QUASISTATIC EVOLUTION PROBLEMS FOR PRESSURE-SENSITIVE PLASTIC MATERIALS

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Abstract. We study quasistatic evolution problems for pressure-sensitive plastic materials in the context of small strain associative perfect plasticity.

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Contents

1. Introduction

Several materials of interest for applications, such as concrete, granular media, metallic foams, and porous metals, exhibit a pressure-sensitive yield behavior. There exists a large literature focusing on yield criteria for these materials, which identify the onset of irreversible inelastic behavior with the fact that a suitable measure of the state of internal stress reaches a threshold. Examples include the Gurson criterion for porous ductile materials [8], the criterion of Ottosen for concrete [14], the Desphande-Fleck criterion for metallic foams [7], and, for soils, Cam-Clay and the many subsequent variants (see, e.g., [6] and the references quoted therein). These criteria and several others are discussed in detail in [3].

Following the engineering literature, we work for simplicity in the framework of associative elasto-plasticity. Moreover we limit our analysis to the case of no hardening (perfect plasticity). With reference to a domain $\Omega \subset \mathbb{R}^n$, the problem can be formulated as follows. The linearized strain Eu, defined as the symmetric part of the spatial gradient of the displacement u, is decomposed as the sum $Eu = e + p$, where e and p are the elastic and plastic strains. The stress σ is determined only by e, through the formula $\sigma = \mathbb{C}e$, where $\mathbb{C}$ is the elasticity tensor. It is constrained to lie in a prescribed convex subset $\mathbb{K}$ of the space

$\mathbb{M}^{n\times n}_{sym}$ of $n\times n$ symmetric matrices, whose boundary $\partial\mathbb{K}$ is referred to as the yield surface. In this context, pressure sensitivity of the yield criterion leads to the hypothesis that $\mathbb{K}$ is bounded.

The data of our problem are a time-dependent body force $f(t,x)$, defined for $t \in [0,T]$ and $x \in \Omega$, a time-dependent surface force $g(t,x)$ acting on a portion Γ_1 of the boundary $\partial\Omega$, and a time-dependent displacement prescribed on the complementary porion Γ_0 of $\partial\Omega$. The classical formulation of the quasistatic evolution problem consists in finding functions $u(t,x)$, $e(t,x)$, $p(t,x)$, $\sigma(t,x)$ satisfying the following conditions for every $t \in [0,T]$ and every $x \in \Omega$:

$$\begin{gathered} \text{additive decomposition: } Eu(t,x) = e(t,x) + p(t,x)\,, \\ \text{constitutive equation: } \sigma(t,x) = \mathbb{C}e(t,x)\,, \\ \text{equilibrium: } -\operatorname{div}\sigma(t,x) = f(t,x)\,, \\ \text{associative flow rule: } \dot p(t,x) \in N_{\mathbb{K}}(\sigma(t,x))\,, \end{gathered} \tag{1.1}$$

where $N_{\mathbb{K}}(\xi)$ is the normal cone to $\mathbb{K}$ at ξ. The problem is supplemented by initial conditions at time $t = 0$, by displacement boundary conditions $u(t,x) = w(t,x)$ for $t \in [0,T]$ and $x \in \Gamma_0$, and traction boundary conditions $\sigma(t,x)\nu(x) = g(t,x)$ for $t \in [0,T]$ and $x \in \Gamma_1$, where $\nu(x)$ is the outer unit normal to $\partial\Omega$.

In recent work [5], a similar problem was considered for the pressure-insensitive case where $\mathbb{K}$ is a cylinder in $\mathbb{M}^{n\times n}_{sym}$ containing all scalar multiples of the identity matrix. There, the existence of a suitably defined weak solution was obtained by time-discretization. According to a general energy approach, see e.g. [13], the discrete time formulation consists in solving a chain of incremental minimum problems which are quadratic in e and have linear growth in p. Thus, the direct methods of the calculus of variations lead to a weak formulation with $u \in BD(\Omega)$, the space of functions with bounded deformation, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $p \in M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym})$, the space of bounded Radon measures on $\Omega \cup \Gamma_0$ with values in $\mathbb{M}^{n\times n}_{sym}$.

Notice that allowing for measure-valued plastic strains is also natural from the point of view of mechanics, see [16]: indeed, localization of plastic deformation and formation of shear bands are often observed experimentally in the materials to which the models we analyze should apply.

In this work, we extend this approach to the case where $\mathbb{K}$ is an arbitrary convex bounded subset of $\mathbb{M}^{n\times n}_{sym}$ with nonempty interior. To adapt the technique to the new situation, we have to introduce a suitable duality product $\langle\sigma,p\rangle$, between stress and plastic strain, defined for every $\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ with $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$ and for every $p \in M_b(\Omega\cup\Gamma_0, \mathbb{M}^{n\times n}_{sym})$ of the form $p = Eu - e$ with $u \in BD(\Omega)$ and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. This is done in Section 3, using results from [11, 1].

After the properties of this duality have been established, we follow the lines of the proof of [5], and obtain, under suitable hypotheses on the data f, g, and w, an existence result (Theorem 4.3) for a weak formulation (Definition 4.1) of problem (1.1), with $u \in AC([0,T];BD(\Omega))$, $e \in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, and $p \in AC([0,T];M_b(\Omega\cup\Gamma_0,\mathbb{M}^{n\times n}_{sym}))$. Moreover, we prove that e, and hence σ, are uniquely determined by the initial conditions.

