An existence and uniqueness result for the motion of self-propelled micro-swimmers

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Abstract

We present an analytical framework to study the motion of micro-swimmers in a viscous fluid. Our main result is that, under very mild regularity assumptions, the change of shape determines uniquely the motion of the swimmer. We assume that the Reynolds number is very small, so that the velocity field of the surrounding, infinite fluid is governed by the Stokes system and all inertial effects can be neglected. Moreover, we enforce the self propulsion constraint (no external forces and torques). Therefore, Newton's equations of motion reduce to the vanishing of the viscous drag force and torque acting on the body. By exploiting an integral representation of viscous force and torque, the equations of motion can be reduced to a system of six ordinary differential equations. Variational techniques are used to prove the boundedness and measurability of its coefficients, so that classical results on ordinary differential equations can be invoked to prove existence and uniqueness of the solution.

Contents

1 Introduction

The study of swimming strategies of living organisms is attracting increasing attention, starting from seminal works by G. I. Taylor [19], M. J. Lighthill [14], and S. Childress [6]. We refer the reader to the recent review [13] for a comprehensive list of references. Among the more mathematical contributions we quote [12], [17], and [4].

Swimming consists in the ability to change position by changing shape periodically and exploiting the interaction with the surrounding liquid. Shape change induces a flow in the fluid. The propulsive effect arises from the action and reaction principle: the swimmer must exert forces to set the fluid in motion and hence it receives from the fluid a propulsive force. In the absence of other actions on its body, this is the only force the swimmer can exploit (self propulsion). In what follows we will focus on the case in which the swimmer is completely immersed in the liquid.

Flows generate both inertial and viscous forces. In a Newtonian fluid, their relative importance is measured by the Reynolds number. Typical swimmers move with a speed which is of the order of some body-lengths per second, so the Reynolds number of a swimming induced flow in a given fluid is determined only by the swimmer's size: at small sizes viscous effects dominate, at large scales the opposite is true.

Thus, a fish swims by accelerating the surrounding water, while bacteria and other unicellular organisms move by exploiting viscous resistance. The striking difference between these two strategies and the subtleties that follow are beautifully illustrated in [16].

In this paper we deal with micro-swimmers immersed in a viscous liquid, therefore the fluid dynamics is governed by the Stokes system. We assume self propulsion and neglect all external forces acting on the fluid and on the swimmer, including gravity. By a suitable choice of the units, we may assume that the viscosity of the fluid is equal to 1.

Our point of view is similar to the one proposed in [18] where the authors exploit a gauge field theory approach in the space of shapes. They give explicit examples in the two-dimensional case and in the case of infinitesimal deformations of a sphere. In the same spirit, axisymmetric swimmers described by finitely many shape parameters have been studied in [2], [3], [1], where energetically optimal strokes are also computed numerically. The novelty in the present work is that we develop a theoretical framework to study swimmers whose shape changes are completely general and genuinely infinite dimensional.

The *motion* of a swimmer is described by a map $t \mapsto \varphi_t$, where, for every fixed t, the *state* φ_t is an orientation preserving bijective C^2 map from the *reference* configuration $A \subset \mathbb{R}^3$ into the *current* configuration $A_t \subset \mathbb{R}^3$.

Given a distinguished point $x_0 \in A$, for every fixed t, we consider the following factorization

$$\varphi_t = r_t \circ s_t \,, \tag{1.1}$$

where the *position* function r_t is a rigid deformation and the *shape* function s_t is such that

$$s_t(x_0) = x_0 \,, \tag{1.2a}$$

$$\nabla s_t(x_0) \quad \text{is symmetric.} \tag{1.2b}$$

In the applications we have in mind, one can choose the map $t \mapsto s_t$ in a suitable class of admissible shape changes and use it as a control to achieve propulsion as a consequence of the viscous reaction of the fluid. By contrast, $t \mapsto r_t$ is *a priori* unknown and it must be determined by imposing that the resulting $\varphi_t = r_t \circ s_t$ satisfies the equations of motion.

The factorization (1.1) of the motion into data (the freely adjustable shapes s_t) and unknowns (the position and orientation r_t achieved by the swimmer as a consequence of having executed some strokes) is conceptually appealing and has far reaching consequences in the analysis of biological and engineered systems. Moreover, it simplifies the problem, reducing it to a system of ordinary differential equations since $r_t(z) = y_t + R_t z$ is finite dimensional; here y_t and R_t are the translation and rotation characterizing the rigid motion r_t. Finally it is natural, because $t \mapsto s_t$ represents the motion as seen by an observer moving with the swimmer, while $t \mapsto r_t$ represents the motion of this observer with respect to a fixed frame. To establish a link with the language of [18], notice that conditions (1.2) select one special gauge for the description of the system, that s_t describes the *standard* (unlocated) shape of the swimmer, and φ_t gives its *located* shape.

The equations of motion that the map $t \mapsto \varphi_t$ must satisfy are the balance of linear and angular momentum, which, since inertia is negligible, reduce to the vanishing of total force and total torque acting on the swimmer A_t. Since we assume self propulsion, there are no external forces applied to A_t, so that the total force and torque reduce to the ones arising from the viscous drag exerted by the fluid on the boundary ∂A_t:

$$0 = F_{A_t,\dot\varphi_t} := \int_{\partial A_t} \sigma_t(y) n(y)\, \mathrm{d}S(y), \tag{1.3a}$$

$$0 = M_{A_t,\dot\varphi_t} := \int_{\partial A_t} y \times \sigma_t(y) n(y)\, \mathrm{d}S(y). \tag{1.3b}$$

Here σ_t is the stress tensor, n is the outer unit normal to ∂A_t, $\mathrm{d}S$ indicates the integration with respect to the surface measure, and $\times$ is the cross product in $\mathbb{R}^3$. Since the Reynolds number is low, stresses are computed by solving the outer Stokes problem in $A_t^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{A}_t$

$$\begin{cases} \Delta u_t(y) = \nabla p_t(y) & \text{in } A_t^{\mathrm{ext}}, \\ \operatorname{div} u_t(y) = 0 & \text{in } A_t^{\mathrm{ext}}, \\ u_t(y) = \dot\varphi_t(x)\big|_{x=\varphi_t^{-1}(y)} & \text{on } \partial A_t, \\ u_t(y) \to 0 & \text{for } |y| \to \infty, \end{cases}$$

where u_t is the velocity and p_t is the pressure, so that $\sigma_t n = -p_t n + (\nabla u_t + (\nabla u_t)^T) n$ (recall that the viscosity is assumed to be 1).

Our main result is Theorem 6.4 stating that for every sufficiently smooth shape change $t \mapsto s_t$, the position functions $t \mapsto r_t$ are uniquely determined by the initial conditions at $t = 0$. More precisely, there exists a unique family of rigid motions $t \mapsto r_t$ such that the state functions $t \mapsto \varphi_t := r_t \circ s_t$ satisfy the equations of motion (1.3), and φ_t (or equivalently r_t) takes a prescribed value at $t = 0$. This result provides a rigorous mathematical justification for the viewpoint pioneered in [18]: the motion of a micro-swimmer is uniquely determined by the history of its shapes.

The main ingredients in the proof are the following. By exploiting the linearity of the Stokes system, we reduce the equations of motion (1.3) to (4.6), namely,

$$\dot y_t = R_t b_t, \qquad \dot R_t = R_t \Omega_t,$$

a system of ordinary differential equations involving the translational and rotational velocities associated with the rigid motion $t \mapsto r_t$. The coefficients b_t and Ω_t of these equations, given in (4.5), depend only on s_t and $\dot s_t$. They are obtained from the shape function $t \mapsto s_t$ by solving some auxiliary outer Stokes problems on A_t^{ext}.

The main difficulty is to prove the continuity, or at least the measurability, of these coefficients. To this aim, we have to obtain the continuous dependence of the solutions of the outer Stokes problems on their domains and on their boundary data; the main technical issue is the fact that they both depend on time.

Once continuity of the coefficients and measurability of the data of the equations of motion are proved, our existence and uniqueness problem can be solved by using classical techniques for ordinary differential equations.

2 Stokes problem

In this section we recall some known results on the Stokes problem. In addition, we introduce a weak definition of the viscous drag force and torque, which does not require any regularity assumption on the velocity field. Finally, we prove that the solutions depend continuously on the domains for special boundary conditions.

We begin with the case of a bounded open set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. Given a function U defined on $\partial\Omega$, the strong formulation of the Stokes problem in Ω is

$$\begin{cases} \Delta u = \nabla p & \text{in } \Omega\,, \\ \operatorname{div} u = 0 & \text{in } \Omega\,, \\ u = U & \text{on } \partial\Omega\,. \end{cases}$$

The corresponding weak formulation is given by

$$\begin{cases} u \in H^1(\Omega;\mathbb{R}^3), & \operatorname{div} u = 0 \text{ in } \Omega, \quad u = U \text{ on } \partial\Omega, \\ \displaystyle\int_\Omega \mathrm{E}u : \mathrm{E}w \, \mathrm{d}x = 0, & \text{for every } w \in H^1_0(\Omega) \text{ with } \operatorname{div} w = 0 \text{ in } \Omega, \end{cases} \tag{2.1}$$

where $\mathrm{E}u$ denotes the *symmetric gradient* of u, defined by $\mathrm{E}u := \frac{1}{2}(\nabla u + (\nabla u)^T)$. The following theorem holds [21, Lemma 2.1 and Theorem 2.4].

Theorem 2.1. *Let Ω be a bounded connected open subset of $\mathbb{R}^3$ with Lipschitz boundary. Given $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ such that*

$$\int_{\partial\Omega} U \cdot n \, \mathrm{d}S = 0, \tag{2.2}$$

there exists a unique solution u to the Stokes problem (2.1)*. Moreover, there exists $p \in L^2(\Omega)$ such that $\Delta u = \nabla p$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

In the rest of this section Ω will be an *exterior domain* with Lipschitz boundary, i.e., Ω is an unbounded, connected open set whose boundary $\partial\Omega$ is bounded and Lipschitz. In this case, the strong formulation of the Stokes problem is

$$\begin{cases} \Delta u = \nabla p & \text{in } \Omega\,, \\ \operatorname{div} u = 0 & \text{in } \Omega\,, \\ u = U & \text{on } \partial\Omega\,, \\ u = 0 & \text{at } \infty, \end{cases} \tag{2.3}$$

which includes a decay condition at infinity.

To write the weak formulation of this problem, we consider the *Deny-Lions space*

$$D^{1,2}(\Omega;\mathbb{R}^3) := \{u \in L^6(\Omega;\mathbb{R}^3) : \nabla u \in L^2(\Omega;\mathbb{M}^{3\times3})\},$$

where $\mathbb{M}^{3\times3}$ is the Hilbert space of 3×3 real matrices endowed with the Euclidean norm $\sigma : \xi := \sum_{i,j} \sigma_{ij}\xi_{ij}$. The space $D^{1,2}(\Omega;\mathbb{R}^3)$ is endowed with the norm

$$\|u\|_{D^{1,2}(\Omega;\mathbb{R}^3)} := \|\nabla u\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\,. \tag{2.4}$$

It is well known that $D^{1,2}(\Omega;\mathbb{R}^3)$ is a Hilbert space and that there exists a constant $C(\Omega)$ such that

$$\|u\|_{L^6(\Omega;\mathbb{R}^3)} \leqslant C(\Omega)\, \|u\|_{D^{1,2}(\Omega;\mathbb{R}^3)}\,,$$

for all $u \in D^{1,2}(\Omega;\mathbb{R}^3)$. For a thorough exposition on these spaces, see the classical work by Deny and Lions [7].

The inequality

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})} \leqslant C(\Omega)\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})}\,, \tag{2.5}$$

proved in Appendix A, equation (A.5), shows that $\|\mathrm{E}u\|_{L^2(\Omega;\mathbb{M}^{3\times3})}$ is an equivalent norm on $D^{1,2}(\Omega;\mathbb{R}^3)$. Since $\partial\Omega$ is bounded, for every $u \in D^{1,2}(\Omega;\mathbb{R}^3)$ the trace of u on $\partial\Omega$, still denoted by u, belongs to $H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and the trace operator is continuous between these two spaces.

We also use the space

$$D^{1,2}_0(\Omega;\mathbb{R}^3) := \{u \in D^{1,2}(\Omega;\mathbb{R}^3) : u = 0 \text{ on } \partial\Omega\},$$

which is closed in $D^{1,2}(\Omega;\mathbb{R}^3)$. The following density result, proved in [11], plays a crucial role in the theory.

Theorem 2.2 (Density). *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then the space*

$$\{u \in C^\infty_c(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \textit{ in } \Omega\}$$

is dense in $\{u \in D^{1,2}_0(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0$ in $\Omega\}$ for the norm (2.4).

