SECOND ORDER SINGULAR PERTURBATION MODELS FOR PHASE TRANSITIONS

IRENE FONSECA* AND CARLO MANTEGAZZA†

Abstract. Singular perturbation models involving a penalization of the first order derivatives have provided a new insight into the role played by surface energies in the study of phase transitions problems. It is known that if $W : \mathbb{R}^d \to [0, +\infty)$ grows at least linearly at infinity and it has exactly two potential wells of level zero at $a, b \in \mathbb{R}^d$, then the $\Gamma(L^1)$– limit of the family of functionals

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon|\nabla u|^2 \right) dx & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^d), \\ \\ +\infty & \text{if } u \in L^1(\Omega; \mathbb{R}^d) \setminus W^{1,2}(\Omega; \mathbb{R}^d), \end{cases}$$

where Ω is a bounded, open set in $\mathbb{R}^N$, is given by

$$\mathcal{F}(u) := \begin{cases} \mathbf{m} \operatorname{Per}_\Omega(\{u = a\}) & \text{if } u \in BV(\Omega; \{a, b\}), \\ +\infty & \text{otherwise}, \end{cases}$$

for a suitable constant $\mathbf{m}$ depending on the energy density W. In this paper, and motivated by the study of phase transitions for nonlinear elastic materials, the $\Gamma(L^1)$– limit is obtained in the case where in $\mathcal{F}_\varepsilon(u)$ the penalization term $\varepsilon|\nabla u|^2$ is replaced by $\varepsilon^3|\nabla^2 u|^2$, for $u \in W^{2,2}(\Omega; \mathbb{R}^d)$. The resulting functional is of the same form as $\mathcal{F}(u)$ above.

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1. Introduction.

In this paper we show that the $\Gamma(L^1)$– limit of the family of singular perturbations

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^3|\nabla^2 u|^2 \right) dx & \text{if } u \in W^{2,2}(\Omega; \mathbb{R}^d), \\ \\ +\infty & \text{if } u \in L^1(\Omega; \mathbb{R}^d) \setminus W^{2,2}(\Omega; \mathbb{R}^d) \end{cases}$$

where $W : \mathbb{R}^d \to [0, +\infty)$ grows at least linearly at infinity and has exactly two potential wells of zero level at $a, b \in \mathbb{R}^d$, is given by

$$\mathcal{F}(u) := \begin{cases} \mathbf{m} \operatorname{Per}_\Omega(\{u = a\}) & \text{if } u \in BV(\Omega; \{a, b\}), \\ +\infty & \text{otherwise}, \end{cases}$$

with

$$\mathbf{m} := \min \left\{ \int_\mathbb{R} (W(f) + |f''|^2\, dt : f \in W^{2,2}_{\text{loc}}(\mathbb{R}; \mathbb{R}^d), \lim_{t\to+\infty} f(t) = b, \lim_{t\to-\infty} f(t) = a \right\}.$$

Singular perturbations of nonconvex, multiple-well variational problems may be found in gradient strain theories in plasticity, ferromagnetics, and other areas of materials science and engineering. In particular, within the context of phase transitions of nonlinear elastic materials, let $W : \mathbb{R}^{d\times N} \to [0, +\infty)$ be the stored energy density

*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA (fonseca@andrew.cmu.edu).

†Scuola Normale Superiore, Pisa, 56126, Italy (mantegaz@sns.it).

of a material with reference configuration an open, bounded set $\Omega \subset \mathbb{R}^N$, and which may undergo a phase transformation. This material instability may be due, in part, to the multiple well profile of W. For simplicity, assume that

$$W(\xi) = 0 \text{ if and only if } \xi \in \{A, B\},$$

where $\operatorname{rank}(A - B) = 1$. Let us assume further that equilibria for fixed phase volume fraction is determined by minimum energy; physically, this model is oversimplified since it is incompatible with the frame indifference requirement, it does not take into account material symmetries, evolution is neglected, and there is no heat diffusion. Then we are led to (see [20, 22])

$$\min\left\{\int_\Omega W(\nabla u)\,dx : u \in W^{1,1}(\Omega;\mathbb{R}^d), \int_\Omega \nabla u\,dx = \mathcal{L}^N(\Omega)(\theta A + (1-\theta)B)\right\},$$

where $\theta \in (0,1)$ is a fixed volume fraction, and $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure in $\mathbb{R}^N$. Due to the rank-one compatibility between A and B, there are infinitely many laminates with strain gradients alternating between A and B which will minimize the total bulk energy. As in the Cahn-Hilliard model for liquid-liquid phase transformations with underlying variational formulation

$$\min\left\{\int_\Omega W(u)\,dx : u \in L^1(\Omega;\mathbb{R}^d), \int_\Omega u\,dx = \mathcal{L}^N(\Omega)(\theta a + (1-\theta)b)\right\},$$

where $\{W = 0\} = \{a, b\}$, and with corresponding family of singular perturbations (see [10, 12, 13, 22, 25, 28, 29, 30, 32, 33])

$$\int_\Omega (W(u) + \varepsilon^2|\nabla u|^2)\,dx,$$

we attempt to resolve the lack of uniqueness by considering higher gradient penalizations. This is conform to the higher strain gradient theories in plasticity. To this end, for any open set $A \subset \Omega$ we introduce the family

$$J_\varepsilon(u; A) := \int_A (W(\nabla u) + \varepsilon^2|\nabla^2 u|)\,dx.$$

The characterization of the $\Gamma(L^1)$– limit of these functionals, and, in particular, of the asymptotic behavior of minimizers as $\varepsilon \to 0^+$, is work under progress by Fonseca and Tartar [21]. Here the main difficulties are, essentially, the need to use intrinsically vectorial techniques, and the proof of the locality of the $\Gamma(L^1)$– limit. The use of vectorial techniques was successfully exploited in the variational study of the eikonal equation, seen as a partial differential constraint on finite limiting energy fields when in J_ε the density W is allowed to vanish on the sphere (see [5, 8, 9, 18, 26, 27, 24]). In the attempts to ascertain locality, the problems encountered seem to stem from the higher order derivative in the model. Precisely, if we knew that the subadditivity property

$$\Gamma(L^1) - \lim_{\varepsilon\to 0^+} J_\varepsilon(u; A) \leq \Gamma(L^1) - \lim_{\varepsilon\to 0^+} J_\varepsilon(u; B) + \Gamma(L^1) - \lim_{\varepsilon\to 0^+} J_\varepsilon(u; A \setminus \overline{C})$$

holds whenever A, B, C are open subsets of Ω with $C \subset\subset B \subset\subset A$, then we would be able to ensure that $\Gamma(L^1) - \lim_{\varepsilon\to 0^+} J_\varepsilon(u; \cdot)$ is a measure, and so Radon-Nikodym

Theorem and a blow-up argument around points on the laminate surfaces would easily yield

$$\Gamma(L^1) - \lim_{\varepsilon\to 0^+} J_\varepsilon(u;\Omega) := \begin{cases} \widehat{\mathbf{m}}\ \mathrm{Per}_\Omega(\{\nabla u = A\}) & \text{if } \nabla u \in BV(\Omega;\{A,B\}), \\ +\infty & \text{otherwise,} \end{cases}$$

for an appropriate surface energy density $\widehat{\mathbf{m}}$. This program may be carried out successfully in the Cahn-Hilliard model where the penalization is of first-order. For this reason, and motivated in part by the need to isolate the understanding of the role played by higher order penalizations and the obstacles that they may introduce, we take a step further in the simplification of the original model for J_ε, and we consider the family $\mathcal{F}_\varepsilon$ as defined above.

We note that higher order perturbations of nonconvex problems have been studied recently within the framework of free discontinuity problems. In particular, elliptic regularizations with second order terms were proposed for the approximation of free discontinuity problems related to the Mumford-Shah model for image segmentation in computer vision (see, e.g., [3, 4, 15, 14]).

The one-dimensional problem encapsules the main features of the model. Indeed the relevant contributions of this paper may be found in the next section where, using apriori bounds provided by Gagliardo and Nirenberg inequalities, we are able to show that the limiting energy minimizers are two-phase fields with minimal interfacial perimeter. Further, the resulting interfacial energy per unit area, $\mathbf{m}$, may be computed explicitly as the solution of an auxiliary minimization problem, corresponding to the one-dimensional energetically efficient profiles which connect a at $-\infty$ to b at $+\infty$. The extension of these results to the N-dimensional case follows a standard slicing argument that enables us to reduce it to the one-dimensional setting.

2. The One-Dimensional Problem. Let $W : \mathbb{R}^d \to \mathbb{R}$ be a continuous function satisfying the hypotheses:
(H1) $W(u) = 0$ if and only if $u \in \{a, b\}$;
(H2) there exist constants $C > 0$, $R > \max\{|a|, |b|\}$, such that if $|u| > R$ then $W(u) \geq C|u| - 1/C$.

Let $I := (\alpha, \beta)$ be a fixed open interval in $\mathbb{R}$. Consider the family of functionals indexed by the parameter $\varepsilon > 0$, and defined as

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_I \left(\frac{W(u)}{\varepsilon} + \varepsilon^3 |u''|^2\right) dt & \text{if } u \in W^{2,2}(I;\mathbb{R}^d), \\ +\infty & \text{if } u \in L^1(I;\mathbb{R}^d) \setminus W^{2,2}(I;\mathbb{R}^d). \end{cases}$$

We seek to identify the limiting states corresponding to sequences of minimizers for $\mathcal{F}_\varepsilon(\cdot)$, and to this purpose we will use the notion of $\Gamma(L^1)$– convergence. We recall some basic notions on $\Gamma(L^1)$– convergence (for a detailed, comprehensive study we refer the reader to [17]). Let Ω be an open, bounded subset of $\mathbb{R}^N$.