We emphasize that our results are obtained under very general qualitative hypotheses on the yield surfaces $\partial\mathbb{K}$ and on the elasticity tensor $\mathbb{C}$. Namely, we just assume that $\mathbb{K}$ is a convex, bounded set with nonempty interior, and that $\mathbb{C}$, regarded as a linear operator acting on $\mathbb{M}^{n\times n}_{sym}$ is symmetric and positive definite. In particular no assumption of isotropy is required.

2. Preliminaries

2.1. **Mathematical preliminaries.** Given a locally compact subset X of $\mathbb{R}^n$ and a finite dimensional Hilbert space Ξ, the space of bounded Ξ-valued Borel measures on X is denoted by $M_b(X;\Xi)$ and is endowed with the norm $\|\mu\|_1 := |\mu|(X)$, where $|\mu| \in M_b(X;\mathbb{R})$ is the variation of the measure μ. By Riesz representation theorem (see, e.g., [15, Theorem 6.19]) $M_b(X;\Xi)$ can be identified with the dual of $C_0(X;\Xi)$, the space of continuous functions $\varphi\colon X\to\Xi$ such that $\{|\varphi|\geq\varepsilon\}$ is compact for every $\varepsilon>0$. This defines the weak* topology in $M_b(X;\Xi)$.

For every $\mu\in M_b(X;\Xi)$ we consider the Lebesgue decomposition $\mu=\mu^a+\mu^s$, where μ^a is absolutely continuous and μ^s is singular with respect to Lebesgue measure $\mathcal{L}^n$. The space $L^1(X;\Xi)$ of Ξ-valued $\mathcal{L}^n$-integrable functions is regarded as a subspace of $M_b(X;\Xi)$, with the induced norm. When $\Xi=\mathbb{R}$, the indication of the space Ξ is omitted.

The L^p norm, $1\leq p\leq\infty$, is denoted by $\|\cdot\|_p$. The brackets $\langle\cdot|\cdot\rangle$ denote the duality product between conjugate L^p spaces, as well as between other pairs of spaces, according to the context.

The space of *symmetric* $n\times n$ *matrices* is denoted by $\mathbb{M}^{n\times n}_{sym}$; it is endowed with the euclidean scalar product $\xi:\zeta:=\mathrm{tr}(\xi\zeta)=\sum_{ij}\xi_{ij}\zeta_{ij}$ and with the corresponding euclidean norm $|\xi|:=(\xi:\xi)^{1/2}$. The *symmetrized tensor product* $a\odot b$ of two vectors $a,\ b\in\mathbb{R}^n$ is the symmetric matrix with entries $(a_ib_j+a_jb_i)/2$.

For every $u\in L^1(U;\mathbb{R}^n)$, with U open in $\mathbb{R}^n$, let Eu be the $\mathbb{M}^{n\times n}_{sym}$-valued distribution on U, whose components are defined by $E_{ij}u=\frac{1}{2}(D_ju_i+D_iu_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u\in L^1(U;\mathbb{R}^n)$ such that $Eu\in M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm $\|u\|_1+\|Eu\|_1$. It is possible to prove that $BD(U)$ is the dual of a normed space (see [12] and [18]), and this defines the weak* topology of $BD(U)$. A sequence u_k converges to u weakly* in $BD(U)$ if and only if $u_k\rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^n)$ and $Eu_k\xrightharpoonup{*}Eu$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. For the general properties of $BD(U)$ we refer to [17].

In our problem $u\in BD(U)$ represents the *displacement* of an elasto-plastic body and Eu is the corresponding linearized *strain*.

We recall that a function f from $[0,T]$ into a Banach space Y is said to be absolutely continuous if for every $\varepsilon>0$ there exists $\delta>0$ such that $\sum_i\|f(t_i)-f(s_i)\|_Y<\varepsilon$, whenever $\sum_i(t_i-s_i)<\delta$ and $0\leq s_1<t_1\leq s_2<t_2\leq\cdots\leq s_k<t_k\leq T$. The space of these functions is denoted by $AC([0,T];Y)$. For the general properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [4, Appendix]. When Y is the dual of a separable Banach space, one can prove (see [5, Theorem 7.1]) that for a.e. $t\in[0,T]$ there exists the weak*-limit

$$\dot f(t):=w^*\text{-}\lim_{s\to t}\frac{f(s)-f(t)}{s-t}.$$

Note that in this general situation it may happen that $\dot f$ is not Bochner integrable.

2.2. **Mechanical preliminaries. The reference configuration.** Throughout the paper the *reference configuration* Ω is a *bounded connected open set* in $\mathbb{R}^n$, with *Lipschitz boundary* $\partial\Omega=\Gamma_0\cup\Gamma_1$. We assume that $\Gamma_0\neq\emptyset$, Γ_1 is closed, and $\Gamma_0\cap\Gamma_1=\emptyset$.

The constraint and its support function. The *constraint on the stress* is given by a closed convex set $\mathbb{K}\subset\mathbb{M}^{n\times n}_{sym}$ with nonempty interior. Its boundary $\partial\mathbb{K}$ plays the role of *yield suface*. For the energy formulation of problem (1.1) it is convenient to introduce the support function $H:\mathbb{M}^{n\times n}_{sym}\to\mathbb{R}$ of $\mathbb{K}$ defined by

$$H(\xi)=\sup_{\zeta\in\mathbb{K}}\xi:\zeta\,. \tag{2.1}$$

H is convex and positively homogeneous of degree one.

