Nonuniqueness for crystalline curvature flow

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Abstract

We discuss some examples of nonuniqueness for the crystalline curvature flow, when the Wulff shape is a square not centered at the origin.

1 Introduction

In this short note we present some examples of anisotropic curvature flow of planar curves. Such evolutions can be considered as the L^2–gradient flow of the energy

$$\int_{\partial E} \varphi(\nu) d\mathcal{H}^1, \tag{1}$$

where E is a subset of $\mathbb{R}^2$, ν is the exterior unit normal to ∂E, and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a convex, positive and positively one-homogeneous function, called *anisotropy* in the sequel. The anisotropic curvature flow is defined as

$$\begin{aligned} v &= -\mathrm{div}_\tau n_\varphi \, n_\varphi, \\ n_\varphi &\in \partial\varphi(\nu), \end{aligned} \tag{2}$$

where v denotes the velocity, n_φ represents a sort of anisotropic normal to ∂E, and div_τ denotes the tangential divergence on ∂E. Due to the fact that φ is not necessarily differentiable, we denote by $\partial\varphi$ its (possibly multivalued) subdifferential. The quantity $\kappa_\varphi := \mathrm{div}_\tau n_\varphi$ is also called the anisotropic curvature of ∂E. Letting φ^* be the dual function of φ:

$$\varphi^*(\xi) := \sup_{\varphi(x) \le 1} \xi \cdot x \,,$$

we define the Wulff shape W_φ as the unit ball of φ^*, i.e.

$$W_\varphi := \{x \in \mathbb{R}^2 : \ \varphi^*(x) \le 1\}.$$

We notice that the anisotropic curvature of ∂W_φ is constantly equal to 1, hence the family of sets

$$E(t) := \sqrt{1 - 2t} \; W_\varphi \qquad t \in [0, 1/2) \,,$$

defines a self-similar shrinking solution of (2). The anisotropy is called *crystalline* if the Wulff shape is a polygon. In the crystalline case the notion of regular solution of (2) has to be suitably adapted (see [25, 3, 15] for precise definitions).

It is well-known that the fattening phenomenon does not happen for isotropic curvature flow. In this case disjoint curves cannot intersect for positive times, and each compact connected component becomes convex and then shrinks to a point in finite time, with its shape approaching a circle (see for instance [13, 17]). The behavior is similar in the anisotropic setting provided the anisotropy is *symmetric*, i.e.

$$\varphi(\nu) = \varphi(-\nu) \qquad \forall \nu \in \mathcal{S}^1. \tag{3}$$

Indeed, from the maximum principle (which also holds in the crystalline case [16, 3]) two disjoint curves remain disjoint if (3) holds. Moreover, at least when the anisotropy is smooth or purely crystalline, it has been proved in [9, 15] that each compact connected component becomes convex and eventually shrinks to a point, while its shape converges to the boundary of W_φ (see [14, 11, 23, 2, 24]), unless the Wulff shape is a quadrilateral [23]. On the other hand, if (3) is violated, the behavior is quite different and, for example, disjoint connected components of ∂E may intersect during the flow, since the evolution law is no more orientation-free. It has been proved in [26] (see also the Introduction of [1]) that, for some smooth anisotropies, there exist convex self-similar shrinking solutions which are different from W_φ (in the crystalline case, there are also nonconvex self-similar solutions [19]). In Section 2 we show that the same holds for the nonsymmetric crystalline flow, when the Wulff shape is a square not centered at the origin.

The picture changes significantly by adding a forcing term or by considering the evolution law (2) in dimension greater than 2. Indeed, it has been shown in [6] that even with a constant forcing term the fattening phenomenon may occur when the initial set is the union of two disjoint circles in $\mathbb{R}^2$ (the case of two liked circles in $\mathbb{R}^3$, evolving by curvature, has been considered in [5]). In both cases, the fattening phenomenon is a consequence of the fact that disjoint components of the boundary of the inital set may collide during the evolution.

This is precisely what can happen also for the anisotropic curvature flow, when the anisotropy is not symmetric. Therefore, one could expect that similar nonuniqueness results hold also for such evolution. Indeed, the numerical simulations presented in [21] show that a fattening phenomenon, similar to the one discussed in [6], occurrs when the Wulff shape is a unit circle, not centered at the origin, and the initial set is a ring. In Section 4 we show that the same holds for the nonsymmetric crystalline flow, providing an example of fattening when the Wulff shape is a square (not centered at the origin).

The main advantage of the crystalline case is the fact that the evolution (2) reduces to a simple ODE, the solutions of which can be often explicitly computed.