Definition 2.1. *Let $F_n : L^1(\Omega;\mathbb{R}^d) \to [-\infty, +\infty]$ and $u \in L^1(\Omega;\mathbb{R}^d)$. We define*

$$\Gamma(L^1) - \liminf F_n(u) := \inf\left\{\liminf_{n\to\infty} F_n(u_n) : u_n \to u \quad \text{in } L^1(\Omega;\mathbb{R}^d)\right\},$$

and

$$\Gamma(L^1) - \limsup F_n(u) := \inf\left\{\limsup_{n\to\infty} F_n(u_n) : u_n \to u \quad \text{in } L^1(\Omega;\mathbb{R}^d)\right\}.$$

If $\Gamma(L^1)-\liminf F_n(u) = \Gamma(L^1)-\limsup F_n(u)$*, then the common value is called the* $\Gamma(L^1)$*– limit of* F_n *at* u*, and is denoted by* $\Gamma(L^1)-\lim F_n(u)$*.*

Moreover, given a family $F_\varepsilon : L^1(\Omega;\mathbb{R}^d)\to[-\infty,+\infty]$*,* $\varepsilon>0$*, if* $u\in L^1(\Omega;\mathbb{R}^d)$ *then we say that* $\Gamma(L^1)-\lim F_\varepsilon(u)=F(u)$ *if* $F(u)=\Gamma(L^1)-\lim F_{\varepsilon_n}(u)$ *for every sequence* $\varepsilon_n\to 0^+$*.*

It can be shown that $F(u)=\Gamma(L^1)$– limit of F_ε at u if and only if
(i) for every sequences $\{u_n\}$ and $\{\varepsilon_n\}$ such that $u_n\to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $\varepsilon_n\to 0^+$

$$F(u)\le \liminf_{n\to\infty} F_{\varepsilon_n}(u_n);$$

(ii) for every sequence $\{\varepsilon_n\}$ converging to 0^+ there exists a sequence $\{u_n\}$ such that $u_n\to u$ in $L^1(\Omega;\mathbb{R}^d)$ and

$$F(u)=\lim_{n\to\infty} F_{\varepsilon_n}(u_n).$$

In what follows C denotes a generic positive constant which may vary from one formula to the next, and from line to line. Also, $\mathcal{L}^N$ stands for the Lebesgue measure in $\mathbb{R}^N$, and $B(x,\delta)$ is the ball centered at the point x and with radius $\delta>0$.

Define

$$\mathcal{A}:=\left\{f\in W^{2,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d) : f(t)=b \text{ if } t>C, f(t)=a \text{ if } t<-C, \text{ for some } C>0\right\}$$

and

$$\mathbf{m}:=\inf\left\{\int_{\mathbb{R}}(W(f)+|f''|^2)\,dt : f\in\mathcal{A}\right\}. \tag{2.1}$$

The main theorem of this section is

THEOREM 2.2. *For every* $u\in L^1(I;\mathbb{R}^d)$

$$\Gamma(L^1)-\lim_{\varepsilon\to 0^+}\mathcal{F}_\varepsilon(u)=\begin{cases}\mathbf{m}\,\mathrm{Per}_I(\{u=a\}) & \text{if } u\in BV(I;\{a,b\}),\\ +\infty & \text{otherwise.}\end{cases}$$

Compactness for energy bounded sequences will rely heavily on the following interpolation inequality due to Gagliardo [23] and Nirenberg [31].

PROPOSITION 2.3. *Let* Ω *be a bounded, open, Lipschitz subset of* $\mathbb{R}^N$*. If* $u\in L^1(\Omega;\mathbb{R}^d)$ *and* $\nabla^2 u\in L^2(\Omega;\mathbb{R}^d)$ *then* $u\in W^{2,2}(\Omega;\mathbb{R}^d)$ *and*

$$\|\nabla u\|_{L^{4/3}}\le C\left(\|u\|^{1/2}_{L^1}\|\nabla^2 u\|^{1/2}_{L^2}+\|u\|_{L^1}\right) \tag{2.2}$$

where $C=C(\Omega,N,d)$*.*

In sequel we will also use the following interpolation inequality.

LEMMA 2.4. *Let* $\varphi:(0,+\infty)\to\mathbb{R}$ *be a convex, nondecreasing function in* $\mathbb{R}^+$*, and let* J *be* $\mathbb{R}$ *or a half-line. Then for every function* $u\in L^1_{\mathrm{loc}}(J;\mathbb{R}^d)$ *with* $u''\in L^1_{\mathrm{loc}}(J;\mathbb{R}^d)$ *we have*

$$\int_J\varphi\left(\frac{|u'|}{4d}\right)dt\le\frac{3}{4}\int_J[\varphi(|u|)+\varphi(|u''|)]\,dt\,. \tag{2.3}$$

Proof. The case where $\varphi(t) = |t|^p$ may be found in Adams [1] (Lemma 4.10). First consider the real valued case where $d = 1$. Given $u \in W^{2,1}(0,1)$, fix $\theta \in (0, 1/3)$ and $\eta \in (2/3, 1)$, and by virtue of the Mean Value Theorem, find $\xi \in (0,1)$ such that

$$u'(\xi) = \frac{u(\theta) - u(\eta)}{\theta - \eta};$$

hence,

$$|u'(x)| \leq \frac{|u(\theta) - u(\eta)|}{|\theta - \eta|} + \int_\xi^x |u''|\, dt \leq \frac{|u(\theta) - u(\eta)|}{|\theta - \eta|} + \int_0^1 |u''|\, dt$$

for all $x \in (0,1)$, which, by the choice of θ and η, implies that

$$|u'(x)| \leq 3|u(\theta)| + 3|u(\eta)| + \int_0^1 |u''|\, dt \quad \text{ for all } \ x \in (0,1).$$

Integrating in θ and η and multiplying both sides by 9 we get

$$\begin{aligned} |u'(x)| &\leq 3\int_0^{1/3} |u|\, dt + 3\int_{2/3}^1 |u|\, dt + \int_0^1 |u''|\, dt \\ &\leq 3\int_0^1 |u|\, dt + \int_0^1 |u''|\, dt \end{aligned}$$

for all $x \in (0,1)$. Now, dividing both sides by 4, and using the convexity and monotonicity properties of φ, together with Jensen's Inequality, we obtain

$$\begin{aligned} \varphi\left(\frac{|u'(x)|}{4}\right) &\leq \varphi\left(\frac{3}{4}\int_0^1 |u|\, dt + \frac{1}{4}\int_0^1 |u''|\, dt\right) \\ &\leq \frac{3}{4}\varphi\left(\int_0^1 |u|\, dt\right) + \frac{1}{4}\varphi\left(\int_0^1 |u''|\, dt\right) \\ &\leq \frac{3}{4}\int_0^1 [\varphi(|u|) + \varphi(|u''|)]\, dt \end{aligned}$$

for all $x \in (0,1)$.

Finally, integrating in x we have

$$\int_0^1 \varphi\left(\frac{|u'|}{4}\right) dt \leq \frac{3}{4}\int_0^1 [\varphi(|u|) + \varphi(|u''|)]\, dt\, .$$

Dividing J in disjoint intervals of length 1 and applying this argument to each one of them we conclude that

$$\int_J \varphi\left(\frac{|u'|}{4}\right) dt \leq \frac{3}{4}\int_J [\varphi(|u|) + \varphi(|u''|)]\, dt\, .$$

If u takes values in $\mathbb{R}^d$, $d \geq 2$, then

$$\begin{aligned}\int_J \varphi\left(\frac{|u'|}{4d}\right) dt &\leq \int_J \varphi\left(\sum_{i=1}^d \frac{|u_i'|}{4d}\right) dt\\ &\leq \frac{1}{d}\sum_{i=1}^d \int_J \varphi\left(\frac{|u_i'|}{4}\right) dt\\ &\leq \frac{3}{4d}\sum_{i=1}^d \int_J [\varphi(|u_i|) + \varphi(|u_i''|)]\, dt\\ &\leq \frac{3}{4d}\sum_{i=1}^d \int_J [\varphi(|u|) + \varphi(|u''|)]\, dt\\ &= \frac{3}{4}\int_J [\varphi(|u|) + \varphi(|u''|)]\, dt\end{aligned}$$

which proves the lemma. □

In the sequel we will exploit the auxiliary functions $G, H : \mathbb{R}^{2d} \to \mathbb{R}$, which take into account the energy stored on an interfacial layer:

$$G(w,z) := \inf\left\{ \int_0^1 (W(g) + |g''|^2)\, dt : \; g \in C^2([0,1];\mathbb{R}^d),\, g(0) = w,\, g(1) = b,\right.$$
$$\left. g'(0) = z,\, g'(1) = 0 \right\},$$

$$H(w,z) = \inf\left\{ \int_0^1 (W(h) + |h''|^2)\, dt : \; h \in C^2([0,1];\mathbb{R}^d),\, h(0) = a,\, h(1) = w,\right.$$
$$\left. h'(0) = 0,\, h'(1) = z \right\}.$$

Testing G and H with third degree polynomials g and h, respectively, satisfying the boundary conditions, it can be shown that

$$\lim_{(w,z)\to(b,0)} G(w,z) = 0\,, \qquad \lim_{(w,z)\to(a,0)} H(w,z) = 0\,. \tag{2.4}$$

Lemma 2.5. *The constant* $\mathbf{m}$ *is positive and*

$$\mathbf{m} = \min\left\{ \int_{\mathbb{R}} (W(f) + |f''|^2)\, dt : f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d),\, \lim_{t\to+\infty} f(t) = b,\, \lim_{t\to-\infty} f(t) = a \right\}.$$

Proof. Step 1. We start by proving that $\mathbf{m} > 0$. Suppose that $\mathbf{m} = 0$ and let $\{f_n\}$ be a minimizing sequence of admissible functions in $\mathcal{A}$. Let

$$S := \left\{ x \in \mathbb{R}^d : |x - a| = \frac{|b-a|}{2} \right\}.$$

By Sobolev Embedding Theorem each function f_n belongs to $C^1(\mathbb{R};\mathbb{R}^d)$, and since $f_n(t) = b$ for $t > M_n$ and $f_n(t) = a$ if $t < -M_n$ for a suitable $M_n > 0$, there must exists a point $t_n \in \mathbb{R}$ such that $f_n(t_n) \in S$. By performing a simple translation in the variable, with no loss of generality we may assume that $f_n(0) \in S$. As $\mathbf{m} = 0$, we have that $\|f_n''\|_{L^2} \to 0$; moreover, by (H2), fixed a bounded interval $J \subset \mathbb{R}$ containing the origin, $\{f_n\}$ is equibounded in $L^1(J;\mathbb{R}^d)$, and Proposition 2.3 implies that $\{f_n'\}$ is equibounded in $L^{4/3}(J;\mathbb{R}^d)$. Therefore, by Sobolev Embedding Theorem it follows that $\{f_n\}$ is bounded in $W^{2,2}(J;\mathbb{R}^d)$, and we may extract a subsequence $f_{n_i}|_J$ of restricted functions converging in $W^{1,\infty}$ to an affine function $f : J \to \mathbb{R}^d$ such that $f(0) =: c \in S$. Setting $f(t) := c + tv$ for some $v \in \mathbb{R}^d$, we have

$$\begin{aligned}\mathbf{m} &= \lim_{n\to\infty} \int_{\mathbb{R}} (W(f_{n_i}) + |f_{n_i}''|^2)\, dt \\ &\geq \lim_{n\to\infty} \int_J (W(f_{n_i}) + |f_{n_i}''|^2)\, dt \\ &\geq \int_J W(c+tv)\, dt > 0\,,\end{aligned}$$

because if $\int_J W(c+tv)\,dt = 0$ then $c+tv$ should belong to $\{a,b\}$ for all $t \in J$, and so $v = 0$ and $c \in \{a,b\}$ which is not possible since $c \in S$. We arrived at a contradiction, and thus $\mathbf{m} > 0$.