For every $\mu\in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$, let $\mu/|\mu|$ be the Radon-Nikodym derivative of μ with respect to its total variation $|\mu|$. According to the general theory of convex functions of measures, we introduce the nonnegative Radon measure $H(\mu)\in M_b(\Omega\cup\Gamma_0)$ defined by

$$H(\mu)(B)=\int_B H\Big(\frac{\mu}{|\mu|}\Big)\,d|\mu| \tag{2.2}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$. Finally we consider the functional $\mathcal{H}\colon M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$ defined by

$$\mathcal{H}(\mu) := H(\mu)(\Omega \cup \Gamma_0)\,. \tag{2.3}$$

We refer to [10] and [17, Chapter II, Section 4] for the properties of $H(\mu)$ and $\mathcal{H}(\mu)$.

The data of the problem. Let us fix a time interval $[0,T]$. We assume that the body force f, the surface force g and the prescribed boundary displacement w satisfy the following assumptions:

$$\begin{gathered} f \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)),\\ g \in AC([0,T]; L^\infty(\Gamma_1;\mathbb{R}^n)),\\ w \in AC([0,T]; H^1(\Omega;\mathbb{R}^n)). \end{gathered} \tag{2.4}$$

Stress and strain. For a given displacement $u \in BD(\Omega)$ and a boundary datum $w \in H^1(\Omega;\mathbb{R}^n)$, the elastic and plastic strains $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_{sym})$ satisfy the relation

$$Eu = e + p \text{ in } \Omega, \tag{2.5}$$

$$p = (w-u)\odot \nu\,\mathcal{H}^{n-1} \text{ on } \Gamma_0, \tag{2.6}$$

so that $e = E^a u - p^a$ a.e. on Ω and $p^s = E^s u$ on Ω. The stress $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ is defined by

$$\sigma := \mathbb{C}e. \tag{2.7}$$

The stored elastic energy $\mathcal{Q} : L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$ is given by

$$\mathcal{Q}(e) = \frac{1}{2}\int_\Omega \mathbb{C}e : e\,dx = \frac{1}{2}\int_\Omega \sigma : e\,dx.$$

For a $w \in H^1(\Omega;\mathbb{R}^n)$, the set of admissible displacements for the boundary datum w on Γ_0 is denoted by $A(w)$ and it is defined as:

$$A(w) := \Big\{(u,e,p) \in BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}) : \text{ (2.5), (2.6) hold}\Big\}. \tag{2.8}$$

The space $\Pi_{\Gamma_0}(\Omega)$ of admissible plastic strains is the set of all $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ for which there exist $u \in BD(\Omega), w \in H^1(\Omega,\mathbb{R}^n)$ and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, such that $(u,e,p) \in A(w)$.

The following lemma, that can be proved as in [5, Lemma 2.1] shows, that the multi-valued map $w \mapsto A(w)$ is closed.

Lemma 2.1. *Let $w_k \in H^1(\Omega;\mathbb{R}^n)$ and let $(u_k,e_k,p_k) \in A(w_k)$. If*

$$u_k \overset{*}{\rightharpoonup} u_\infty \ \textit{weakly}^* \ \textit{in } BD(\Omega), \quad e_k \rightharpoonup e_\infty \ \textit{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),$$

$$p_k \overset{*}{\rightharpoonup} p_\infty \ \textit{weakly}^* \ \textit{in } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}), \quad w_k \rightharpoonup w_\infty \ \textit{weakly in } H^1(\mathbb{R}^n;\mathbb{R}^n),$$

then $(u_\infty,e_\infty,p_\infty) \in A(w_\infty)$.

The traces of the stress. If $\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$, then one can define a distribution $[\sigma\nu]$ on $\partial\Omega$ by

$$\langle[\sigma\nu]|\psi\rangle = \int_\Omega \operatorname{div}\sigma\cdot\psi\,dx + \int_\Omega \sigma : E\psi\,dx \tag{2.9}$$

for $\psi \in W^{1,1}(\Omega;\mathbb{R}^n)$. By Gagliardo's extension result [9, Theorem 1.II], it is easy to see that $[\sigma\nu] \in L^\infty(\partial\Omega;\mathbb{R}^n)$ and that

$$[\sigma_k\nu] \overset{*}{\rightharpoonup} [\sigma\nu] \quad \text{weakly}^* \text{ in } L^\infty(\partial\Omega;\mathbb{R}^n), \tag{2.10}$$

whenever $\sigma_k \overset{*}{\rightharpoonup} \sigma$ weakly* in $L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^n)$.

Uniform safe-load condition. We assume that there exist a function ϱ in the space $AC([0,T];L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}))$ and a compact set $\mathbb{K}_0 \subset \mathrm{int}\mathbb{K}$, such that for every $t \in [0,T]$

$$\mathrm{div}\ \varrho(t) = -f(t) \text{ in } \Omega, \quad [\varrho(t)\nu] = g(t) \text{ on } \Gamma_1, \quad \varrho(t,x) \in \mathbb{K}_0 \text{ in } \Omega. \tag{2.11}$$

3. Stress-strain duality

In this section we develop the notion of duality between the stress and the plastic part of the strain. We begin with the definition and properties of the duality between stress and strain in the spirit of [11], where only the deviatoric part of the stress is bounded, and [1], where a similar problem is studied in $BV(\Omega)$.

