A Blow-up Theorem for regular hypersurfaces on nilpotent groups

Valentino Magnani
Scuola Normale Superiore
Piazza dei Cavalieri 7, 56126 Pisa, Italy
e-mail magnani@cibs.sns.it

Abstract

We obtain an intrinsic Blow-up Theorem for regular hypersurfaces on graded nilpotent groups. This procedure allows us to represent explicitly the Riemannian surface measure in terms of the spherical Hausdorff measure with respect to an intrinsic distance of the group, namely homogeneous distance. We apply this result to get a version of the Riemannian coarea forumula on sub-Riemannian groups, that can be expressed in terms of arbitrary homogeneous distances. We introduce the natural class of horizontal isometries in sub-Riemannian groups, giving examples of rotational invariant homogeneous distances and rotational groups, where the coarea formula takes a simpler form. By means of the same Blow-up Theorem we obtain an optimal estimate for the Hausdorff dimension of the characteristic set relative to $C^{1,1}$ hypersurfaces in 2-step groups and we prove that it has finite $Q-2$ Hausdorff measure, where Q is the homogeneous dimension of the group.

Contents

Mathematics Subject Classification: 28A75 (22E25)

keywords: stratified groups, characteristic points, coarea formula, Hausdorff measure

Introduction

Stratified Lie groups, also known as "Carnot groups", have been the object of extensive studies in connection with different areas of Mathematics, e.g. the theory of Subelliptic Partial Differential Equations, Sobolev Spaces and Optimal Control Theory. Actually, the project to develop classical tools of Geometric Measure Theory in these groups and in the more general Carnot-Carathéodory spaces is at an embryonal stage. However, only recently there has been some progress in this direction, [2], [6], [13], [14], [15], [17], [18], [20], [21], [22], [23], [24], [25], [30], [31], [33], but the list is surely incomplete.

The initial question which motivated this paper was finding the geometrical meaning of the following integral

$$\int_\Sigma \left(\sum_{i=1}^m \langle X_i, \nu \rangle^2 \right)^{1/2} d\sigma \tag{1}$$

in terms of an arbitrary homogeneous distance of the group, where Σ is a hypersurface of class C^1, ν is a unit normal vector to Σ, and σ is the classical surface measure. The integral term (1) appears in isoperimetric inequalities formulated in stratified groups and in general Carnot-Carathéodory spaces, when Σ is the boundary of a regular open set, see [11], [12], [32]. Moreover, regular sets of finite perimeter in Carnot-Carathéodory spaces fulfill the same formula for the perimeter measure, see [5]. Hence formula (1) represents a natural notion of surface measure in "stratified geometries". But notice that in a stratified group the Euclidean scalar product in (1) prevents a natural representation of this surface measure in terms of intrinsic objects of the group. This is due to the fact that Euclidean metric in some representation of $\mathbb{G}$ as $\mathbb{R}^q$ is not left invariant. So we are forced to employ left invariant Riemannian metrics.

We reformulate (1) in terms of a left invariant Riemannian metric and a homogeneous distance of the group, where the latter can be considered analogously to a Banach norm in vector spaces. Precisely, we consider graded metrics (see Definition 1.1). We give an explicit representation of (1) with respect to the spherical Hausdorff measure $\mathcal{S}^{Q-1}$ built using a homogeneous distance, where Q is the homogeneous dimension of the group. We state our formula as follows

$$\int_\Sigma |\nu_H|_g \, d\sigma = \int_\Sigma \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x). \tag{2}$$

The map $\nu_H(x)$ denotes the Riemannian projection of the unit normal $\nu(x)$ on the horizontal subbundle and the *metric factor* $\theta_{Q-1}(\nu_H(x))$ takes into account both the anisotropy of the homogeneous distance and the position of the tangent space of Σ at x (Definition 1.11). The main tool to get formula (2) is the Blow-up Theorem (Theorem 2.1) on the Riemannian surface measure with respect to a fixed homogeneous distance. This theorem on regular hypersurfaces can be interpreted as the counterpart of the Blow-up Theorem on points of the reduced boundary for sets of finite perimeter [14], [16]. In the proof of Theorem 2.1 we assume the existence of a continuous homogeneous distance, instead of the classical Hörmander condition, which always guarantees such assumption. In formula (2) we consider the hypersurface Σ with $\mathcal{S}^{Q-1}$-negligible characteristic set, (Definition 1.14). Recently, Z. Balogh proved in [2] that this assumption is always verified in Heisenberg groups for C^1 hypersurfaces, [2]. We mention that the size of the characteristic set is of great importance in connection with isoperimetric estimates and trace theorems in stratified groups as well as in general Carnot-Carathéodory spaces, [6], [7], [12]. As a variant of the Blow-up Theorem (Theorem 3.1) we obtain that the characteristic set of a $C^{1,1}$ hypersurface in a 2-step graded group has finite $Q-2$ Hausdorff measure and its Hausdorff dimension does not exceed $Q-2$ (Theorem 3.2 and Remark 3.3). Due to Theorem 1.4(1) of [2] this estimate is optimal, i.e. it cannot be improved with a number less than $Q-2$. In the Heisenberg group our upper bound on the Hausdorff dimension fits into the case $\alpha = 1$ of Theorem 1.1(2) in [2], where a different method, based on a covering argument, is employed.

By virtue of the Blow-up Theorem, we also derive a version of the Riemannian coarea formula on stratified groups. Let $f : \mathbb{M} \longrightarrow \mathbb{R}$ be a Lipschitz map with respect to the Riemannian distance of the group $\mathbb{M}$. Our coarea formula reads as follows

$$\int_E |D_H f|_g \, d\mu_g = \int_{\mathbb{R}} \Big(\int_{E\cap f^{-1}(t)} \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \,. \tag{3}$$

In order to prove (3) we use also a general coarea estimate, [22], which implies that the set of singular points in a.e. level set of f is $\mathcal{S}^{Q-1}$-negligible. Formula (3) was first obtained by P. Pansu in the Heisenberg group, using the Carnot-Carathéodory distance, [26], and in general stratified groups it was first proved for smooth functions by J. Heinonen, [18]. We point out that the problem of extending the validity of (3) to Lipschitz maps with respect to a homogeneous distance is an open problem, nevertheless some particular cases have been considered in [22]. In the case $\mathbb{M}$ is an Euclidean space $\mathbb{E}^n$, with the classical Riemannian metric, formula (3) gives an extension of the classical Euclidean coarea formula

$$\int_E |Df| \, d\mu = \int_{\mathbb{R}} \Big(\int_{E\cap f^{-1}(t)} \theta_{n-1}(\nu(x)) \, d\mathcal{H}^{n-1}(x) \Big) \, dt \,,$$

where $\mathcal{H}^{n-1}$ is the Hausdorff measure with respect to a norm η of $\mathbb{E}^n$ and $\theta_{n-1}(\nu(x))$ is the corrisponding metric factor with respect to the canonical Riemannian metric (Corollary 2.8). We mention that another type of coarea formula for metric space valued Lipschitz maps on Euclidean normed spaces (or rectifiable subsets) is proved in [1], where the role of the metric factor is replaced by an "intrinsic" notion of jacobian.