To write the weak formulation of the exterior Stokes problem, we introduce the spaces

$$\begin{aligned}\mathcal{V}(\Omega) &:= \{u \in D^{1,2}(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\},\\ \mathcal{V}_0(\Omega) &:= \{u \in \mathcal{V}(\Omega) : u = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

Given a function $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, which plays the role of the boundary condition, the weak formulation of (2.3) is given by

$$\begin{cases} u \in \mathcal{V}(\Omega), \quad u = U & \text{on } \partial\Omega,\\ \displaystyle\int_\Omega \mathrm{E}u : \mathrm{E}w \,\mathrm{d}x = 0 & \text{for every } w \in \mathcal{V}_0(\Omega). \end{cases} \tag{2.6}$$

We now prove the following trace result.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary, and let $\gamma : D^{1,2}(\Omega;\mathbb{R}^3) \to H^{1/2}(\partial\Omega;\mathbb{R}^3)$ be the trace operator. Then $\gamma(\mathcal{V}(\Omega)) = H^{1/2}(\partial\Omega;\mathbb{R}^3)$. Moreover there exists a continuous linear operator $\mathcal{T}\colon H^{1/2}(\partial\Omega;\mathbb{R}^3) \to \mathcal{V}(\Omega)$ such that $\gamma(\mathcal{T}(\psi)) = \psi$ for every $\psi \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$.*

Proof. Let $\psi \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$. Consider a sufficiently large open ball Σ_ρ such that $\partial\Omega \subset \Sigma_\rho$, and let $\Omega_\rho := \Omega \cap \Sigma_\rho$. We apply Theorem 2.1 to the Stokes problem

$$\begin{cases} \Delta v = \nabla p & \text{in } \Omega_\rho\,,\\ \operatorname{div} v = 0 & \text{in } \Omega_\rho\,,\\ v = \psi & \text{on } \partial\Omega,\\ v = \lambda x/|x|^3 & \text{on } \partial\Sigma_\rho\,, \end{cases} \tag{2.7}$$

where

$$\lambda := -\frac{1}{4\pi}\int_{\partial\Omega} \psi\cdot n \,\mathrm{d}S. \tag{2.8}$$

Problem (2.7) admits a unique solution $v \in H^1(\Omega_\rho;\mathbb{R}^3)$ since its boundary condition satisfies (2.2) by the choice of λ. We extend v to Ω by setting $v(x) = \lambda x/|x|^3$ for $x \in \Omega\setminus\Sigma_\rho$. It is easy to see that $v \in L^6(\Omega;\mathbb{R}^3)$, $\nabla v \in L^2(\Omega;\mathbb{M}^{3\times3})$, and $\operatorname{div} v = 0$. We define $\mathcal{T}\colon H^{1/2}(\partial\Omega;\mathbb{R}^3) \to \mathcal{V}(\Omega)$ by $\mathcal{T}(\psi) = v$. Then $\mathcal{T}$ is linear and continuous and $\gamma(\mathcal{T}(\psi)) = \psi$ for every $\psi \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$. $\square$

For Stokes problems in exterior domains condition (2.2) is not needed, as shown in the following theorem.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary and let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$. Then problem* (2.6) *has a solution. Moreover, there exists $p \in L^2_{\rm loc}(\Omega)$, with $p \in L^2(\Omega \cap \Sigma_\rho)$ for every ball Σ_ρ centered at the origin and of radius $\rho > 0$, such that $\Delta u = \nabla p$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

Proof. By Proposition 2.3 there exists $w \in \mathcal{V}(\Omega)$ such that $w = U$ on $\partial\Omega$. Setting $u = v + w$, our problem is equivalent to finding v such that

$$\begin{cases} v \in \mathcal{V}_0(\Omega), \\ \displaystyle\int_\Omega \mathrm{E}v : \mathrm{E}z \,\mathrm{d}x = -\int_\Omega \mathrm{E}w : \mathrm{E}z \,\mathrm{d}x \quad \text{for every } z \in \mathcal{V}_0(\Omega). \end{cases}$$

The solution can be obtained by using the Lax-Milgram Lemma in $\mathcal{V}_0(\Omega)$, taking into account (2.5). The existence of p can be deduced as in [21, Theorem 2.3]. □

If u and p are the velocity and pressure fields of problem (2.3), the stress tensor is given by

$$\sigma := -p\,\mathrm{I} + 2\mathrm{E}u, \tag{2.9}$$

where I is the identity matrix (recall, again, that the viscosity is equal to 1). Note that if u satisfies (2.6), then

$$\operatorname{div}\sigma = -\nabla p + \Delta u + \nabla(\operatorname{div} u) = 0. \tag{2.10}$$

If σn has a trace in $L^1(\partial\Omega;\mathbb{R}^3)$, then the *drag force*, defined as the resultant of the forces acting on the boundary $\partial\Omega$, is given by

$$F := \int_{\partial\Omega} \sigma(x)n(x)\,\mathrm{d}S(x), \tag{2.11}$$

while the *torque*, defined as the resultant of the corresponding momenta with respect to the origin, is given by

$$M := \int_{\partial\Omega} x \times \sigma(x)n(x)\,\mathrm{d}S(x). \tag{2.12}$$

A technical problem arises from the fact that σn has not, in general, a trace in $L^1(\partial\Omega;\mathbb{R}^3)$, even if u satisfies the outer Stokes problem as in Theorem 2.1, so that F and M cannot be defined via (2.11) and (2.12). Thanks to (2.10), the following definition allows us to introduce the trace of σn as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$. Through this we can define in a consistent way the *power* of the drag force and of the torque.

Let $\mathbb{M}^{3\times3}_{\rm sym}$ be the space of 3×3 symmetric matrices. Every $\sigma \in \mathbb{M}^{3\times3}_{\rm sym}$ can be orthogonally decomposed as

$$\sigma = \tfrac{\operatorname{tr}\sigma}{3}\,\mathrm{I} + \sigma_D\,,$$

where the *deviatoric part* σ_D satisfies $\operatorname{tr}\sigma_D = 0$.

Definition 2.5. *Let Ω be an exterior domain with Lipschitz boundary and let $\sigma \in L^1_{\rm loc}(\Omega;\mathbb{M}^{3\times3}_{\rm sym})$ be such that $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})$ and $\operatorname{div}\sigma \in L^{6/5}(\Omega;\mathbb{R}^3)$. We define the trace of σn on $\partial\Omega$, still denoted by σn, as the unique element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ satisfying*

$$\langle \sigma n, V\rangle_\Omega := \int_\Omega (\operatorname{div}\sigma)\cdot v \,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v \,\mathrm{d}x, \tag{2.13}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality pairing between $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ and $H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and v is any function in $\mathcal{V}(\Omega)$ such that $v = V$ on $\partial\Omega$.

We will drop the subscript Ω whenever the domain of integration is understood. If σ is sufficiently smooth, then an integration by parts shows that

$$\langle \sigma n, V\rangle_\Omega = \int_{\partial\Omega} \sigma n \cdot V \,\mathrm{d}S,$$

for every $V \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$.

Returning to the general case, it is easy to see that the right-hand side of (2.13) is well defined, since $\operatorname{div}\sigma \in L^{6/5}(\Omega;\mathbb{R}^3)$, $v \in L^6(\Omega;\mathbb{R}^3)$, $\sigma : \mathrm{E}v = \sigma_D : \mathrm{E}v$, $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$, and $\mathrm{E}v \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$. Moreover, the definition of σn does not depend on the choice of v, since the right-hand side of (2.13) vanishes whenever $v \in \mathcal{V}_0(\Omega)$. This follows from the distributional definition of $\operatorname{div}\sigma$ whenever $v \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^3)$ and $\operatorname{div} v = 0$, and can be obtained by approximation in the general case using the Density Theorem 2.2. Finally, by choosing $v = \mathcal{T}(V)$, where $\mathcal{T}$ is the lifting operator introduced in Proposition 2.3, we conclude that (2.13) defines a continuous linear functional on $H^{1/2}(\partial\Omega;\mathbb{R}^3)$.

Let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and let u be the solution to the Stokes problem (2.6) with boundary datum U and let σ be the corresponding stress tensors defined by (2.9). Since $\sigma \in L^2_{\mathrm{loc}}(\Omega;\mathbb{M}^{3\times3})$, $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$, and $\operatorname{div}\sigma = 0$ by (2.10), we can apply Definition 2.5 and for every $V \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ we obtain

$$\begin{aligned}\langle \sigma n, V\rangle &= \int_\Omega \sigma : \mathrm{E}v \,\mathrm{d}x = \int_\Omega [-pI : \mathrm{E}v + 2\mathrm{E}u : \mathrm{E}v]\,\mathrm{d}x \\ &= -\int_\Omega p \operatorname{div} v \,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v \,\mathrm{d}x = 2\int_\Omega \mathrm{E}u : \mathrm{E}v\,\mathrm{d}x,\end{aligned} \tag{2.14}$$

where v is an arbitrary element of $\mathcal{V}(\Omega)$ such that $v = V$ on $\partial\Omega$. In particular, we can take as v the solution to the Stokes problem (2.6) with boundary datum V. This leads to the *reciprocity condition*,

$$\langle \sigma n, V\rangle = \langle \tau n, U\rangle,$$

where τ is the stress tensor corresponding to v. By taking $U = V$ in (2.14), we get

$$\langle \sigma n, U\rangle = 2\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})}\,. \tag{2.15}$$

We now show that the quadratic form $\langle \sigma n, U\rangle$ is positive definite. Indeed, if $\langle \sigma n, U\rangle = 0$, by (2.15) we obtain $\mathrm{E}u = 0$ almost everywhere on Ω. This implies that that $u(x) = c + Ax$, where $c \in \mathbb{R}^3$ and A is a skew symmetric 3×3 matrix. Since $u \in L^6(\Omega;\mathbb{R}^3)$, we have $c = 0$ and $A = 0$, so that $U = 0$.

By using the duality product $\langle \sigma n, V\rangle$ for a suitable choice of V, one can define the viscous force F and the torque M in a rigorous way, extending (2.11) and (2.12) to the general case where the trace σn is not necessarily integrable on $\partial\Omega$.

Definition 2.6. *Let Ω be an exterior domain with Lipschitz boundary, let $u \in \mathcal{V}(\Omega)$ be the solution of the Stokes problem* (2.6) *with boundary datum $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, let σ be the corresponding stress tensor defined by* (2.9)*, and let $\sigma n \in H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ be the trace on $\partial\Omega$ introduced in Definition 2.5. The* drag force *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $F \in \mathbb{R}^3$ such that*

$$F \cdot V = \langle \sigma n, V\rangle \quad \textit{for every } V \in \mathbb{R}^3. \tag{2.16}$$

The torque *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $M \in \mathbb{R}^3$ such that*

$$M \cdot \omega = \langle \sigma n, W_\omega\rangle \quad \textit{for every } \omega \in \mathbb{R}^3, \tag{2.17}$$

where $W_\omega(x) := \omega \times x$ is the velocity field generated by the angular velocity ω.

We conclude this section by proving the continuous dependence on the domains of the solutions to the Stokes problems. To this aim, we introduce a notion of convergence for subsets of $\mathbb{R}^3$. We say that a sequence of sets $(S_k)_k$ converges to S_∞, and we write $S_k \to S_\infty$, if for every $\varepsilon > 0$ there exists m such that for every $k \geqslant m$

$$S_\infty^{-\varepsilon} \subset S_k \subset S_\infty^{+\varepsilon}, \tag{2.18}$$

where $S_\infty^{-\varepsilon} = \{y \in \mathbb{R}^3 : \mathrm{dist}(y, \mathbb{R}^3 \setminus S_\infty) \geqslant \varepsilon\}$ and $S_\infty^{+\varepsilon} = \{y \in \mathbb{R}^3 : \mathrm{dist}(y, S_\infty) \leqslant \varepsilon\}$.

Theorem 2.7. *For $k = 1, 2, \dots, \infty$, let S_k be a bounded connected open set of class C^1, and let w_k be the solution to the minimum problem*

$$\min \left\{ \int_{\mathbb{R}^3} |\mathrm{E}w|^2 \, \mathrm{d}x \,:\, w \in \mathcal{V}(\mathbb{R}^3),\, w = W \text{ on } \partial S_k \right\}, \tag{2.19}$$

where W denotes either a constant vector $a \in \mathbb{R}^3$ or the affine function $W_\omega(x) = \omega \times x$, for some $\omega \in \mathbb{R}^3$. Assume that $S_k \to S_\infty$ in the sense of (2.18)*. Then $w_k \to w_\infty$ strongly in $\mathcal{V}(\mathbb{R}^3)$.*

Notice that w_k coincides in $S_k^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{S}_k$ with the solution to the Stokes problem (2.6) in $\Omega = S_k^{\mathrm{ext}}$ with boundary condition $w_k = W$ on ∂S_k, while $w_k = W$ in S_k.

Proof. Consider a ball Σ_ρ centered at 0 and containing the closures of all the S_k's. It is possible to find a solenoidal function $\Psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\Psi = W$ in ∂S_k.

When W is a constant vector a, we consider a smooth closed curve Γ passing through the origin, whose tangent vector coincides with a in all points of $\Gamma \cap \Sigma_\rho$, and with curvature less than $1/(2\rho)$. In the tubular neighborhood $\Gamma + \Sigma_{2\rho}$, we consider the vector field $\Psi(x) := \psi(\mathrm{dist}(x,\Gamma))\tau(\pi_\Gamma(x))$, where π_Γ is the projection on Γ, τ returns the tangential component, and $\psi \in C_{\mathrm{c}}^\infty([0, 2\rho[)$ with $\psi(r) = 1$ for $0 \leqslant r \leqslant \rho$. It is easy to see that Ψ is solenoidal, coincides with a on Σ_ρ, and vanishes near the boundary of the tubular neighborhood. Its extension by 0 provides the required function in $C_{\mathrm{c}}^\infty(\mathbb{R}^3;\mathbb{R}^3)$.

In the case $W = W_\omega$, it is enough to take $\Psi(x) = \omega \times \phi(x) x$, with ϕ a radial scalar function with compact support such that $\phi(x) = 1$ for $x \in \Sigma_\rho$.

By minimality,

$$\int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}\Psi|^2 \, \mathrm{d}x, \quad \text{for } k = 1, 2, \dots, \infty.$$

It follows that the sequence $(w_k)_k$ admits a weak limit w^* in $\mathcal{V}(\mathbb{R}^3)$.

Notice that $\Delta W = 0$ and $\operatorname{div} W = 0$ on S_k, hence $w_k = W$ on S_k for $k = 1, 2, \dots, \infty$. Since $S_\infty^{-\varepsilon} \subset S_k$ for k large enough by the first inclusion in (2.18), we get $w^* = W$ on $S_\infty^{-\varepsilon}$. As ε is arbitrary, we conclude $w^* = W$ on S_∞, which implies that the same equality holds for the traces on ∂S_∞. Therefore, w^* is a competitor in the problem for ∂S_∞.

We now show it is also the minimum. For this, consider an admissible function v for the problem (2.19) for $k = \infty$. Then $v - \Psi \in \mathcal{V}(\mathbb{R}^3)$; it follows that $v - \Psi = 0$ on ∂S_∞. In particular, $v - \Psi \in \mathcal{V}_0(S_\infty^{\mathrm{ext}})$ and by Theorem 2.2 there exist functions $\varphi_\eta \in C_{\mathrm{c}}^\infty(S_\infty^{\mathrm{ext}};\mathbb{R}^3)$ such that $\varphi_\eta \to v - \Psi$ when $\eta \to 0$. For every $\eta > 0$ the function $v_\eta := \varphi_\eta + \Psi$ coincides with W in a neighborhood of ∂S_∞. By (2.18), this implies that v_η is a competitor for problem (2.19) on ∂S_k, for k large enough. Therefore, by the minimality of w_k

$$\int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}v_\eta|^2 \, \mathrm{d}x.$$

Taking the limit first as $k \to \infty$ and then as $\eta \to 0$, we get

$$\limsup_{k\to\infty} \int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}v|^2 \,\mathrm{d}x.$$

By the lower semicontinuity of the norm in $\mathcal{V}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} |\mathrm{E}w^*|^2 \,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \,\mathrm{d}x \leqslant \limsup_{k\to\infty} \int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}v|^2 \,\mathrm{d}x,$$

thus proving the minimality of w^*. By uniqueness, we have $w_\infty = w^*$. The last chain of inequalities, applied with $v = w_\infty$, shows also that $\|w_k\|_{D^{1,2}} \to \|w_\infty\|_{D^{1,2}}$, hence $w_k \to w_\infty$ strongly in $\mathcal{V}(\mathbb{R}^3)$. □

3 Kinematics

In this section we fix the notation and the assumptions for the kinematics of the swimmer. As mentioned in the introduction, we show that it is possible to decompose the deformation into a pure shape change followed by a time-dependent rigid motion, whose rotations and translations are Lipschitz continuous with respect to time.