In the sequel we will make use of the following space

$$\Sigma(\Omega) = \{\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}) : \ \mathrm{div}\,\sigma \in L^n(\Omega;\mathbb{R}^n)\}.$$

3.1. **Duality between stress and strain.** For every $u \in BD(\Omega)$ and $\sigma \in \Sigma(\Omega)$ we can define a distribution $[\sigma : Eu]$ on Ω by

$$\langle [\sigma : Eu] | \varphi \rangle = -\int_\Omega \varphi\, u \cdot \mathrm{div}\,\sigma\, dx - \int_\Omega \sigma : (u \odot \nabla\varphi)\, dx \tag{3.1}$$

for every $\varphi \in C^\infty_c(\Omega)$. Arguing as in [11, Theorem 3.2] one can prove that the distribution $[\sigma : Eu]$ is a bounded measure on Ω and its variation satisfies

$$|[\sigma : Eu]| \le \|\sigma\|_\infty\, |Eu| \quad \text{in } M_b(\Omega). \tag{3.2}$$

Moreover [2, Corollary 3.2], with obvious changes, implies that

$$[\sigma : Eu]^a = \sigma : E^a u \quad \text{a.e. in } \Omega. \tag{3.3}$$

From the definition (3.1) it follows that

$$[\psi\, \sigma : Eu] = \psi[\sigma : Eu] \quad \text{in } M_b(\Omega) \tag{3.4}$$

for every $\psi \in C^1(\overline{\Omega})$.

We define the measure $[\sigma : E^s u]$ on Ω by putting

$$[\sigma : E^s u] := [\sigma : Eu]^s = [\sigma : Eu] - \sigma : E^a u. \tag{3.5}$$

Inequality (3.2) yields

$$|[\sigma : E^s u]| \le \|\sigma\|_\infty\, |E^s u| \quad \text{in } M_b(\Omega). \tag{3.6}$$

Remark 3.1. This inequality implies that $[\sigma_1 : E^s u_1] = [\sigma_2 : E^s u_2]$ in $M_b(\Omega)$ whenever $\sigma_1 = \sigma_2$ a.e. in Ω and $E^s u_1 = E^s u_2$.

As in [11, Theorem 3.2] one can prove the following stability property: if

$$\sigma_k \stackrel{*}{\rightharpoonup} \sigma \quad \text{weakly}^* \text{ in } L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}),$$

$$\mathrm{div}\,\sigma_k \rightharpoonup \mathrm{div}\,\sigma \quad \text{weakly in } L^n(\Omega;\mathbb{R}^n),$$

then for every $u \in BD(\Omega)$

$$[\sigma_k : Eu] \stackrel{*}{\rightharpoonup} [\sigma : Eu] \quad \text{and} \quad [\sigma_k : E^s u] \stackrel{*}{\rightharpoonup} [\sigma : E^s u] \quad \text{weakly}^* \text{ in } (C_b(\Omega))'$$

that is, for each bounded continuous function $\varphi \colon \Omega \to \mathbb{R}$ one has

$$\int_\Omega \varphi\, d[\sigma_k : Eu] \to \int_\Omega \varphi\, d[\sigma : Eu], \qquad \int_\Omega \varphi\, d[\sigma_k : E^s u] \to \int_\Omega \varphi\, d[\sigma : E^s u]. \tag{3.7}$$

3.2. **Duality between stress and plastic strain.** Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, fix $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$, satisfying (2.5) and (2.6). Then we define a measure $[\sigma : p] \in M_b(\Omega \cup \Gamma_0)$ by setting

$$[\sigma : p] := \sigma : p^a + [\sigma : E^s u] = [\sigma : Eu] - \sigma : e \quad \text{on } \Omega,$$

$$[\sigma : p] := [\sigma\nu] \cdot (w - u)\mathcal{H}^{n-1} \quad \text{on } \Gamma_0,$$

so that

$$\int_{\Omega\cup\Gamma_0} \varphi \, d[\sigma : p] = \int_\Omega \varphi \, d[\sigma : Eu] - \int_\Omega \varphi \, \sigma : e \, dx + \int_{\Gamma_0} \varphi \, [\sigma\nu] \cdot (w - u) \, d\mathcal{H}^{n-1} \tag{3.8}$$

for every $\varphi \in C_b(\Omega \cup \Gamma_0)$, the space of bounded continuous functions on $\Omega \cup \Gamma_0$. In this case Remark 3.1 shows that the measure $[\sigma : p]$ is well defined, that is, it does not depend upon the particular choice of u, e and w.

It follows from the definition that

$$[\sigma : p]^a = \sigma : p^a \quad \text{a.e. on } \Omega, \qquad [\sigma : p]^s = [\sigma : E^s u] \quad \text{in } M_b(\Omega)$$

and

$$|[\sigma : p]| \le \|\sigma\|_\infty |p| \quad \text{in } M_b(\Omega \cup \Gamma_0), \qquad |[\sigma : p]^s| \le \|\sigma\|_\infty |p^s| \quad \text{in } M_b(\Omega \cup \Gamma_0). \tag{3.9}$$

Moreover (3.4) implies that

$$[\psi\sigma : p] = \psi[\sigma : p] \quad \text{in } M_b(\Omega \cup \Gamma_0)$$

for every $\psi \in C^1(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and using the definitions one can deduce that

$$\int_{\Omega\cup\Gamma_0} \varphi \, d[\sigma : p] = \int_{\Omega\cup\Gamma_0} \varphi \, \sigma \, dp \tag{3.10}$$

for every $\sigma \in C^1(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and every $\varphi \in C^1(\overline{\Omega})$. By (3.9) we deduce, that (3.10) holds for all $\sigma \in C(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and $\varphi \in C(\overline{\Omega})$. Therefore for every $\sigma \in C(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and $p \in \Pi_{\Gamma_0}(\Omega)$ we have

$$[\sigma : p] = \sigma : p \quad \text{in } M_b(\Omega \cup \Gamma_0),$$

where the right-hand side denotes the measure defined by

$$(\sigma : p)(B) = \int_B \sigma_{ij} \, dp_{ij}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$.