Another aspect which naturally comes up, is finding particular conditions on both the Riemannian metric and the homogeneous distance such that the metric factor becomes a dimensional constant independent of the direction. In Subsection 2.1 we introduce $\mathcal{R}$-invariant distances and $\mathcal{R}$-rotational groups that possess these symmetry properties and we present some important examples where these properties hold.

1 Definitions and notation

In this section we present the notation we are going to use throughout the paper and we recall the main definitions concerning Carnot groups.

Let us consider a simply connected *graded nilpotent Lie group* $\mathbb{G}$, i.e. its Lie algebra $\mathcal{G}$ admits the grading $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$, with the inclusions $[V_j, V_1] \subset V_{j+1}$, for any $j \geq 1$ and $V_i = \{0\}$ iff $i > \iota$. If we assume further that the preceding inclusions are equalities we say that the group is a *stratified group*, or a *Carnot group*, see [10], [27]. We we will denote by $\mathbb{M}$ all the stratified groups. Notice that the stratification hypothesis amounts to the so-called *Hormander condition* on the left invariant vector fields which span V_1. The integer ι is called *degree of nilpoterncy* or *step* of the group. The subspace V_1 is called the *horizontal space*. We denote the translations of the group as $l_x : \mathbb{G} \longrightarrow \mathbb{G}$, $l_x(y) = xy$. Via the differential of translations we can move V_1 to any point x of $\mathbb{G}$, denoting as $H_x \subset T_x\mathbb{G}$ the horizontal fiber. This family of subspaces generates what we call an *horizontal subbundle* of $\mathbb{G}$, denoted by H.

By virtue of the graded structure we can define one parameter group of dilations $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$, $r > 0$, defined as follows

$$\delta_r\Big(\sum_{j=1}^{\iota} v_j\Big) = \sum_{j=1}^{\iota} r^j v_j \,,$$

where $\sum_{j=1}^{\iota} v_j = v$ and $v_j \in V_j$ for each $j = 1, \ldots \iota$. To any element of V_j we associate the integer j, which is called the *degree* of the vector.

Since $\mathbb{G}$ is simply connected and nilpotent it follows that $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism. The inverse function of the exponential map is denoted by ln. By means of these maps there is a canonical way to transpose dilations from $\mathcal{G}$ to $\mathbb{G}$. We will use the same symbol to denote dilations of the group. The following standard properties hold

1. $\delta_r(x \cdot y) = \delta_r x \cdot \delta_r y \quad$ for any $x, y \in \mathbb{G}$ and $r > 0$,
2. $\delta_r(\delta_s x) = \delta_{rs} x \quad$ for any $r, s > 0$ and $x \in \mathbb{G}$.

To provide a metric structure on the group we fix a "natural" Riemannian metric g on $\mathbb{G}$, which is compatible with the algebraic structure of the group.

Definition 1.1 Let $\mathbb{G}$ be a stratified group. We say that a Riemannian metric g on $\mathbb{G}$ is *left invariant* if translations of the group are isometries. A *graded metric* g on $\mathbb{G}$ is a left invariant metric such that all the subspaces $V_j \subset \mathcal{G}$ of the grading are orthogonal each other.

These metrics will be always understood throughout the paper. Stratified groups endowed with a graded metrics are also called *sub-Riemannian groups.* The Riemannian norm of a vector $W \in T_x\mathbb{G}$ will be denoted by $|W|_g$. We will denote with σ the Riemannian measure of hypersurfaces, which can be represented precisely as the $q-1$ dimensional Hausdorff measure with repsect to the Riemannian geodesic distance, where

q is the topological dimension of $\mathbb{G}$. In a stratified group $\mathbb{M}$ it is always possible to construct a left invariant distance such that it is 1-homogeneous with respect to dilations. The standard way to do this is to consider the class of *horizontal curves*, i.e. the absolutely continuous curves whose derivatives belong to H a.e. The conditions on commutators of H guarantee that each couple of points in $\mathbb{M}$ can be connected by an horizontal curve. Hence, it is possible to define the infimum among all the Riemannian lengths of horizontal curves which connect the two points. The outcome is a distance, named the Carnot-Carathéodory distance, which is continuous and satisfies the following properties

1. $d(x,y) = d(ux,uy)$ for every $u,x,y \in \mathbb{G}$,
2. $d(\delta_r x, \delta_r y) = r\, d(x,y)$ for every $r > 0$.

Any continuous distance on a graded group $\mathbb{G}$ which has the above properties is called *homogeneous distance*. All the homogeneous distances are bi-Lipschitz equivalent and induce the topology of the group. We denote by Q the Hausdorff dimension of $\mathbb{G}$ with respect to a homogeneous distance. By properties of dilations it is not difficult to prove that $Q = \sum_{j=1}^{\iota} j \dim(V_j)$.

Definition 1.2 We define the set $B_{x,r} \subset \mathbb{G}$ as the open ball of center x and radius $r > 0$ with respect to a homogeneous distance. We will omit the center of the ball if it coincides with the unit element of the group.

Using properties of homogeneous distances we have

$$B_{x,r} = xB_r = x\delta_r B_1 .$$

Definition 1.3 (Hausdorff measures) For each $a \geq 0$ and $E \subset \mathbb{G}$ we define the *a-dimensional spherical Hausdorff measure* of E as

$$\mathcal{S}^a(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_{j=1}^{\infty} r_i^a \mid E \subset \bigcup_{i=1}^{\infty} B_{x_i,r_i} \,,\ r_i \leq \varepsilon \Big\}$$

and the *a-dimensional Hausdorff measure* of E as

$$\mathcal{H}^a(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_{j=1}^{\infty} \frac{\operatorname{diam}(F_i)^a}{2^a} \mid E \subset \bigcup_{i=1}^{\infty} F_i \,,\ \operatorname{diam}(F_i) \leq \varepsilon \Big\}$$

where $\{F_i\}$ are subsets of $\mathbb{G}$ and $\operatorname{diam}(F_i) = \sup_{(x,y)\in F_i \times F_i} d(x,y)$.

Definition 1.4 The set of all continuously differentiable real valued functions defined on an open subset $A \subset \mathbb{G}$ will be denoted by $C^1(A, \mathbb{R})$.

