Regularity for solutions of the total variation denoising problem

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Abstract

The main purpose of this paper is to prove a local Hölder regularity result for the solutions of the total variation based denoising problem assuming that the datum is locally Hölder continuous. We also prove a global estimate on the modulus of continuity of the solution in convex domains of $\mathbb{R}^N$ and some extensions of this result for the total variation minimization flow.

1 Introduction

We study the local regularity properties of a local minimizer of the functional

$$\int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega |u(x) - f(x)|^2\,dx \tag{1}$$

where Ω is an open set in $\mathbb{R}^N$, $\lambda > 0$, and $f : \Omega \to \mathbb{R}$ is locally Hölder continuous. Our main purpose is to prove that u is also locally Hölder continuous (with the same exponent).

The previous functional was introduced as a model for image denoising by Rudin, Osher, and Fatemi in [19]. In that context, Ω is a bounded domain and f represents the observed image which we assume to be related to the undistorted image by

$$f \;=\; u + n, \tag{2}$$

where n represents a Gaussian white noise of zero mean and standard deviation σ. The parameter $\lambda > 0$ may be interpreted as a regularization parameter or as a Lagrange multiplier ir order to adjust the constraint $\int_\Omega (u - f)^2\,dx \le |\Omega|\sigma^2$ determined by (2).

One of the main features of total variation denoising (1), confirmed by numerical experiments, is its ability to restore the discontinuities of the image [19, 12, 13]. The underlying a priori assumption is that functions of bounded variation (the BV model [3]) are a reasonable functional model for many problems in image processing, in particular, for denoising and restoration problems. Typically, functions of bounded variation admit a set of discontinuities which is countably rectifiable [3], being continuous in some sense (in the measure theoretic sense) away from discontinuities. The discontinuities could be identified with edges. The ability of total variation regularization to recover edges is one of the main features which advocates for the use of this model which had a strong influence in image processing (its ability to describe textures is less clear, even if some textures can be recovered, up to a certain scale of oscillation).

Motivated by the experimental evidence in image processing, we initiated the study of the local regularity properties of (1) in [11] where we proved that for any $f \in BV(\Omega) \cap L^\infty(\Omega)$ the set of jumps of u (in the BV sense) is contained in the set of jumps of f. In other words, model (1) does not create any new discontinuity besides the existing ones. This has to be combined with results describing which discontinuities are preserved. No general statement in this sense exist but many examples are described

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in the papers [5, 7, 8, 1]. The preservation of a jump discontinuity depends on the curvature of the level line at the given point, the size of the jump and the regularization parameter λ. The examples support the idea that total variation may be a reasonable model in order to restore discontinuities.

In the present work we continue our analysis of the local regularity properties of (1) by proving that if the datum f is locally Hölder continuous with exponent $\beta \in (0,1]$ in some region $\Omega' \subset \Omega$, then its local minimizer u is also locally Hölder continuous in Ω' with the same exponent.

Recall that a function $u \in BV(\Omega)$ is a local minimizer of (1) if for any $v \in BV(\Omega)$ such that $u - v$ has support in a compact subset $K \subset \Omega$, we have

$$\int_K |Du| + \frac{1}{2}\int_K |u(x) - f(x)|^2\,dx \;\le\; \int_K |Dv| + \frac{1}{2}\int_K |v(x) - f(x)|^2\,dx \tag{3}$$

It follows that u satisfies the equation [10]

$$-\mathrm{div}\, z + u \;=\; f \tag{4}$$

with $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \le 1$, and $z \cdot Du = |Du|$ (see Section 2).

As in [11], our analysis of the regularity of the local minimizers of u will be based on the following observation: for any $t \in \mathbb{R}$, the level sets $\{u > t\}$ (resp., $\{u \ge t\}$) are solutions (the minimal and maximal, indeed) of the minimal surface problem

$$\min_{E \subseteq \Omega} P(E, \Omega) + \frac{1}{\lambda}\int_E (t - f(x))\,dx \tag{5}$$

(whose solution is defined in the class of finite-perimeter sets and hence up to a Lebesgue-negligible set). The local regularity of u can be described in terms of the distance of any two of its level sets. This is the main idea in [11] which is further refined here. We proved in [11] that, outside the jump discontinuities of f (modulo an $\mathcal{H}^{N-1}$-null set), any two level sets at different heights cannot touch and hence the function u is continuous there. To be able to assert a Hölder type regularity property for u we need to prove a local estimate of the distance of the boundaries of two level sets. This will be done here under the assumption of local Hölder regularity for f.

Let us describe the plan of the paper. In section 2 we recall some basic facts about functions of bounded variation. In section 3 we collect some basic regularity results when $f \in L^N(\Omega)$. In Section 4 we prove the main result of the paper, namely the local Hölder regularity of the local minimizers of (1) in any subdomain Ω' of Ω when the datum f is locally Hölder continuous in Ω'. We also consider in Section 5 the case of global regularity of solutions of (4) with Neumann boundary conditions in convex domains of $\mathbb{R}^N$ and we then extend this result to the case of the total variation flow.

2 Notation and preliminaries on BV functions

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector-valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\sup\left\{\int_\Omega u\ \mathrm{div}\, z\ dx : z \in C_0^\infty(\Omega; \mathbb{R}^N),\ |z(x)| \le 1\ \forall x \in \Omega\right\}, \tag{6}$$

where for a vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$, and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. The map $u \to |Du|(\Omega)$ is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous. $BV(\Omega)$ is a Banach space when endowed with the norm $\|u\| := \int_\Omega |u|\,dx + |Du|(\Omega)$.