Also it is easy to see that the relation

$$[\sigma : p] = (\sigma : p) \cdot \mathcal{L}^n \quad \text{in } M_b(\Omega)$$

holds in the case $\sigma, p \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

It follows from the definition and from (2.10) and (3.7) that

$$[\sigma_k : p] \overset{*}{\rightharpoonup} [\sigma : p] \quad \text{weakly}^* \text{ in } (C_b(\Omega \cup \Gamma_0))' \tag{3.11}$$

whenever $\sigma_k \overset{*}{\rightharpoonup} \sigma$ weakly* in $L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div} \sigma_k \rightharpoonup \operatorname{div} \sigma$ weakly in $L^n(\Omega; \mathbb{R}^n)$.

Finally, for every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we define

$$\begin{aligned} \langle \sigma | p \rangle_{\Sigma,\Pi} :&= [\sigma : p](\Omega \cup \Gamma_0) = \\ &= \int_\Omega \sigma : p^a \, dx + [\sigma : E^s u](\Omega) + \int_{\Gamma_0} [\sigma\nu] \cdot (w - u) \, d\mathcal{H}^{n-1} = \\ &= [\sigma : Eu](\Omega) - \int_\Omega \sigma : e \, dx + \int_{\Gamma_0} [\sigma\nu] \cdot (w - u) \, d\mathcal{H}^{n-1}. \end{aligned}$$

where $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfy (2.5) and (2.6).

Let us now prove the integration by parts formula for stresses and displacements:

Proposition 3.2. *Let* $\sigma \in \Sigma(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^n)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$ *and let* $(u,e,p) \in A(w)$. *Assume that* $-\mathrm{div}\,\sigma = f$ *in* Ω *and* $[\sigma\nu] = g$ *on* Γ_1. *Then*

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} \varphi\, d[\sigma : p] + \int_\Omega \varphi\,\sigma : (e - Ew)\, dx + \int_\Omega \sigma : \Big((u-w)\odot\nabla\varphi\Big)\, dx = \\ = \int_\Omega \varphi\, f\cdot(u-w)\, dx + \int_{\Gamma_1} \varphi\, g\cdot(u-w)\, d\mathcal{H}^{n-1}\end{aligned} \tag{3.12}$$

for every $\varphi \in C^1(\overline{\Omega})$.

Proof: First, let us establish the following formula for $\sigma \in \Sigma(\Omega)$, $v \in BD(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$:

$$\int_{\partial\Omega} \varphi\,[\sigma\nu]\cdot v\, d\mathcal{H}^{n-1} = \int_\Omega \varphi\,\mathrm{div}\,\sigma\cdot v\, dx + \int_\Omega \sigma : (v\odot\nabla\varphi)\, dx + \int_\Omega \varphi\, d[\sigma : Ev]. \tag{3.13}$$

Arguing as in [5, Lemma 2.3] we can find a sequence σ_k in $C^\infty(\overline{\Omega})$, such that

$$\sigma_k \to \sigma \quad \text{strongly in } L^p(\Omega;\mathbb{M}^{n\times n}_{sym}), \quad \mathrm{div}\,\sigma_k \to \mathrm{div}\,\sigma \quad \text{strongly in } L^n(\Omega;\mathbb{R}^n)$$

for every $1 \le p < \infty$. By the integration by parts formula for $BD(\Omega)$, formula (3.13) holds for every σ_k. The left-hand side converges to that of (3.13) by (2.10), while the convergence of the right-hand side follows from (3.7). This proves (3.13).

By the assumptions of the theorem, for $v = u - w \in BD(\Omega)$ formula (3.13) takes the form:

$$\begin{aligned}-\int_\Omega \varphi\, f\cdot(u-w)\, dx + \int_\Omega \sigma : \Big((u-w)\odot\nabla\varphi\Big)\, dx + \int_\Omega \varphi\, d[\sigma : E(u-w)] = \\ = \int_{\Gamma_0} \varphi\,[\sigma\nu]\cdot(u-w)\, d\mathcal{H}^{n-1} + \int_{\Gamma_1} \varphi\, g\cdot(u-w)\, d\mathcal{H}^{n-1}.\end{aligned} \tag{3.14}$$

On the other hand, (3.8) gives

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} \varphi\, d[\sigma : p] + \int_\Omega \varphi\,\sigma : (e - Ew)\, dx + \int_\Omega \sigma : \Big((u-w)\odot\nabla\varphi\Big)\, dx = \\ = \int_\Omega \varphi\, d[\sigma : E(u-w)] + \int_\Omega \sigma : \Big((u-w)\odot\nabla\varphi\Big)\, dx - \int_{\Gamma_0} \varphi\,[\sigma\nu]\cdot(u-w)\, d\mathcal{H}^{n-1}.\end{aligned}$$

Thus, the last relation together with (3.14) yields (3.12). ☐

Let

$$\mathcal{K}(\Omega) := \{\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}) : \sigma(x) \in \mathbb{K} \text{ for a.e. } x \in \Omega\}.$$

The following proposition can pe proved as in [5, Proposition 2.2].