Definition 1.5 Let $f \in C^1(A, \mathbb{R})$, where A is an open subset of $\mathbb{G}$ and $x \in A$. We define the *horizontal differential* $d_H f(x) : \mathcal{G} \longrightarrow \mathbb{R}$ as follows

$$d_H f(x)(W) = \lim_{r \to 0} \frac{f\left(x \cdot \exp(\delta_r W)\right) - f(x)}{r} . \tag{4}$$

Remark 1.6 The differentiability of f implies the existence of the limit (4). One can prove that $d_H f(x)$ is a linear map which vanishes on vectors of degree higher than one and it has the following homogeneity

$$d_H f(x)(\delta_r W) = r\, d_H f(x)(W) .$$

In fact, $D_H f(x)$ can be realized as the composition of $df(x)$ with the projection of $\mathcal{G}$ in H_e. See [14], [21], [27], [33] for more information on the notion of *Pansu differentiability*, or *horizontal differentiability*, which generalizes (4).

The unique vector of $T_x\mathbb{G}$ which represents the linear map $d_H f(x)$ with respect to the Riemannian metric is denoted by $D_H f(x)$ and it is called *horizontal gradient*.

Definition 1.7 We fix an orthonormal basis $(W_1, \ldots, W_q)$ of $\mathcal{G}$ and define the map $\mathcal{J} : \mathbb{R}^q \longrightarrow \mathbb{G}$ as

$$\mathcal{J}(y) = \exp\Big(\sum_{i=1}^{q} y_i W_i \Big) .$$

We call the couple $(\mathcal{J}, y)$ a *system of normal coordinates* associated to the basis.

Definition 1.8 Let us denote $n_j = \dim V_j$ for any $j = 1, \ldots, \iota$, $m_0 = 0$ and $m_i = \sum_{j=1}^{i} n_j$ for any $i = 1, \ldots \iota$. We say that a basis $(W_1, \ldots, W_q)$ of $\mathcal{G}$ is an *adapted basis*, if

$$(W_{m_{j-1}+1}, W_{m_{j-1}+2}, \ldots, W_{m_j})$$

is a basis of V_j for any $j = 1, \ldots \iota$.

Definition 1.9 (Weighted coordinates) A system of normal coordinates associated to an adapted basis will be called a system *weighted coordinates*. Assuming to have $\mathcal{J}(y) = \exp\Big(\sum_{i=1}^{q} y_i W_i \Big)$, we define the *weight* of the coordinate y_i as $d_i = j + 1$ if $m_j \le i \le m_{j+1}$, for any $i = 1, \ldots, q$.

Notice that any graded metric admits weighted coordinates. We mention that Definition 1.9 has a natural generalization in Carnot-Carathéodory spaces, see [3], [23].

Lemma 1.10 *Let B_1 be the open unit ball with respect to a homogeneous distance of the group and let $\mathcal{L} \subset \mathcal{G}$ be a hyperplane. We read the hyperplane on $\mathbb{G}$ as $L = \exp(\mathcal{L})$. Then for any couple of normal coordinates $(\mathcal{J}, y)$ and $(\mathcal{T}, z)$ we have*

$$\mathcal{H}^{q-1}_{|\cdot|}\left(\mathcal{J}^{-1}(L \cap B_1)\right) = \mathcal{H}^{q-1}_{|\cdot|}\left(\mathcal{T}^{-1}(L \cap B_1)\right) ,$$

where $\mathcal{H}^{q-1}_{|\cdot|}$ denotes the $q-1$ dimensional Hausdorff measure with respect to the Euclidean norm in $\mathbb{R}^q$.

PROOF. It is sufficient to observe that the composition $\mathcal{J}^{-1}\circ\mathcal{T}$ transforms coordinates with respect to different orthonormal bases. Then $\mathcal{J}^{-1}\circ\mathcal{T}$ is an isometry of $\mathbb{R}^q$ with respect to the Euclidean norm. The identity

$$\mathcal{J}^{-1}(L\cap B_1)=\left(\mathcal{J}^{-1}\circ\mathcal{T}\right)\left(\mathcal{T}^{-1}(L\cap B_1)\right)$$

leads us to the claim. □

Definition 1.11 Consider a vector $\nu\in\mathcal{G}\setminus\{0\}$ and its orthogonal hyperplane $\mathcal{L}\subset\mathcal{G}$, with $L=\exp(\mathcal{L})$. We fix a system of normal coordinates $(\mathcal{J},y)$ and define

$$\theta^g_{Q-1}(\nu)=\mathcal{H}^{q-1}_{|\cdot|}\left(\mathcal{J}^{-1}(L\cap B_1)\right)\ . \tag{5}$$

We call $\theta^g_{Q-1}(\nu)$ the *metric factor* of the homogeneous distance d with respect to the direction ν.

Remark 1.12 In view of the Lemma 1.10, the above definition does not depend on the choice of normal coordinates. We observe further that the number $\theta^g_{Q-1}(\nu)$ depends only on the direction of ν and the left invariant Riemannian metric on $\mathbb{G}$. Furthermore, it is not difficult to see from Definition 1.11 that the function $\theta^g_{Q-1}(\nu)$ is uniformly bounded from below by a positive constant.

Now we present a simple case which shows that in absence of particular symmetry properties the metric factor depends on the horizontal direction ν.

Example 1.13 Let us consider the Euclidean space $\mathbb{E}^2$, with homogenous distance $\eta(x)=\max\{|x_1|,|x_2|\}$, where (x_1,x_2) are Euclidean coordinates. We observe that $\mathbb{E}^2$ is an abelian 2-dimensional stratified group, where the canonical Riemannian metric is obviously left invariant. We denote by $L(\alpha)$ the straight line which contains the origin and whose direction is $\alpha\in\mathbb{T}^1$, where $\mathbb{T}^1$ is the 1-dimensional torus. In this case, by definition of $\theta_1(\alpha)$, we have

$$\theta_1(\alpha)=\mathcal{H}^1_{|\cdot|}\left(L(\alpha+\frac{\pi}{2})\cap\{x\in E^2\mid\max\{|x_1|,|x_2|\}<1\}\right)\ ,$$

By a direct computation we have

$$\theta_1(\alpha)=\begin{cases}2(\cos\alpha)^{-1} & -\frac{\pi}{4}\le\alpha\le\frac{\pi}{4}\\ 2(\sin\alpha)^{-1} & \frac{\pi}{4}\le\alpha\le\frac{3}{4}\pi\\ 2|\cos\alpha|^{-1} & -\frac{3}{4}\pi\le\alpha\le\frac{5}{4}\pi\\ 2|\sin\alpha|^{-1} & \frac{5}{4}\pi\le\alpha\le\frac{7}{4}\pi\end{cases}\ .$$

In Subsection 2.1 we will see some important cases where the metric factor is constant and can still be computed explicitly.