STEP 2. Next we prove that $\mathbf{m} = \widetilde{\mathbf{m}}$, where

$$\widetilde{\mathbf{m}} := \inf\left\{\int_{\mathbb{R}} (W(f) + |f''|^2)\, dt : f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d),\ \lim_{t\to+\infty} f(t) = b,\ \lim_{t\to-\infty} f(t) = a\right\}. \tag{2.5}$$

It is clear that $\mathbf{m} \geq \widetilde{\mathbf{m}}$.

Conversely, fix $\delta > 0$ and let f be a function admissible for $\widetilde{\mathbf{m}}$ and such that

$$\widetilde{\mathbf{m}} + \delta \geq \int_{\mathbb{R}} (W(f) + |f''|^2)\, dt.$$

We claim that we may find two sequences $\{x_i\}$ and $\{y_i\}$ converging to $+\infty$ and $-\infty$, respectively, such that

$$|f'(x_i)| + |f'(y_i)| + |f(x_i) - b| + |f(y_i) - a| \to 0 \tag{2.6}$$

as $i \to \infty$. Indeed, fix $\tau < |b-a|/2$ and consider a convex, nondecreasing function $\varphi : \mathbb{R} \to [0,+\infty)$ such that $\varphi(t) \leq t^2$ for every $t \in \mathbb{R}$, $\varphi(|y|) \leq W(y+b)$ for every $y \in B(0,\tau) \subset \mathbb{R}^d$, and $\varphi(t) = 0$ if and only if $t = 0$. To prove the existence of φ it suffices to set

$$\begin{aligned}\varphi(t) := \sup\{g : \mathbb{R} \to [0,+\infty) : g \text{ is convex, nondecreasing,} \\ g(t) \leq t^2 \text{ for all } t \in \mathbb{R},\ g(|y|) \leq W(y+b) \text{ for all } y \in B(0,\tau)\}\end{aligned}$$

and use hypothesis (H1). Let $R > 0$ be such that $|f(t) - b| < \tau$ whenever $t > R$. Applying Lemma 2.4 to the function $f - b$, and using the properties of the function φ, we obtain

$$\begin{aligned}\int_R^{+\infty} \varphi\left(\frac{|f'|}{4d}\right) dt &\leq \frac{3}{4} \int_R^{+\infty} [\varphi(|f-b|) + \varphi(|f''|)]\, dt \\ &\leq \frac{3}{4} \int_R^{+\infty} (W(f) + |f''|^2)\, dt \leq \frac{3\,(\widetilde{\mathbf{m}} + \delta)}{4}\,.\end{aligned}$$

Thus $\varphi\left(\frac{|f'|}{4d}\right)$ is integrable on $[R,+\infty)$, and so there exists a sequence of points $x_n \to +\infty$ such that $\lim_{n\to\infty} \varphi\left(\frac{|f'(x_n)|}{4d}\right) = 0$, and since φ is monotone nondecreasing on $[0,+\infty)$, with $\varphi(t) = 0$ if and only if $t = 0$, we conclude that $\lim_{n\to\infty} f'(x_n) = 0$. Repeating this argument with the point a in place of b, we are now in a position to assert the existence of two sequences satisfying (2.6).

Set

$$g_i(t) := g(t - x_i), \qquad h_i(t) := h(t - y_i + 1)$$

where g and h are admissible for G and H, respectively, and

$$\int_0^1 (W(g) + |g''|^2)\,dt \le G(f(x_i), f'(x_i)) + \delta,$$

$$\int_0^1 (W(h) + |h''|^2)\,dt \le H(f(y_i), f'(y_i)) + \delta.$$

We define

$$\widetilde{f}_i(t) := \begin{cases} b & \text{if } t \ge x_i + 1, \\ g_i(t) & \text{if } t \in [x_i, x_i + 1], \\ f(t) & \text{if } t \in [y_i, x_i], \\ h_i(t) & \text{if } t \in [y_i - 1, y_i], \\ a & \text{if } t \le y_i - 1. \end{cases}$$

Clearly $\widetilde{f}_i$ is admissible for $\mathbf{m}$, and we have

$$\begin{aligned} \widetilde{\mathbf{m}} + \delta &\ge \int_{\mathbb{R}} (W(f) + |f''|^2)\,dt \ge \int_{y_i}^{x_i} (W(f) + |f''|^2)\,dt \\ &= \int_{\mathbb{R}} (W(\widetilde{f}_i) + |\widetilde{f}_i''|^2)\,dt - \int_{x_i}^{x_i+1} (W(g_i) + |g_i''|^2)\,dt \\ &\quad - \int_{y_i-1}^{y_i} (W(h_i) + |h_i''|^2)\,dt \\ &\ge \mathbf{m} - G(f(x_i), f'(x_i)) - H(f(y_i), f'(y_i)) - 2\delta. \end{aligned}$$

The inequality $\widetilde{\mathbf{m}} \ge \mathbf{m}$ now follows by letting $\delta \to 0^+$, $i \to \infty$, and using (2.4).

Step 3. Finally, we prove that $\mathbf{m}$ is attained, or, equivalently, that $\widetilde{\mathbf{m}}$ admits a minimizer. Let $\{f_n\}$ be a minimizing sequence for $\widetilde{\mathbf{m}}$. Possibly passing to a subsequence, and making a translation change of variables, we may assume as before that $f_n(0) \in S$, where S was defined in Step 1, and that the sequence of C^1 functions $\{f_n\}$ converges in $W^{1,\infty}_{\mathrm{loc}}$ to a C^1 function $f : \mathbb{R} \to \mathbb{R}^d$. If the function f is admissible, then it realizes the infimum, since

$$\int_{\mathbb{R}} (W(f) + |f''|^2)\,dt \le \varliminf_{n\to\infty} \int_{\mathbb{R}} (W(f_n) + |f_n''|^2)\,dt,$$

where we have used Fatou's Lemma and the lower semicontinuity of the L^2 norm of the second derivative. In order to prove that f approaches a and b at infinity, set

$$L := \left\{ l \in \mathbb{R}^d \mid \ l \text{ is a limit point of } f(t) \text{ when } t \to +\infty \ \right\}.$$

The integrability of $W(f)$ and (H1) imply that a or b must belong to L. Suppose that $b \in L$, and that there is another limiting value $l \in L$. Note that, without loss of generality, we may assume that $l \neq a$, for if $l = a$ then, by the continuity of f, there would exist a sequence $y_i \to +\infty$ such that $f(y_i) \in S$; hence, for a subsequence (not relabelled) $f(y_i) \to l' \in S$. Consider two monotone sequences of points $\{x_i\}$ and $\{z_i\}$ such that $x_{i+1} - x_i \geq 3$, $z_i \in [x_i + 1, x_{i+1} - 1]$, $f(x_i) \to b$ and $f(z_i) \to l$, and for $0 < \delta < \min\{|l-a|, |l-b|\}$ we introduce still another constant $\widehat{\mathbf{m}}$ defined as follows:

$$\widehat{\mathbf{m}} := \inf \Big\{ \int_x^y (W(g) + |g''|^2)\, dt : y - x \geq 3,\, z \in [x+1, y-1], \\ g \in W^{2,2}((x,y);\mathbb{R}^d),\, |g(z) - l| \leq \delta \Big\}.$$

We claim that $\widehat{\mathbf{m}} = 0$. Indeed, if $\widehat{\mathbf{m}} > 0$ then there would exist n_0 such that, for $n \geq n_0$, $|f(z_n) - l| \leq \delta$, and it would follow that

$$\begin{aligned}\int_{\mathbb{R}} (W(f) + |f''|^2)\, dt &= \int_{-\infty}^{x_{n_0}} (W(f) + |f''|^2)\, dt + \sum_{i=n_0}^{\infty} \int_{x_i}^{x_{i+1}} (W(f) + |f''|^2)\, dt \\ &\geq \int_{-\infty}^{x_{n_0}} (W(f) + |f''|^2)\, dt + \sum_{i=n_0}^{\infty} \widehat{\mathbf{m}} = +\infty.\end{aligned}$$

On the other hand, we can show that the assertion $\widehat{\mathbf{m}} = 0$ leads to a contradiction. The reasoning is similar to the one used in Step 1 for the constant $\mathbf{m}$. Let $g_n \in W^{2,2}((x_n, y_n);\mathbb{R}^d)$ minimize $\widehat{\mathbf{m}}$. Translating the intervals, without loss of generality we can suppose that $z_n = 0$, thus $x_n \leq -1$ and $y_n \geq 1$, and possibly passing to a subsequence (not relabelled), we may assume that the functions g_n converge in $W^{1,\infty}([-1,1];\mathbb{R}^d)$ to an affine function $g(t) = d + tv$. Therefore

$$\begin{aligned}\widehat{\mathbf{m}} &\geq \lim_{n\to\infty} \int_{x_n}^{y_n} (W(g_n) + |g_n''|^2)\, dt \\ &\geq \lim_{n\to\infty} \int_{-1}^{1} (W(g_n) + |g_n''|^2)\, dt \\ &\geq \int_{-1}^{1} W(d + tv)\, dt > 0\,,\end{aligned}$$

because if $\int_{-1}^{1} W(d+tv)\, dt = 0$ then $d + tv$ should belong to $\{a, b\}$ for all $t \in (-1, 1)$, i.e. $v = 0$ and $d \in \{a, b\}$. This is not possible since $g_n(0) \to d$, $g_n(0) \in B(l, \delta)$, and $a, b \notin B(l, \delta)$. We conclude that $f(t) \to b$ as $t \to +\infty$.