A measurable set $E \subseteq \Omega$ is said to be of finite perimeter in Ω if (6) is finite when u is substituted with the characteristic function χ_E of E. The perimeter of E in Ω is defined as $P(E, \Omega) := |D\chi_E|(\Omega)$. We denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$, respectively, the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

If $E \subseteq \mathbb{R}^N$ is a measurable set and $x \in \mathbb{R}^N$, we define the upper density of E at x by

$$\overline{D}(E,x) := \limsup_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|}.$$

Given $u \in BV(\Omega)$, we define

$$u^+(x) := \inf\{t : \overline{D}(\{u > t\}, x) = 0\} \quad \text{and} \quad u^-(x) := \sup\{t : \overline{D}(\{u < t\}, x) = 0\}.$$

Then, we say that u is approximately continuous at $x \in \Omega$ if and only if $u^+(x) = u^-(x)$. The set of points where u is not approximately continuous is called the singular set of u and denoted by S_u.

For a comprehensive treatment of functions of bounded variation, we refer the reader to [3].

If $z \in L^\infty_{\rm loc}(\Omega, \mathbb{R}^N)$ with $\operatorname{div} z \in L^p_{\rm loc}(\Omega)$, and $w \in BV_{\rm loc}(\Omega) \cap L^q_{\rm loc}(\Omega)$ (with $\frac{1}{p} + \frac{1}{q} = 1$), we define the functional $z \cdot Dw : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle z \cdot Dw, \varphi\rangle := -\int_\Omega w\,\varphi \operatorname{div} z\,dx - \int_\Omega w\, z \cdot \nabla\varphi\,dx \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{7}$$

It is a Radon measure in Ω, which is of course finite if $z \in L^\infty(\Omega, \mathbb{R}^N)$, $\operatorname{div} z \in L^p(\Omega)$ and $w \in BV(\Omega) \cap L^q(\Omega)$. Moreover, we have $z \cdot Dw = z \cdot \nabla w\,dx$ for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. We denote by $\nu_\Omega(x)$ the outer unit normal to a point $x \in \partial\Omega$ is denoted by $\nu_\Omega(x)$. The following integration by parts formula can be found in [6]. Let $z \in L^\infty(\Omega, \mathbb{R}^N)$ with $\operatorname{div} z \in L^p(\Omega)$. Then there exists a function $[z \cdot \nu_\Omega] \in L^\infty(\partial\Omega)$ satisfying $\|[z \cdot \nu_\Omega]\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and such that for any $u \in BV(\Omega) \cap L^q(\Omega)$ we have

$$\int_\Omega u \operatorname{div} z\,dx + \int_\Omega z \cdot Du = \int_{\partial\Omega} [z \cdot \nu_\Omega] u\,d\mathcal{H}^{N-1}.$$

3 Basic regularity

Proposition 3.1. *Assume $f \in L^N(\Omega)$. Then there exists $\delta > 0$ and $\rho_0 > 0$ such that if $\rho < \rho_0$ and $B_\rho(x) \subset \Omega$, then, for almost any $t \in \mathbb{R}$,*

$$|\{u > t\} \cap B_s(x)| > 0\ \forall s > 0 \quad \Rightarrow\ |\{u > t\} \cap B_\rho(x)| \ge \delta\rho^N. \tag{8}$$

The same holds for the sets $\{u < t\}$, $\{u \ge t\}$, and $\{u \le t\}$.

Proof. We only sketch the proof (see also [2, Th.4.2.3]). Comparing $\{u > t\}$ with $\{u > t\} \setminus B_s(x)$, by minimality we have

$$\mathcal{H}^{N-1}(B_s \cap \partial^*\{u > t\}) \ \le\ \mathcal{H}^{N-1}(\partial B_s \cap \{u > t\}) \ +\ \int_{B_s \cap \{u>t\}} f(x) - t\,dx$$

hence, if $h(s) = |B_s \cap \{u > t\}|$, then $h'(s) = \mathcal{H}^{N-1}(\partial B_s \cap \{u > t\})$, and adding $\mathcal{H}^{N-1}(\partial B_s \cap \{u > t\})$ to both sides of the above inequality we get

$$Per(B_s \cap \{u > t\}) \ \le\ 2h'(s) \ +\ \|f - t\|_{L^N(B_s)} h(s)^{1-1/N}.$$

By equiintegrability there exists $\rho_0 > 0$ such that if $s < \rho_0$, $\|f - t\|_{L^N(B_s)}$ is less than half the isoperimetric constant in $\mathbb{R}^N$ so that

$$ch(s)^{1-1/N} \le h'(s)$$

and we conclude using Gronwall's Lemma. □

As a consequence of Proposition 3.1, if we identify the set $E_t = \{u > t\}$ with the set of points where it has density one, then E_t is an open set and

$$\mathcal{H}^{N-1}\left(\partial E_t \setminus \partial^* E_t\right) = 0. \tag{9}$$

Indeed, if $x \in \partial E_t$, then it may be approximated by points in E_t and points in $\mathbb{R}^N \setminus E_t$. Then, by Proposition 3.1, both the upper density of x in E_t and in $\mathbb{R}^N \setminus E_t$ is positive. That is, x belongs to the measure theoretic boundary of E_t and (9) follows.