Definition 1.14 Let $\Sigma\subset\mathbb{G}$ be a hypersurface of class C^1, with $x\in\Sigma$. Consider a unit normal $\nu(x)$ of Σ at x. We denote by ν_H the Riemannian projection of $\nu(x)$ on the horizontal space H_x. We call the vector $\nu_H(x)$ the *horizontal normal* of Σ at x. We say that $x\in\Sigma$ is a *characteristic point* of Σ if $|\nu_H(x)|_g=0$. We denote by $C(\Sigma)$ the set of all characteristic points of Σ, namely the *characteristic set.*

2 Blow-up and coarea formula

In this section we prove the Blow-up Theorem on graded nilpotent groups. Its main application is the coarea formula for Riemannian Lipschitz maps on sub-Riemannian groups with respect to arbitrary homogeneous distances (Theorem 2.6).

Next, we introduce the notions of $\mathcal{R}$-invariant distances and $\mathcal{R}$-rotational groups and we prove that coarea formula can have a simpler form in this class of groups and distances (Theorem 2.22).

Theorem 2.1 (Blow-up) *Let $\Sigma \subset \mathbb{G}$ be a hypersurface of class C^1 and let $x \in \Sigma$ be a noncharacteristic point. Then we have*

$$\lim_{r \to 0} \frac{\sigma(\Sigma \cap xB_r)}{r^{Q-1}} = \frac{\theta^g_{Q-1}(\nu_H(x))}{|\nu_H(x)|_g}\,, \tag{6}$$

PROOF. Let $(X_1, \ldots, X_m)$ be an orthonormal frame of the horizontal subbundle H. We can represent Σ in a neighbourhood of x as $A \cap f^{-1}(t) \subset \Sigma$, where $t \in \mathbb{R}$, $A \subset \mathbb{G}$ is an open subset and $f \in C^1(A, \mathbb{R})$. Since the point x is noncharacteristic, the map $d_H f(x)$ is surjective, then $D_H f(x) = \sum_{i=1}^m X_i f(x)\, X_i \neq 0$. Let us define the unit vector $Y_1(x) = D_H f(x)/|D_H f(x)|_g$ and consider the corresponding left invariant vector field $Y_1 \in \mathcal{G}$. We can choose a left invariant orthonormal basis $(Y_1, \ldots, Y_m)$ which span the horizontal subbundle, hence $Y_j f(x) = 0$, for any $j \geq 2$. Next, we complete $(Y_1, \ldots, Y_m)$ to an orthonormal basis $(Y_1, \ldots, Y_m, Z_1, \ldots, Z_s)$ adapted to the grading of $\mathbb{G}$. We represent f with respect to the associated weighted coordinates centered at x. Precisely, we consider an open neighbourhood of the origin $V \subset \mathbb{R}^q$ with weighted coordinates $(y, z) \in V$ such that $\exp(V) \subset A$ and we define $F : V \longrightarrow \mathbb{R}$ as follows

$$F(y, z) = f\Big(x \exp\Big(\sum_{i=1}^m y_i Y_i + \sum_{j=1}^s z_j Z_j\Big)\Big) = f\left(x \mathcal{J}(y, z)\right)\,.$$

We have $D_y F(0) = (Y_1 f(x), 0, \ldots, 0) = (|D_H f(x)|_g, 0, \ldots, 0)$. By the Implicit Function Theorem we get a coordinate hyperplane $\Pi_x = \{(u_1, \ldots, u_q) \in \mathbb{R}^q \mid u_1 = 0\}$, an open neighbourhood of the origin $U \subset V \cap \Pi_x$ and a map $\phi \in C^1(U, V)$ such that $F(\phi(u)) = t$ for any $u \in U$, with $\phi(0) = 0$. Notice that $v = (u_1, \ldots, u_q) = (y, z)$ is a system of weighted coordinates with $d_i = 1$ for any $i = 1, \ldots, m$ and $d_i \geq 2$ for any $i = m+1, \ldots, q$, where d_i is the weight of u_i. We define the set $\Sigma_0 = \exp(\phi(U))$, observing that $x\Sigma_0$ is an open neighbourhood of x in Σ and denote by $x\phi$ the map $(\ln x) \cdot \phi : U \longrightarrow \ln(\Sigma)$. Thus, for any suitable small $r > 0$ we have

$$\sigma(\Sigma \cap xB_r) = \sigma(x\Sigma_0 \cap xB_r) = \int_{(x\phi)^{-1}\left(\ln x \cdot \tilde{B}_r\right)} \sqrt{\det\left(g_{ij}(x\phi(u))\right)}\, du\,,$$

where $\tilde{B}_r = \mathcal{J}^{-1}(B_r) \subset \mathbb{R}^q$. Let us observe that $(x\phi)^{-1}(\ln x \cdot \tilde{B}_r) = \phi^{-1}(\tilde{B}_r)$. Now, taking into account the weight of coordinates $u = (u_2, \ldots, u_q)$, the dilation δ_r reads

$$\delta_r u = \sum_{j=2}^{q} r^{d_j}\, u_j \, .$$

Hence, the restriction of δ_r to Π_x has jacobian r^{Q-1}. Then, we make a change of variable $u = \delta_r u'$, obtaining

$$\sigma(\Sigma \cap xB_r) = r^{Q-1} \int_{\delta_{1/r}\phi^{-1}(\tilde{B}_r)} \sqrt{\det\left(g_{ij}(x\phi(\delta_r u'))\right)}\, du' \, . \tag{7}$$

Now, we analyze the domain of integration $\delta_{1/r}\phi^{-1}(\tilde{B}_r) \subset \Pi_x$ as $r \to 0$. We can write $\phi(u) = (\varphi(u), u)$, with $\varphi : U \longrightarrow \mathbb{R}$, obtaining

$$\delta_{1/r}\phi^{-1}(B_r) = \{ u \in \Pi_x \mid \big(\varphi(\delta_r u)\, r^{-1}, u\big) \in \tilde{B}_1 \} \, .$$