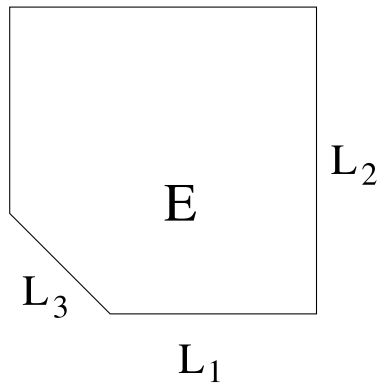

Figure 1: the self-similar solution.

2 The examples of self–similar evolutions

Example 1.

Let us consider the crystalline anisotropy having the Wulff shape equal to a square of sidelength 2, centered at (δ, δ), for $\delta \in (0,1)$. Let the initial set E be as in Figure 1. The edges L_1, L_2 and L_3 of E move with (inward) velocity respectively given by $v_1 = 2(1-\delta)/|L_1|$, $v_2 = 2(1+\delta)/|L_2|$ and $v_3 = 0$. Notice that $|\dot{L}_1| = v_1 - v_2$, $|\dot{L}_2| = -(v_1 + v_2)$. Letting $x := |L_1|/|L_2| \in (0,1)$, we have

$$\dot{x} = \frac{\frac{2(1-\delta)}{x} - 4\delta + 2(1+\delta)x}{|L_2|^2} = \frac{f(x)}{|L_2|^2}. \tag{4}$$

Thus, if we have

$$x^{\pm} = \frac{\delta \pm \sqrt{2\delta^2 - 1}}{1+\delta},$$

we get that the ratio $|L_1|/|L_2|$ is invariant under the evolution, which implies that the evolution is self-similar (this is possible only if $\delta \geq \sqrt{2}/2$).
Observe that the volume $V(t) := |E(t)|$ satisfies $\dot{V}(t) = -4$, hence we get

$$V(t) = \left(x + \frac{1-x^2}{2}\right)|L_2|^2 = V(0) - 4t \qquad t \in [0,T), \tag{5}$$

where $T := V(0)/4$ is the extinction time of the evolution. Substituting (5) into (4), we obtain

$$\dot{x} = \frac{f(x)}{V(0) - 4t}\left(x + \frac{1-x^2}{2}\right) \qquad t \in [0,T). \tag{6}$$

Notice that this equation become autonomous with the change of variable $\tau = \ln V(0) - \ln(V(0) - 4t) \in [0, +\infty)$.

Notice also that $f(x) < 0$ if and only if $x \in (x^-, x^+)$, therefore $x(t) \equiv x^-$ (resp. $x(t) \equiv x^+$) is a stable (resp. unstable) solution of (4). In particular, if we start with $x(0) \in (0, x^+)$ at time 0, the solution $x(t)$ will converge to x^- as $t \to T$. On the other hand, if we start with $x(0) \in (x^+, 1)$ the solution will hit 1 before T, and the evolving set $E(t)$ will converge to the Wulff shape, up to rescaling.

Example 2.

Let now the Wulff shape be equal to the Wulff shape of the previous example, intersected with the half-plane $\{x + y \geq k\}$, with $k \in (2\delta - 2, 0)$, so that the Wulff shape is an irregular pentagon. Letting E be as in Figure 1, in this case we have $v_1 = (2\delta - k)(1 - \delta)/|L_1|$, $v_2 = 2(1 + \delta)/|L_2|$ and $v_3 = k(2\delta - k - 2)/|L_3|$. Notice that $|\dot{L}_1| = v_1 - v_2 - \sqrt{2}v_3$, $|\dot{L}_2| = -(v_1 + v_2)$. Letting $x := |L_1|/|L_2| \in (0, 1)$, we have

$$\begin{aligned} \dot{x} &= \frac{\frac{(2\delta-k)(1-\delta)}{x} - 2(1+\delta)(1-x) - \frac{k(2\delta-k-2)}{1-x} + (2\delta-k)(1-\delta)}{|L_2|^2} \\ &= \frac{f(x)}{V(0) - ct}\left(x + \frac{1-x^2}{2}\right) = \frac{g(x)}{V(0) - ct}\,\frac{1 + 2x - x^2}{2x(1-x)} \qquad t \in [0, T), \end{aligned} \tag{7}$$

where $c = 4 - (2 - \sqrt{2})(2 - 2\delta + k)$ is the perimeter of the Wulff shape, $T = V(0)/c$ is the extinction time of the evolution, and

$$\begin{aligned} g(x) &= (2\delta - k)(1 - \delta)(1 - x) - 2(1 + \delta)x(1 - x)^2 \\ &\quad -k(2\delta - k - 2)x + (2\delta - k)(1 - \delta)(x - x^2) \\ &= (2\delta - k - 2x)\big((1 + \delta)x^2 - (2 + k)x + (1 - \delta)\big). \end{aligned}$$

We recall that, from [2, Prop. 7.1] it follows that the evolving set $E(t)$ has bounded isoperimetric ratio and converges (up to rescaling) to a self-similar solution, possibly different from the Wulff shape.