Corollary 3.2. *u^- is l.s.c. and u^+ is u.s.c.*

Proof. If $x \in \{u^- > t\}$ for some $t \in \mathbb{R}$, then there exists $t' > t$ such that $u^-(x) \geq t'$. Hence,

$$\lim_{\rho \to 0} \frac{|\{u < t'\} \cap B_\rho(x)|}{\rho^N} = 0,$$

which implies by the previous result that there exists $\rho > 0$ small such that $B_\rho(x) \subset \{u \geq t'\}$, up to a negligible set. Hence, $u^- \geq t' > t$ on $B_\rho(x)$, so that $\{u^- > t\}$ is open, and u^- is l.s.c.. The statement for u^+ follows at once, since $u^+ = -(-u)^-$. □

In particular, it follows that u is continuous out of $S_u = \{x \in \Omega : u^-(x) < u^+(x)\}$. We can give a more precise statement:

Proposition 3.3. *For any $\beta \in [0, 1)$, let A_β be the set of points where u is not β-Hölder continuous at x. Then, it holds*

$$dim_{\mathcal{H}}(A_\beta) \leq N - 1 + \beta.$$

Proof. Let $B_{2\rho}(x) \subset \Omega$, with $\rho < \rho_0$. For almost any $s \in (\inf_{B_\rho(x)} u, \sup_{B_\rho(x)} u)$, there is some $y \in B_\rho(x)$ such that $|\{u > s\} \cap B_r(y)| > 0$ for all $r > 0$. Then, by Proposition 3.1, we have that $|\{u > s\} \cap B_r(y)| \geq \delta r^N$ for all $r > 0$. Hence $|\{u > s\} \cap B_{2\rho}(y)| \geq \delta \rho^N$ and we deduce that

$$\begin{aligned} osc_{B_\rho(x)}(u) &\leq \frac{1}{\delta \rho^N} \int_{\inf_{B_\rho(x)} u}^{\sup_{B_\rho(x)} u} \min\{|B_{2\rho(x)} \cap \{u > s\}|, |B_{2\rho(x)} \cap \{u < s\}|\}\, ds \\ &\leq \frac{C}{\rho^{N-1}} \int_{\inf_{B_\rho(x)} u}^{\sup_{B_\rho(x)} u} Per(B_{2\rho(x)} \cap \{u > s\})\, ds \ \leq \frac{C|Du|(B_{2\rho}(x))}{\rho^{N-1}}. \end{aligned} \tag{10}$$

where the relative isoperimetric inequality was used in the second inequality. The thesis follows from (10) and [3, Th. 2.56]. □

In particular, it follows again from (10) that u is continuous at each x such that

$$\frac{|Du|(B_{2\rho}(x))}{\rho^{N-1}} \ \to \ 0$$

as $\rho \to 0$.

We recall that in [11], it is proven that in dimension $N \leq 7$, if f is continuous in Ω, then also u is. We will try now to extend this result to slightly higher regularity. It is clear, though, that the highest possible regularity is Lipschitz. This can be shown by trivial examples, for instance in 1D if $\Omega = [-1, 1]$, $f = \lambda|x|^2/2$, λ large enough, and u is the global minimizer of (1).

4 Interior regularity of solutions

We will now show the local regularity of the function u, at least in dimension $N \leq 7$, whenever f is regular. By regular, we mean either Hölder with some exponent β, or Lipschitz-continuous ($\beta = 1$). Our proofs could be adapted to more general moduli of continuity.

The proof relies on an "exclusion" principle for the level sets of u, which is valid near any sufficiently regular level line. However, in order to make this argument uniform, we need quite strong results of regularity for solutions of the prescribed curvature problem (i.e., the problem which our level lines satisfy). The restriction on the dimension is due to these results and the existence of singular minimal cones in dimension 8 or more. We believe, however, that this is technical and that the regularity of u should be preserved near the possible singular points of the level lines.

4.1 Local regularity of the level sets of u

Let us first quote the following theorem of I. Tamanini, which is shown in [20] and relies on the previous works of Massari [16, 17], and Massari and Pepe [18]:

Theorem 1 (Theorem 1 in [20]). *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 2$, and let E be a Caccioppoli set satisfying for $\alpha \in (0,1)$:*

$$\psi(E, B_\rho(x)) := |D\chi_E|(B_\rho(x)) - \inf_{F \triangle E \Subset B_\rho(x)} |D\chi_F|(B_\rho(x)) \leq c\rho^{N-1+2\alpha} \tag{11}$$

for every $x \in \Omega$ and every $\rho \in (0, R)$ with c and R local positive constants.

Then the reduced boundary $\partial^* E$ is a $C^{1,\alpha}$-hypersurface in Ω, and $\mathcal{H}^s((\partial E \setminus \partial^* E) \cap \Omega) = 0$ for any $s > N - 8$.

Moreover, assuming that (11) holds uniformly for any E_h, with $(E_h)_{h\geq 1}$ locally convergent in Ω to some limit set E_∞ as $h \to \infty$, we have that if $x_h \in \partial E_h$ for every h, with $(x_h)_{h\geq 1}$ convergent to some point $x_\infty \in \Omega$, then $x \in \partial E_\infty$; while, if $x_\infty \in \partial^* E_\infty$, then there exists $\bar{h}$ such that $x_h \in \partial^* E_h$ for every $h \geq \bar{h}$, and the unit outward normal to ∂E_h at x_h converges to the unit outward normal to ∂E_∞ at x_∞.

Consider now a solution u of (4) and assume $f \in L^p(\Omega)$ with $p > N$. Let $t \in \mathbb{R}$, $E_t = \{u > t\}$ and let $x \in \Omega$. If $\rho > 0$ and $F \triangle E_t \Subset B_\rho(x)$, then

$$|D\chi_{E_t}|(B_\rho(x)) + \int_{E_t} (t - f(x))\, dx \;\leq\; |D\chi_F|(B_\rho(x)) + \int_F (t - f(x))\, dx\,,$$

and we deduce easily

$$\psi(E_t, B_\rho(x)) \;\leq\; \int_{B_\rho(x)} |t - f(x)|\, dx \;\leq\; \|t - f\|_{L^p(B_\rho(x))} |B_\rho(x)|^{1-1/p}.$$

Hence, (11) holds for E_t in $B_\rho(x)$, for $\alpha = (1 - N/p)/2$. Moreover, this estimate is uniform. We deduce the following corollary:

Corollary 4.1. *Let $f \in L^p(\Omega)$, $p > N$, and let $\alpha = (1 - N/p)/2$. Let u solve (4) and, for any $t \in \mathbb{R}$, let $E_t = \{u > t\}$. Let $\bar{x} \in \partial^* E_t$. Then, there exists an open neigborhood A of $\bar{x}$ such that for any $s \in \mathbb{R}$, $\partial E_s \cap A$ is a $C^{1,\alpha}$ hypersurface, moreover, one may assume that $\nu_{E_s}(x) \cdot \nu_{E_t}(\bar{x}) \geq \sqrt{2}/2$ for any $x \in \partial E_s \cap A$.*

Proof. If the corollary were not true, there would exist a sequence x_h, $x_h \to \bar{x}$, such that either $x_h \in \partial E_t \setminus \partial^* E_t$ (the singular set) for all h large, or such that $\nu_{E_{u(x_h)}}(x_h) \cdot \nu_{E_t}(\bar{x}) < \sqrt{2}/2$ for all h large. This would contradict Theorem 1. □

Corollary 4.2. *Let $t \in \mathbb{R}$, $\bar{x} \in \partial^* E_t$ and A be the set given by Corollary 4.1. Choose a system of coordinates such that the last component x_N is along the vector $\nu_{E_t}(\bar{x})$. Then, for any R, and $x^0 \in A$, if we let C_R be the cylinder $C_R = B'_R \times (-R, R) = \{x : \sum_{i=1}^{N-1}(x_i - x_i^0)^2 \leq R^2, -R \leq x_N - x_N^0 \leq R\}$, then if $C_R \subset A$, $E_{u(x^0)} \cap C_R$ is the supergraph $\{x_N > v(x_1, \ldots, x_{N-1})\}$ of a 1-Lipschitz (and $C^{1,\alpha}$) function $v : B'_R \to [-R, R]$.*

Proof. We just need to observe that $\nu_{E_{u(x^0)}}$ is given by $(\nabla v, -1)/\sqrt{1 + |\nabla v|^2}$. Hence, the condition $\nu_{E_{u(x^0)}}(x) \cdot \nu_{E_t}(\bar{x}) \geq \sqrt{2}/2$, which holds in A thanks to Corollary 4.1, becomes $1/\sqrt{1 + |\nabla v|^2} \geq \sqrt{2}/2$, that is, $|\nabla v| \leq 1$. □

Corollary 4.3. *Under the same assumptions, assume also that f is Hölder-continuous with exponent $\beta > 0$ in Ω (hence, in particular, bounded, so that the above holds with any $\alpha < 1$). Then, in addition, we have that the function v in Corollary 4.2 is $C^{2,\beta}$, with a norm which does not depend on $x^0 \in A$.*

Proof. Since the graph of v satisfies a mean curvature type equation with Hölder continuous right hand side, the corollary follows as a consequence of Corollary 4.2 and standard regularity results [15]. □

4.2 Local regularity of u

We consider here the case $f \in C^{0,\beta}(\Omega)$, with $0 \le \beta \le 1$ (we include the Lipschitz case for $\beta = 1$).

Theorem 2. *Let $N \le 7$ and let u be a solution of (4). Assume $f \in C^{0,\beta}$ locally in Ω, for some $\beta \in [0,1]$. Then u is also $C^{0,\beta}$ locally in Ω.*

Proof. We divide the proof into four Steps.

Step 1. The case $\beta = 0$ has been treated in [11, Theorem 1 and Remark 3.4]. We therefore consider here the case where $\beta > 0$. Since $N \le 7$, all points of any level set are regular. From Corollaries 4.2 and 4.3 we know that, uniformly in the neighborhood of any regular point $x^0 \subset \{u = t_1\}$ (as long as the neighborhood is inside Ω), and after an appropriate change of coordinates, the set $\{u > t_1\}$ is the supergraph $x_N > v_1(x')$, in a suitable cylinder $C_R = B'_R \times (-R, R) = \{x : \sum_{i=1}^{N-1}(x_i - x_i^0)^2 \le R^2, -R \le x_N - x_N^0 \le R\}$, where we have denoted $x' = (x_1, \dots, x_{N-1})$. We can also assume that $\|\nabla' v_1\|_\infty \le 1$ in B'_R, and $\|D'^2 v_1\|_\infty \le \kappa < \infty$. We have denoted the derivatives with respect to the first $N-1$ variables with a prime ("$'$"). To simplify, we assume that $x^0 = 0$.

We denote $F(p) = \sqrt{1+|p|^2}$ (and $F^*(q) = \sup_p q \cdot p - F(p)$ its Legendre-Fenchel conjugate), and consider, for $\gamma \in (0,1)$, the function $w_\gamma = v_1 + (\gamma/2)(R^2 - |x'|^2)$. Then

$$-\mathrm{div}'\nabla F(\nabla' w_\gamma) \;\le\; t_1 - f(x', v_1(x')) \,+\, C\gamma \qquad \text{in } B'_R, \tag{12}$$

where C is an absolute constant (it is a bound on the second and third derivatives of $F(p) = \sqrt{1+|p|^2}$). Indeed, we have

$$\begin{aligned} \mathrm{div}'\nabla F(\nabla' w_\gamma) &= \mathrm{div}'\nabla F(\nabla' v_1) \,-\, \gamma\,\mathrm{div}' \int_0^1 D^2 F(\nabla' v_1 - \gamma s x') \cdot x'\, ds \\ &= -t_1 + f(x', v_1(x')) \,-\, \gamma(\mathcal{R}) \end{aligned}$$

where the rest $(\mathcal{R})$ is given by (the sums range from 1 to $N-1$):