Proposition 3.3. *Let* $p \in \Pi_{\Gamma_0}(\Omega)$. *Then*

$$H(p) \ge [\sigma : p] \quad \text{in } M_b(\Omega\cup\Gamma_0) \tag{3.15}$$

for every $\sigma \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)$, *and*

$$\mathcal{H}(p) = \sup\{\langle\sigma|p\rangle \ : \ \sigma \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)\}. \tag{3.16}$$

Moreover, if $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$ *and there exists* $\varrho \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)$ *such that* $[\varrho\nu] = g$ *on* Γ_1, *then*

$$\mathcal{H}(p) = \sup\{\langle\sigma|p\rangle \ : \ \sigma \in \Sigma(\Omega)\cap\mathcal{K}(\Omega),\ [\sigma\nu] = g \text{ on } \Gamma_1\}. \tag{3.17}$$

4. Quasistatic evolution

4.1. **Definition and existence result.** From assumptions (2.4) it follows, that for the functional $\mathcal{L}(t) \in BD(\Omega)'$, defined by

$$\langle \mathcal{L}(t)|u\rangle = \int_\Omega f(t)\, u\, dx + \int_{\Gamma_1} g(t)\, u, \tag{4.1}$$

the weak* limit

$$\dot{\mathcal{L}}(t) = w^*\text{-}\lim_{s\to t} \frac{\mathcal{L}(s) - \mathcal{L}(t)}{s-t}$$

exists in $BD(\Omega)'$ for a.e. $t \in [0,T]$, and that

$$\langle \dot{\mathcal{L}}(t)|u\rangle = \int_\Omega \dot{f}(t)\, u\, dx + \int_{\Gamma_1} \dot{g}(t)\, u. \tag{4.2}$$

Therefore the function $t \mapsto \langle \dot{\mathcal{L}}(t)|u(t)\rangle$ belongs to $L^1([0,T])$ whenever $t \mapsto u(t)$ is in $L^\infty([0,T]; BD(\Omega))$.

A function $p : [0,T] \to M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ will be regarded as a function defined on the time interval $[0,T]$ with values in the dual of the separable Banach space $C_0(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{sym})$. Its variation $\mathcal{V}$ and $\mathcal{H}-$variation $\mathcal{D}_\mathcal{H}$ are defined as

$$\mathcal{V}(p; s, t) = \sup\Big\{ \sum_{j=1}^N \|p(t_j) - p(t_{j-1})\|_1 \ :\ s = t_0 \le \cdots \le t_N = t,\ N \in \mathbb{N} \Big\},$$

$$\mathcal{D}_\mathcal{H}(p; s, t) = \sup\Big\{ \sum_{j=1}^N \mathcal{H}(p(t_j) - p(t_{j-1})) \ :\ s = t_0 \le \cdots \le t_N = t,\ N \in \mathbb{N} \Big\}.$$

The notation $D_\mathcal{H}$ for the $\mathcal{H}$-variation is motivated by the more standard case in which the set $\mathbb{K}$ of admissible stresses contains the origin in its interior. In this case, $\mathcal{H}$ is positive and the $\mathcal{H}$-variation of p has the physical interpretation of plastic dissipation in the time interval (s,t).

Next we give a variational formulation of the quasistatic problem.

Definition 4.1. A quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ which satisfies the following conditions:

(qs1) global stability: for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$ and

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t)|u(t)\rangle \le \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle \mathcal{L}(t)|v\rangle \tag{4.3}$$

for every $(v, \eta, q) \in A(0)$.

(qs2) energy balance: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned} \mathcal{Q}(e(t)) + \mathcal{D}_\mathcal{H}(p; 0, t) - \langle \mathcal{L}(t)|u(t)\rangle = \mathcal{Q}(e(0)) + \langle \mathcal{L}(0)|u(0)\rangle + \\ + \int_0^t \Big(\langle \sigma(s)|E\dot{w}(s)\rangle - \langle \mathcal{L}(s)|\dot{w}(s)\rangle - \langle \dot{\mathcal{L}}(s)|u(s)\rangle \Big)\, ds, \end{aligned} \tag{4.4}$$

where $\sigma(t) = \mathbb{C}e(t)$.

Remark 4.2. Since the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{sym})$ has bounded variation, it is bounded and the set of its discontinuity points (in the strong topology) is at most countable. As the estimates of [5, Theorem 3.8] are true also in this case, the same continuity property holds for $t \mapsto e(t)$ and $t \mapsto \sigma(t)$ from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and for $t \mapsto u(t)$ from $[0,T]$ into $BD(\Omega)$. Therefore

$$e,\ \sigma \in L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})) \quad \text{and } u \in L^\infty([0,T]; BD(\Omega)).$$

Finally, as $\dot{w} \in L^1([0,T]; W^{1,2}(\Omega; \mathbb{R}^n))$ and $E\dot{w} \in L^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, the integral in the right-hand side of (4.4) is well-defined.

Theorem 4.3. *Assume* (2.4) *and* (2.11). *If* $(u_0, e_0, p_0) \in A(w(0))$ *satisfy the stability condition*

$$\mathcal{Q}(e(0)) - \langle \mathcal{L}(0)|u_0\rangle \le \mathcal{Q}(e) + \mathcal{H}(p - p_0) - \langle \mathcal{L}(0)|u\rangle$$

for every $(u, e, p) \in A(w(0))$, *then there exists a quasistatic evolution* $t \mapsto (u(t), e(t), p(t))$ *such that* $u(0) = u_0,\ e(0) = e_0,\ p(0) = p_0$.

PROOF: The proof can be obtained by time discretization. For every $k \in \mathbb{N}$ we fix a subdivision $0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T$, satisfying (4.11) of [5]. At each time step we solve the incremental minimum problem (4.12) of [5], adopting the definitions of $A(w)$ and $\mathcal{H}$ of the present paper. Then we define the piecewise constant interpolations $u_k(t)$, $e_k(t),\ p_k(t),\ \sigma_k(t)$ as in (4.15) of [5], and we prove that for every $t \in [0,T]$ $u_k(t) \overset{*}{\rightharpoonup} u(t)$ weakly* in $BD(\Omega)$, $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $p_k(t) \overset{*}{\rightharpoonup} p(t)$ weakly* in $M_b(\Omega \cup \Gamma_0; \mathrm{M}^D)$, where $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution.