We now look for the zeros of the function f (or equivalently g), which correspond to self-similar solutions of (2). Notice that the solution $\bar{x}_k = (2\delta - k)/2$ corresponds to the Wulff shape. The solutions different from the Wulff shape are are given by

$$x_k^{\pm} = \frac{2 + k \pm \sqrt{k^2 + 4k + 4\delta^2}}{2(1 + \delta)},$$

with the necessary condition $k \geq -2 + 2\sqrt{1 - \delta^2}$. Notice that, since $k > -2 + 2\delta$, this condition is automatically satisfied whenever $\delta \geq \sqrt{2}/2$. Moreover, we always have $0 < x_k^- \leq x_k^+ < 1$.

We now analyze the stability of the solutions $x^{\pm}, \bar{x}$, by considering the sign of f, as in the previous example. Notice that $\lim_{x \to 0^+} f(x) = +\infty$, $\lim_{x \to 1^-} f(x) = -\infty$, and that $f(x)$ reduces to the expression in (4) for $k \to 2\delta - 2$, which implies that $\bar{x}_k$ goes to 1, and $x_k^{\pm}$ converge to $\left(\delta \pm \sqrt{2\delta^2 - 1}\right)/(1+\delta)$, provided $\delta > \sqrt{2}/2$. Similarly, $\bar{x}_k$ goes to δ, and $x_k^\pm$ converge to $(1 \pm \delta)/(1+\delta)$, as $k \to 0$.

As a consequence, for all $\delta \in (\sqrt{2}/2, 1)$ there exist two self-similar solutions different from the Wulff shape, moreover we have $x_k^- < x_k^+ < \bar{x}_k$ for k close enough to $2\delta - 2$, and $x_k^- < \bar{x}_k < x_k^+$ for k close enough to 0. In the first case, as in Example 1, x_k^- and $\bar{x}_k$ are stable solutions of (7), and x_k^+ is unstable, whereas in the latter case x_k^- and $\bar{x}_k$ are unstable, and x_k^+ is stable. In particular, this implies that the Wulff shape corresponds to an unstable solution, if k is close enough to 0.

This is in sharp contrast with the (two-dimensional) symmetric case, where the Wulff shape is always stable, and it is also the unique self-similar solution unless is equal to a quadrilateral [23, 2]. A similar phenomenon has been shown for the crystalline mean curvature flow in $\mathbb{R}^3$, even in the symmetric case [22, 20].

3 Weak solutions

Since the fattening phenomenon is related to formation of singularities, to give it a precise meaning it is necessary to define a weak solution of the evolution (2). The notion which is more adapted to this situation is the so-called geometric viscosity solution, defined as zero level-set of the viscosity solution of an appropriate (level-set) formulation of (2) (see [12, 8, 15] for precise difinitions).

We shall use here a more geometric presentation of this solution, defined through the *minimal barrier* theory of De Giorgi [10]. The equivalence between this two notions has been proved in [18, 7, 4].

We recall the definition of minimal barrier in the sense of De Giorgi. We denote by $\mathcal{F}$ the family of all the regular solution of (2), so that an element of $\mathcal{F}$ will be a set-valued function $\Sigma : [a,b] \to \mathcal{P}(\mathbb{R}^n)$, $[a,b] \subset [0,+\infty)$.

Definition 3.1. *We say that a function $\Phi : [0,+\infty) \to \mathcal{P}(\mathbb{R}^n)$ is a* barrier *with respect to $\mathcal{F}$ if for any $\Sigma(t) \in \mathcal{F}$, $t \in [a,b] \subset [0,+\infty)$, the inclusion $\Sigma(a) \subseteq \Phi(a)$ implies $\Sigma(b) \subseteq \Phi(b)$.*

In the following we denote by $\mathcal{B}$ the class of all barriers with respect to $\mathcal{F}$, defined on $[0,+\infty)$.