$$\begin{aligned} (\mathcal{R}) &= \int_0^1 \partial_i(\partial^2_{i,j} F(\nabla' v_1 - \gamma s x') x_j \, ds \\ &= \int_0^1 \mathrm{Tr} D^2 F(\nabla' v_1 - \gamma s x')\, ds \,+\, \int_0^1 \partial^3_{i,j,k} F(\nabla' v_1 - \gamma s x')(\partial^2_{i,k} v_1 - \gamma s \delta_{i,k}) x_j \, ds \\ &\le \int_0^1 \mathrm{Tr} D^2 F(\nabla' v_1 - \gamma s x')\, ds \,+\, CR\|D'^2 v_1\|_{L^\infty(B'_R)} \\ &+ \int_0^1 s\partial_s[\mathrm{Tr} D^2 F(\nabla' v_1 - \gamma s x')]\, ds \\ &= CR\|D'^2 v_1\|_{L^\infty(B'_R)} + \mathrm{Tr} D^2 F(\nabla' v_1 - \gamma x') \;\le\; C(R\|D'^2 v_1\|_{L^\infty(B'_R)} + 1). \end{aligned}$$

Hence, using $|D'^2 v_1| \le \kappa$ in B'_R, we get a uniform bound for $(\mathcal{R})$, and we obtain (12).

Step 2. Now we build, in $C_R^+ = \{x \in C_R : x_N > v_1(x')\}$ the vector field

$$\sigma(x', w_\gamma(x')) \;= \begin{pmatrix} \nabla F(\nabla' w_\gamma(x')) \\ F^*(\nabla F(\nabla' w_\gamma(x'))) \end{pmatrix} \tag{13}$$

which has everywhere norm equal to 1. We show that, in C_R^+,

$$\mathrm{div}\,\sigma(x) \;\ge\; -t_1 + f(x', v_1(x')) - C\gamma(x) \tag{14}$$

where $\gamma(x)$ is the unique value of γ such that $w_\gamma(x') = x_N$, that is, $\gamma(x) = 2(x_N - v_1(x'))/(R^2 - |x'|^2)$.

Let $\sigma = (\sigma_i)_{i=1}^N$ and σ' denote the first $N-1$ coordinates of σ. For fixed $\gamma > 0$, we have using (12)

$$\mathrm{div}'[\sigma'(x', w_\gamma(x'))] \;=\; \mathrm{div}'\nabla F(\nabla' w_\gamma) \;\ge\; -t_1 + f(x', v_1(x')) \,-\, C\gamma. \tag{15}$$

Observe that

$$\operatorname{div}'[\sigma'(x', w_\gamma(x'))] \;=\; \sum_{i=1}^{N-1} (\partial_i \sigma_i)(x', w_\gamma(x')) \;+\; \sum_{i=1}^{N-1} (\partial_N \sigma_i)(x', w_\gamma(x')) \partial_i w_\gamma(x'). \tag{16}$$

Now, for fixed $x' \in B'_R$, we have on one hand

$$\partial_\gamma[\sigma_N(x', w_\gamma(x'))] \;=\; (\partial_N \sigma_N)(x', w_\gamma(x')) \frac{R^2 - |x'|^2}{2}.$$

On the other hand, using $\nabla F^*(\nabla F(p)) = p$ for any $p \in \mathbb{R}^N$, we have

$$\begin{aligned}
\partial_\gamma[\sigma_N(x', w_\gamma(x'))] &= \nabla F^*(\nabla F(\nabla' w_\gamma(x'))) \cdot \partial_\gamma[\nabla F(\nabla' w_\gamma(x')] \\
&= \sum_{i=1}^{N-1} \partial_i w_\gamma(x') \partial_\gamma[\sigma_i(x', w_\gamma(x'))] \\
&= \sum_{i=1}^{N-1} \partial_i w_\gamma(x') (\partial_N \sigma_i)(x', w_\gamma(x')) \frac{R^2 - |x'|^2}{2}.
\end{aligned}$$

Then, dividing by $(R^2 - |x'|^2)/2$ we find that

$$(\partial_N \sigma_N)(x', w_\gamma(x')) \;=\; \sum_{i=1}^{N-1} \partial_i w_\gamma(x') (\partial_N \sigma_i)(x', w_\gamma(x')).$$

Hence, combining this with (15) and (16), we find that

$$(\operatorname{div} \sigma)(x', w_\gamma(x')) \;\geq\; -t_1 + f(x', v_1(x')) \;-\; C\gamma$$

which yields (14).

Step 3. Let $t_2 > t_1$ and consider the set $E_2 = \{u > t_2\}$. Let C_f be the Hölder constant of f, $\gamma \leq 1$ and $1 - \beta \geq 0$. Let $W_\gamma = \{v_1(x') < x_N < w_\gamma(x')\}$. Then $E_2 \cap W_\gamma = \emptyset$ as soon as $\gamma^\beta \leq (t_2 - t_1)/(C + C_f(R^2/2)^\beta)$.

Observe that $E_2 \cap C_R \subset C_R^+$. Assume that γ is such that $E_2 \cap W_\gamma$ is not empty. Then, by minimality of E_2, we have

$$\mathcal{H}^{N-1}(\partial E_2 \cap W_\gamma) \;+\; \int_{E_2 \cap W_\gamma} (t_2 - f(x))\, dx \;\leq\; \mathcal{H}^{N-1}(\partial W_\gamma \cap E_2).$$

Using the fact that, by construction, σ is the inner normal to W_γ, from Gauss-Green's formula and (14) we deduce

$$\begin{aligned}
\mathcal{H}^{N-1}(\partial W_\gamma \cap E_2) \;-\; \mathcal{H}^{N-1}(\partial E_2 \cap W_\gamma) &\leq \int_{\partial W_\gamma \cap E_2} (-\sigma \cdot \nu)\, d\mathcal{H}^{N-1} \\
&+ \int_{W_\gamma \cap \partial E_2} (-\sigma \cdot \nu)\, d\mathcal{H}^{N-1} = \int_{\partial (W_\gamma \cap E_2)} (-\sigma \cdot \nu)\, d\mathcal{H}^{N-1} \\
&\leq \int_{W_\gamma \cap E_2} (t_1 - f(x', v_1(x')) + C\gamma)\, dx.
\end{aligned}$$