The reference configuration $A \subset \mathbb{R}^3$ is a bounded connected open set of class C^2. The time-dependent deformation of A from the point of view of an external observer is described by a function $\varphi_t : \overline{A} \to \mathbb{R}^3$. We assume that, for every t,

$$\varphi_t \in C^2(\overline{A};\mathbb{R}^3), \tag{3.1a}$$

$$\varphi_t \quad \text{is injective,} \tag{3.1b}$$

$$\det \nabla\varphi_t(x) > 0 \quad \text{for all } x \in \overline{A}. \tag{3.1c}$$

Here and henceforth ∇ denotes the gradient with respect to the space variable. Under these hypotheses the set $A_t := \varphi_t(A)$ is a bounded connected open set of class C^2 and

$$\text{the inverse} \quad \varphi_t^{-1} : \overline{A}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{A}_t;\mathbb{R}^3).$$

We assume in addition that

$$\text{the sets } \mathbb{R}^3 \setminus \overline{A}_t \text{ are connected for all } t \in [0,T]. \tag{3.2}$$

Concerning the regularity in time, we require that

$$\text{the map } t \mapsto \varphi_t \text{ belongs to } \mathrm{Lip}([0,T];C^1(\overline{A};\mathbb{R}^3)) \cap L^\infty([0,T];C^2(\overline{A};\mathbb{R}^3)), \tag{3.3}$$

so that $\|\varphi_{t+h} - \varphi_t\|_{C^1} \leqslant L\,|h|$, for a suitable constant $L > 0$.

We now prove that for almost every t there exists $\dot\varphi_t \in \mathrm{Lip}(\overline{A};\mathbb{R}^3)$ such that

$$\frac{\varphi_{t+h} - \varphi_t}{h} \to \dot\varphi_t\,, \quad \text{uniformly on } \overline{A} \text{ as } h \to 0. \tag{3.4}$$

Indeed, condition (3.3) implies that $t \mapsto \varphi_t$ belongs to $\mathrm{Lip}([0,T];W^{1,4}(A;\mathbb{R}^3))$. Therefore, the general theory of Lipschitz functions with values in reflexive Banach spaces (see, e.g., [5, Appendix]) implies that for almost every t the difference quotient in (3.4) converges strongly in $W^{1,4}(A;\mathbb{R}^3)$ to some element $\dot\varphi_t$ of $W^{1,4}(A;\mathbb{R}^3)$. The embedding of $W^{1,4}(A;\mathbb{R}^3)$ into $C^0(A;\mathbb{R}^3)$ implies the uniform convergence considered in (3.4). Finally the bound $\|\varphi_t - \varphi_s\|_{C^1} \leqslant L\,|t-s|$ implies that $\mathrm{Lip}(\dot\varphi_t) = L$ in $\overline{A}$, where, for every function f, $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f.

It turns out that the Eulerian velocity on the boundary ∂A_t, defined by

$$U_t := \dot{\varphi}_t \circ \varphi_t^{-1}$$

belongs to $\mathrm{Lip}(\partial A_t;\mathbb{R}^3)$ with Lipschitz constant independent of t.

We now describe the kinematics from the point of view of the swimmer. We fix a point $x_0 \in A$ and we look for a factorization of φ_t of the form (1.1), where $s_t : A \to \mathbb{R}^3$ satisfies properties (1.2) and $r_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid motion of the form

$$r_t(z) = y_t + R_t z, \tag{3.5}$$

with $y_t \in \mathbb{R}^3$ and $R_t \in \mathrm{SO}(3)$, the set of orthogonal matrices with positive determinant. Conditions (1.2) allow us to interpret s_t as a pure shape change from the point of view of an observer located at x_0. Therefore, the deformation φ_t, from the point of view of an external observer, is decomposed into a shape change followed by a rigid motion.

It follows from (1.1), (3.1), and (3.5) that, for every t,

$$s_t \in C^2(\overline{A};\mathbb{R}^3), \tag{3.6a}$$
$$s_t \quad \text{is injective}, \tag{3.6b}$$
$$\det \nabla s_t(x) > 0 \quad \text{for all } x \in \overline{A}, \tag{3.6c}$$

and, consequently, that

$$\text{the inverse} \quad s_t^{-1} : \overline{B}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{B}_t;\mathbb{R}^3), \tag{3.7}$$

where $B_t := s_t(A)$, see Fig. 1. Note that B_t is a bounded connected open set

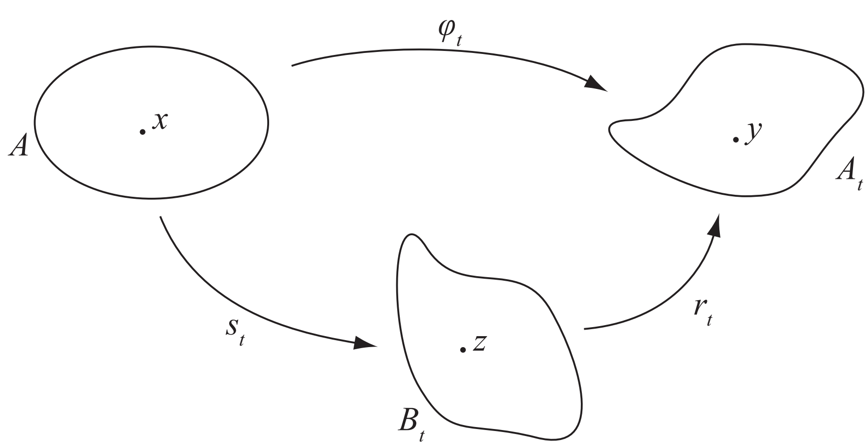

Figure 1: Notation for the kinematics.

of class C^2 and that $s_t(B_t) = A_t$ and $s_t(\partial B_t) = \partial A_t$. Notice that, since A is bounded and s_t is continuous, there exists a ball Σ_ρ centered at 0 with radius ρ such that

$$A \subset\subset \Sigma_{\rho-1} \qquad \text{and} \qquad B_t \subset\subset \Sigma_{\rho-1}\,. \tag{3.8}$$

It follows from (3.2) that

$$\text{the sets } \Sigma_\rho \setminus \overline{B}_t \text{ are connected for all } t \in [0,T]. \tag{3.9}$$

Conditions (1.1), (1.2), and (3.5) imply that

$$R_t = \nabla\varphi_t(x_0)\Big[\sqrt{\nabla\varphi_t(x_0)^T\nabla\varphi_t(x_0)}\Big]^{-1}, \tag{3.10a}$$
$$y_t = \varphi_t(x_0) - R_t x_0\,. \tag{3.10b}$$

The existence of a factorization (1.1) satisfying (1.2) and (3.5) is obtained by setting $s_t := r_t^{-1} \circ \varphi_t$, where r_t is given by (3.5) with y_t and R_t defined by (3.10). Moreover, (3.3) together with (3.10), implies that

$$t \mapsto R_t \quad \text{and} \quad t \mapsto y_t \quad \text{are Lipschitz continuous.} \tag{3.11}$$

Finally, since $s_t = r_t^{-1} \circ \varphi_t$,

$$\text{the map } t \mapsto s_t \text{ belongs to } \mathrm{Lip}([0,T]; C^1(\overline{A}; \mathbb{R}^3)) \cap L^\infty([0,T]; C^2(\overline{A}; \mathbb{R}^3)), \tag{3.12}$$

so that $\|s_{t+h} - s_t\|_{C^1} \leqslant L\,|h|$, for a suitable constant $L > 0$. Properties (3.6c) and (3.12) imply that

$$\left\|s_t^{-1}\right\|_{C^2(\overline{B}_t;\mathbb{R}^3)} \leqslant C, \tag{3.13}$$

where $C < +\infty$ is a constant independent of t.

As for function φ_t, we can exploit condition (3.12) to prove that there exists $\dot{s}_t \in \mathrm{Lip}(\overline{A}; \mathbb{R}^3)$ such that

$$\frac{s_{t+h} - s_t}{h} \to \dot{s}_t\,, \quad \text{uniformly on } \overline{A}, \text{ as } h \to 0.$$

Notice that

$$\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T]; W^{1,p}(\overline{A}; \mathbb{R}^3)) \text{ for every } p \in [2, \infty[,$$

therefore, by the Sobolev immersions,

$$\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T]; C^0(\overline{A}; \mathbb{R}^3)),$$

and, by the continuous immersion of $H^1(A; \mathbb{R}^3)$ into $H^{1/2}(\partial A; \mathbb{R}^3)$,

$$\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T]; H^{1/2}(\partial A; \mathbb{R}^3)).$$

Again as for $\dot{\varphi}_t$, we can prove that

$$\mathrm{Lip}(\dot{s}_t) \leqslant L, \qquad \text{with } L \text{ independent of } t. \tag{3.14}$$

Moreover, for any fixed $x \in \overline{A}$, the map $t \mapsto \dot{s}_t(x)$ is measurable.

Define now $V_t(z) := R_t^T U_t(r_t(z))$ and $W_t(z) := \dot{s}_t(s_t^{-1}(z))$, for every $z \in \partial B_t$. An elementary computation shows that for almost every $t \in [0,T]$

$$V_t(z) = R_t^T \dot{y}_t + R_t^T \dot{R}_t z + W_t(z) \quad \text{for every } z \in \partial B_t\,.$$

4 The equations of motion

The motion $t \mapsto \varphi_t$ determines for almost every $t \in [0,T]$ the Eulerian velocity U_t through the formula

$$U_t(y) := \dot{\varphi}_t(\varphi_t^{-1}(y)) \quad \text{for almost every } y \in \partial A_t\,.$$

As shown in Section 3, A_t is of class C^2 and

$$U_t \in H^{1/2}(\partial A_t; \mathbb{R}^3) \quad \text{for almost every } t \in [0,T].$$

We can apply Theorem 2.4 with $\Omega = A_t^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{A}_t$ and, for almost every $t \in [0,T]$, we obtain a unique solution u_t to the problem

$$\begin{cases} u_t \in \mathcal{V}(A_t^{\mathrm{ext}}), \quad u_t = U_t & \text{on } \partial A_t\,, \\ \displaystyle\int_{A_t^{\mathrm{ext}}} \mathrm{E}u_t : \mathrm{E}w \,\mathrm{d}y = 0 & \text{for every } w \in \mathcal{V}_0(A_t^{\mathrm{ext}}). \end{cases}$$

Let F_{A_t,U_t} and M_{A_t,U_t} be the drag force and torque determined by the velocity field U_t according to (2.16) and (2.17). Since we are neglecting inertia and imposing the self-propulsion constraint, the equations of motion reduce to the vanishing of the viscous force and torque, i.e.,

$$F_{A_t,U_t} = 0 \quad \text{and} \quad M_{A_t,U_t} = 0 \quad \text{for almost every } t \in [0,T]. \tag{4.1}$$

We assume that φ_t is written as $\varphi_t = r_t \circ s_t$, where r_t is a rigid motion as in (3.5) and $t \mapsto s_t$ is a prescribed shape function. Our aim is to find $t \mapsto r_t$ so that the equations of motion (4.1) are satisfied. More precisely, we prove Theorem 4.1 below, which shows that (4.1) is equivalent to a system of ordinary differential equations where the unknown functions are the translation $t \mapsto y_t$ and the rotation $t \mapsto R_t$ appearing in (3.5).

To define the coefficients of these differential equations, we consider the sets $B_t = s_t(A)$ introduced in Section 3 and the $3{\times}3$ matrices K_t , C_t , J_t , depending only on the geometry of B_t , whose entries are defined by

$$(K_t)_{ij} := \langle \sigma[e_j]n, e_i \rangle_{B_t^{\rm ext}} , \tag{4.2a}$$

$$(C_t)_{ij} := \langle \sigma[e_j]n, e_i{\times}z \rangle_{B_t^{\rm ext}} , \tag{4.2b}$$

$$(J_t)_{ij} := \langle \sigma[e_j{\times}z]n, e_i{\times}z \rangle_{B_t^{\rm ext}} , \tag{4.2c}$$

where $B_t^{\rm ext} := \mathbb{R}^3 \setminus \overline{B}_t$, the duality product is given in Definition 2.5, and $\sigma[W]$ denotes the stress tensor associated to the outer Stokes problem in $B_t^{\rm ext}$ with boundary datum W. The notation $\sigma[W]$ emphasizes that, by the linearity of Stokes system, the dependence of σ on W is linear. Formula (2.14) shows that K_t and J_t are symmetric. The matrix

$$\begin{bmatrix} K_t & C_t^T \\ C_t & J_t \end{bmatrix}$$

is often called in the literature *grand resistance matrix*, and is invertible. Let

$$\begin{bmatrix} H_t & D_t^T \\ D_t & L_t \end{bmatrix} := \begin{bmatrix} K_t & C_t^T \\ C_t & J_t \end{bmatrix}^{-1} \tag{4.3}$$

be its inverse. For almost every $t \in [0,T]$, let $W_t := \dot{s}_t \circ s_t^{-1}$, and let $F_t^{\rm sh}$ and $M_t^{\rm sh}$ be the drag force and torque on ∂B_t determined by the boundary value W_t. According to (2.16) and (2.17), the components of $F_t^{\rm sh}$ and $M_t^{\rm sh}$ are given by

$$(F_t^{\rm sh})_i = \langle \sigma[W_t]n, e_i \rangle_{B_t^{\rm ext}} , \tag{4.4a}$$

$$(M_t^{\rm sh})_i = \langle \sigma[W_t]n, e_i{\times}z \rangle_{B_t^{\rm ext}} . \tag{4.4b}$$

Let $\mathcal{A} : \mathbb{R}^3 \to \mathbb{M}^{3\times3}$ be the linear operator that associates to every $\omega \in \mathbb{R}^3$ the only antisymmetric matrix $\mathcal{A}(\omega)$ such that $\mathcal{A}(\omega)z = \omega{\times}z$. In other words, ω is the axial vector of $\mathcal{A}(\omega)$. Finally, we define

$$b_t := H_t F_t^{\rm sh} + D_t^T M_t^{\rm sh}, \qquad \Omega_t := \mathcal{A}(D_t F_t^{\rm sh} + L_t M_t^{\rm sh}), \tag{4.5}$$

which depend on s_t via (4.4) and the definition of W_t.