Similarly, if $a \in L$ then $f(t)$ converges to a as $t \to -\infty$.

If the limits of f at $+\infty$ and $-\infty$ are, respectively, a and b, then $f(-t)$ is still a minimizer and it converges to b and a at, respectively, $+\infty$ and $-\infty$.

It remains to exclude the possibility that the two limits coincide. Suppose that $\lim_{t\to\pm\infty} f(t) = a$. As in Step 2, by virtue of Lemma 2.4 we can find a sequence of points $x_n \to +\infty$ such that

$$|f'(x_n)| + |f(x_n) - a| \to 0$$

and, due to the convergence of f_n to f in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$, there exists a subsequence

$\{f_{k_n}\}$ such that $f_{k_n}(x_n) \to a$ and $f'_{k_n}(x_n) \to 0$. Hence, we have

$$\begin{aligned}\int_{\mathbb{R}} (W(f_{k_n}) + |f''_{k_n}|^2)\,dt &= \int_{-\infty}^{x_n} (W(f_{k_n}) + |f''_{k_n}|^2)\,dt + \int_{x_n}^{+\infty} (W(f_{k_n}) + |f''_{k_n}|^2)\,dt \\ &\geq \int_{-\infty}^{x_n} (W(f_{k_n}) + |f''_{k_n}|^2)\,dt + \widetilde{\mathbf{m}} - H(f_{k_n}(x_n), f'_{k_n}(x_n))\,,\end{aligned}$$

and letting $n \to \infty$ we deduce that

$$\begin{aligned}\widetilde{\mathbf{m}} &\geq \limsup_{n\to\infty} \int_{-\infty}^{x_n} (W(f_{k_n}) + |f''_{k_n}|^2)\,dt + \widetilde{\mathbf{m}} \\ &\geq \limsup_{n\to\infty} \int_{-\infty}^{x_n} W(f_{k_n})\,dt + \widetilde{\mathbf{m}} \\ &= \int_{\mathbb{R}} W(f)\,dt + \widetilde{\mathbf{m}}\,.\end{aligned}$$

This would imply that f is constantly equal to a, but since $f_n(0) \in S$ for every n we have also $f(0) = \lim_{n\to\infty} f_n(0) \in S$ which is in contradiction with $a \notin S$. The case where $\lim_{t\to\pm\infty} f(t) = b$ is treated in an analogous way. ☐

Remark 2.6. A simple rescaling argument provides equi-partition of energy. Precisely, if f realizes the minimum $\mathbf{m}$ then

$$\int_{\mathbb{R}} W(f)\,dt = 3\int_{\mathbb{R}} |f''|^2\,dt\,.$$

It suffices to use the fact that

$$\int_{\mathbb{R}} (W(f) + |f''|^2)\,dt \leq \int_{\mathbb{R}} (W(f_\lambda) + |f''_\lambda|^2)\,dt$$

for all $\lambda > 0$, where $f_\lambda(x) := f(\lambda x)$.

We now state and prove the compactness result for sequences with finite energy.

Proposition 2.7. *If $u_\varepsilon \in W^{2,2}(I;\mathbb{R}^d)$ satisfy $\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$ then there exists a subsequence $\{u_{\varepsilon_n}\} \subset \{u_\varepsilon\}$ and $u \in BV(I;\{a,b\})$ such that $u_{\varepsilon_n} \to u$ in $L^1(I;\mathbb{R}^d)$. Moreover,*

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \mathbf{m}\,\mathrm{Per}_I(\{u = a\}).$$

Proof. Suppose that $\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) =: K < +\infty$. We claim that there exists a function $u \in BV(I;\{a,b\})$ such that, up to a subsequence, $u_\varepsilon \to u$. Extract a subsequence from the start (not relabelled) realizing $\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon)$. We have

$$W(u_\varepsilon) \to 0 \text{ in } L^1, \quad ||u''||_{L^2} \leq C\varepsilon^{-3/2}.$$

By (H2)

$$||u_\varepsilon||_{L^1} \leq R\,\mathcal{L}^1(I) + \int_{\{|u_\varepsilon|>R\}} \left(\frac{1}{C^2} + \frac{1}{C} W(u_\varepsilon)\right) dt \leq \widetilde{C},$$

therefore, by the Gagliardo and Nirenberg inequality (2.2) we conclude that

$$||u'_\varepsilon||_{L^{4/3}} \leq C(||u_\varepsilon||_{L^1}^{1/2} ||u''_\varepsilon||_{L^2}^{1/2} + ||u_\varepsilon||_{L^1}) \leq \widetilde{C}\varepsilon^{-3/4}. \tag{2.7}$$

Also

$$\mathcal{L}^1(\{|u_\varepsilon| > R\}) \to 0 \text{ as } \varepsilon \to 0^+. \tag{2.8}$$

Indeed, if $\tau := \inf\{W(\xi) : |\xi| > R\}$ then by (H1) we have $\tau > 0$, and so (2.8) follows from the fact that

$$0 = \lim_{\varepsilon\to 0^+} \int_I W(u_\varepsilon)\, dt \geq \limsup_{\varepsilon\to 0^+} \int_{\{|u_\varepsilon|>R\}} W(u_\varepsilon)\, dt \geq \tau \limsup_{\varepsilon\to 0^+} \mathcal{L}^1(\{|u_\varepsilon| > R\}).$$

Since $\{u_\varepsilon\}$ is bounded in L^1, we may extract a further subsequence (not relabelled) generating a Young measure $\{\nu_t\}_{t\in I}$. In particular, if $f : I \times \mathbb{R}^d \to [0, +\infty)$ is a Carathéodory function such that $\{f(\cdot, u_\varepsilon(\cdot))\}$ is equi-integrable, then $f(\cdot, u_\varepsilon(\cdot)) \rightharpoonup \bar{f}$ in L^1 where (see [11, 34])

$$\bar{f}(t) := \int_{\mathbb{R}} f(t, y) d\nu_t(y), \quad \text{a. e. } t \in I.$$

Setting $f(y) := \min\{W(y), 1\}$, it follows that

$$0 = \lim \int_I f(u_\varepsilon)\, dt = \int_I \int_{\mathbb{R}} f(y) d\nu_t(y)\, dt;$$

hence, since $f(y) = 0$ if and only $y \in \{a, b\}$, we have

$$\nu_t = \theta(t)\delta_{y=a} + (1 - \theta(t))\delta_{y=b}$$

for some $\theta \in L^\infty(I, [0, 1])$. We claim that

$$\theta \in \{0, 1\} \text{ a. e. in } I, \quad \text{i.e. } \theta = \chi_E \text{ for some measurable subset } E \subset I. \tag{2.9}$$

Suppose that the claim holds. Define

$$u(t) := a\chi_E(t) + b(1 - \chi_E(t)).$$

Then $u_\varepsilon \to u$ strongly in L^1. Indeed, let

$$\varphi(y) := \begin{cases} R\frac{y}{|y|} & \text{if } |y| > R \\ y & \text{if } |y| \leq R. \end{cases}$$

Note that $u = \varphi(u)$, and recall that $W(y) \geq C|y| - 1/C$ if $|y| > R$, with $R > \max\{|a|, |b|\}$. Then

$$\begin{aligned} \int_I |u_\varepsilon - u|\, dt &\leq \int_I |\varphi(u_\varepsilon) - u|\, dt + 2\int_{|u_\varepsilon|>R} |u_\varepsilon|\, dt \\ &\leq \int_I |\varphi(u_\varepsilon) - u|\, dt + \frac{2}{C}\int_I W(u_\varepsilon)\, dt + \frac{2}{C^2}\mathcal{L}^1(\{|u_\varepsilon| > R\}). \end{aligned}$$

Therefore, by (2.8) and (2.9) we conclude that

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \int_I |u_\varepsilon - u|\, dt &= \int_I \int_{\mathbb{R}} |\varphi(y) - u(t)| d\nu_t(y) dt \\ &= \int_E |\varphi(a) - u(t)|\, dt + \int_{I\setminus E} |\varphi(b) - u(t)|\, dt \\ &= \int_E |a - u(t)|\, dt + \int_{I\setminus E} |b - u(t)|\, dt = 0. \end{aligned}$$

To prove (2.9), define

$$X := \left\{ t \in I : \frac{1}{|B(t,\delta)|} \int_{B(t,\delta)} \theta(s)ds \in (0,1) \;\text{ for all }\; \delta > 0 \right\}.$$

We show that the cardinality of X (call it L) cannot exceed the integer part of $K/\mathbf{m}$; hence $\theta \in \{0,1\}$ a. e. and $u \in BV(I;\{a,b\})$ with $\mathbf{m}\,\mathrm{Per}_I(\{u=a\}) = \mathbf{m}\,L \leq K$ which gives the result.