Hence

$$\int_{W_\gamma \cap E_2} (t_2 - f(x) - t_1 + f(x', v_1(x')) - C\gamma)\; dx \;\leq\; 0$$

and since $|f(x', v_1(x')) - f(x)| \leq C_f(\gamma R^2/2)^\beta$, this is impossible as soon as

$$\left(C\gamma^{1-\beta} + C_f \left(\frac{R^2}{2} \right)^\beta \right) \gamma^\beta \;<\; t_2 - t_1\,.$$

By continuity, and using $\gamma \leq 1$ and $1 - \beta \geq 0$, we deduce that $E_2 \cap W_\gamma = \emptyset$ as soon as $\gamma^\beta \leq (t_2 - t_1)/(C + C_f(R^2/2)^\beta)$.

Step 4. Conclusion. For any for any $x \in C_{R/2}$, $\bar{x} \in \{u = t_1\} \cap C_{R/2}$, we have

$$|u(x) - u(\bar{x})| \ \leq \ \left(\frac{8}{3R^2}\right)^{\beta} (C + C_f(R^2/2)^{\beta})|x - \bar{x}|^{\beta}. \tag{17}$$

Assume that $u(x) = t_2 > t_1$ (a symmetric construction can be done below the graph of v_1). By Step 3 we have that $x \in \{u > t_2'\} \setminus W_\gamma$ where $t_2' = t_2 - \epsilon$ and $\gamma^\beta = (t_2' - t_1)/(C + C_f(R^2/2)^\beta)$. Then

$$|x - \bar{x}| \geq w_\gamma(x') \geq \frac{3}{8}\gamma R^2.$$

Hence

$$|x - \bar{x}|^\beta \geq \left(\frac{3R^2}{8}\right)^{\beta} \frac{|t_2' - t_1|}{C + C_f(R^2/2)^\beta}$$

Letting $t_2' \to t_2$, this shows that in $C_{R/2}$, the distance between $\{u = t_1\}$ and $\{u = t_2\}$ is bounded from below by $(3/8)R^2[(t_2 - t_1)/(C + C_f(R^2/2)^\beta)]^{1/\beta}$, i.e., (17) holds.

Since R and C can be chosen uniform in the neighborhood $A' \Subset \Omega$ of any open set $A \Subset \Omega$, this yields that u is β-Hölder (Lipschitz, when $\beta = 1$) in $\{x \in A' : \mathrm{dist}(x, \partial A) > \sqrt{2}R\}$, which contains A if R was chosen small enough. □

We are able to extend the above result to the total variation flow in case that we have a uniform convergence of the implicit in time Euler scheme. This can be proved for instance for the total variation flow with Neumann boundary conditions in convex domains and this is the purpose of our next Section. We expect the local regularity result for the total variation flow to be true in general.

5 Global minimizers on convex domains

In this section we assume that $\Omega \subset \mathbb{R}^N$ is a convex domain.

Let $f : \overline{\Omega} \to \mathbb{R}$ be a uniformly continuous function, with modulus of continuity $\omega_f : [0, +\infty) \to [0, +\infty)$. We consider the solution u of (4) with homogeneous Neumann boundary condition, that is, such that (3) for any compact set $K \subset \overline{\Omega}$ and any $v \in BV(\Omega)$ such that $v = u$ out of K. This solution is unique, as can be shown adapting the proof of [10, Cor. C.2.] (see also [4] for the required adaptations to deal with the boundary condition), which deals with the case $\Omega = \mathbb{R}^N$.

Then, the following result holds true:

Theorem 3. *Assume $N \leq 7$. Then, the function u is uniformly continuous in Ω, with modulus $\omega_u \leq \omega_f$.*

Again, is quite likely here that the assumption $N \leq 7$ is not necessary for this result.

Proof. We first assume that Ω is bounded, smooth and uniformly convex. Let us also assume that f is smooth up to the boundary. Let $a \in \mathbb{R}$ and consider the set $E_a = \{u > a\}$. Then, we now have that

$$Per(E_a, \Omega) \ + \ \int_{E_a} (a - f(x))\, dx \ \leq \ Per(E, \Omega) \ + \ \int_E (a - f(x))\, dx \tag{18}$$

for any finite-perimeter set $E \subset \Omega$. In particular, ∂E_a is smooth up to the boundary, and orthogonal to $\partial\Omega$ at the contact points. Then ∂E_a satisfies the prescribed mean curvature equation

$$\mathbf{H}_{\partial E_a} = f - a \qquad \text{on } \partial E_a \tag{19}$$

where $\mathbf{H}_{\partial E_a}$ is the mean curvature of ∂E_a with the convention that we oriented the surface with the outer unit normal.

Choose now $t > s$ and consider the sets $E_t \subseteq E_s$. For simplicity, let us write $\tilde{\partial} E_t = \overline{\partial E_t \cap \Omega}$, $\tilde{\partial} E_s = \overline{\partial E_s \cap \Omega}$. Let $\delta = \mathrm{dist}(\tilde{\partial} E_t, \tilde{\partial} E_s) = \min\{|x - y| : x \in \tilde{\partial} E_t,\, y \in \tilde{\partial} E_t\} \geq 0$, and choose $x_t \in \tilde{\partial} E_t$ and $x_s \in \tilde{\partial} E_s$ such that $|x_t - x_s| = \delta$. We let e be the outer normal to $\tilde{\partial} E_t$, at x_t, which is also the outer normal to $\tilde{\partial} E_s$ at x_s and is given by $e = (x_s - x_t)/\delta$ whenever $\delta > 0$.