Theorem 4.1. *Assume that the shape function $t \mapsto s_t$ satisfies* (3.6)*,* (3.7)*, and* (3.12) *and that the position function $t \mapsto r_t$ satisfies* (3.5) *and* (3.11)*. Then the following conditions are equivalent:*

(i) the deformation function $t \mapsto \varphi_t := r_t \circ s_t$ satisfies the equations of motion (4.1)*;*

(ii) the functions $t \mapsto y_t$ and $t \mapsto R_t$ satisfy the system

$$\dot{y}_t = R_t b_t \,, \qquad \dot{R}_t = R_t \Omega_t \,, \quad \textit{for almost every } t \in [0,T], \tag{4.6}$$

where b_t and Ω_t are defined in (4.5).

Proof. It is convenient to set the problem in the intermediate configuration B_t, thus assuming the point of view of the coordinate system of the shape functions.

After performing the change of variables $y = r_t(z)$, $z \in B_t^{\mathrm{ext}}$, it turns out that the velocity field $v_t(z) := R_t^T u_t(r_t(z))$ is the solution of the Stokes problem

$$\begin{cases} v_t \in \mathcal{V}(B_t^{\mathrm{ext}}), \quad v_t = V_t & \text{on } \partial B_t \,, \\ \displaystyle\int_{B_t^{\mathrm{ext}}} \mathrm{E} v_t : \mathrm{E} w \, \mathrm{d}z = 0, & \text{for every } w \in \mathcal{V}_0(B_t^{\mathrm{ext}}), \end{cases}$$

where $V_t(z) = R_t^T U_t(r_t(z))$, see Fig. 2.

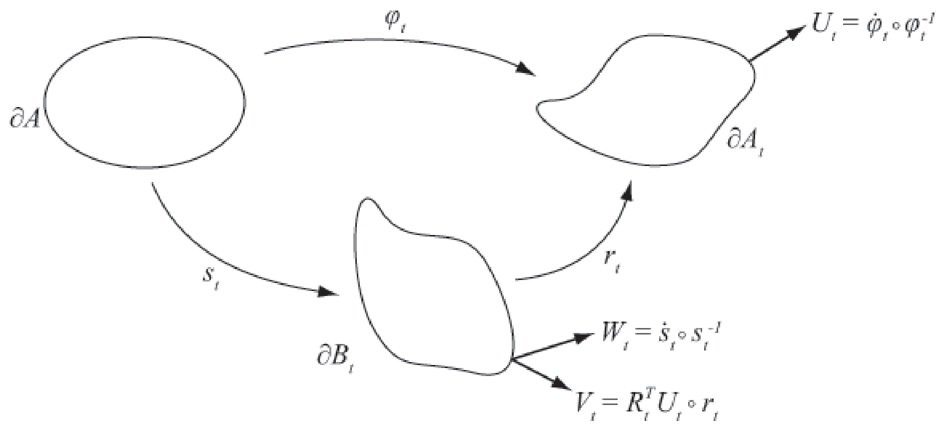

Figure 2: Notation for the boundary velocities.

Let F_{B_t,V_t} and M_{B_t,V_t} be the drag force and torque on ∂B_t determined by v_t according to (2.16) and (2.17), with $\Omega = B_t^{\mathrm{ext}}$. It is easy to check that $F_{B_t,V_t} = R_t^T F_{A_t,U_t}$ and $M_{B_t,V_t} = R_t^T M_{A_t,U_t}$, so that the equations of motion (4.1) reduce to

$$F_{B_t,V_t} = 0 \quad \text{and} \quad M_{B_t,V_t} = 0 \quad \text{for almost every } t \in [0,T]. \tag{4.7}$$

Let ω_t be the *axial vector* of $\dot{R}_t R_t^T$, i.e., the unique vector $\omega_t \in \mathbb{R}^3$ such that $\omega_t \times z = \dot{R}_t R_t^T z$. It is easy to see that $R_t^T \dot{R}_t z = (R_t^T \omega_t) \times z$, so that

$$V_t(z) = W_t(z) + R_t^T \dot{y}_t + (R_t^T \omega_t) \times z \quad \text{for almost every } z \in \partial B_t,$$

where $W_t(z) = \dot{s}_t(s_t^{-1}(z))$. Let $(F_t^{\mathrm{tr}}, M_t^{\mathrm{tr}})$ and $(F_t^{\mathrm{rot}}, M_t^{\mathrm{rot}})$ be the pairs drag force–torque on ∂B_t corresponding to the boundary values $R_t^T \dot{y}_t$ and $(R_t^T \omega_t) \times z$, respectively. It is well known, see, e.g., [10] that

$$\begin{aligned} F_t^{\mathrm{tr}} &= -K_t R_t^T \dot{y}_t \,, & F_t^{\mathrm{rot}} &= -C_t^T R_t^T \omega_t \,, \\ M_t^{\mathrm{tr}} &= -C_t R_t^T \dot{y}_t \,, & M_t^{\mathrm{rot}} &= -J_t R_t^T \omega_t \,, \end{aligned}$$

where K_t, C_t, and J_t are the matrices defined in (4.2). Recalling the linearity of the equations, we get

$$\begin{bmatrix} F_{B_t,V_t} \\ M_{B_t,V_t} \end{bmatrix} = - \begin{bmatrix} K_t R_t^T & C_t^T R_t^T \\ C_t R_t^T & J_t R_t^T \end{bmatrix} \begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} + \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix},$$

hence the equations of motion (4.7) become

$$\begin{bmatrix} K_t & C_t^T \\ C_t & J_t \end{bmatrix} \begin{bmatrix} R_t^T & 0 \\ 0 & R_t^T \end{bmatrix} \begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix}. \tag{4.8}$$

It follows from (4.3) and (4.8) that the equations of motion (4.7) are equivalent to

$$\begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} R_t & 0 \\ 0 & R_t \end{bmatrix} \begin{bmatrix} H_t & D_t^T \\ D_t & L_t \end{bmatrix} \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix} \quad \text{for almost every } t \in [0, T].$$

The first equation reads

$$\dot{y}_t = R_t b_t \,, \quad \text{with } b_t = H_t F_t^{\mathrm{sh}} + D_t^T M_t^{\mathrm{sh}}. \tag{4.9}$$

To write the second equation in the form (4.6), we use the equality $\mathcal{A}(\omega_t) = \dot{R}_t R_t^T$. In order to rewrite the second equation

$$\omega_t = R_t(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}}) \tag{4.10}$$

in a more useful way, we need a formula for $\mathcal{A}(R\omega)$ when R is an arbitrary rotation. In view of the following equalities

$$\mathcal{A}(R\omega)z = (R\omega)\times z = R\omega \times RR^T z = R(\omega \times R^T z) = R\mathcal{A}(\omega)R^T z,$$

we can conclude that $\mathcal{A}(R\omega) = R\mathcal{A}(\omega)R^T$. Therefore, by applying $\mathcal{A}$ to both members of (4.10), we get

$$\dot{R}_t R_t^T = \mathcal{A}(\omega_t) = \mathcal{A}(R_t(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}})) = R_t \mathcal{A}(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}}) R_t^T,$$

so that, eventually, equation (4.10) reads

$$\dot{R}_t = R_t \Omega_t \,, \quad \text{with } \Omega_t = \mathcal{A}(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}}). \tag{4.11}$$

This concludes the proof. □

Remark 4.2. We claim that every absolutely continuous solution to the second equation in (4.6) belongs to $\mathrm{SO}(3)$, whenever $R_0 \in \mathrm{SO}(3)$. Indeed, by differentiating $R_t R_t^T$ with respect to time, we get

$$(R_t R_t^T)^{\cdot} = \dot{R}_t R_t^T + R_t \dot{R}_t^T = R_t \Omega_t R_t^T - R_t \Omega_t R_t^T = 0,$$

where we used the fact that Ω_t is skew symmetric. This shows that the matrix $R_t R_t^T$ is constant in time and the claim follows.

The standard theory of ordinary differential equations with possibly discontinuous coefficients [9], ensures that the Cauchy problem for (4.6) has one and only one Lipschitz solution $t \mapsto R_t$, $t \mapsto y_t$, provided that the functions $t \mapsto \Omega_t$ and $t \mapsto b_t$ are measurable and bounded. By (4.9) and (4.11), this happens when the functions

$$t \mapsto H_t \,, \quad t \mapsto D_t \,, \quad t \mapsto L_t \,, \quad t \mapsto F_t^{\mathrm{sh}}, \quad t \mapsto M_t^{\mathrm{sh}} \tag{4.12}$$

are measurable and bounded. This property for the first three functions follows from the continuity of the block elements of the grand resistance matrix

$$t \mapsto K_t \,, \quad t \mapsto C_t \,, \quad t \mapsto J_t \,, \tag{4.13}$$

which will be proved in the last part of this section. The proof of the measurability and boundedness of the last two functions in (4.12) requires some technical tools that will be developed in Sections 5 and 6.

To prove the continuity of the function in (4.13) we will use Theorem 2.7. To this aim, in the next lemma, we prove a continuity property of the set-valued function $t \mapsto B_t$.

Lemma 4.3. *Let s_t satisfy* (3.12)*. Then if $t \to t_\infty$ the sets B_t converge to the set B_{t_∞} in the sense of* (2.18)*.*

Proof. We recall that $B_t = s_t(A)$ for all $t \in [0, T]$. Let us prove the two inclusions separately. To see that $s_t(A) \subset (s_{t_\infty}(A))^{+\varepsilon}$, consider a point $y \in s_t(A)$: then, there exists a point $x \in A$ such that $y = s_t(x)$. We conclude if we prove that $|s_{t_\infty}(x) - s_t(x)| \leqslant \varepsilon$, for all $x \in A$ and for all t sufficiently close to t_∞.

$$\sup_{x \in A} |s_t(x) - s_{t_\infty}(x)| \leqslant \|s_t - s_{t_\infty}\|_{C^1(A;\mathbb{R}^3)} \leqslant L\,|t - t_\infty| \leqslant \varepsilon,$$

provided that $|t - t_\infty| \leqslant \varepsilon/L$. For the inclusion $(s_{t_\infty}(A))^{-\varepsilon} \subset s_t(A)$, a simple topological degree argument can be applied, so we can conclude the proof. □

We are now in a position to prove the continuity of the elements of the grand resistance matrix.

Proposition 4.4. *Assume that s_t satisfies* (3.6)*,* (3.7)*, and* (3.12)*. Then the functions*

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \tag{4.14a}$$

$$t \mapsto H_t\,, \quad t \mapsto D_t\,, \quad t \mapsto L_t \tag{4.14b}$$

are continuous.

Proof. Recalling (4.2) and (2.14), we can write

$$(K_t)_{ij} = 2 \int_{\mathbb{R}^3} \mathrm{E}v_t^j : \mathrm{E}v_t^i \,\mathrm{d}z, \tag{4.15a}$$

$$(C_t)_{ij} = 2 \int_{\mathbb{R}^3} \mathrm{E}v_t^j : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z, \tag{4.15b}$$

$$(J_t)_{ij} = 2 \int_{\mathbb{R}^3} \mathrm{E}\hat{v}_t^j : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z, \tag{4.15c}$$

where v_t^j and $\hat{v}_t^j$ are the solutions to problem (2.19) for $S_k = B_t$, with $W = e_j$ and $W = e_j \times z$, respectively. Since the convergence of the sets B_t is guaranteed by Lemma 4.3, we can now apply Theorem 2.7 and we obtain that the functions in (4.14a) are continuous. The continuity in (4.14b) follows from (4.3). □

The proof of the measurability and boundedness of $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ requires much more work, due to the fact that both the domains B_t and the boundary data $W_t = \dot{s}_t \circ s_t^{-1}$ depend on time. Moreover, the boundary value W_t might be discontinuous with respect to t, so that we cannot expect the functions $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ to be continuous.

To prove the measurability we start from an integral representation of F_t^{sh} and M_t^{sh}, similar to (4.15). As $\int_{\partial B_t} W_t \cdot n \,\mathrm{d}S$ is not necessarily zero, we have to replace $\mathbb{R}^3$ in (4.15) by the complement of an open ball $\Sigma_\varepsilon^0 \subset\subset B_t$. Since, in general, this inclusion holds only locally in time, we first fix $t_0 \in [0, T]$ and $z^0 \in B_{t_0}$ and select $\delta > 0$ and $\varepsilon > 0$ so that the open ball $\Sigma_\varepsilon^0 := \Sigma_\varepsilon(z^0)$ of radius ε centered at z^0 satisfies

$$\Sigma_\varepsilon^0 \subset\subset B_t\,, \quad \text{for all } t \in I_\delta(t_0) := [0, T] \cap (t_0 - \delta, t_0 + \delta). \tag{4.16}$$

This is possible thanks to the continuity properties of $t \mapsto s_t$ listed in the previous Section.

Next we consider the solution w_t to the problem

$$\min \int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |\mathrm{E}w|^2 \,\mathrm{d}z,$$

where the minimum is taken over all functions $w \in \mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ such that $w = W_t$ on ∂B_t and $w = \lambda_t(z - z^0)/\varepsilon^3$ on $\partial\Sigma_\varepsilon^0$, where

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t} W_t \cdot n \,\mathrm{d}S.$$

The value of λ_t is chosen so that the flux condition (2.2) on $\partial B_t \cup \partial\Sigma_\varepsilon^0$ is satisfied.