We note that

$$\partial_{u_i}\varphi(0) = -\frac{\partial_{y_i}F(0)}{\partial_{y_1}F(0)} = 0 \quad \text{for } i = 2, \ldots, m \, ,$$

hence, by Taylor formula we get

$$\varphi(\delta_r u) r^{-1} = \sum_{i=m+1}^{q} \partial_{y_i}F(0) r^{d_i - 1} u_i + R(\delta_r u) r^{-1} \, ,$$

where $R(v)|v|^{-1} \to 0$ as $|v| \to 0$ and $|\cdot|$ is a norm on the space Π_x. For any $i > m$ we have $d_i \geq 2$, then $\varphi(\delta_r u) r^{-1} \to 0$ as $r \to 0$, uniformly in u which varies in a bounded set. Hence, for any $u \in \tilde{B}_1 \cap \Pi_x$ we have

$$\mathbf{1}_{\delta_{1/r}\phi^{-1}(B_r)}(u) \longrightarrow 1 \quad \text{as} \quad r \to 0 \, ,$$

whereas for any $u \in \Pi_x \setminus \overline{\tilde{B}_1}$ we get

$$\mathbf{1}_{\delta_{1/r}\phi^{-1}(B_r)}(u) \longrightarrow 0 \quad \text{as} \quad r \to 0 \, ,$$

so by formula (7) and Lebesgue Convergence Theorem it follows

$$\lim_{r \to 0} \frac{\sigma(\Sigma \cap xB_r)}{r^{Q-1}} = \int_{\tilde{B}_1 \cap \Pi_x} \sqrt{\det\left(g_{ij}(x)\right)}\, du \, . \tag{8}$$

Let us compute explicitly the left invariant Riemannian metric restricted to $x\Sigma_0$ with respect to our coordinates $u \in U$. We have

$$g_{ij}(x\phi(u)) = g(x\phi(u))\left(\frac{\partial(x\phi)}{\partial u_i}, \frac{\partial(x\phi)}{\partial u_j} \right) =$$

$$g(x\phi(u))\Big(dl_x \frac{\partial\phi}{\partial u_i}, dl_x \frac{\partial\phi}{\partial u_j} \Big) = g(\phi(u))\Big(\frac{\partial\phi}{\partial u_i}, \frac{\partial\phi}{\partial u_j} \Big) \,.$$

The metric g_{ij} with respect to the coordinates u coincides with δ_{ij} at the origin, then we get

$$\sqrt{\det\left(g_{ij}(x)\right)} = \sqrt{\det\left(\left\langle \frac{\partial\phi}{\partial u_i}, \frac{\partial\phi}{\partial u_j} \right\rangle\right)} = \frac{|DF(0)|}{|\partial_{y_1}F(0)|} = \frac{|Df(x)|_g}{|D_Hf(x)|_g} \,.$$

Finally, observing that

$$\nu_H(x) = \frac{D_Hf(x)}{|Df(x)|_g} \tag{9}$$

and $\mathcal{L}^{q-1}(\tilde{B}_1 \cap \Pi_x) = \theta^g_{Q-1}(\nu_H(x))$, equation (8) gives us the thesis. $\Box$

Remark 2.2 A version of Theorem 2.1 can be obtained on less regular hypersurfaces, such as reduced boundaries of sets of finite perimeter, see [14], [16]. In this case is required an isoperimetric inequality, which comes from the stratification of the group. Due to the fact that we are considering a C^1 smooth surface, our approach can be accomplishd whenever there exists a continuous homogeneous distance on $\mathbb{G}$. Clearly, in the case of stratified groups we always have the Carnot-Carathéodry distance, which is in particular a continuous homogeneous distance.

Theorem 2.3 *Consider a hypersurface $\Sigma \subset \mathbb{G}$ of class C^1 such that the characteristic set $C(\Sigma)$ is negligible with respect to the measure $\mathcal{S}^{Q-1}$. Then for any measurable set $E \subset \Sigma$ we have*

$$\int_E |\nu_H(x)|_g \, d\sigma = \int_E \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1} \quad \text{and} \quad \mathcal{S}^{Q-1}(E) = \int_E \frac{|\nu_H(x)|_g}{\theta^g_{Q-1}(\nu_H(x))} \, d\sigma \,. \tag{10}$$

Proof. Theorem 2.1 implies that for any $x \in \Sigma \setminus C(\Sigma)$ we have

$$\lim_{r\to 0} \frac{\sigma(\Sigma \cap xB_r)}{r^{Q-1}} = \frac{\theta^g_{Q-1}(\nu(x))}{|\nu_H(x)|_g} \,.$$

Now, using theorems on measure derivatives, see for instance Theorems 2.10.17 (2) and 2.10.18 (1) of [9], and observing that the characteristic set is negligible, the proof follows by a standard argument. $\Box$

Corollary 2.4 *Let A be an open subset of $\mathbb{G}$. Consider a map $f \in C^1(A, \mathbb{R})$ with regular value $t \in f(A)$. If the characteristic set*

$$C\left(f^{-1}(t)\right) = \left\{ x \in f^{-1}(t) \mid d_Hf : H_x \longrightarrow \mathbb{R} \, vanishes \right\}$$

is negligible with respect to $\mathcal{S}^{Q-1}$, then for any measurable subset $E \subset A$ we have

$$\int_{E\cap f^{-1}(t)} \frac{|D_Hf(x)|_g}{|Df(x)|_g} \, d\sigma = \int_{E\cap f^{-1}(t)} \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \,.$$

PROOF. It is enough to use Theorem 2.3 and the following equation

$$\nu_H(x) = \frac{D_H f(x)}{|Df(x)|_g} . \quad \square \tag{11}$$

Now we state the classical Riemannian coarea formula, see section 13.4 of [4].

Theorem 2.5 *Let $f : \mathbb{G} \longrightarrow \mathbb{R}$ be a Riemannian Lipschitz function. Then for any summable map $u : \mathbb{G} \longrightarrow \mathbb{R}$, the following formula holds*

$$\int_{\mathbb{G}} u \, |Df|_g \, d\mu_g = \int_{\mathbb{R}} \Big(\int_{f^{-1}(t)} u \, d\sigma \Big) \, dt \, , \tag{12}$$

where μ_g is the Riemannian volume measure and σ is the Riemannian surface measure.

The following result is an important application of Theorem 2.1.

Theorem 2.6 (Coarea formula) *Let E be a measurable subset of a sub-Riemannian group $\mathbb{M}$ and consider a Lipschitz map $f : E \longrightarrow \mathbb{R}$ with respect to the Riemannian distance of $\mathbb{M}$. Then we have*

$$\int_E |D_H f|_g \, d\mu_g = \int_{\mathbb{R}} \Big(\int_{E \cap f^{-1}(t)} \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \, , \tag{13}$$

where the spherical Hausdorff measure and the metric factor are understood with respect to the same homogeneous distance.