Since we already know that u is continuous inside Ω [11], then the equality $x_s = x_t$ could only happen in $\partial\Omega$. But this cannot happen since both ∂E_s and ∂E_t satisfy the prescribed mean curvature equation (19) classically up to the boundary, with Neumann boundary condition, and $t > s$. Thus we have that $\delta > 0$. Notice also that, since Ω is strictly convex, none of the points x_s, x_t can lie on its boundary: indeed if for instance we had $x_s \in \partial\Omega$ (and $x_t \in \overline{\Omega}$), we would have $(x_t - x_s)\cdot\nu_\Omega(x_s) < 0$, and since $-\nu_\Omega(x_s)$ is tangent to $\tilde{\partial}E_s$, pointing towards its interior, it would contradict the minimality of $\|x_t - x_s\|$ with respect to x_s.

Let C be the connected component of $(\tilde{\partial}E_t + \delta e)\cap\tilde{\partial}E_s$ containing x_s. Since Ω is strictly convex, we have that C is a compact subset of $(x_s + e^\perp)\cap\Omega$. In particular, if $\varepsilon > 0$ is small enough, then the open set $(E_t + (\delta+\varepsilon)e)\setminus\overline{E}_s$ has a connected component W_ε with $C \subset \partial W_\varepsilon$ and which is strictly contained in Ω.

We use (18), comparing E_t with $E_t \setminus (W_\varepsilon - (\delta+\varepsilon)e)$ and E_s with $E_s \cup W_\varepsilon$:

$$\begin{aligned} Per(E_t,\Omega) \;+\; \int_{E_t}(t - f(x))\,dx \;&\le\; Per(E_t\setminus(W_\varepsilon-(\delta+\varepsilon)e),\Omega) \;+\; \int_{E_t\setminus(W_\varepsilon-(\delta+\varepsilon)e)}(t-f(x))\,dx \\ Per(E_s,\Omega) \;+\; \int_{E_s}(s - f(x))\,dx \;&\le\; Per(E_s\cup W_\varepsilon,\Omega) \;+\; \int_{E_s\cup W_\varepsilon}(s-f(x))\,dx. \end{aligned} \tag{20}$$

Now, if we let $L_t = \mathcal{H}^{N-1}(\partial W_\varepsilon \setminus \tilde{\partial}E_s)$ and $L_s = \mathcal{H}^{N-1}(\partial W_\varepsilon \cap \tilde{\partial}E_s)$, we have that

$$Per(E_t\setminus(W_\varepsilon-(\delta+\varepsilon)e),\Omega) \;=\; Per(E_t,\Omega) - L_t + L_s$$

and

$$Per(E_s\cup W_\varepsilon,\Omega) \;=\; Per(E_s,\Omega) + L_t - L_s\,,$$

so that, summing both equations in (20), we deduce

$$\int_{W_\varepsilon-(\delta+\varepsilon)e}(t-f(x))\,dx \;\le\; \int_{W_\varepsilon}(s-f(x))\,dx.$$

Hence,

$$(t-s)|W_\varepsilon| \;\le\; \int_{W_\varepsilon}(f(x+(\delta+\varepsilon)e)-f(x))\,dx \;\le\; |W_\varepsilon|\omega_f(\delta+\varepsilon)\,.$$

Dividing both sides by $|W_\varepsilon| > 0$ and sending then ε to zero, we deduce

$$t - s \;\le \omega_f(\delta)\,.$$

The regularity of u follows, that is, $\omega_u \le \omega_f$. Now, if f is continuous in $\overline{\Omega}$ we may approximate it uniformly by smooth functions $f_\epsilon \in C^\infty(\overline{\Omega})$. If u_ϵ is the corresponding solution of (4) with Neumann boundary condition, then we already proved that $\omega_{u_\epsilon} \le \omega_{f_\epsilon}$. In particular, this gives us the equicontinuity of u_ϵ. By uniqueness of solutions of the problem (4) with Neumann boundary condition, we have that $u_\epsilon \to u$ in $C(\overline{\Omega})$ where u is the solution corresponding to f and we get that $\omega_u \le \omega_f$.

Finally, if Ω is an arbitrary convex subset of $\mathbb{R}^N$, we can approximate Ω by a sequence of smooth, strictly bounded uniformly convex sets Ω_n and consider u_n the solution of (4) with homogeneous Neumann boundary condition in Ω_n. Then, each u_n is uniformly continuous with $\omega_{u_n} \le \omega_f$. Passing to the limit and recalling the uniqueness of the Neumann solution u of (4), we get the thesis. □

Let us recall some basic definitions in order to state the analogous of Theorem 3 for the total variation flow. For brevity, we shall only sketch the results. As above, we assume that Ω is an open convex set in $\mathbb{R}^N$. Let us consider the minimizing total variation flow

$$\begin{aligned} &\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{Du}{|Du|}\right) && \text{in } Q_T =]0,T[\times\Omega, \\ &\frac{Du}{|Du|}\cdot\nu_\Omega = 0 && \text{in } Q_T =]0,T[\times\partial\Omega \end{aligned} \tag{21}$$

with the initial condition

$$u(0,x) = f(x), \qquad x \in \Omega. \tag{22}$$

Let us recall that, in the Hilbertian framework (in L^2), it is the gradient flow of the total variation as defined in [9]. In the general case we shall follow [5, 7, 10] (where the case of the total variation flow unbounded domains is considered).