Definition 3.2. *Let $E \subseteq \mathbb{R}^n$. The* minimal barrier *$\mathcal{M}(E) : [0,+\infty) \to \mathcal{P}(\mathbb{R}^n)$ starting from E at time 0 is defined as*

$$\mathcal{M}(E)(t) := \bigcap \left\{ \Phi(t) : \ \Phi \in \mathcal{B}, \ \Phi(0) \supseteq E \right\}.$$

We also define the upper and lower regularized barriers as

$$\mathcal{M}_*(E)(t) \ := \ \bigcup_{\rho>0} \mathcal{M}(E_\rho^-)(t) \quad \mathcal{M}^*(E)(t) := \bigcap_{\rho>0} \mathcal{M}(E_\rho^+)(t),$$

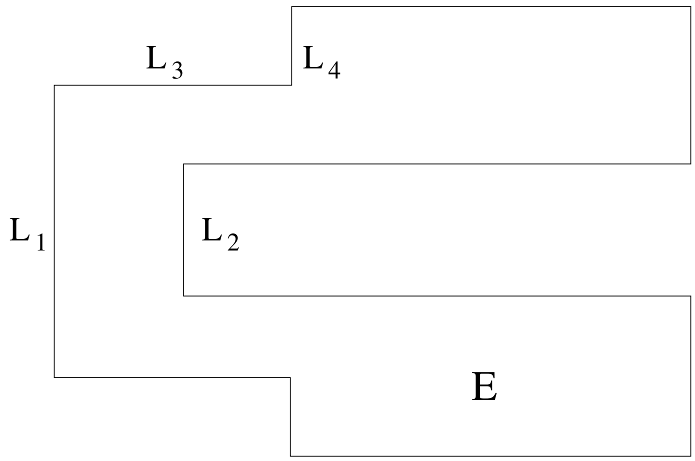

Figure 2: the initial set.

where $E_\rho^\pm := \{d_E \le \pm\rho\}$ *and* $d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathbb{R}^n \setminus E)$.

The set-valued map $t \mapsto \mathcal{M}^*(E)(t) \setminus M_*(E)(t)$ will be called the (weak) anisotropic curvature flow, starting from ∂E at time 0. We say that the evolution develops fattening at time $\bar{t} \ge 0$ if

$$|\mathcal{M}^*(E)(t_n) \setminus M_*(E)(t_n)| > 0,$$

for some sequence $t_n \downarrow \bar{t}$.

4 The example of fattening

Let us consider the crystalline anisotropy having the Wulff shape equal to a square of sidelength 2, centered at $(-1/2, 0)$. We let the initial set E be as in Figure 2. The edges L_1 and L_2 of E move with velocity (in the right direction) respectively given by $v_1 = 3/|L_1|$ and $v_2 = 1/|L_2|$, whereas L_3 and L_4 do not move, even if $|L_3|$ decreases and eventually reduces to 0. Let $E(t)$ be the evolution of the set E at time $t > 0$, and let $L_i(t)$, $i \in \{1, 2, 3\}$, be the edge of $E(t)$ corresponding to L_i. The evolution of $E(t)$ is given by an ODE (see [25, 16]), before the possible collision of the edges $L_1(t)$ and $L_2(t)$ (the disappearence of $L_3(t)$ does not create any additional difficulty). However, we may choose the initial set E in such a way that $v_1 > v_2$ at time zero, and $L_1(t)$ collides with $L_2(t)$ *precisely* at the time $\bar{t}$ when $L_3(t)$ disappears. In this case, the length of $L_1(\bar{t})$ suddenly increases, and its velocity jumps down to a value stricly less than v_2, provided $|L_4|$ is big enough. As depicted in Figure 3 (in dashed lines), starting from $\partial E(\bar{t})$ there are two different regular solutions of (2), which are both contained in $\mathcal{M}^*(E)(t) \setminus M_*(E)(t)$, for $t > \bar{t}$. Moreover, the weak evolution also

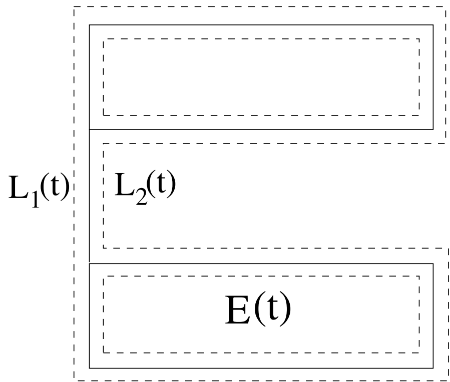

Figure 3: the evolving set for $t > \bar{t}$.

contains the "singular" solutions obtained by keeping the edges $L_1(t)$ and $L_2(t)$ glued together and evolving with velocity $v \in [v_1, v_2]$. Starting from these singular evolutions at each time $t > \bar{t}$, we can construct as above two regular solutions, which are still contained in the weak evolution. This implies that the weak evolution $\mathcal{M}^*(E)(t) \setminus M_*(E)(t)$ coincides with the set delimited by the two regular evolutions starting from $\partial E(\bar{t})$, and shows in particular the occurrence of fattening at time $\bar{t}$.

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