PROOF. Without loss of generality, we can assume that E is a bounded set and that f is extended to a Lipschitz map on $\mathbb{M}$. The Whitney Extension Theorem (see 3.1.15 of [9]) ensures that for any $\varepsilon > 0$ there exists a map $\tilde{f} : \mathbb{M} \longrightarrow \mathbb{R}$ of class C^1 such that, defining

$$E' = \Big\{ x \in \mathbb{M} \mid f(x) = \tilde{f}(x) \Big\} \, ,$$

we have $\mu_g(E \setminus E') \leq \varepsilon$. Thus, the gradients of f and $\tilde{f}$ coincide a.e. on E'. In view of formulae (9) and (12) we obtain

$$\int_E |D_H f|_g \, d\mu_g = \int_{\mathbb{R}} \Big(\int_{E \cap f^{-1}(t)} |\nu_H|_g \, d\sigma \Big) \, dt \, ,$$

for any measurable subset $E \subset \mathbb{M}$. Hence, the general inequality 2.10.25 of [9] implies

$$0 \leq \int_E |Df|_g \, d\mu_g - \int_{\mathbb{R}} \Big(\int_{E' \cap \tilde{f}^{-1}(t)} |\tilde{\nu}_H|_g \, d\sigma \Big) \, dt \leq C \, Lip(f) \, \varepsilon \, ,$$

where C is a dimensional constant and $\tilde{\nu}_H$ is the horizontal normal relative to the level sets of $\tilde{f}$. By virtue of Theorem 2.7 of [22] we know that the set of characteristic points is $\mathcal{S}^{Q-1}$-negligible for a.e. level set of $\tilde{f}$. Thus, we can apply formula (10), getting

$$0 \leq \int_E |Df|_g \, d\mu_g - \int_{\mathbb{R}} \Big(\int_{E' \cap \tilde{f}^{-1}(t)} \theta^g_{Q-1}(\tilde{\nu}_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \leq C \, Lip(f) \, \varepsilon \,. \qquad (14)$$

Let us observe that $E' \cap f^{-1}(t) = E' \cap \tilde{f}^{-1}(t)$ and for a.e. level set we have $Df = D\tilde{f}$ outside of a $\mathcal{S}^{Q-1}$-negligible set. Thus, for a.e. $t \in \mathbb{R}$ the following equality holds for $\mathcal{S}^{Q-1}$-a.e. $x \in f^{-1}(t)$

$$\theta_{Q-1}(\tilde{\nu}_H(x)) = \theta_{Q-1}(\nu_H(x)) \,.$$

Hence, inequality (14) becomes

$$0 \leq \int_E |Df|_g \, d\mu_g - \int_{\mathbb{R}} \Big(\int_{E' \cap f^{-1}(t)} \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \leq C \, Lip(f) \, \varepsilon \,.$$

Again, using the general inequality 2.10.25 of [9] and observing that in view of (5) the function $\theta^g_{Q-1}(\cdot)$ is bounded, we get

$$\int_{(E \setminus E') \cap f^{-1}(t)} \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \leq C' \, Lip(f) \, \varepsilon \,.$$

Finally, joining the last two inequalities we arrive at

$$-C' Lip(f) \, \varepsilon \leq \int_E |Df|_g \, d\mu_g - \int_{\mathbb{R}} \Big(\int_{E \cap f^{-1}(t)} \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \leq C Lip(f) \varepsilon \,.$$

Letting $\varepsilon \to 0$, the proof is complete. $\square$

Remark 2.7 It is natural to ask whether it is possible to get a coarea formula where only the restriction of the left invariant metric g on the horizontal subbundle is involved. The left invariance of μ_g and $\mathcal{S}^Q$ implies $\mu_g = c \, \mathcal{S}^Q$, where $c = \mu_g(B_1)/\mathcal{S}^Q(B_1)$. Then formula (13) becomes

$$\int_E |D_H f|_g \, d\mathcal{S}^Q = \int_{\mathbb{R}} \Big(\int_{E \cap f^{-1}(t)} \sigma_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \,,$$

where $\sigma_{Q-1}(\nu) = \mathcal{S}^Q(B_1) \, \theta_{Q-1}(\nu)/\mu_g(B_1)$. Now, by a standard but a bit long calculation one can check that the quotient $\sigma_{Q-1}(\nu)$ is constant over all left invariant metrics which coincide on the horizontal subbundle.

Corollary 2.8 *Let E be a measurable subset of $\mathbb{E}^n$ and let $f : E \longrightarrow \mathbb{R}$ be a Lipschitz map. Consider a norm $\eta : \mathbb{E}^n \longrightarrow [0, +\infty[$ and the Hasudorff measure $\mathcal{H}^{n-1}$ relative to this norm. Then we have*

$$\int_E |Df| \, d\mu = \int_{\mathbb{R}} \Big(\int_{E \cap f^{-1}(t)} \theta_{n-1}(\nu(x)) \, d\mathcal{H}^{n-1}(x) \Big) \, dt \,, \qquad (15)$$

where $|Df|$ is the length of the Euclidean gradient of f, ν is the normal direction to the level set and $\theta_{n-1}(\nu(x)) = \mathcal{H}^{n-1}_{|\cdot|}\left(\Pi_x \cap \{y \in \mathbb{E}^n \mid \eta(y) < 1\}\right)$, with Π_x equal to the hyperplane normal to $\nu(x)$.

Proof. Formula (15) follows directly from (13), observing that the intrinsic Hausdorff dimension Q of $\mathbb{E}^n$ coincides with n and that any direction in $\mathbb{E}^n$ is horizontal. Thus, the horizontal gradient coincides with the Euclidean gradient and the horizontal normal ν_H coincides with the normal ν to the level set. Now, we recall that the spherical Hausdorff measure coincides with the Hasudorff measure on rectifiable subsets of a normed space. This fact follows by a isodiametric inequality which holds on any finite dimensional normed space, see [4]. Thus, for a.e. level set of f we can replace the $\mathcal{S}^{n-1}$ in formula (13) with $\mathcal{H}^{n-1}$. This completes the proof. □

2.1 Invariant metrics and horizontal isometries

In this subsection we introduce the notion of "horizontal isometry", that respects both the Riemannian and the algebraic structure of the group. This concept allows us to distinguish a class of sub-Riemannian groups that have particular symmetry properties, namely "rotational groups". Furthermore, it is always possible to define homogeneous distances which are compatible with horizontal isometries (see Remark 2.14). The metric factor $\theta^g_{Q-1}(\nu)$ with respect to these particular matrics and homogeneous distances is independent of $\nu \in H \setminus \{0\}$, so it becomes a dimensional constant in the coarea formula, see Theorem 2.22. In Remark 2.23 we give some applications of this theorem.

The following definition generalizes the concept of horizontal isometry first introduced in the particular case of Heisenberg groups, [22].