Indeed, suppose that there were l distinct points of I in X, $s_1 < s_2 < \ldots < s_l$. Let $\delta_0 := \min\{|s_i - s_{i+1}| : i = 1,\ldots,l-1\}$. Choose $\delta_1 < \delta_0/2$ such that for all $\delta \leq \delta_1$, and all $i \in \{1,\ldots,l\}$,

$$\int_{B(s_i,\delta)} \theta(s)ds > 0, \qquad \int_{B(s_i,\delta)} (1-\theta(s))ds > 0.$$

Fix $0 < \eta < |b-a|/2$, let φ_η be a cut-off function with support on $B(a,\eta)$, $\varphi_\eta(a) = 1$, ψ_η is a cut-off function with support on $B(0,\eta)$, $\psi_\eta(0) = 1$, and γ_η is a cut-off function with support on $B(b,\eta)$, $\gamma_\eta(b) = 1$. By (2.7) $\psi_\eta(\varepsilon u_\varepsilon')$ converges strongly to $\psi_\eta(0)$ in L^1, $\varphi_\eta(u_\varepsilon)$ converges in L^∞ weak-* to $\theta\varphi_\eta(a) + (1-\theta)\varphi_\eta(b)$, and we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \int_{B(s_i,\delta_1)} \psi_\eta(\varepsilon u_\varepsilon')\varphi_\eta(u_\varepsilon)dt &= \int_{B(s_i,\delta_1)} \psi_\eta(0)[\theta(t)\varphi_\eta(a) + (1-\theta(t))\varphi_\eta(b)]dt \\ &= \int_{B(s_i,\delta_1)} \theta(t)\,dt > 0,\end{aligned}$$

and, similarly,

$$\lim_{\varepsilon\to 0^+} \int_{B(s_i,\delta_1)} \psi_\eta(\varepsilon u_\varepsilon')\gamma_\eta(u_\varepsilon)\,dt = \int_{B(s_i,\delta_1)} (1-\theta(t))\,dt > 0\,.$$

Thus, for each $i \in \{1,\ldots,l\}$ and each $\varepsilon > 0$, we may find $x_{\varepsilon,i}^+, x_{\varepsilon,i}^- \in B(s_i,\delta_1)$ such that

$$u_\varepsilon(x_{\varepsilon,i}^+) \in B(b,\eta), u_\varepsilon(x_{\varepsilon,i}^-) \in B(a,\eta), |\varepsilon u_\varepsilon'(x_{\varepsilon,i}^+)| < \eta, |\varepsilon u_\varepsilon'(x_{\varepsilon,i}^-)| < \eta\,. \tag{2.10}$$

Define

$$g_{\varepsilon,i}(t) := \widehat{g_{\varepsilon,i}}\left(t - \frac{x_{\varepsilon,i}^+}{\varepsilon}\right), \quad h_{\varepsilon,i}(t) := \widehat{h_{\varepsilon,i}}\left(t - \frac{x_{\varepsilon,i}^-}{\varepsilon} + 1\right),$$

where the functions $\widehat{g_{\varepsilon,i}}$ and $\widehat{h_{\varepsilon,i}}$ are admissible for $G(u_\varepsilon(x_{\varepsilon,i}^+), \varepsilon u_\varepsilon'(x_{\varepsilon,i}^+))$ and for $H(u_\varepsilon(x_{\varepsilon,i}^-), \varepsilon u_\varepsilon'(x_{\varepsilon,i}^-))$, respectively, with

$$\int_0^1 (W(\widehat{g_{\varepsilon,i}}) + |\widehat{g_{\varepsilon,i}}''|^2)\,dt \leq G(u_\varepsilon(x_{\varepsilon,i}^+), \varepsilon u_\varepsilon'(x_{\varepsilon,i}^+)) + \varepsilon,$$

and

$$\int_0^1 (W(\widehat{h_{\varepsilon,i}}) + |\widehat{h_{\varepsilon,i}}''|^2)\,dt \leq H(u_\varepsilon(x_{\varepsilon,i}^-), \varepsilon u_\varepsilon'(x_{\varepsilon,i}^-))) + \varepsilon.$$

Construct the functions

$$v_{\varepsilon,i}(t) := \begin{cases} b & \text{if } \ t \geq \frac{x^+_{\varepsilon,i}}{\varepsilon} + 1, \\ g_{\varepsilon,i}(t) & \text{if } t \in \left[\frac{x^+_{\varepsilon,i}}{\varepsilon}, \frac{x^+_{\varepsilon,i}}{\varepsilon} + 1\right], \\ u_\varepsilon(\varepsilon t) & \text{if } t \in \left[\frac{x^-_{\varepsilon,i}}{\varepsilon}, \frac{x^+_{\varepsilon,i}}{\varepsilon}\right], \\ h_{\varepsilon,i}(t) & \text{if } t \in \left[\frac{x^+_{\varepsilon,i}}{\varepsilon} - 1, \frac{x^+_{\varepsilon,i}}{\varepsilon}\right], \\ a & \text{if } \ t \leq \frac{x^+_{\varepsilon,i}}{\varepsilon} - 1. \end{cases}$$

Then $v_{\varepsilon,i}$ are admissible for $\mathbf{m}$, and since the intervals $[x^-_{\varepsilon,i}, x^+_{\varepsilon,i}]$ are disjoint we have

$$\begin{aligned} K &\geq \liminf_{\varepsilon \to 0^+} \sum_{i=1}^{l} \int_{x^-_{\varepsilon,i}}^{x^+_{\varepsilon,i}} \left(\frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon^3 |u''_\varepsilon|^2\right) dt \\ &= \liminf_{\varepsilon \to 0^+} \sum_{i=1}^{l} \int_{\frac{x^-_{\varepsilon,i}}{\varepsilon}}^{\frac{x^+_{\varepsilon,i}}{\varepsilon}} (W(v_{\varepsilon,i}) + |v''_{\varepsilon,i}|^2)\, dt \\ &\geq \mathbf{m} l - \limsup_{\varepsilon \to 0^+} \sum_{i=1}^{l} [H(u_\varepsilon(x^-_{\varepsilon,i}), \varepsilon u'_\varepsilon(x^-_{\varepsilon,i})) + G(u_\varepsilon(x^+_{\varepsilon,i}), \varepsilon u'_\varepsilon(x^+_{\varepsilon,i}))]. \end{aligned}$$

Letting $\eta \to 0^+$, we conclude that $K \geq \mathbf{m}\, l$, where we have used (2.4) and (2.10). The result now follows from the arbitrariness of $l \leq L$ and the fact that $\mathbf{m} > 0$ (see Lemma 2.5). □

As an immediate consequence of the previous result we have,

Corollary 2.8. *If $u \in L^1(I; \mathbb{R}^d)$ and $\Gamma(L^1) - \liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u) < +\infty$ then the function u belongs to $BV(I; \{a, b\})$ and*

$$\Gamma(L^1) - \liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u) \geq \ \mathbf{m} \, \mathrm{Per}_I(\{u = a\}).$$

Now we turn our attention to the $\Gamma(L^1) - \limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon$.

Proposition 2.9. *If $u \in BV(I; \{a, b\})$ then*

$$\Gamma(L^1) - \limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u) \leq \ \mathbf{m} \, \mathrm{Per}_I(\{u = a\}).$$

Proof. Suppose that $S(u) = \{s_1, \ldots, s_l\} \subset I = (\alpha, \beta)$ is the jump set of the function u, with $\alpha < s_1 < \cdots < s_l < \beta$. Set $\delta_0 := \min\{s_{j+1} - s_j : j = 0, \ldots, l\}$, with $s_0 := \alpha$ and $s_{l+1} := \beta$, and let $I_i := \left[\frac{s_{i-1}+s_i}{2}, \frac{s_i+s_{i+1}}{2}\right]$ for $i = 1, \ldots l$. Fix $\delta \in (0, \delta_0)$ and let $f \in \mathcal{A}$ be an admissible function for $\mathbf{m}$, with $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^d)$, $f(t) = b$ if $t > M$, $f(t) = a$ if $t < -M$,

$$\int_{\mathbb{R}} (W(f) + |f''|^2)\, dt \leq \mathbf{m} + \frac{\delta}{l}. \tag{2.11}$$

Consider a sequence $\varepsilon_n \to 0^+$, and choose n sufficiently large so that $\frac{\delta}{2\varepsilon_n} > M$.

Define

$$u_n(t) := \begin{cases} f\left(\frac{t-s_i}{\varepsilon_n}\right) & \text{if } t \in \left[\frac{s_{i-1}+s_i}{2}, \frac{s_{i+1}+s_i}{2}\right] \text{ and if } [u](s_i) = b-a, \\ \\ f\left(-\frac{t-s_i}{\varepsilon_n}\right) & \text{if } t \in \left[\frac{s_{i-1}+s_i}{2}, \frac{s_{i+1}+s_i}{2}\right] \text{ and if } [u](s_i) = a-b, \\ \\ u(t) & \text{otherwise} \end{cases}$$

where $[u](s_i) := u(s_i) - u(s_{i-1})$ for $i = 1, \dots, l$. We note that $u_n \in W^{2,2}(I;\mathbb{R}^d)$. Indeed, if $[u](s_i) = b-a$ then $u\left(\frac{s_{i-1}+s_i}{2}\right) = a$, $u\left(\frac{s_{i+1}+s_i}{2}\right) = b$, and since

$$\frac{s_{i-1}-s_i}{2\varepsilon_n} < -\frac{\delta}{2\varepsilon_n} < -M, \quad \frac{s_{i+1}-s_i}{2\varepsilon_n} > \frac{\delta}{2\varepsilon_n} > M,$$

we have that

$$f\left(\frac{s_{i-1}-s_i}{2\varepsilon_n}\right) = a, \quad f\left(\frac{s_{i+1}-s_i}{2\varepsilon_n}\right) = b, \quad f'\left(\frac{s_{i\pm1}-s_i}{2\varepsilon_n}\right) = 0.$$

A similar argument applies to the case where $[u](s_i) = a-b$.