Finally, recalling (4.4) and (2.14), we can write the following explicit integral representation of F_t^{sh} and M_t^{sh}

$$(F_t^{\mathrm{sh}})_i = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}v_t^i \,\mathrm{d}z = 2\int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}v_t^i \,\mathrm{d}z, \tag{4.17a}$$

$$(M_t^{\mathrm{sh}})_i = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z = 2\int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z, \tag{4.17b}$$

where v_t^i and $\hat{v}_t^i$ have been defined in the proof of Proposition 4.4 and where the last equalities are due to the fact that $\mathrm{E}v_t^i = \mathrm{E}\hat{v}_t^i = 0$ in B_t. We deduce from Theorem 2.7 and Lemma 4.3 that the functions $t \mapsto v_t^i$ and $t \mapsto \hat{v}_t^i$ are continuous from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$. Therefore, the measurability and boundedness of $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ will be proved if we show that the function $t \mapsto w_t$ from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ is measurable and bounded.

Even the boundedness of $\|\nabla w_t\|_{L^2}$ is an issue, since all estimates for a solenoidal extension of W_t considered so far in the literature depend on the geometry of ∂B_t. In Section 5 we make this dependence explicit and conclude that under our assumptions on $t \mapsto s_t$ the L^2 bound for the gradient of the solenoidal extension is uniform with respect to t. This result will be used in Section 6 to prove the measurability of the function $t \mapsto w_t$.

5 Extension operators

We give now two extension results of a function defined on ∂B_t to an open region containing ∂B_t. Lemma 5.2 is classical, but for our future purposes we need a solenoidal version, as stated in Proposition 5.3. Its proof requires a number of preliminary lemmas that are proved beforehand. The next lemma shows that, locally in time, the sets $\Sigma_\rho \setminus \overline{B}_t$ are C^2 diffeomorphic to each other.

Lemma 5.1. *Assume that s_t satisfies* (3.6), (3.7), *and* (3.12), *and let Σ_ρ be as in* (3.8). *Let $t_0 \in [0,T]$. Then, there exists a neighborhood $I_\delta(t_0) = [0,T] \cap (t_0 - \delta, t_0 + \delta)$ of t_0 with the following property: for every $t \in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Phi_t^{t_0} : \Sigma_\rho \to \Sigma_\rho$, coinciding with the identity on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, such that $\Phi_t^{t_0} = s_{t_0} \circ s_t^{-1}$ on B_t. In particular, we have*

$$\Phi_t^{t_0}(B_t) = B_{t_0} \quad \text{and} \quad \Phi_t^{t_0}(\Sigma_\rho \setminus \overline{B}_t) = \Sigma_\rho \setminus \overline{B}_{t_0}. \tag{5.1}$$

Moreover,

$$\left\|\Phi_t^{t_0}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} + \left\|(\Phi_t^{t_0})^{-1}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant C, \tag{5.2}$$

where C is a constant independent of t_0, t.

Proof. Recall that $B_t \subset\subset \Sigma_{\rho-1}$ by (3.8), so that $B_t \cup (\Sigma_\rho \setminus \overline{\Sigma}_{\rho-1})$ has a C^2 boundary. Therefore, it is possible to find a function $\Psi_t^{t_0} \in C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)$ such that $\Psi_t^{t_0} = s_{t_0} \circ s_t^{-1} - I$ on B_t, $\Psi_t^{t_0} = 0$ on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, and $\left\|\Psi_t^{t_0}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant C\left\|s_{t_0} \circ s_t^{-1} - I\right\|_{C^2(\overline{B}_t;\mathbb{R}^3)}$, where I is the identity map and C is a constant depending only on ρ and t_0 (see, e.g., [8, Theorem 6.37, page 136]). Since $s_{t_0} \circ$

$s_t^{-1} - I \to 0$ in $C^2(\overline{B}_t;\mathbb{R}^3)$ as $t \to t_0$, there exists a neighborhood $I_\delta(t_0)$ of t_0 such that $\left\|\Psi_t^{t_0}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant 1/2$.

For every $t \in I_\delta(t_0)$ let us define $\Phi_t^{t_0} := I + \Psi_t^{t_0}$. Then $\Phi_t^{t_0} = I$ on $\Sigma_\rho \setminus \Sigma_{\rho-1}$ and $\Phi_t^{t_0} = s_{t_0} \circ s_t^{-1}$ on B_t, which proves the first equality in (5.1). Notice that $\left|\Phi_t^{t_0}(x) - x\right| \leqslant 1/2$ for every $x \in \Sigma_\rho$; this implies $\Phi_t^{t_0}(\Sigma_{\rho-1}) \subset \Sigma_\rho$. Since $\Phi_t^{t_0}(\Sigma_\rho \setminus \Sigma_{\rho-1}) = \Sigma_\rho \setminus \Sigma_{\rho-1}$, we conclude that $\Phi_t^{t_0}(\Sigma_\rho) \subset \Sigma_\rho$.

Let us prove that $\Sigma_\rho \subset \Phi_t^{t_0}(\Sigma_\rho)$. Since $\Phi_t^{t_0} = I$ on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, it is enough to show that $\Sigma_{\rho-1} \subset \Phi_t^{t_0}(\Sigma_\rho)$. To this aim we fix $y \in \Sigma_{\rho-1}$. We want to show that there exists $x \in \Sigma_\rho$ such that $x + \Psi_t^{t_0}(x) = y$. This is equivalent to solve the fixed point problem $x = y - \Psi_t^{t_0}(x)$. Since $\left\|\Psi_t^{t_0}\right\|_{C^1(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant 1/2$, the map $x \mapsto y - \Psi_t^{t_0}(x)$ is a contraction of $\overline{\Sigma}_{\rho-1/2}$ into itself. This implies the existence of a fixed point and concludes the proof of the inclusion $\Sigma_{\rho-1} \subset \Phi_t^{t_0}(\Sigma_\rho)$.

The injectivity of $\Phi_t^{t_0}$ follows easily from the inequality $\left\|\Psi_t^{t_0}\right\|_{C^1(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant 1/2$. Therefore, $\Phi_t^{t_0}\colon \Sigma_\rho \to \Sigma_\rho$ is bijective. Its inverse is of class C^2 by the Local Invertibility Theorem. The second equality in (5.1) follows now from the first one.

Estimate (5.2) is a consequence of (3.12) and (3.13). □

Given two Banach spaces X and Y, the symbol $\mathcal{L}(X;Y)$ denotes the Banach space of continuous linear maps from X into Y. Given a function $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$, recalling (2.8), let us define

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t} (\Phi \circ s_t^{-1}) \cdot n \,\mathrm{d}S,$$

for every $t \in [0,T]$. The constant λ_t is chosen so that if $u|_{\partial B_t} = \Phi \circ s_t^{-1}$ and $u|_{\partial\Sigma_\rho} = \lambda_t z/|z|^3$, then

$$\int_{\partial(B_t^{\mathrm{ext}} \cap \Sigma_\rho)} u \cdot n \,\mathrm{d}S = 0.$$

Lemma 5.2 (Extension operators). *Under the assumptions of Lemma 5.1, there exists a continuous function $t \mapsto \mathcal{S}_t$ from $I_\delta(t_0)$ into $\mathcal{L}(H^{1/2}(\partial A;\mathbb{R}^3); H^1(\Sigma_\rho;\mathbb{R}^3))$ such that*

$$\begin{aligned} \mathcal{S}_t(\Phi) &= \Phi \circ s_t^{-1} \quad \textit{on } \partial B_t\,, \\ \mathcal{S}_t(\Phi) &= \lambda_t \frac{z}{|z|^3} \quad \textit{on } \partial\Sigma_\rho\,, \\ \|\mathcal{S}_t(\Phi)\|_{H^1(\Sigma_\rho;\mathbb{R}^3)} &\leqslant C\,\|\Phi\|_{H^{1/2}(\partial A;\mathbb{R}^3)}\,, \end{aligned}$$

where the constant C is independent of t and Φ.

Proof. By known results on Sobolev spaces [15, Theorem 5.7, page 103], there exists $\mathcal{S}_{t_0} \in \mathcal{L}(H^{1/2}(\partial A;\mathbb{R}^3); H^1(\Sigma_\rho;\mathbb{R}^3))$ such that $\mathcal{S}_{t_0}(\Phi) = \Phi \circ s_{t_0}^{-1}$ on ∂B_{t_0}. Let $\Phi_t^{t_0}$ be the function given in the proof of Lemma 5.1. It is easy to show that $[\mathcal{S}_{t_0}(\Phi)] \circ \Phi_t^{t_0} = \Phi \circ s_t^{-1}$ on ∂B_t. It is enough to define $\mathcal{S}_t(\Phi) = [\mathcal{S}_{t_0}(\Phi)] \circ \Phi_t^{t_0}$. □

Proposition 5.3 (Solenoidal extension operators). *Under the assumptions of Lemma 5.1, let $t_0 \in [0,T]$ and let $z^0 \in B_{t_0}$. Let $\delta > 0$ and $\varepsilon > 0$ be such that* (4.16) *holds true. Then there exists a uniformly bounded family $(\mathcal{T}_t)_{t \in I_\delta(t_0)}$ of continuous linear operators*

$$\mathcal{T}_t\colon H^{1/2}(\partial A;\mathbb{R}^3) \to H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$$

such that

(i) for all $t \in I_\delta(t_0)$ and for all $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$,

$$\mathcal{T}_t(\Phi) = \Phi \circ s_t^{-1} \quad \text{on } \partial B_t\,, \tag{5.3a}$$

$$\mathcal{T}_t(\Phi) = \lambda_t \frac{z}{|z|^3} \quad \text{on } \partial\Sigma_\rho\,, \tag{5.3b}$$

$$\operatorname{div}(\mathcal{T}_t(\Phi)) = 0 \quad \text{in } \Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon\,; \tag{5.3c}$$

(ii) for every $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$ the map $t \mapsto \mathcal{T}_t(\Phi)$ is continuous from $I_\delta(t_0)$ into $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$.

In particular, the following estimate holds

$$\|\mathcal{T}_t(\Phi)\|_{H^1(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)} \leqslant C\,\|\Phi\|_{H^{1/2}(\partial A;\mathbb{R}^3)}\,, \tag{5.4}$$

where the constant C is independent of t and Φ.

The proof of Proposition 5.3 requires the estimates contained in the following lemmas.

Lemma 5.4. *For every bounded open set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, there exists a constant $C_1(\Omega) > 0$ such that*

$$\|p\|_{L^2(\Omega)} \leqslant C_1(\Omega)\big(\,\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)} + \|p\|_{H^{-1}(\Omega)}\,\big)$$

for every $p \in L^2(\Omega)$.

The proof can be found in Lemma 7.1 in [15, page 187].

Lemma 5.5. *For every bounded connected open set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, there exists a constant $C_2(\Omega) > 0$ such that*

$$\inf_{t\in\mathbb{R}} \|p - t\|_{H^{-1}(\Omega)} \leqslant C_2(\Omega)\,\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)}\,, \tag{5.5}$$

for every $p \in L^2(\Omega)$.

Proof. First, let us notice that the infimum in (5.5) is attained. Indeed,

$$\|p - t\|_{H^{-1}(\Omega)} = |t|\left\|\frac{p}{t} - 1\right\|_{H^{-1}(\Omega)} \to +\infty \quad \text{as } t \to \pm\infty.$$

Let us assume that (5.5) does not hold. Then, there exists a sequence $(p_k)_k$ such that

$$\|p_k - t_k\|_{H^{-1}(\Omega)} > k\,\|\nabla p_k\|_{H^{-1}(\Omega;\mathbb{R}^3)}\,,$$

where t_k is the optimal constant for p_k. It is not restrictive to assume the left hand side of the last inequality to be equal to 1, and that $t_k = 0$ for every k. Indeed, we can take $\widetilde{p}_k := \|p_k - t_k\|^{-1}_{H^{-1}(\Omega)}\,(p_k - t_k)$ and for every $\tau \in \mathbb{R}$ we obtain

$$\begin{aligned}
\|\widetilde{p}_k - \tau\|_{H^{-1}(\Omega)} &= \left\|\frac{p_k - t_k}{\|p_k - t_k\|_{H^{-1}(\Omega)}} - \tau\right\|_{H^{-1}(\Omega)} \\
&= \frac{1}{\|p_k - t_k\|_{H^{-1}(\Omega)}}\Big\|p_k - t_k - \tau\,\|p_k - t_k\|_{H^{-1}(\Omega)}\Big\|_{H^{-1}(\Omega)} \\
&\geqslant \frac{\|p_k - t_k\|_{H^{-1}(\Omega)}}{\|p_k - t_k\|_{H^{-1}(\Omega)}} = 1 = \|\widetilde{p}_k\|_{H^{-1}(\Omega)}\,,
\end{aligned} \tag{5.6}$$

so that the minimum for $\widetilde{p}_k$ is attained at $\tau = 0$. Moreover, we have

$$1 = \|\widetilde{p}_k\|_{H^{-1}(\Omega)} > k\,\|\nabla\widetilde{p}_k\|_{H^{-1}(\Omega;\mathbb{R}^3)}\,.$$

Therefore, $\|\nabla\widetilde{p}_k\|_{H^{-1}(\Omega;\mathbb{R}^3)} \to 0$, $\|\widetilde{p}_k\|_{H^{-1}(\Omega)} = 1$, and $\|\widetilde{p}_k\|_{L^2(\Omega)} \leqslant C < +\infty$ by Lemma 5.4. This implies that $(\widetilde{p}_k)_k$ converges strongly in $H^{-1}(\Omega)$ and weakly in $L^2(\Omega)$ to a function $\widetilde{p}$ with $\|\widetilde{p}\|_{H^{-1}(\Omega)} = 1$. Notice that $\nabla\widetilde{p}_k \to \nabla\widetilde{p}$ in $H^{-1}(\Omega;\mathbb{R}^3)$. We conclude that $\nabla\widetilde{p} = 0$, so that $\widetilde{p} = K$, a constant. If $K = 0$, this contradicts the equality $\|\widetilde{p}\|_{H^{-1}(\Omega)} = 1$. Without loss of generality, we may assume $K > 0$. From (5.6) we get

$$(K-\tau)\,\|1\|_{H^{-1}(\Omega)} = \|\widetilde{p}-\tau\|_{H^{-1}(\Omega)} \geqslant \|\widetilde{p}\|_{H^{-1}(\Omega)} = K\,\|1\|_{H^{-1}(\Omega)}\,,$$

and this is clearly false for $0 < \tau < K$. The lemma is proved. □

Lemma 5.6. *For every bounded connected open set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, there exists a constant $C_3(\Omega) > 0$ such that*

$$\|p\|_{L^2(\Omega)} \leqslant C_3(\Omega)\,\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)}\,,$$

for every $p \in L^2(\Omega)$ with $\int_\Omega p\,\mathrm{d}x = 0$.