Definition 2.9 We say that a map $T : \mathbb{G} \longrightarrow \mathbb{G}$ is a *horizontal isometry* if the following properties hold

1. $T(x \cdot y) = T(x) \cdot T(y)$ for any $x, y \in \mathbb{G}$ (T is a group homomorphism)
2. $T(\delta_r x) = \delta_r T(x)$ for any $x \in \mathbb{G}$ and $r > 0$ (T is 1-homogeneous)
3. $dT(e) : \mathcal{G} \longrightarrow \mathcal{G}$ is an isometry, where $e \in \mathbb{G}$ is the unit element.

Notice that conditions 1 and 3 imply that $T : \mathbb{G} \longrightarrow \mathbb{G}$ is an isometry of $\mathbb{G}$ in the sense of Riemannian Geometry. Furthermore, conditions 1 and 2 say that T is a G-linear map (see [21]), so it is bi-Lipschitz with respect to any homogeneous distance of the group.

Remark 2.10 We point out that the existence of horizontal isometries is strongly related to the compatibility of the left invariant Riemannian metric with the algebraic structure of the group. This fact will appear evident in the following examples.

Definition 2.11 (Invariant distances) Let $\mathcal{R}$ be a set of horizontal isometries. We say that a homogeneous distance is *$\mathcal{R}$-invariant* if for any $T \in \mathcal{R}$ we have $T(B_1) = B_1$, where B_1 is the open unit ball with respect to the homogeneous distance.

Definition 2.12 (Rotational groups) A *vertical hyperplane* $\mathcal{L} \subset \mathcal{G}$ is the orthogonal space of a horizontal vector of $\mathcal{G}$. We say that a stratified group $\mathbb{G}$ is *$\mathcal{R}$-rotational*, if there exists a class $\mathcal{R}$ of horizontal isometries such that for any couple of vertical hyperplanes $\mathcal{L}$ and $\mathcal{L}'$ there exists $T \in \mathcal{R}$ with $dT(e)(\mathcal{L}) = \mathcal{L}'$. We will simply say *rotational group*, when the class $\mathcal{R}$ is understood.

Example 2.13 (Rotational Euclidean spaces) The Euclidean space $\mathbb{E}^n$ with the canonical Riemannian metric is a rotational group. In fact, any hyperplane is vertical, so it is natural to choose the class of all Euclidean isometries of $\mathbb{E}^n$ as $\mathcal{R}$. Hence, Euclidean spaces are $\mathcal{R}$-rotational, with $\mathcal{R}$-invariant Euclidean distance.

Remark 2.14 Notice that if $\mathcal{R}$ is the class of all horizontal isometries and ρ is the Carnot-Carathéodory distance built with respect to the same Riemannian metric, then ρ is $\mathcal{R}$-invariant. In fact, any horizontal isometry transforms horizontal curves into horizontal curves and it preserves their length. So, there is a natural class of $\mathcal{R}$-invariant distances associated to a stratified group with respect to its Riemannian metric. A nontrivial question for a general stratified group is to get the existence of a sufficiently large class of horizontal isometries. In Example 2.15 we will show that horizontal isometries cannot always be obtained starting from isometries of $\mathcal{G}$. In other words, if we consider an isometry $I : \mathcal{G} \longrightarrow \mathcal{G}$, it may not exist a 1-homogemeous group homomorphism $T : \mathbb{G} \longrightarrow \mathbb{G}$ such that $dT(e) = I$.

Example 2.15 Let us consider the isometry $A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ represented by the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We define the rotation $T : \mathbb{C}^2 \times \mathbb{R} \longrightarrow \mathbb{C}^2 \times \mathbb{R}$ as $T(x + iy, s) = (A(x, y), s)$, where $x, y \in \mathbb{R}^2$ and $s \in \mathbb{R}$. The map T cannot be a horizontal isometry of the Heisenberg group $\mathbb{H}^2 = \mathbb{C}^2 \times \mathbb{R}$ with the group operation

$$(z, s) \cdot (w, t) = (z + w, s + t + 2\mathrm{Im}(z \cdot \overline{w}))\,, \tag{16}$$

where $z, w \in \mathbb{C}^2$, $z_j = x_j + iy_j$, $w_j = \xi_j + i\eta_j$, with $j = 1, 2$. In fact, the homomorphism property would imply

$$\mathrm{Im}(z \cdot \overline{w}) = \mathrm{Im}(Az \cdot \overline{Aw})\,, \tag{17}$$

which gives

$$y_2\eta_1 - y_1\eta_2 - x_2\xi_1 + x_1\xi_2 = -x_1\eta_1 + y_1\xi_1 - x_2\eta_2 + y_2\xi_2\,.$$

The last equality fails for $x_i = 0$, $y_i = \xi_i = \eta_i = 1$, with $i = 1, 2$, so T is not a group homomorphism.

However, Heisenberg groups are important examples of rotational Carnot groups. This fact will be a consequence of the following example.

Example 2.16 (Rotational Heisenberg groups) In order to emphasize the crucial role played by the metric structure associated to a stratified group we consider an intrinsic version of the Heisenberg group. Let $\psi : V_1 \times V_1 \longrightarrow \mathbb{R}$ be a symplectic map, where V_1 is a vector space of dimension $2n$. We define a Lie product on $\mathfrak{h}_{2n+1} = V_1 \oplus \mathbb{R}\omega$ as follows

$$[(u + t\omega), (v + \tau\omega)] = \psi(u, v)\,\omega$$

for any $u, v \in V_1$ and $t, \tau \in \mathbb{R}$. Then, any homogeneous algebra isomomorphism T can be written as $T = S \times (\alpha \mathrm{Id}_{\mathbb{R}\omega})$, where $T(\omega) = \alpha\omega$ and

$$\alpha\psi(u, v) = \psi(Su, Sv) \tag{18}$$

for some $\alpha \neq 0$. When $\alpha = 1$ the maps S satisfying (18) are the well known *symplectic transformations*. We recall that ψ admits a symplectic basis $(e_1, \ldots, e_{2n})$ of V_1, i.e. $\psi(e_i, e_{n+j}) = -4\delta_{ij}$, $\psi(e_i, e_j) = 0$ and $\psi(e_{n+i}, e_{n+j}) = 0$ for any $i, j = 1, \ldots, n$. Hence, a metric compatible with the symplectic structure of $\mathfrak{h}_{2n+1}$ has to make the basis $(e_i) \cup (\omega)$ orthonormal. Such a metric is called *symplectic metric*. With this particular choice, we will be able to show the existence of a large class of horizontal isometries on $\mathfrak{h}_{2n+1}$. In fact, the symplectic metric allows us to get an isometric identification between $\mathfrak{h}_{2n+1}$ and $\mathbb{H}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$, where the group operation of the latter is defined analogously as in (16). Moreover, the map ψ in these coordinates is represented as $\psi(z, w) = 4\mathrm{Im}(z \cdot \overline{w})$. Now, consider a unitary operator $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and define the map $T : \mathbb{C}^n \times \mathbb{R} \longrightarrow \mathbb{C}^n \times \mathbb{R}$ as $T(z, s) = (U(z), s)$ for any $(z, s) \in \mathbb{C}^n \times \mathbb{R}$. The invariance of the complex scalar product under unitary trasformations gives condition (18) with $\alpha = 1$, therefore T is a group isomomorphism. Properties 2 and 3 of Definition 2.9 are easily verified, so it follows that T is a horizontal isometry.