The details can be recovered by repeating the arguments of [5, Section 4], with obvious modifications due to the new definitions introduced in Section 3 of the present paper. □

The next theorem shows, that the convergence of elastic strains and stresses takes place in the strong topology of $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. See [5, Theorem 4.8] for the proof.

Theorem 4.4. *Assume that*

$$p_k(t) \overset{*}{\rightharpoonup} p(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}) \tag{4.5}$$

for every $t \in [0,T]$. *Then* $e_k(t) \to e(t)$ *and* $\sigma_k(t) \to \sigma(t)$ *strongly in* $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. *Moreover,*

$$\lim_k \sum_{0 < t_k^r \le t} \Big\{ \mathcal{H}(p_k(t_k^r) - p_k(t_k^{r-1})) - \langle \varrho(t_k^r) | p_k(t_k^r) - p_k(t_k^{r-1})\rangle \Big\} =$$
$$= \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle \varrho(t)|p(t)\rangle + \langle \varrho(0)|p(0)\rangle + \int_0^t \langle \dot\varrho(s)|p(s)\rangle\, ds$$

for every $t \in [0,T]$.

4.2. **Regularity and uniqueness.** The next statement shows that the quasistatic evolution is absolutely continuous with respect to time. We refer to [5, Theorem 5.2] for the proof.

Theorem 4.5. *Let* $t \mapsto (u(t), e(t), p(t))$ *be a quasistatic evolution. Then*

$$e \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})), \quad p \in AC([0,T]; M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}), \quad u \in AC([0,T]; BD(\Omega)).$$

Moreover, for a.e. $t \in [0,T]$

$$\|\dot e(t)\|_2 \le C_1(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2), \tag{4.6}$$
$$\|\dot p(t)\|_1 \le C_2(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2), \tag{4.7}$$
$$\|E\dot u(t)\|_1 \le C_1(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2), \tag{4.8}$$
$$\|\dot u(t)\|_1 \le C_1(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2 + \|\dot w(t)\|_2). \tag{4.9}$$

Remark 4.6. Assume that $u \in AC([0,T]; BD(\Omega))$, $e \in AC([0,T]; L^2(\Omega\mathbb{M}^{n\times n}_{sym}))$, and $p \in AC([0,T]; M_b(\Omega \cup \Gamma_0); \mathbb{M}^{n\times n}_{sym})$. Assume that $(u(t), e(t), p(t)) \in A(w(t))$ for every $t \in [0,T]$. Then $(\dot u(t), \dot e(t), \dot p(t)) \in A(\dot w(t))$ for a.e. $t \in [0,T]$. Indeed, it is enough to apply Lemma 2.1 to the difference quotients.

As in [5, Theorem 5.9] we can prove that $t \mapsto e(t)$ (and, consequently, $t \mapsto \sigma(t)$) is uniquely determined by its initial condition.

Theorem 4.7. *Let* $t \mapsto (u(t), e(t), p(t))$ *and* $t \mapsto (v(t), \eta(t), q(t))$ *be two quasistatic evolutions and let* $\sigma(t) := \mathbb{C}e(t)$ *and* $\tau(t) := \mathbb{C}\eta(t)$. *If* $e(0) = \eta(0)$, *then* $e(t) = \eta(t)$ *for every* $t \in [0,T]$. *Equivalently, if* $\sigma(0) = \tau(0)$, *then* $\sigma(t) = \tau(t)$ *for every* $t \in [0,T]$.

4.3. **Equivalent formulations in rate form.** Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. Suppose for a moment that $\dot{p}(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. In this section we want to prove that

$$\dot{p}(t,x) \in N_{\mathbb{K}}(\sigma(t,x)) \quad \text{for a.e.} x \in \Omega, \tag{4.10}$$

which represents the classical formulation of the flow rule. By the definition of $N_{\mathbb{K}}$ it is easy to see that (4.10) is equivalent to saying that

$$\langle \sigma(t) - \tau(t) | \dot{p}(t) \rangle \geq 0 \tag{4.11}$$

for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ with $[\tau\, \nu] = g(t)$ on Γ_1. Indeed, the implication (4.10)$\Rightarrow$(4.11) is straightforward, while the converse one is obtained by considering the test functions of the form $\tau = \varphi\xi + (1-\varphi)\sigma$, with a cut-off $\varphi \in C^\infty_c(\Omega)$, $0 \leq \varphi \leq 1$ and arbitrary $\xi \in \mathbb{K}$.

Note, that the variational inequality (4.11) makes sense even if one considers the duality between $\Sigma(\Omega)$ and $\Pi_{\Gamma_0}(\Omega)$, defined in Section 3, since $\dot{p}(t) \in \Pi_{\Gamma_0}(\Omega)$ by Remark 4.6. We will regard (4.11) as the weak formulation of the inclusion (4.10) when $\dot{p}(t) \in M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{sym})$.

The following theorem collects three different sets of conditions, including (4.11) and expressed in terms of the time derivatives $\dot{p}(t)$, $\dot{e}(t)$, and $\dot{u}(t)$, which are equivalent to the conditions considered in Definition 4.1. For its proof we refer to [5, Theorem 6.1], with obvious modifications.