Since $u_n \to u$ in $L^1(I;\mathbb{R}^d)$ we conclude that

$$\begin{aligned} \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) &= \lim_{n\to\infty} \sum_{i=1}^{l} \int_{I_i} \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^3 |u_n''|^2\right) dt \\ &= \lim_{n\to\infty} \left\{ \sum_{i=1,\dots,l,\ [u](s_i)=b-a} \int_{\frac{s_{i-1}+s_i}{2\varepsilon_n}}^{\frac{s_{i+1}+s_i}{2\varepsilon_n}} (W(f(t)) + |f''(t)|^2)\, dt \right. \\ &\qquad \left. + \sum_{i=1,\dots,l,\ [u](s_i)=a-b} \int_{\frac{s_{i-1}+s_i}{2\varepsilon_n}}^{\frac{s_{i+1}+s_i}{2\varepsilon_n}} (W(f(-t)) + |f''(-t)|^2)\, dx \right\} \\ &= \left\{ \sum_{i=1,\dots,l,\ [u](s_i)=b-a} \int_{\mathbb{R}} (W(f(t)) + |f''(t)|^2)\, dt \right. \\ &\qquad \left. + \sum_{i=1,\dots,l,\ [u](s_i)=a-b} \int_{\mathbb{R}} (W(f(-t)) + |f''(-t)|^2)\, dt \right\} \\ &= l \int_{\mathbb{R}} (W(f) + |f''|^2)\, dt \\ &\le \mathbf{m}\, l + \delta \\ &= \mathbf{m}\, \mathrm{Per}_I(\{u = a\}) + \delta, \end{aligned}$$

where we have used (H1) and (2.11). It suffices to let $\delta \to 0^+$. □

Remark 2.10. The arguments used in the proof of Theorem 2.2 may be easily adapted to generalize the model above to the case where

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^{2p-1} |\nabla^2 u|^p\right) dx & \text{if } u \in W^{2,p}(\Omega;\mathbb{R}^d), \\ \\ +\infty & \text{if } u \in L^1(\Omega;\mathbb{R}^d) \setminus W^{2,p}(\Omega;\mathbb{R}^d) \end{cases}$$

for $1 < p < +\infty$, with

$$\Gamma(L^1) - \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = \begin{cases} \mathbf{m}\, \mathrm{Per}_I(\{u=a\}) & \text{if } u \in BV(I;\{a,b\}), \\ +\infty & \text{otherwise,} \end{cases}$$

and now

$$\mathbf{m} := \inf\left\{ \int_{\mathbb{R}} (W(f) + |f''|^p)\, dt : f \in \mathcal{A} \right\}.$$

As it is usual, the scaling ε^{2p-1} in $\mathcal{F}_\varepsilon$ is the natural one obtained by testing the finiteness of energy with admissible fields u which are a and b in most of the domain, except on a transition layer of width ε. Note that here Proposition 2.3 still applies provided (2.2) is modified to read

$$\|\nabla u\|_{L^q} \le C\left(\|u\|_{L^1}^{1/2} \|\nabla^2 u\|_{L^p}^{1/2} + \|u\|_{L^1} \right)$$

with $2/q = 1 + 1/p$.

Naturally, the next step is to try to understand higher than two perturbations, i.e., how to treat

$$\mathcal{F}_\varepsilon^k(u) := \begin{cases} \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^{2k-1}|\nabla^k u|^2 \right) dx & \text{if } u \in W^{k,2}(\Omega;\mathbb{R}^d), \\ \\ +\infty & \text{if } u \in L^1(\Omega;\mathbb{R}^d) \setminus W^{k,2}(\Omega;\mathbb{R}^d) \end{cases}$$

where $k \in \mathbb{N}$. Although the methods involved may stay close the the ones introduced in this paper, this generalization does not seem to follow as immediately as the one above : last, but not least, the corresponding G and H will now require matching of all derivatives up to order $(k-1)$, and a new version of Lemma 2.4 will be in order. This analysis will be carried on in a forthcoming paper.

3. The N-Dimensional Case. Let Ω be an open, bounded, Lipschitz domain in $\mathbb{R}^N$, and consider the functionals

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon^3|\nabla^2 u|^2 \right) dx & \text{if } u \in W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^d) \\ \\ +\infty & \text{if } u \in L^1(\Omega;\mathbb{R}^d) \setminus W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^d) \end{cases}$$

for $\varepsilon > 0$, where W satisfies hypotheses (H1) and (H2). We recall that the constant $\mathbf{m}$ was defined in (2.1).

We start by proving L^1 compactness for energy bounded sequences.

Proposition 3.1. *If $u_\varepsilon \in W^{2,2}(\Omega;\mathbb{R}^d)$ satisfy $\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$ then there exists a subsequence $\{u_{\varepsilon_n}\} \subset \{u_\varepsilon\}$ and $u \in BV(\Omega;\{a,b\})$ such that $u_{\varepsilon_n} \to u$ in $L^1(\Omega;\mathbb{R}^d)$.*

The proof of this result uses the L^1-slicing compactness criterion introduced by Alberti, Bouchitté and Seppecher in [2], Theorem 6.6 (see Proposition 3.2 below). Here two sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ are said to be *δ-close* if $||u_\varepsilon - v_\varepsilon|| < \delta$, $\delta > 0$. When Ω is a rectangle of the form $I \times J$, with I, J open intervals, we write $x \in \Omega$ as $x = (y,z)$ with $y \in I$, $z \in J$. For every function u defined on Ω and every $y \in I$ we denote by u^y the function on J defined by $u^y(z) := u(y,z)$, and for every $z \in J$ we set $u^z(y) := u(y,z)$ for $y \in I$. The functions u^y and u^z are called *one-dimensional slices* of u.

PROPOSITION 3.2. *Assume that the sequence $\{u_\varepsilon\}$ is equi-integrable, and suppose that for every $\delta > 0$ there exist sequences $\{v_\varepsilon\}$ and $\{w_\varepsilon\}$ δ-close to $\{u_\varepsilon\}$, and such that, $\{v_\varepsilon^y\}$ is precompact in $L^1(J;\mathbb{R}^d)$ for a. e. $y \in I$, and $\{w_\varepsilon^z\}$ is precompact in $L^1(I;\mathbb{R}^d)$ for a. e. $z \in J$. Then $\{u_\varepsilon\}$ is precompact in $L^1(\Omega;\mathbb{R}^d)$.*

Remark 3.3. We note that the original statement of Theorem 6.6 in [2] assumes that $\{u_\varepsilon\}$ is bounded in L^∞. However, it is easy to verify that the main tool involved is the use of Fréchet-Kolmogorov Theorem for pre-compactness in L^1, which clearly holds as well when $\{u_\varepsilon\}$ is equi-integrable.

Proof of Proposition 3.1. For simplicity we suppose $N = 2$; the higher dimensional case is treated in an analogous way.

Assume first that Ω is a rectangle of the form $I \times J$, with I, J open intervals.

We denote by $\mathcal{F}^1_\varepsilon(u,A)$ the one-dimensional functional

$$\mathcal{F}^1_\varepsilon(u,A) := \begin{cases} \int_A \left(\frac{W(u)}{\varepsilon} + \varepsilon^3|u''|^2\right) dt & \text{if } u \in W^{2,2}(A;\mathbb{R}^d), \\ +\infty & \text{if } u \in L^1(A;\mathbb{R}^d) \setminus W^{2,2}(A;\mathbb{R}^d), \end{cases}$$

for every open interval A and every $u \in L^1(A;\mathbb{R}^d)$. We recall that if $u \in W^{2,2}(\Omega;\mathbb{R}^d)$ then $u^y \in W^{2,2}(J;\mathbb{R}^d)$ for a. e. $y \in I$ and $u^z \in W^{2,2}(I;\mathbb{R}^d)$ for a. e. $z \in J$, and

$$\frac{\partial^2 u}{\partial z^2}(x) = \frac{d^2 u^y}{dz^2}(z)\,, \quad \frac{\partial^2 u}{\partial y^2}(x) = \frac{d^2 u^z}{dy^2}(y) \quad \text{for a. e. } x \in \Omega$$

(see [19], Section 4.9.2). Since $|\nabla^2 u|^2 \geq \max\left\{\left|\frac{\partial^2 u}{\partial z^2}\right|^2, \left|\frac{\partial^2 u}{\partial y^2}\right|^2\right\}$, we immediately obtain the following *slicing inequalities* :

$$\mathcal{F}_\varepsilon(u) \geq \int_I \mathcal{F}^1_\varepsilon(u^y, J)\,dy, \qquad \mathcal{F}_\varepsilon(u) \geq \int_J \mathcal{F}^1_\varepsilon(u^z, I)\,dz\,. \tag{3.1}$$

Now consider a family of functions $\{u_\varepsilon\}$ such that $\mathcal{F}_\varepsilon(u_\varepsilon) \leq C < +\infty$. Since $\int_\Omega W(u_\varepsilon)\,dx \leq C\varepsilon$, we have that $W(u_\varepsilon) \to 0$ in L^1, and so equi-integrability of $\{u_\varepsilon\}$ follows from (H2). Therefore, fix $\delta > 0$ and let $\delta' \in (0,\delta)$ be such that

$$\mathcal{L}^2(S) \leq \delta'|J| \quad \Longrightarrow \quad \sup_{\varepsilon>0} \int_S (|u_\varepsilon(x)| + |b|)\,dx \leq \delta\,.$$

For $\varepsilon > 0$ we define $v_\varepsilon : \Omega \to \mathbb{R}^d$ by

$$v^y_\varepsilon(z) := \begin{cases} u^y_\varepsilon(z) = u_\varepsilon(y,z) & \text{if } \mathcal{F}^1_\varepsilon(u^y_\varepsilon, J) \leq C/\delta', \\ b & \text{otherwise.} \end{cases}$$

We claim that $v^y_\varepsilon = u^y_\varepsilon$ for all $y \in I$ except at most on a set $Z_\varepsilon \subset I$ of measure smaller than δ'. Indeed, by (3.1) we have

$$C \geq \sup_{\varepsilon>0} \int_I \mathcal{F}^1_\varepsilon(u^y_\varepsilon, J)\,dy, \tag{3.2}$$

and so

$$|Z_\varepsilon| \leq |\{\mathcal{F}^1_\varepsilon(u^y_\varepsilon, J) > C/\delta'\}| \leq \frac{\delta'}{C} \int_I \mathcal{F}^1_\varepsilon(u^y_\varepsilon, J)\,dy \leq \delta',$$

and we have

$$\|u_\varepsilon - v_\varepsilon\|_1 \le \int_{Z_\varepsilon\times J} |u_\varepsilon(x) - b|\,dx \le \int_{Z_\varepsilon\times J} (|u_\varepsilon(x)| + |b|)\,dx \le \delta$$

for every $\varepsilon > 0$, since $\mathcal{L}^2(Z_\varepsilon \times J) \le \delta'|J|$. Hence the sequence $\{v_\varepsilon\}$ is δ-close to $\{u_\varepsilon\}$. Moreover, for every $y \in I$ there holds $\mathcal{F}^1_\varepsilon(v^y_\varepsilon, J) \le C/\delta'$, where we have used the fact that that $\mathcal{F}^1_\varepsilon(b, J) = 0$, and so Proposition 2.7 yields $L^1(J;\mathbb{R}^d)$ precompactness of $\{v^y_\varepsilon\}$. Similarly, we way can construct a sequence $\{w_\varepsilon\}$ δ-close to $\{u_\varepsilon\}$ so that $\{w^z_\varepsilon\}$ is precompact in $L^1(I;\mathbb{R}^d)$ for every $z \in J$, and it suffices to now use Proposition 3.2 to conclude that the sequence $\{u_\varepsilon\}$ is pre-compact in $L^1(\Omega;\mathbb{R}^d)$.

