = \langle Eu\zeta, \zeta\rangle$ in the sense of measures and pointwise if u is regular.

The following density result [17], [18] is what we need for the proof of the upper bound for the Γ-limsup.

Proposition 6.1 *Let ϕ be a norm in $\mathbf{R}^2$; for every $u \in SBD(\Omega) \cap L^\infty(\Omega, \mathbf{R}^2)$ there exists a sequence u_n in $SBD(\Omega)$ converging to u in $L^1(\Omega, \mathbf{R}^2)$, in such a way that $||u_n||_\infty \leq ||u||_\infty$, $u_n \in W^{k,\infty}(\Omega \setminus J(u_n), \mathbf{R}^2)$ (with k arbitrary large) and $\boldsymbol{\varepsilon}(u_n) \to \boldsymbol{\varepsilon}(u)$ in $L^2(\Omega, \mathbf{R}^{2\times 2})$. Moreover $J(u_n)$ is the union of a finite number of disjoint segments and*

$$\lim_{n\to+\infty} \int_{J(u_n)} \phi(\nu) d\mathcal{H}^1 \leq \int_{J(u)} \phi(\nu) d\mathcal{H}^1.$$

6.2 Supremum of measures

We conclude recalling a very useful tool from measure theory, which can be used for the Γ-liminf inequality; we denote by $\mathcal{A}(\Omega)$ be the set of all open subsets of Ω.

Proposition 6.2 *Let $\mu\colon \mathcal{A}(\Omega) \to [0,+\infty)$ be such that $\mu(A\cup B) \geq \mu(A)+\mu(B)$ for every $A, B \in \mathcal{A}(\Omega)$ with $A\cap B=\emptyset$ and let λ be a positive Borel measure on Ω. If ψ_i is a family of positive Borel functions with*

$$\int_A \psi_i d\lambda \leq \mu(A), \quad \forall A \in \mathcal{A}(\Omega)$$

then

$$\int_A \sup_i \psi_i d\lambda \leq \mu(A), \quad \forall A \in \mathcal{A}(\Omega).$$

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