We define the operator $\mathcal{A}_p \subseteq L^p(\Omega) \times L^p(\Omega)$, $\frac{N}{N-1} \leq p \leq \infty$, by

$$\begin{gathered}(u,v) \in \mathcal{A}_p \quad \text{if and only if } u, v \in L^p(\Omega),\ u \in BV(\Omega), \text{ and} \\ \text{there exists } z \in X(\Omega)_p \text{ with } \|z\|_\infty \leq 1 \text{ such that } (z \cdot Du) = |Du|,\ [z \cdot \nu_\Omega] = 0, \text{ and} \\ v = -\operatorname{div}(z) \qquad \text{in } \mathcal{D}'(\Omega).\end{gathered}$$

By $v \in \mathcal{A}_p u$ we mean that $(u,v) \in \mathcal{A}_p$.

Proposition 5.1. *[5, 7, 10] The operator $\mathcal{A}_p$ is m-accretive in $L^p(\Omega)$; that is, for any $f \in L^p(\Omega)$ and any $\lambda > 0$ there is a unique solution $u \in L^p(\Omega)$ of the problem*

$$u + \lambda \mathcal{A}_p u \ni f. \tag{23}$$

Moreover, if $u_1, u_2 \in L^p(\Omega)$ are the solutions of (23) corresponding to the right-hand sides $f_1, f_2 \in L^p(\Omega)$, then

$$\|u_1 - u_2\|_p \leq \|f_1 - f_2\|_p.$$

Moreover, the domain of $\mathcal{A}_p$ is dense in $L^p(\Omega)$ when $p < \infty$ and its closure contains $BUC(\overline{\Omega})$ (the space of bounded uniformly continuous functions) when $p = \infty$

We denote by $\mathcal{R}_\lambda f$ the solution of (23), and by $\mathcal{R}_\lambda^k f$ its k-iterate, for any $k \geq 1$. Recall the notion of strong solution for nonlinear semigroups generated by accretive operators.

Definition 5.2. *A function u is called a strong solution in the sense of semigroups of $\frac{du}{dt} + \mathcal{A}_p u \ni 0$ with $u(0) = f$ if*

$$\begin{cases} u \in C([0,T]; L^p(\Omega)) \cap W^{1,1}_{loc}(]0,T[; L^p(\Omega)), \\ \\ u(t) \in \operatorname{Dom}(\mathcal{A}_p) \quad \text{a.e. in } t > 0 \text{ and } u' + \mathcal{A}_p u(t) \ni 0 \text{ a.e. } t \in]0,T[, \\ \\ u(0) = f. \end{cases} \tag{24}$$

By $L^1_w(]0,T[; BV(\Omega))$ we denote the space of weakly measurable functions $w : [0,T] \to BV(\Omega)$ (i.e., $t \in [0,T] \to \langle w(t), \phi \rangle$ is measurable for any $\phi \in BV(\Omega)^*$, where $BV(\Omega)^*$ denote the dual of $BV(\Omega)$) such that $\int_0^T \|w(t)\|\, dt < \infty$.

Definition 5.3. *A function $u \in C([0,T]; L^p(\Omega))$ is called a strong solution of (21) if $u \in W^{1,1}_{\mathrm{loc}}(0,T; L^p(\Omega)) \cap L^1_w(]0,T[; BV(\Omega))$ and there exists $z \in L^\infty\left(]0,T[\times \Omega; \mathbb{R}^N\right)$ with $\|z\|_\infty \leq 1$ such that*

$$\int_\Omega (z(t) \cdot Du(t)) = \int_\Omega |Du(t)| \qquad \text{for a.e. } t > 0, \tag{25}$$

$$[z(t) \cdot \nu_\Omega] = 0 \qquad \text{in } \partial\Omega \text{ for a.e. } t > 0, \tag{26}$$

and

$$u_t = \operatorname{div} z \qquad \text{in } \mathcal{D}'\left(]0,T[\times \Omega\right).$$

Using Proposition 5.1, by Crandall and Ligget's semigroup generation theorem [14] we obtain the following result.

Theorem 4. *Let $f \in L^p(\Omega)$ if $\frac{N}{N-1} \le p < \infty$, or let $f \in BUC(\overline{\Omega})$ if $p = \infty$. Then there is a unique strong solution in the sense of semigroups $u(t) = S(t)f := \lim_{\lambda\downarrow 0,\, k\lambda\to t} \mathcal{R}_\lambda^k f \in C([0,T], L^p(\Omega))$ of the problem*

$$\frac{du}{dt} + \mathcal{A}_p u \ni 0, \qquad u(0) = f. \tag{27}$$

Moreover, the semigroup solution is a strong solution of (21)*, and, conversely, any strong solution of* (21) *is a strong solution in the sense of semigroups of* (27)*.*

Remark 5.4. Notice that given $p \in [\frac{N}{N-1}, \infty]$ the limit $\lim_{\lambda\downarrow 0,\, k\lambda\to t} \mathcal{R}_\lambda^k f$ is taken in $L^p(\Omega)$.

Let us point out that a more general existence and uniqueness result of solutions of 21 for initial data in $L^p_{loc}(\overline{\Omega})$ for any $p \in [1,\infty]$ holds [7, 10]. Indeed, it is proved in $\mathbb{R}^N$ in [7, 10] but it can be easily adapted to the case of Neumann boundary condition in $\overline{\Omega}$ using the techniques in [4].

As a consequence of Theorems 3 and 4, we deduce:

Corollary 5.5. *Let f be a uniformly continuous function in $\overline{\Omega}$ and $u(t,x)$ be the Total Variation flow starting from f, with Neumann boundary condition. Then, for any $t \ge 0$, $u(t,\cdot)$ is uniformly continuous with modulus $\omega_{u(t,\cdot)} \le \omega_f$.*

Proof. If $f \in BUC(\overline{\Omega})$, this is a consequence of Theorem 3 and Theorem 4. If we only assume that f is uniformly continuous in $\overline{\Omega}$, we may approximate it by functions in $BUC(\overline{\Omega})$, apply the result in this case, and use the uniqueness result for data in $L^\infty_{\rm loc}(\overline{\Omega})$ mentioned before the statement of the corollary. □

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