The case where Ω is a general open subset of $\mathbb{R}^N$ is obtained by decomposing Ω into a countable union of closed rectangles with disjoint interiors.

The fact that the limit function u belongs to $BV(\Omega;\{a,b\})$ is showed in the proof of Proposition 3.5. □

Theorem 3.4. *If* $u \in L^1(\Omega;\mathbb{R}^d)$ *then*

$$\Gamma(L^1) - \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = \begin{cases} \mathbf{m}\,\mathrm{Per}_\Omega(\{u = a\}) & \text{if } u \in BV(\Omega;\{a,b\}),\\ +\infty & \text{otherwise.}\end{cases}$$

We divide the proof of this theorem into two propositions concerning, respectively, the $\Gamma(L^1) - \liminf$ and the $\Gamma(L^1) - \limsup$. Although nowadays these arguments may be considered to be quite standard, and we refer the reader to [7, 16], and to [4] for the treatment of second derivatives in the study of the $\Gamma(L^1) - \limsup$, we included here the proofs of Proposition 3.5 and Proposition 3.6 for completeness and for the convenience of the reader.

Proposition 3.5. *Let* $u \in L^1(\Omega;\mathbb{R}^d)$. *If* $\Gamma(L^1) - \liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) < +\infty$ *then* u *belongs to* $BV(\Omega;\{a,b\})$ *and*

$$\Gamma(L^1) - \liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) \ge \mathbf{m}\,\mathrm{Per}_\Omega(\{u = a\}).$$

Proof. Suppose that $\varepsilon_n \to 0^+$, $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $\mathcal{F}_{\varepsilon_n}(u_n)$ converges to $\Gamma(L^1) - \liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) < +\infty$. Fixing an unit vector $\nu \in \mathbb{S}^{N-1}$, possibly passing to a subsequence (not relabelled), we may assume that $u_n|_{L_{y,\nu}\cap\Omega} \to u|_{L_{y,\nu}\cap\Omega}$ in $L^1(L_{y,\nu}, \mathcal{H}^1)$ for almost every line $L_{y,\nu}$ parallel to ν, where $L_{y,\nu} := \{y + s\nu : s \in \mathbb{R}\}$, $y \in \mathbb{R}^N$. By Proposition 2.8, and setting

$$u^{y,\nu}_n(t) := u_n(y + t\nu) \ \text{ for } \ \mathcal{H}^{N-1} a.\ e.\ y \in \nu^\perp,$$

we have

$$\mathbf{m}\,\frac{|Du^{y,\nu}|(L_{y,\nu}\cap\Omega)}{|b-a|} \le \liminf_{n\to\infty} \int_{L_{y,\nu}\cap\Omega} \left(\frac{W(u^{y,\nu}_n)}{\varepsilon_n} + \varepsilon_n^3 \left| \frac{d^2 u^{y,\nu}_n}{dt^2} \right|^2 \right) dt.$$

Thus, by Fatou's Lemma and the slicing properties of BV functions (see [19], Section 5.10.2),

$$
\begin{aligned}
\mathbf{m}\,\mathrm{Per}_\Omega(\{u=a\}) &= \mathbf{m}\,\frac{|Du|(\Omega)}{|b-a|}\\
&= \frac{\mathbf{m}}{|b-a|}\int_{\{y\in\nu^\perp\}} |Du^{y,\nu}|(L_{y,\nu}\cap\Omega)\,d\mathcal{H}^{N-1}(y)\\
&\le \int_{\{y\in\nu^\perp\}} \liminf_{n\to\infty}\int_{L_{y,\nu}\cap\Omega}\left(\frac{W(u_n^{y,\nu})}{\varepsilon_n}+\varepsilon_n^3\left|\frac{d^2u_n^{y,\nu}}{dt^2}\right|^2\right)dt\,d\mathcal{H}^{N-1}\\
&\le \liminf_{n\to\infty}\int_{\{y\in\nu^\perp\}}\int_{L_{y,\nu}\cap\Omega}\left(\frac{W(u_n)}{\varepsilon_n}+\varepsilon_n^3|\nabla^2u_n|^2\right)dt\,d\mathcal{H}^{N-1}\\
&= \liminf_{n\to\infty}\int_\Omega\left(\frac{W(u_n)}{\varepsilon_n}+\varepsilon_n^3|\nabla^2u_n|^2\right)dx\\
&= \Gamma(L^1)-\liminf_{\varepsilon\to0^+}\mathcal{F}_\varepsilon(u). \qquad \square
\end{aligned}
$$

PROPOSITION 3.6. *For every function $u\in BV(\Omega;\{a,b\})$ we have*

$$\Gamma(L^1)-\limsup_{\varepsilon\to0^+}\mathcal{F}_\varepsilon(u)\le \mathbf{m}\,\mathrm{Per}_\Omega(\{u=a\}).$$

Proof. Let $u\in BV(\Omega;\{a,b\})$, with $u=a\chi_E+b(1-\chi_E)$, and where E is a set of finite perimeter, i.e. $\mathrm{Per}_\Omega(E)=|D\chi_E|(\Omega)<+\infty$. Since E can be approximated by a sequence of smooth sets $E_i=\widetilde{E}_i\cap\Omega$ such that $\widetilde{E}_i$ is a smooth bounded set in $\mathbb{R}^N$, $\chi_{E_i}\to\chi_E$ in $L^1(\Omega)$ and $|D\chi_{E_i}|(\Omega)\to|D\chi_E|(\Omega)$ (see Lemma 4.3 in [6]), in order to study the $\Gamma(L^1)-\limsup$ it suffices to consider a function $u:\Omega\to\mathbb{R}$ such that

$$u(x)=\begin{cases} a & \text{if } x\in E,\\ b & \text{if } x\in\Omega\setminus E,\end{cases}$$

where $E=\widetilde{E}\cap\Omega$ and $\widetilde{E}$ is a compact subset of $\mathbb{R}^N$ with a C^2 boundary. We claim that

$$\Gamma(L^1)-\limsup_{\varepsilon\to0^+}\mathcal{F}_\varepsilon(u)\le \mathbf{m}\,\mathrm{Per}_\Omega(\{u=a\}).$$

Since $M:=\partial\widetilde{E}$ is a C^2 manifold in $\mathbb{R}^N$, there exists $\delta_0>0$ such that for all $\delta<\delta_0$ the points in the tubular neighborhood U_δ of the manifold M admit a unique smooth projection onto M, where $U_\delta:=\{x\in\mathbb{R}^N:\ \mathrm{dist}(x,M)<\delta\}$.

Let $\varepsilon_n\to0^+$, and consider a sequence of functions $v_n\in W^{2,2}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^d)$ such that

$$v_n(t)=\begin{cases} a & \text{if } t\le -\frac{1}{\sqrt{\varepsilon_n}},\\ b & \text{if } t\ge \frac{1}{\sqrt{\varepsilon_n}},\end{cases}$$

and

$$\lim_{n\to\infty}\int_\mathbb{R}(W(v_n)+|v_n''|^2)\,dt=\mathbf{m}\,.$$

We define the sequence of functions $u_n:\Omega\to\mathbb{R}$

$$u_n(x):=\begin{cases} v_n\left(\frac{\widetilde{d}_M(x)}{\varepsilon_n}\right) & \text{if } x\in U_n\cap\Omega,\\ a & \text{if } x\in E\setminus U_n,\\ b & \text{if } x\in\Omega\setminus(E\cup U_n),\end{cases}$$

where $\widetilde{d}_M : \mathbb{R}^N \to \mathbb{R}$ is the signed distance function from the boundary of $\widetilde{E}$, negative inside $\widetilde{E}$, and $U_n := U_{\sqrt{\varepsilon_n}}$. We have

$$
\begin{aligned}
\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) &= \limsup_{n\to\infty} \int_\Omega \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^3 \left|\nabla^2 u_n\right|^2 \right) dx \\
&= \limsup_{n\to\infty} \left\{ \int_{U_n} \frac{W(v_n(\widetilde{d}_M(x)/\varepsilon_n))}{\varepsilon_n}\, dx + \int_{U_n} e_n^3 \left| v_n'' \nabla \widetilde{d}_M \times \nabla \widetilde{d}_M / \varepsilon_n^2 + v_n' H/\varepsilon_n \right|^2 dx \right\},
\end{aligned}
$$

where H is the Hessian matrix of $\widetilde{d}_M$. Change variables via the diffeomorphism $x := F(y,t)$, where $F : M \times (-\delta_0/2, \delta_0/2) \to U_{\delta_0/2}$, $F(y,t) := y + t\nu(y)$, with $\nu(y)$ the normal vector to M at y pointing outside $\widetilde{E}$. We indicate by $J(y,t)$ the Jacobian of this transformation. Then