Proof. Let us fix $p \in L^2(\Omega)$ with $\int_\Omega p\,\mathrm{d}x = 0$. By Lemma 5.4 for every $t \in \mathbb{R}$, we have

$$\|p\|_{L^2(\Omega)} \leqslant \|p-t\|_{L^2(\Omega)} \leqslant C_1(\Omega)\left(\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)} + \|p-t\|_{H^{-1}(\Omega)}\right).$$

Taking the infimum with respect to t and using Lemma 5.5, we obtain

$$\|p\|_{L^2(\Omega)} \leqslant C_3(\Omega)\,\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)}\,,$$

where $C_3(\Omega) := C_1(\Omega)(1 + C_2(\Omega))$. □

The constant $C_3(\Omega)$ plays a crucial role in the following result concerning the estimate of a particular solution of the equation $\operatorname{div} u = g$ in Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$.

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded connected open set with Lipschitz boundary and let $g \in L^2(\Omega)$ with $\int_\Omega g\,\mathrm{d}x = 0$. Then there exists a unique $u \in H^1_0(\Omega;\mathbb{R}^3)$ such that*

(i) $\operatorname{div} u = g$ *in* Ω,

(ii) $\int_\Omega \nabla u : \nabla v\,\mathrm{d}x = 0$ *for all* $v \in H^1_0(\Omega;\mathbb{R}^3)$ *with* $\operatorname{div} v = 0$ *in* Ω.

Moreover, the following estimate holds

$$\|u\|_{H^1_0(\Omega;\mathbb{R}^3)} \leqslant C_3(\Omega)\,\|g\|_{L^2(\Omega)}\,, \tag{5.7}$$

where $C_3(\Omega)$ is the constant in Lemma 5.6.

Proof. Let X be the subspace of $L^2(\Omega)$ determined by the condition $\int_\Omega g\,\mathrm{d}x = 0$, and let $Y = H^1_0(\Omega;\mathbb{R}^3)$, endowed with the scalar product $(u\,|\,v)_Y := \int_\Omega \nabla u : \nabla v\,\mathrm{d}x$, so that $Y' = H^{-1}(\Omega;\mathbb{R}^3)$. Let $A : X \to Y'$ be the operator defined by $Ag = \nabla g$, and let $A' : Y \to X$ be its conjugate operator, given by $A'u = -\operatorname{div} u$. Let $J : Y' \to Y$ be the Riesz operator, defined by $\langle f, u\rangle = (Jf\,|\,u)_Y$ for every $u \in Y$ and for every $f \in Y'$, where $\langle\cdot,\cdot\rangle$ denotes the duality product between $Y' = H^{-1}(\Omega;\mathbb{R}^3)$ and $Y = H^1_0(\Omega;\mathbb{R}^3)$. Notice that in our case $u = Jf$ if and only if u is the weak solution of the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u \in H^1_0(\Omega;\mathbb{R}^3). \tag{5.8}$$

Let $(JA)^* : Y \to X$ be the Hilbert space adjoint of the operator $JA : X \to Y$. Let us prove that $(JA)^* = A'$. For every $g \in X$ and every $u \in Y$, we have

$$((JA)^* u \mid g)_X = (JAg \mid u)_Y = \langle Ag, u \rangle = (A'u \mid g)_X \,,$$

where $(\cdot \mid \cdot)_X$ denotes the scalar product of $L^2(\Omega)$ on X. This implies $(JA)^* = A'$.

By Lemma 5.6 the range $R(A)$ of A is closed in $Y' = H^{-1}(\Omega; \mathbb{R}^3)$. Since $R(JA) = J(R(A))$, we conclude that $R(JA)$ is closed in Y, hence $R(JA) = N((JA)^*)^\perp$, where N denotes the kernel and $\perp$ denotes the orthogonal complement in $Y = H^1_0(\Omega; \mathbb{R}^3)$.

Since $R(A)$ is closed, $R(A') = R((JA)^*)$ is closed too, by Banach's Closed Range Theorem [22, page 205].

We now want to prove that $A' = (JA)^*$ is an isomorphism from $R(JA) = N((JA)^*)^\perp = N(A')^\perp$ into X. Indeed, if $u \in N((JA)^*)^\perp$, the equality $(JA)^* u = 0$ implies that $u \in N((JA)^*)$, so that $u = 0$; this proves injectivity. Moreover, it is clear that the image of $N((JA)^*)^\perp$ under the map $(JA)^*$ coincides with $R((JA)^*)$. Since this is closed, we have $R((JA)^*) = N(JA)^\perp = \{0\}^\perp = X$, where the second equality follows from the injectivity of A, and $\perp$ denotes now the orthogonal complement in X.

This concludes the proof the fact that A' is an isomorphism from $N(A')^\perp$ into X, and this implies (i) and (ii).

To achieve estimate (5.7), let us fix u and g satisfying (i) and (ii). By (i), $u \in N(A')^\perp = R(JA)$, hence there exists $h \in X$ such that $u = JAh$. It follows that

$$\begin{aligned} \|u\|^2_{H^1_0(\Omega)} &= \langle Ah, u \rangle = (h \mid A'u)_{L^2(\Omega)} = (h \mid g)_{L^2(\Omega)} \\ &\leqslant \|h\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} \\ &\leqslant C_3(\Omega) \|\nabla h\|_{H^{-1}(\Omega;\mathbb{R}^3)} \|g\|_{L^2(\Omega)} = C_3(\Omega) \|u\|_{H^1_0(\Omega)} \|g\|_{L^2(\Omega)} \,, \end{aligned}$$

where the last equality follows from (5.8) with $f = \nabla h$. This implies (5.7). □

To prove Proposition 5.3 we have to show that the constants $C_3(B_t)$ and $C_3(\Sigma_\rho \setminus \overline{B}_t)$ are uniformly bounded with respect to t. The following lemma will be used, together with Lemma 5.1, to obtain a uniform bound for the constants $C_3(B_t)$ and $C_3(\Sigma_\rho \setminus \overline{B}_t)$ under C^2 diffeomorphisms. Let $\Omega \subset \mathbb{R}^3$ be a bounded open subset with C^2 boundary, and let $\Phi \in C^2(\mathbb{R}^3; \mathbb{R}^3)$. Let $\Lambda := \Phi(\Omega)$ and let $\Psi \in C^2(\mathbb{R}^3; \mathbb{R}^3)$ be the inverse of Φ, so that $\Omega = \Psi(\Lambda)$. Then, the following estimate holds.

Lemma 5.8. *There exists a non decreasing function $a : [0, +\infty) \to [0, +\infty)$ such that the constants C_i introduced in Lemmas 5.4, 5.5, and 5.6 satisfy the estimate*

$$C_i(\Lambda) \leqslant a(C_\Phi + C_\Psi) C_i(\Omega), \qquad \textit{for } i = 1, 2, 3,$$

where $C_\Phi := \max\{|\Phi|, |\nabla\Phi|, |\nabla^2\Phi|\}$ and $C_\Psi := \max\{|\Psi|, |\nabla\Psi|, |\nabla^2\Psi|\}$.

Proof. We prove the lemma only for C_2. The proof for C_1 is similar. The result for C_3 follows from the equality $C_3 = C_1(1 + C_2)$. Let $q = p \circ \Psi$ and $p = q \circ \Phi$. Then,

$$\begin{aligned} \|q - t\|_{H^{-1}(\Lambda)} &= \sup_{\substack{g \in H^1_0(\Lambda) \\ \|g\|=1}} \langle q - t, g \rangle = \sup_{\substack{g \in H^1_0(\Lambda) \\ \|g\|=1}} \langle p \circ \Psi - t, g \rangle \\ &= \sup_{\substack{g \in H^1_0(\Lambda) \\ \|g\|=1}} \langle p - t, g(\Phi) \, |\det \nabla\Phi| \rangle \\ &\leqslant b(C_\Phi) \|p - t\|_{H^{-1}(\Omega)} \,, \end{aligned}$$

where b is a suitable increasing function. By (5.5), there exists an increasing function a such that

$$\begin{aligned}\inf_{t\in\mathbb{R}} \|q-t\|_{H^{-1}(\Omega)} &\leqslant b(C_\Phi)C_2(\Omega)\,\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)}\\ &\leqslant a(C_\Phi+C_\Psi)C_2(\Omega)\,\|\nabla q\|_{H^{-1}(\Lambda;\mathbb{R}^3)}\,.\end{aligned}$$

This concludes the proof. ◻

Let Σ_ρ be as in (3.8) and t_0, z^0, δ, ε, $I_\delta(t_0)$, and Σ^0_ε be as in Proposition 5.3. For every $t\in I_\delta(t_0)$ let $\mathcal{U}_t:\big\{g\in L^2(B_t\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3):\int_{B_t\setminus\overline{\Sigma}^0_\varepsilon} g\,\mathrm{d}z=0\big\}\to H^1_0(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ be the linear operator defined by $\mathcal{U}_t(g)=u$, where $u|_{B_t\setminus\overline{\Sigma}^0_\varepsilon}$ is the unique function in $H^1_0(B_t\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ such that

$$\operatorname{div} u = g \quad\text{in } B_t\setminus\overline{\Sigma}^0_\varepsilon, \tag{5.9a}$$

$$\int_{B_t\setminus\overline{\Sigma}^0_\varepsilon} \nabla u:\nabla v\,\mathrm{d}z=0 \quad\text{for all } v\in H^1_0(B_t\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3):\operatorname{div} v=0 \text{ in } B_t\setminus\overline{\Sigma}^0_\varepsilon, \tag{5.9b}$$

and $u=0$ in $(\Sigma_\rho\setminus\overline{B}_t)$. By Lemmas 5.1, 5.7, and 5.8, there exists a constant M, independent of t, such that

$$\|\mathcal{U}_t\|_{\mathcal{L}_t}\leqslant M, \tag{5.10}$$

where $\mathcal{L}_t$ is the Banach space of continuous linear operators from $\big\{g\in L^2(B_t\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3):\int_{B_t\setminus\overline{\Sigma}^0_\varepsilon} g\,\mathrm{d}z=0\big\}$ into $H^1_0(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$.

Lemma 5.9. *Assume* (3.6)*,* (3.7)*,* (3.9)*, and* (3.12)*. Let $t_0\in[0,T]$ and let $t_k\in I_\delta(t_0)$, $k=1,2,\dots,\infty$, and let $g\in L^2(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon)$ with $\int_{\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon} g\,\mathrm{d}z=0$ and*

$$\operatorname{supp}(g)\subset\subset B_{t_k}\setminus\overline{\Sigma}^0_\varepsilon \quad \textit{for every } k. \tag{5.11}$$

Assume that $t_k\to t_\infty$ as $k\to\infty$. Then $\mathcal{U}_{t_k}(g)\to\mathcal{U}_{t_\infty}(g)$ strongly in $H^1_0(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. A similar result holds if we exchange the roles of $B_{t_k}\setminus\overline{\Sigma}^0_\varepsilon$ and $\Sigma_\rho\setminus\overline{B}_{t_k}$ in the definition of $\mathcal{U}_t$ and in (5.11)*.*

Proof. For $k=1,2,\dots,\infty$, let $u_{t_k}:=\mathcal{U}_{t_k}(g)$. By (5.10), the sequence $(u_{t_k})_k$ is bounded in $H^1_0(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. Therefore a subsequence, still denoted by $(u_{t_k})_k$, converges weakly in $H^1_0(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ to some function u^*.

We claim that $u^*\in H^1_0(B_{t_\infty}\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. First notice that $u_{t_k}\circ(s_{t_k}\circ s^{-1}_{t_\infty})=0$ on ∂B_{t_∞}, hence $u_{t_k}\circ(s_{t_k}\circ s^{-1}_{t_\infty})\in H^1_0(B_{t_\infty}\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. Since $s_{t_k}\circ s^{-1}_{t_\infty}\to I$ in $C^1(\overline{B}_{t_\infty}\setminus\Sigma^0_\varepsilon;\mathbb{R}^3)$ as $k\to\infty$, and $u_{t_k}\rightharpoonup u^*$ weakly in $H^1(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$, we obtain $u_{t_k}\circ(s_{t_k}\circ s^{-1}_{t_\infty})\rightharpoonup u^*$ weakly in $H^1(B_{t_\infty};\mathbb{R}^3)$. This implies that $u^*\in H^1_0(B_{t_\infty}\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ and proves the claim.

Since $\operatorname{supp}(g)\subset\subset B_{t_k}\setminus\overline{\Sigma}^0_\varepsilon$ for every k, condition (i) in Lemma 5.7 gives $\operatorname{div} u_{t_k}=g$ in $\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon$ for every k, hence $\operatorname{div} u^*=g$ in Σ_ρ.

If $v\in C^\infty_{\mathrm{c}}(B_{t_\infty}\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ with $\operatorname{div} v=0$, from (ii) we have

$$\int_{B_{t_k}\setminus\overline{\Sigma}^0_\varepsilon}\nabla u_{t_k}:\nabla v\,\mathrm{d}z=0,\quad\text{for } k \text{ large enough.}$$

Passing to the limit as $k\to\infty$, we get

$$\int_{B_{t_\infty}\setminus\overline{\Sigma}^0_\varepsilon}\nabla u^*:\nabla v\,\mathrm{d}z=0.$$

An approximation argument based on Theorem 2.2 gives the same equality for every $v\in H^1_0(B_{t_\infty}\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ with $\operatorname{div} v=0$. By the uniqueness result proved in Lemma 5.7, we have $u^*=u_{t_\infty}$.

To prove the strong convergence of $(u_{t_k})_k$ in $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$, we fix a connected open set B with Lipschitz boundary such that $\mathrm{supp}(g) \subset\subset B \subset\subset B_{t_k} \setminus \overline{\Sigma}^0_\varepsilon$ for every k. By Lemma 5.7, there exists $w \in H^1_0(B;\mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{div} w = g & \text{on } B, \\ \displaystyle\int_B \nabla w : \nabla v \,\mathrm{d}z = 0 & \text{for every } v \in H^1_0(B;\mathbb{R}^3) \text{ with } \operatorname{div} v = 0. \end{cases}$$

We extend w by setting $w = 0$ on $(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon) \setminus \overline{B}$. Since $\mathrm{supp}(g) \subset\subset B$, we have $\operatorname{div} w = g$ on $\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon$.