Theorem 4.8. *Let $(u, e, p) : [0,T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ and let $\sigma(t) = \mathbb{C}e(t)$. Then the following conditions are equivalent:*

(a) *$t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution;*

(b) *$t \mapsto (u(t), e(t), p(t))$ is absolutely continuous and*

  (b1) *for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\mathrm{div}\,\sigma(t) = f(t)$ in Ω, and $[\sigma(t)\, \nu] = g(t)$ on Γ_1,*

  (b2) *for a.e. $t \in [0,T]$ we have*

$$\mathcal{H}(\dot{p}(t)) = \langle \sigma(t) | \dot{p}(t) \rangle;$$

(c) *$t \mapsto (u(t), e(t), p(t))$ is absolutely continuous and*

  (c1) *for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\mathrm{div}\,\sigma(t) = f(t)$ in Ω, and $[\sigma(t)\, \nu] = g(t)$ on Γ_1,*

  (c2) *for a.e. $t \in [0,T]$ we have*

$$\langle \sigma(t) - \tau | \dot{p}(t) \rangle \geq 0$$

for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ with $[\tau\, \nu] = g(t)$ on Γ_1;

Remark 4.9. By Proposition 3.3 the measure $H(\dot{p}(t)) - [\sigma(t) : \dot{p}(t)]$ is nonnegative on $\Omega \cup \Gamma_0$, so that (b2) implies that

$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \quad \text{on } \Omega \cup \Gamma_0. \tag{4.12}$$

Let us return to the classical formulation of the flow rule, which makes sense for $\dot{p}(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. It can be written equivalently in the form

$$\frac{\dot{p}(t,x)}{|\dot{p}(t,x)|} \in N_K(\sigma(t,x)) \quad \text{for } \mathcal{L}^n - \text{a.e. } x \in \{|\dot{p}(t)| > 0\}. \tag{4.13}$$

When $\dot{p}(t) \in M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{sym})$, we can consider the Radon-Nikodym derivative $\dot{p}(t)/|\dot{p}(t)|$ of $\dot{p}(t)$ with respect to its variation $|\dot{p}(t)|$, which is a function defined $|\dot{p}(t)|$-a.e. on $\Omega \cup \Gamma_0$.

We notice that

$$\frac{\dot{p}(t)}{|\dot{p}(t)|}(x) = \frac{\dot{p}(t,x)}{|\dot{p}(t,x)|} \quad \text{for } \mathcal{L}^n - \text{a.e. } x \in \{|\dot{p}(t)| > 0\}$$

when $p \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Unfortunately, when $p \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ one cannot prove the inclusion

$$\frac{\dot{p}(t)}{|\dot{p}(t)|}(x) \in N_K(\sigma(t,x)), \tag{4.14}$$

that is the natural generalization of (4.13), as a pointwise formulation of the flow rule, since its left-hand side is defined $|\dot p(t)|$-a.e. on $\Omega\cup\Gamma_0$, while its right-hand side is defined only $\mathcal{L}^n$-a.e. on Ω. In the following Theorem this difficulty is overcome by introducing a representative $\hat\sigma(t)$ of $\sigma(t)$, which is defined $\dot p(t)$-a.e. on $\Omega\cup\Gamma_0$. For the proof see [5, Theorem 6.4].

Theorem 4.10. *Let* $(u,e,p):[0,T]\to BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$, $\sigma(t)=\mathbb{C}e(t)$ *and let* $\mu(t)=\mathcal{L}^n+|\dot p(t)|$. *Then* $t\mapsto(u(t),e(t),p(t))$ *is a quasistatic evolution if and only if*

(d) $t\mapsto(u(t),e(t),p(t))$ *is absolutely continuous and*

(d1) *for every* $t\in[0,T]$ *we have* $(u(t),e(t),p(t))\in A(w(t))$, $\sigma(t)\in\Sigma(\Omega)\cup\mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t)=f(t)$ *on* Ω, *and* $[\sigma(t)\nu]=g(t)$ *on* Γ_1,

(d2) *for a.e.* $t\in[0,T]$ *there exists* $\hat\sigma(t)\in L^\infty_{\mu(t)}(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ *such that*

$$\hat\sigma(t)=\sigma(t)\quad \mathcal{L}^n-a.e.\ on\ \Omega, \tag{4.15}$$

$$[\sigma(t):\dot p(t)]=\hat\sigma(t):\frac{\dot p(t)}{|\dot p(t)|}|\dot p(t)|\quad in\ M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}), \tag{4.16}$$

$$\frac{\dot p(t)}{|\dot p(t)|}(x)\in N_{\mathbb{K}}(\hat\sigma(t,x))\quad for\ |\dot p(t)|-a.e.\ x\in\Omega\cup\Gamma_0. \tag{4.17}$$

For every $r>0$ and every $t\in[0,T]$ we consider the function $\sigma^r(t)\in C(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ defined by

$$\sigma^r(t,x)=\frac{1}{\mathcal{L}^n(B(x,r)\cap\Omega)}\int_{B(x,r)\cap\Omega}\sigma(t,y)\,dy. \tag{4.18}$$

When $\mathbb{K}$ is strictly convex, the previous result can be improved by making the definition of $\hat\sigma$ more precise. We refer to [5, Theorem 6.6] for the proof.

Theorem 4.11. *Assume that* $\mathbb{K}$ *is strictly convex. Let* $t\mapsto(u(t),e(t),p(t))$ *be a quasistatic evolution, let* $\mu(t)=\mathcal{L}^n+|\dot p(t)|$, *let* $\sigma(t)=\mathbb{C}e(t)$, *and let* $\sigma^r(t)$ *be defined by* (4.18) . *Then* $\sigma^r(t)\to\hat\sigma(t)$ *strongly in* $L^1_{\mu(t)}(\Omega;\mathbb{M}^{n\times n}_{sym})$ *for a.e.* $t\in[0,T]$, *where* $\hat\sigma(t)$ *satisfies* (4.15)-(4.17).

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