$$
\begin{aligned}
\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \le \liminf_{n\to\infty} \Bigg\{ \int_M \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \Bigg(& \frac{W(v_n(t/\varepsilon_n))}{\varepsilon_n} \\
& + \varepsilon_n^3 \frac{\left|v_n''(t/\varepsilon_n)\right|^2}{\varepsilon_n^4} \left|\nabla \widetilde{d}_M(F(y,t))\right|^2 \Bigg) J(y,t)\, dt\, d\mathcal{H}^{N-1}(y) \\
+ \int_M \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \varepsilon_n^3 \frac{\left|v_n'(t/\varepsilon_n)\right|^2}{\varepsilon_n^2} \left|H(F(y,t))\right|^2 J(y,t)\, dt\, d\mathcal{H}^{N-1}(y) & \\
+ 2 \int_M \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \varepsilon_n^3 \frac{|v_n''(t/\varepsilon_n)||v_n'(t/\varepsilon_n)|}{\varepsilon_n^3} \left|\nabla \widetilde{d}_M(F(y,t))\right| & |H(F(y,t))| J(y,t) dt d\mathcal{H}^{N-1}(y) \Bigg\},
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
& \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \\
& \le \limsup_{n\to\infty} \Bigg\{ \int_M \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \left(\frac{W(v_n(t/\varepsilon_n))}{\varepsilon_n} + \frac{\left|v_n''(t/\varepsilon_n)\right|^2}{\varepsilon_n} \right) J(y,t)\, dt\, d\mathcal{H}^{N-1}(y) \\
& \quad + A \int_M \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \varepsilon_n \left|v_n'(t/\varepsilon_n)\right|^2 dt\, d\mathcal{H}^{N-1}(y) \\
& \quad + B \int_M \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \left|v_n''(t/\varepsilon_n)\right| \left|v_n'(t/\varepsilon_n)\right| dt\, d\mathcal{H}^{N-1}(y) \Bigg\} \\
& =: \limsup_{n\to\infty} \left\{ I_1^{(n)}(u) + I_2^{(n)}(u) + I_3^{(n)}(u) \right\},
\end{aligned}
$$

where we took into account the facts that the gradient of the distance is always equal to one, and that the Jacobian J and the Hessian H of the distance are uniformly bounded. We have

$$
\begin{aligned}
I_1^{(n)}(u) &= \int_M \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \left(\frac{W(v_n(t/\varepsilon_n))}{\varepsilon_n} + \frac{\left|v_n''(t/\varepsilon_n)\right|^2}{\varepsilon_n} \right) J(y,t)\, dt\, d\mathcal{H}^{N-1}(y) \\
&= \int_M \int_{-1/\sqrt{\varepsilon_n}}^{1/\sqrt{\varepsilon_n}} \left(W(v_n(s)) + \left|v_n''(s)\right|^2 \right) J(y, s\varepsilon_n)\, ds\, d\mathcal{H}^{N-1}(y) \\
&\le \left(\sup_{y\in M,\, t\in(-\sqrt{\varepsilon_n}, \sqrt{\varepsilon_n})} J(y,t) \right) \int_M \int_{\mathbb{R}} \left(W(v_n(s)) + \left|v_n''(s)\right|^2 \right) ds\, d\mathcal{H}^{N-1}(y)\,,
\end{aligned}
$$

and passing to the limit in n as $n \to \infty$, we get

$$\begin{aligned}\limsup_{n\to\infty} I_1^{(n)} &\leq \left(\sup_{y\in M,\, t\in(-\sqrt{\varepsilon_n},\sqrt{\varepsilon_n})} J(y,t)\right) \mathbf{m}\, \mathcal{H}^{N-1}(M)\\ &= \left(\sup_{y\in M,\, t\in(-\sqrt{\varepsilon_n},\sqrt{e_n})} J(y,t)\right) \mathbf{m}\, \mathrm{Per}_\Omega(\{u=a\}).\end{aligned}$$

If we show that the other two integrals $I_2^{(n)}(u)$ and $I_3^{(n)}(u)$ go to zero as $n \to \infty$, then we obtain that

$$\begin{aligned}\Gamma(L^1) - \limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) &\leq \limsup_{n\to\infty} \left(\sup_{y\in M,\, t\in(-\sqrt{\varepsilon_n},\sqrt{\varepsilon_n})} J(y,t)\right) \mathbf{m}\, \mathrm{Per}_\Omega(\{u=a\})\\ &= \mathbf{m}\, \mathrm{Per}_\Omega(\{u=a\}),\end{aligned}$$

where we used the fact that $\{\varepsilon_n\}$ is an arbitrary sequence converging to zero, and that since M is compact, $J(y,t)$ goes uniformly to one as $t \to 0$.

Finally,

$$\begin{aligned}I_2^{(n)} + I_3^{(n)} &\leq C \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \Big(\varepsilon_n |v_n'(t/\varepsilon_n)|^2 + |v_n''(t/\varepsilon_n)|\, |v_n'(t/\varepsilon_n)|\Big)\, dt\\ &= C \int_{\mathbb{R}} \Big(\varepsilon_n^2 |v_n'(s)|^2 + \varepsilon_n |v_n''(s)|\, |v_n'(s)|\Big)\, ds\\ &\leq C\, \Big(\varepsilon_n^2 \|v_n'\|_2^2 + \varepsilon_n \|v_n''\|_2\, \|v_n'\|_2\Big)\\ &\leq C\, \Big(\varepsilon_n^2 \|v_n'\|_2^2 + \varepsilon_n\, \|v_n'\|_2\Big) \qquad (3.3)\end{aligned}$$

where we changed variables, used Hölder inequality and the fact that

$$\|v_n''\|_{L^2}^2 \leq \int_{\mathbb{R}} (W(v_n) + |v_n''|^2)\, dt \to \mathbf{m}\,.$$

Set $w_n(t) := v_n\,(t/\varepsilon_n)$. Then

$$\begin{aligned}\limsup_{n\to\infty} &\int_{-1}^{1} \left(\frac{1}{\varepsilon_n} W(w_n) + \varepsilon_n^3 |w_n''|^2\right)\, dt\\ &= \limsup_{n\to\infty} \int_{-\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} \left(\frac{1}{\varepsilon_n} W(w_n) + \varepsilon_n^3 |w_n''|^2\right)\, dt\\ &= \lim_{n\to\infty} \int_{\mathbb{R}} (W(v_n) + |v_n''|^2)\, dt,\end{aligned}$$

and so by Proposition 2.7 and (2.7)

$$||w_n'||_{L^{4/3}(-1,1)} \leq C \varepsilon_n^{-3/4},$$

where the constant C is independent of n. Also, Hölder inequality yields

$$\begin{aligned}||w_n''||_{L^{4/3}(-1,1)} &= ||w_n''||_{L^{4/3}(-\sqrt{\varepsilon_n},\sqrt{\varepsilon_n})}\\ &\leq C ||w_n''||_{L^2(-1,1)}\, \varepsilon_n^{1/6}\\ &\leq C\, \varepsilon_n^{-3/2}\, \varepsilon_n^{1/6}.\end{aligned}$$

Therefore, by Sobolev Embedding Theorem

$$||w_n'||_{L^2(-1,1)} \leq C||w_n'||_{W^{1,4/3}(-1,1)} \leq C(\varepsilon_n^{-3/4} + \varepsilon_n^{-3/2}\,\varepsilon_n^{1/6}),$$

and in view of (3.3), we conclude that

$$\begin{aligned}\varepsilon_n^2 \int_R |v_n'|^2\,dt &= \varepsilon_n^3 \int_{\sqrt{\varepsilon_n}}^{\sqrt{\varepsilon_n}} |w_n'|^2\,dt \\ &\leq C\,\varepsilon_n^3\,(\varepsilon_n^{-3/4} + \varepsilon_n^{-3/2}\,\varepsilon_n^{1/6})^2 \\ &\leq C\,(\varepsilon_n^{3/2} + \varepsilon_n^{1/3}),\end{aligned}$$

and it suffices to let $n \to \infty$. □

4. Final Remarks. As in the singular perturbation model for phase transitions (see [25]), the interfacial energy appears due to the need to go across an energy barrier in order to remain on the zero set of W. Indeed, if the zero set of W is a smooth, connected set, then the $\Gamma(L^1)$– limit may simply reduce to zero. As an example, consider the case where $\{W = 0\} = \mathbb{S}^{d-1}$. Then

$$\Gamma(L^1) - \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = \begin{cases} 0 & \text{if } u \in L^1(\Omega;\mathbb{S}^{d-1}), \\ +\infty & \text{otherwise.}\end{cases}$$

To prove this assertion, fix $u \in L^1(\Omega;\mathbb{S}^{d-1})$ and let $\{u_n\}$ be a sequence of smooth functions with compact support, converging to u in $L^1(\Omega;\mathbb{S}^{d-1})$. The existence of such approximating sequence can be obtained as follows: there exists a point $y \in \mathbb{S}^{d-1}$ such that $u^{-1}(y)$ has zero Lebesgue measure, so we may assume with no loss of generality that u does not take such value y. The manifold $\mathbb{S}^{d-1} \setminus \{y\}$ is diffeomorphic to the open unit ball B of $\mathbb{R}^{d-1}$ via some smooth map Φ; hence it is sufficient to approximate the function $\Phi(u)$ in $L^1(\Omega;B)$ with a sequence of smooth functions $\{v_n\}$ with compact support and then to consider the sequence $u_n := \Phi^{-1}(v_n)$.
If now we choose a positive sequence $\varepsilon_n \to 0^+$ such that

$$\int_\Omega \varepsilon_n^3|\nabla^2 u_n|^2\,dx \leq \frac{1}{n} \qquad \text{for every } n \in \mathbb{N},$$

we get

$$\Gamma(L^1) - \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) \leq \liminf_{n\to\infty} \int_\Omega \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^3|\nabla^2 u_n|^2\right) dx \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

and this proves the claim.

Finally, we remark that if we could prove that for energy bounded sequences $\{u_\varepsilon\}$, with

$$\sup_{\varepsilon>0} \int_\Omega \left(\frac{W(u_\varepsilon)}{\varepsilon} + \varepsilon^3|\nabla^2 u_\varepsilon|^2\right) dx < +\infty,$$

it follows that

$$\sup_{\varepsilon>0} \int_\Omega \varepsilon|\nabla u|^2\,dx < +\infty,$$

then most proofs would be greatly simplified, and, in particular, the compactness in L^1 (see Propositions 2.7, 3.1) would follow immediately from the compactness for the singular perturbations model studied in [10, 12, 13, 22, 25, 28, 29, 32, 33].

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