We take $v = u_{t_k} - w$ as test function in condition (ii) and we obtain

$$\int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} |\nabla u_{t_k}|^2 \,\mathrm{d}z = \int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} \nabla u_{t_k} : \nabla v \,\mathrm{d}z, \quad \text{for } k = 1, 2, \dots, \infty.$$

Since $\nabla u_{t_k} \rightharpoonup \nabla u_{t_\infty}$ in $L^2(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{M}^{3\times 3})$, taking the limit as $k \to \infty$ we get

$$\int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} |\nabla u_{t_k}|^2 \,\mathrm{d}z \to \int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} |\nabla u_{t_\infty}|^2 \,\mathrm{d}z,$$

which concludes the proof of the strong convergence in $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. □

Lemma 5.10. *Under the hypotheses of Lemma 5.9, let $t \mapsto g_t$ be a continuous function from $I_\delta(t_0)$ into $L^2(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon)$, endowed with the strong topology, and let $\mathcal{U}_t$ be the operator defined in* (5.9)*. Assume that*

$$\int_{B_t \setminus \overline{\Sigma}^0_\varepsilon} g_t \,\mathrm{d}z = 0 \quad \textit{for every } t \in I_\delta(t_0). \tag{5.12}$$

Then the function $t \mapsto \mathcal{U}_t(g_t)$ is continuous from $I_\delta(t_0)$ into $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$, endowed with the strong topology. A similar result holds if we exchange the roles of $B_t \setminus \overline{\Sigma}^0_\varepsilon$ and $\Sigma_\rho \setminus \overline{B}_t$ in the definition of $\mathcal{U}_t$ and in (5.12)*.*

Proof. Let us fix $\tau \in I_\delta(t_0)$ and $\eta > 0$. There exists $h \in L^2(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon)$ with compact support in B_τ such that

$$\|h - g_\tau\|_{L^2(B_\tau \setminus \overline{\Sigma}^0_\varepsilon)} < \eta.$$

By continuity, for t sufficiently close to τ we have

$$\|h - g_t\|_{L^2(B_t \setminus \overline{\Sigma}^0_\varepsilon)} < \eta,$$

and $\mathrm{supp}(h) \subset\subset B_t \setminus \overline{\Sigma}^0_\varepsilon$. By (5.10) we have

$$\begin{aligned}
&\|\mathcal{U}_t(g_t) - \mathcal{U}_\tau(g_\tau)\|_{H^1} \\
&\leqslant \|\mathcal{U}_t(g_t - h)\|_{H^1} + \|\mathcal{U}_t(h) - \mathcal{U}_\tau(h)\|_{H^1} + \|\mathcal{U}_\tau(h - g_\tau)\|_{H^1} \\
&\leqslant \|\mathcal{U}_t\|_{\mathcal{L}_t} \|g_t - h\|_{L^2(B_t \setminus \overline{\Sigma}^0_\varepsilon)} + \|\mathcal{U}_t(h) - \mathcal{U}_\tau(h)\|_{H^1} + \|\mathcal{U}_\tau\|_{\mathcal{L}_\tau} \|h - g_\tau\|_{L^2(B_\tau \setminus \overline{\Sigma}^0_\varepsilon)} \\
&\leqslant M\eta + \|\mathcal{U}_t(h) - \mathcal{U}_\tau(h)\| + M\eta.
\end{aligned}$$

Lemma 5.9 yields

$$\limsup_{t \to \tau} \|\mathcal{U}_t(g_t) - \mathcal{U}_\tau(g_\tau)\|_{H^1} \leqslant 2M\eta.$$

As η is arbitrary, we have shown that $\mathcal{U}_t(g_t) \to \mathcal{U}_\tau(g_\tau)$ strongly in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. □

Proof of Proposition 5.3. For all $t \in I_\delta(t_0)$, let $\zeta_t := \mathcal{S}_t(\Phi)$ be the extension given by Lemma 5.2. Define $g_t^{\rm int}$ and $g_t^{\rm ext}$ as $\operatorname{div}(\zeta_t)$ restricted to $B_t \setminus \overline{\Sigma}_\varepsilon^0$ and $\Sigma_\rho \setminus \overline{B}_t$, respectively. An easy computation shows that

$$\int_{B_t\setminus\overline{\Sigma}_\varepsilon^0} g_t^{\rm int}\,\mathrm{d}z = \int_{\Sigma_\rho\setminus\overline{B}_t} g_t^{\rm ext}\,\mathrm{d}z = 0.$$

Therefore, there exist functions $u_t^{\rm int} \in H_0^1(B_t\setminus\overline{\Sigma}_\varepsilon^0;\mathbb{R}^3)$ and $u_t^{\rm ext} \in H_0^1(\Sigma_\rho\setminus\overline{B}_t;\mathbb{R}^3)$ satisfying conditions (i) and (ii) of Lemma 5.7. One can define $u_t = \mathcal{U}_t(g_t)$ as the function defined by $u_t^{\rm int}$ on $B_t\setminus\overline{\Sigma}_\varepsilon^0$ and by $u_t^{\rm ext}$ on $\Sigma_\rho\setminus\overline{B}_t$. Notice that u_t agrees with zero on ∂B_t, on $\partial\Sigma_\rho$, and on $\partial\Sigma_\varepsilon^0$.

Consider now $\mathcal{T}_t(\Phi) := \mathcal{S}_t(\Phi) - \mathcal{U}_t(g_t) = \zeta_t - u_t$. This extension is clearly in $H^1(\Sigma_\rho\setminus\overline{\Sigma}_\varepsilon^0;\mathbb{R}^3)$ and agrees with (5.3) so that (i) is satisfied. Moreover, by the continuity properties of $\mathcal{S}_t$ and $\mathcal{U}_t$, also $\mathcal{T}_t$ is continuous from $I_\delta(t_0)$ into $H^1(\Sigma_\rho\setminus\overline{\Sigma}_\varepsilon^0;\mathbb{R}^3)$, so that (ii) and estimate (5.4) follow. □

6 Dependence on the data

Using the tools developed in the preceding section, we are finally ready to prove some results concerning continuity and measurability properties of the solutions to the Stokes problems. These, eventually, will lead us to the statement of Theorem 6.4 about the existence, uniqueness, and regularity of the rigid motion $t \mapsto r_t$ that causes the swimmer's displacement in the viscous fluid.

Proposition 6.1. *Assume that s_t satisfies* (3.6)*,* (3.7)*, and* (3.12)*. Let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$, and let Σ_ε^0 and $I_\delta(t_0)$ be as in* (4.16)*. Suppose, in addition, that $I_\delta(t_0)$ satisfies Lemma 5.1. Let the map $t\mapsto\Phi_t$ belong to $C^0(I_\delta(t_0);H^{1/2}(\partial A;\mathbb{R}^3))\cap L^\infty(I_\delta(t_0);\mathrm{Lip}(\partial A;\mathbb{R}^3))$. Define*

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t}(\Phi_t\circ s_t^{-1})\cdot n\,\mathrm{d}S.$$

Let w_t be the solution of the problem

$$\min\int_{\Sigma_\varepsilon^{0,\rm ext}}|\mathrm{E}w|^2\,\mathrm{d}z, \tag{6.1}$$

where the minimum is taken over all functions $w \in \mathcal{V}(\Sigma_\varepsilon^{0,\rm ext})$ such that $w = \Phi_t\circ s_t^{-1}$ on ∂B_t and $w = \lambda_t(z-z^0)/\varepsilon^3$ on $\partial\Sigma_\varepsilon^0$. Then $t\mapsto w_t$ belongs to $C^0(I_\delta(t_0);\mathcal{V}(\Sigma_\varepsilon^{0,\rm ext}))$.

Proof. Let $(t_k)_k \subset I_\delta(t_0)$ be a sequence that converges to $t_\infty \in I_\delta(t_0)$. Let ψ_{t_k} be the extension of $\Phi_{t_k}\circ s_{t_k}^{-1}$ provided by Proposition 5.3. It can be further extended by $\lambda_t z/|z|^3$ on $\mathbb{R}^3\setminus\Sigma_\rho$, so that $\psi_{t_k}\in\mathcal{V}(\Sigma_\varepsilon^{0,\rm ext})$ and is a competitor in the minimum problem (6.1) corresponding to $t = t_k$; therefore,

$$\begin{aligned}\int_{\Sigma_\varepsilon^{0,\rm ext}}|\mathrm{E}w_{t_k}|^2\,\mathrm{d}z &\leqslant \int_{\Sigma_\varepsilon^{0,\rm ext}}|\mathrm{E}\psi_{t_k}|^2\,\mathrm{d}z \leqslant \|\psi_{t_k}\|^2_{H^1(\Sigma_\rho\setminus\overline{\Sigma}_\varepsilon^0;\mathbb{R}^3)}\\ &\leqslant C^2(\mathrm{Lip}(\Phi_{t_k})+\max|\Phi_{t_k}|)^2 \leqslant (CM)^2,\end{aligned}$$

where C is the constant in (5.4) and $M > 0$ is a uniform upper bound of $\mathrm{Lip}(\Phi_{t_k})+\max|\Phi_{t_k}|$, whose existence is guaranteed by the fact that $t\mapsto\Phi_t$ belongs to $L^\infty(I_\delta(t_0);\mathrm{Lip}(\partial A;\mathbb{R}^3))$. Thus, the sequence $(w_{t_k})_k$ is equi-bounded in $\mathcal{V}(\Sigma_\varepsilon^{0,\rm ext})$ and, up to a subsequence, it converges weakly to some $w^*\in\mathcal{V}(\Sigma_\varepsilon^{0,\rm ext})$.

We claim that w^* is a competitor in problem (6.1) for $t = t_\infty$. First, notice that $\Phi_{t_k}\circ s_{t_\infty}^{-1} = w_{t_k}\circ(s_{t_k}\circ s_{t_\infty}^{-1})$ on ∂B_{t_∞}. Let $\Phi_{t_\infty}^{t_k}$ be the extension of $s_{t_k}\circ s_{t_\infty}^{-1}$

considered in Lemma 5.1. Arguing as in the proof of that lemma, we find that $\Phi^{t_k}_{t_\infty} \to I$ in $C^1(\overline{\Sigma}_\rho;\mathbb{R}^3)$ as $t_n \to t_\infty$. Since $w_{t_k} \rightharpoonup w^*$ weakly in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$, we obtain that $w_{t_k} \circ \Phi^{t_k}_{t_\infty} \rightharpoonup w^*$ weakly in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. This implies that $w_{t_k} \circ (s_{t_k} \circ s^{-1}_{t_\infty}) \rightharpoonup w^*$ weakly in $H^{1/2}(\partial B_{t_\infty};\mathbb{R}^3)$. On the other hand, $\Phi_{t_k} \circ s^{-1}_{t_\infty} \to \Phi_{t_\infty} \circ s^{-1}_{t_\infty}$ in $H^{1/2}(\partial B_{t_\infty};\mathbb{R}^3)$. As $\Phi_{t_k} \circ s^{-1}_{t_\infty} = w_{t_k} \circ (s_{t_k} \circ s^{-1}_{t_\infty})$ on ∂B_{t_∞}, we deduce that $w^* = \Phi_{t_\infty} \circ s^{-1}_{t_\infty}$ on ∂B_{t_∞}. This concludes the claim.

Let $v \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ be another competitor in problem (6.1) for $t = t_\infty$, and let $\zeta := v - \psi_{t_\infty}$, where $\psi_{t_\infty} := \mathcal{T}_{t_\infty}(\Phi_{t_\infty})$ is the extension provided by Proposition 5.3, extended by zero on $\mathbb{R}^3 \setminus \Sigma_\rho$. The function ζ vanishes on ∂B_{t_∞} and its restrictions to B_{t_∞} and $B^{\mathrm{ext}}_{t_\infty}$ belong to $H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ and $\mathcal{V}_0(B^{\mathrm{ext}}_{t_\infty};\mathbb{R}^3)$, respectively. Then by the Density Theorem 2.2 and by a classical density result in $H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$, for every $\eta > 0$, there exist a function $\zeta^\eta \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$, vanishing in a neighborhood of ∂B_{t_∞}, such that $\|\zeta^\eta - \zeta\|_{D^{1,2}(\Sigma^{0,\mathrm{ext}}_\varepsilon;\mathbb{R}^3)} \leqslant \eta$. Define now $v^\eta_{t_k} := \psi_{t_k} + \zeta^\eta$, and observe that, for k large enough, it is a competitor in the minimum problem (6.1) for $t = t_k$. Therefore,

$$\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w_{t_k}|^2 \,\mathrm{d}z \leqslant \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}v^\eta_{t_k}|^2 \,\mathrm{d}z = \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}\psi_{t_k} + \mathrm{E}\zeta^\eta|^2 \,\mathrm{d}z.$$

Taking the limit first as $k \to \infty$ and then as $\eta \to 0$, we get

$$\begin{aligned}\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w^*|^2 \,\mathrm{d}z &\leqslant \limsup_{k\to\infty} \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w_{t_k}|^2 \,\mathrm{d}z \\ &\leqslant \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}\psi_{t_\infty} + \mathrm{E}\zeta|^2 \,\mathrm{d}z = \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}v|^2 \,\mathrm{d}z,\end{aligned}$$

where the convergence of $\mathrm{E}\psi_{t_k}$ to $\mathrm{E}\psi_{t_\infty}$ is guaranteed as a consequence of (ii) in Proposition 5.3. This proves that w^* is a minimum, so that $w^* = w_{t_\infty}$. By taking $v = w^*$, we get the convergence of the $D^{1,2}$ norms, therefore $w_{t_k} \to w_{t_\infty}$ strongly in $\mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$. This concludes the proof. □

We notice that Theorem 2.7 turns out to be a particular case of Proposition 6.1, for special boundary data not depending on time. Nonetheless, we think it is useful to present both results, since the technique of the proof is much easier in Theorem 2.7.

As we have seen at the end of Section 4, Theorem 2.7 applied to purely linear and purely angular boundary velocities guarantees the continuity of the elements of the matrices in (4.3), while Proposition 6.1 will give the continuity of the known terms F^{sh}_t and M^{sh}_t in (4.8).

Theorem 6.2. *Assume that s_t satisfies* (3.6), (3.7), (3.9), *and* (3.12), *and let $t_0 \in [0,T]$, $z^0 \in B_{t_0}$, and let Σ^0_ε and $I_\delta(t_0)$ be as in* (4.16). *Assume, in addition, that $I_\delta(t_0)$ satisfies Lemma 5.1. Let w_t be the solution of the problem*

$$\min \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w|^2 \,\mathrm{d}z, \tag{6.2}$$

where the minimum is taken over all functions $w \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ such that $w = \dot{s}_t \circ s^{-1}_t$ on ∂B_t and $w = \lambda_t(z - z^0)/\varepsilon^3$ on $\partial\Sigma^0_\varepsilon$. Then the function $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$.

Proof. We approximate the functions $\dot{s}_t$ with the sequence Φ^η_t defined by

$$\Phi^\eta_t(x) := \int_{\mathbb{R}} \kappa_\eta(t-\tau)\dot{s}_\tau(x)\,\mathrm{d}\tau, \tag{6.3}$$

where κ_η is a regularizing kernel supported in the ball Σ_η of radius η and of unit mass. Since the function $\tau \mapsto \dot{s}_\tau$ belongs to $L^\infty(I_\delta(t_0); W^{1,p}(A;\mathbb{R}^3))$ for every $2 \leqslant p < \infty$, the integral in (6.3) can be seen as a Bochner integral in $W^{1,p}(A;\mathbb{R}^3)$. This implies that $t \mapsto \Phi^\eta_t$ belongs to $C^0(I_\delta(t_0); W^{1,p}(A;\mathbb{R}^3))$; in particular, it belongs to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3))$. Moreover, by (3.14), we have $\mathrm{Lip}(\Phi^\eta_t) \leqslant L$. Therefore, the map $t \mapsto \Phi^\eta_t$ belongs to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3)) \cap L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. Moreover, for almost every $t \in I_\delta(t_0)$, $\Phi^\eta_t \to \dot{s}_t$ strongly in $H^{1/2}(\partial A;\mathbb{R}^3)$.

Let w^η_t be the solutions to problems (6.2), where the minimum is now taken over all functions $w \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ such that $w = \Phi^\eta_t \circ s_t^{-1}$ on ∂B_t and $w = \lambda_t(z - z^0)/\varepsilon^3$ on $\partial\Sigma^0_\varepsilon$. By the properties of the functions $t \mapsto \Phi^\eta_t$ mentioned above and by Proposition 6.1, the functions $t \mapsto w^\eta_t$ are continuous from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$.

We recall that, for almost every $t \in I_\delta(t_0)$, $\Phi^\eta_t \to \dot{s}_t$ strongly in $H^{1/2}(\partial A;\mathbb{R}^3)$. This implies that $\Phi^\eta_t \circ s_t^{-1} \to \dot{s}_t \circ s_t^{-1}$ strongly in $H^{1/2}(\partial B_t;\mathbb{R}^3)$. By the continuous dependence of the solutions on the data, we have $w^\eta_t \to w_t$ in $\mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ for almost every $t \in I_\delta(t_0)$. This implies the measurability of $t \mapsto w_t$. □

Theorem 6.3. *Under the hypotheses of Theorem 6.2, the vector b_t and the matrix Ω_t in* (4.5) *are bounded and measurable with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T]; C^1(\overline{A};\mathbb{R}^3))$, then $t \mapsto (b_t\,, \Omega_t)$ belongs to $C^0([0,T]; \mathbb{R}^3 \times \mathbb{M}^{3\times 3})$.*

Proof. As noticed in Section 4, it is enough to prove that the functions in (4.12) are bounded and measurable, and that they are continuous under the additional assumption on $t \mapsto s_t$. Moreover, it is sufficient to prove the measurability and boundedness of these functions in a subinterval of time; the measurability and boundedness on the whole $[0,T]$ will easily follow. As for the first three functions, this property is proved in Proposition 4.4. The function $t \mapsto w_t$ from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ is bounded and measurable by Theorem 6.2. By Proposition 6.1 it is also continuous under the additional assumption. By formulas (4.17), this yields the boundedness and measurability of $t \mapsto F^{\mathrm{sh}}_t$ and $t \mapsto M^{\mathrm{sh}}_t$, and the continuity under the additional assumption on $t \mapsto s_t$, since the functions $t \mapsto v^i_t$ and $t \mapsto \hat{v}^i_t$ are continuous from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ by Theorem 2.7 and Lemma 4.3. □

We are now in a position to prove the main result of the paper.

Theorem 6.4. *Assume that $t \mapsto s_t$ satisfies* (3.6), (3.7), (3.9), *and* (3.12). *Let $y^* \in \mathbb{R}^3$ and $R^* \in \mathrm{SO}(3)$. Then* (4.6) *has a unique absolutely continuous solution $t \mapsto (y_t\,, R_t)$ defined in $[0,T]$ with values in $\mathbb{R}^3 \times \mathrm{SO}(3)$ such that $y_0 = y^*$ and $R_0 = R^*$. In other words, there exists a unique rigid motion $t \mapsto r_t(z) = y_t + R_t z$ such that the deformation function $t \mapsto \varphi_t = r_t \circ s_t$ satisfies the equations of motion* (4.1).

Moreover this solution is Lipschitz continuous with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T]; C^1(\overline{A};\mathbb{R}^3))$, then the solution $t \mapsto (y_t\,, R_t)$ belongs to $C^1([0,T]; \mathbb{R}^3 \times \mathrm{SO}(3))$.

Proof. The existence and uniqueness of the solution of the Cauchy problem for (4.6) follow immediately from Theorem 6.3, by standard results on ordinary differential equations with bounded measurable coefficients, see, e.g., [9, Theorem I.5.1]. The assertion concerning the deformation function $t \mapsto \varphi_t$ and the equation of motion (4.1) follows from the equivalence Theorem 4.1. The Lipschitz continuity of the solution follows from the boundedness of the right-hand sides of the equation in (4.6).

If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T]; C^1(\overline{A}; \mathbb{R}^3))$, then Theorem 6.3 ensures that the coefficients of the equations in (4.6) are continuous with respect to t, and therefore the solutions are of class C^1. □

We notice that assumptions (1.2) are not needed in Theorem 6.4. As a consequence, the theorem holds in a more general setting, when s_t is not a pure shape change. For instance, if s_t were a rigid motion for every t, the unique r_t given by the theorem would be $r_t = s_t^{-1}$. Consequently, φ_t would be the identity for every t and the swimmer would not move.

A Remarks on Korn's inequality

We prove here a version of Korn's inequality in the space $D^{1,2}(\Omega; \mathbb{R}^3)$ for an exterior domain with Lipschitz boundary $\Omega \subset \mathbb{R}^3$. The classical version of Korn's inequality concerns bounded domains $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary and provides the existence of a constant $\gamma(\Omega)$ such that

$$\|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla u\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leqslant \gamma(\Omega) \big[\, \|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} \big], \tag{A.1}$$

for every $u \in H^1(\Omega; \mathbb{R}^3)$, see [20, page 16]. Another version of Korn's inequality holds true in $\Omega = \mathbb{R}^3$ in the form

$$\|\nabla u\|^2_{L^2(\mathbb{R}^3;\mathbb{M}^{3\times 3})} \leqslant 2 \, \|\mathrm{E}u\|^2_{L^2(\mathbb{R}^3;\mathbb{M}^{3\times 3})} \,, \tag{A.2}$$

for every $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ with $\nabla u \in L^2(\mathbb{R}^3; \mathbb{M}^{3\times 3})$. This classical inequality can be obtained from the identity $\|\nabla u\|^2_{L^2(\mathbb{R}^3;\mathbb{M}^{3\times 3})} + \|\mathrm{div}\, u\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} = 2 \, \|\mathrm{E}u\|^2_{L^2(\mathbb{R}^3;\mathbb{M}^{3\times 3})}$, which, in turn, can be proved by Fourier transform.

For the purpose of this paper we need to extend (A.2) to the case of an exterior domain with Lipschitz boundary Ω.

The first step for the proof of this result is the following lemma concerning the projections onto the space Π of infinitesimal rigid motions, defined as the affine functions of the form $v(x) = c + Ax$, where $c \in \mathbb{R}^3$ and A is a skew symmetric 3×3 matrix.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary. Then, there exists a constant $c(\Omega) > 0$ such that*

$$\|u - \pi(u)\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant c(\Omega) \, \|\mathrm{E}u\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \quad \textit{for all } u \in H^1(\Omega; \mathbb{R}^3), \tag{A.3}$$

where π is the orthogonal projection from $L^2(\Omega; \mathbb{R}^3)$ onto Π.

Proof. Suppose by contradiction that (A.3) does not hold. Then, for every k, there exists a function $u_k \in H^1(\Omega; \mathbb{R}^3)$ such that

$$\|u_k - \pi(u_k)\|_{L^2(\Omega;\mathbb{R}^3)} > k \, \|\mathrm{E}u_k\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \,.$$

It is not restrictive to normalize, so that

$$1 = \|u_k - \pi(u_k)\|_{L^2(\Omega;\mathbb{R}^3)} > k \, \|\mathrm{E}u_k\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \,. \tag{A.4}$$

By applying the classical Korn's inequality (A.1), we obtain that $u_k - \pi(u_k)$ is bounded in $H^1(\Omega; \mathbb{R}^3)$; therefore a subsequence, not relabeled, of $u_k - \pi(u_k)$ converges weakly in $H^1(\Omega; \mathbb{R}^3)$ and strongly in $L^2(\Omega; \mathbb{R}^3)$ to a function v enjoying the following properties: $\|v\|_{L^2(\Omega;\mathbb{R}^3)} = 1$ and, using (A.4), $\mathrm{E}v = 0$, so that v is an infinitesimal rigid motion. Invoking the minimality of the projection π and once more (A.4), we are driven to

$$1 = \|u_k - \pi(u_k)\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant \|u_k - \pi(u_k) - v\|_{L^2(\Omega;\mathbb{R}^3)} \to 0,$$

which is clearly impossible. □

We now in a position to prove a version of Korn's inequality for an exterior domain with Lipschitz boundary.

Theorem A.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then there exists a constant $C(\Omega)$ such that*

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})} \leqslant C(\Omega)\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})}\,, \tag{A.5}$$

for every $u \in D^{1,2}(\Omega;\mathbb{R}^3)$.

Proof. Let Σ_ρ be an open ball of radius ρ centered at the origin such that $\Lambda := \mathbb{R}^3 \setminus \overline{\Omega} \subset\subset \Sigma_\rho$. Let $\Omega_\rho := \Omega \cap \Sigma_\rho$ and let $\mathcal{T}\colon H^1(\Omega_\rho;\mathbb{R}^3) \to H^1(\Sigma_\rho;\mathbb{R}^3)$ be a continuous linear extension operator. Then

$$\begin{aligned}\|\mathcal{T}(u)\|^2_{H^1(\Lambda;\mathbb{R}^3)} &\leqslant \|\mathcal{T}\|\,\|u\|^2_{H^1(\Omega_\rho;\mathbb{R}^3)}\\ &\leqslant \|\mathcal{T}\|\,\gamma(\Omega_\rho)\big[\,\|\mathrm{E}u\|^2_{L^2(\Omega_\rho;\mathbb{M}^{3\times3})} + \|u\|^2_{L^2(\Omega_\rho;\mathbb{R}^3)}\,\big],\end{aligned} \tag{A.6}$$

where the last inequality follows from (A.1). Notice that all the preceding estimates hold true because every $u \in D^{1,2}(\Omega;\mathbb{R}^3)$ is of class $H^1(V;\mathbb{R}^3)$ for any bounded open subset $V \subset \Omega$.

We now introduce a slightly different extension operator $\mathcal{S}\colon H^1(\Omega_\rho;\mathbb{R}^3) \to H^1(\Sigma_\rho;\mathbb{R}^3)$, defined by $\mathcal{S}(u) := \mathcal{T}(u - \pi_{\Omega_\rho}(u)) + \pi_{\Omega_\rho}(u)$, where π_{Ω_ρ} is the projection from $L^2(\Omega_\rho;\mathbb{R}^3)$ onto Π. Notice that $\mathrm{E}(\mathcal{S}(u)) = \mathrm{E}(\mathcal{T}(u - \pi_{\Omega_\rho}(u)))$; therefore, by (A.6),

$$\begin{aligned}\|\mathrm{E}(\mathcal{S}(u))\|^2_{L^2(\Lambda;\mathbb{M}^{3\times3})} &\leqslant \left\|\nabla(\mathcal{T}(u - \pi_{\Omega_\rho}(u)))\right\|^2_{L^2(\Lambda;\mathbb{M}^{3\times3})}\\ &\leqslant \|\mathcal{T}\|\,\gamma(\Omega_\rho)\big[\,\|\mathrm{E}u\|^2_{L^2(\Omega_\rho;\mathbb{M}^{3\times3})} + \left\|u - \pi_{\Omega_\rho}(u)\right\|^2_{L^2(\Omega_\rho;\mathbb{R}^3)}\,\big].\end{aligned} \tag{A.7}$$

Define now a function v on the whole $\mathbb{R}^3$ by

$$v = \begin{cases} u & \text{in } \Omega,\\ \mathcal{S}(u) & \text{in } \Omega_\rho\,.\end{cases}$$

Now, combining together (A.1), (A.2), (A.3), and (A.7), we can prove the final estimate

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})} &\leqslant \|\nabla v\|^2_{L^2(\mathbb{R}^3;\mathbb{M}^{3\times3})} \leqslant 2\,\|\mathrm{E}v\|_{L^2(\mathbb{R}^3;\mathbb{M}^{3\times3})}\\ &= 2\,\|\mathrm{E}(\mathcal{S}(u))\|^2_{L^2(\Lambda;\mathbb{M}^{3\times3})} + 2\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})}\\ &\leqslant 2\,\|\mathcal{T}\|\,\gamma(\Omega_\rho)\big[\,\|\mathrm{E}u\|^2_{L^2(\Omega_\rho;\mathbb{M}^{3\times3})} + \left\|u - \pi_{\Omega_\rho}(u)\right\|^2_{L^2(\Omega_\rho;\mathbb{R}^3)}\,\big] + 2\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})}\\ &\leqslant C(\Omega)\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})}\,,\end{aligned}$$

where $C(\Omega) = 2(1 + \|\mathcal{T}\|\,\gamma(\Omega_\rho) + \|\mathcal{T}\|\,\gamma(\Omega_\rho)c(\Omega_\rho))$. □

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