Now, vertical hyperplanes in $\mathbb{H}^n$ can be characterized as products $\Pi \times \mathbb{R}$, where Π is a real $2n-1$ dimensional space of $\mathbb{C}^n$. Furthermore, unitary operators preserve the real scalar product of $\mathbb{R}^{2n}$, so it is not difficult to show that for any couple of hyperplanes $\mathcal{L}$ and $\mathcal{L}'$ of $\mathbb{C}^n$ there exists a unitary map $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $T(\mathcal{L}) = \mathcal{L}'$. Then the product of unitary operators of $\mathbb{C}^n$ with the projection on the last component corresponds to a class $\mathcal{R}$ of horizontal isometries in $\mathfrak{h}_{2n+1}$. Thus, we have proved that Heisenberg groups with a symplectic metric are rotational groups.

Remark 2.17 (Rotational H-type groups) The results obtained in the preceeding example can be achieved also in some more general groups of Heisenberg type. These are 2-step groups endowed with a scalar product $\langle\, , \,\rangle$ and a linear map $J : V_2 \longrightarrow \mathrm{End}(V_1)$ with the following properties

1. $\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle$ for any $X, Y \in V_1$ and $Z \in V_2$

2. $J_Z^2 = -|Z|^2 I$,

see [19] for more information. Let us consider the group

$$G = \left\{(\phi,\psi) \in O(V_2) + O(V_1) \mid J_{\phi(v)}(\psi(x)) = \psi(J_v(x))\right\},$$

where $O(V_1)$ and $O(V_2)$ denote the group of isometries in V_1 and V_2, respectively. In Proposition 5 of [29], C. Riehm proves that the maps of $\phi+\psi$ of G are homomorphisms, hence G corresponds to a group of horizontal isometries according to our definition. Furthermore, denoting by G_{V_1} the projection of G in $O(V_1)$, in [28] there is a precise characterization of H-type groups where G_{V_1} is transitive on the sphere $V_1^* = \{v \in V_1 \mid |v| = 1\}$. In view of Definitio 2.12, groups with this transitive property on V_1^* are $\mathcal{R}$-rotational with $\mathcal{R} = G$.

In the following proposition we show that $\mathcal{R}$-rotational groups and $\mathcal{R}$-invariant distances yield a constant metric factor.

Proposition 2.18 *Let $\mathbb{G}$ be an $\mathcal{R}$-rotational group and let d be an $\mathcal{R}$-invariant distance of $\mathbb{G}$. Then there exists $\alpha_{Q-1} \in \mathbb{R}$ such that*

$$\theta^g_{Q-1}(\nu) = \alpha_{Q-1}$$

for any $\nu \in H \setminus \{0\}$.

Proof. Let us consider the map $\mathcal{J} : \mathbb{R}^q \longrightarrow \mathbb{G}$ associated to a system of normal coordinates. We want to prove that for each vertical hyperplane $\mathcal{L}$, posed $L = \exp(\mathcal{L})$, we have

$$\mathcal{H}^{q-1}_{|\cdot|}\left(\mathcal{J}^{-1}(L \cap B_1)\right) = \mathcal{H}^{q-1}_{|\cdot|}\left(\mathcal{J}^{-1}(T(L) \cap B_1)\right),$$

for any horizontal isometry $T \in \mathcal{R}$. We define the isometry $I = \exp^{-1} \circ \mathcal{J} = \mathbb{R}^q \longrightarrow \mathcal{G}$, observing that $I^{-1} \circ dT(e) \circ I$ is an isometry of $\mathbb{R}^q$ with respect to the Euclidean norm. Then we have

$$\mathcal{H}^{q-1}_{|\cdot|}\left(\mathcal{J}^{-1}(L \cap B_1)\right) = \mathcal{H}^{q-1}_{|\cdot|}\left(I^{-1} \circ dT(e) \exp^{-1}(L \cap B_1)\right)$$

and

$$I^{-1} \circ dT(e) \exp^{-1}(L \cap B_1) = I^{-1} \circ \exp^{-1} \circ T(L \cap B_1) = \mathcal{J}^{-1}\left(T(L) \cap T(B_1)\right).$$

Finally, the $\mathcal{R}$-invariance property $T(B_1) = B_1$ leads us to the conclusion. □

Remark 2.19 The number α_{Q-1} in Proposition 2.18 amounts to the measure of the intersection between the unit ball and a vertical hyperplane. In the next example, we will see that α_{n-1} in $\mathbb{E}^n$ with the Euclidean distance corresponds exactly to the measure ω_{n-1} of the unit ball in $\mathbb{E}^{n-1}$.

Example 2.20 Let us consider $\mathbb{E}^n$ with standard coordinates $x = (x_i)$ and the classical Euclidean norm $\eta(x) = |x| = \sqrt{\sum_{i=1}^n x_i^2}$. In this case we have

$$\theta_{n-1}(\nu(x)) = \mathcal{H}^{n-1}_{|\cdot|}\left(\Pi_x \cap \{y \in \mathbb{E}^n \mid |y| < 1\}\right) = \mathcal{H}^{n-1}_{|\cdot|}\left(\{y \in \mathbb{E}^{n-1} \mid |y| < 1\}\right) = \omega_{n-1}$$

Example 2.21 Let us consider the distance $d([z,t],0) = \max\{|z|, |t|^{1/2}\}$ in the Heisenberg group $\mathbb{H}^{2n+1}$. By calculations of Lemma 4.5 (iii) in [14] we have that the corresponding metric factor is $\alpha_{Q-1} = 2\omega_{2n-1}$.

Theorem 2.22 *Let $\mathbb{G}$ be an $\mathcal{R}$-rotational group and suppose to have an $\mathcal{R}$-invariant distance on $\mathbb{G}$. Thus, if $f : E \longrightarrow \mathbb{R}$ is a Lipschitz map with respect to the Riemannian distance and E is a measurable subset of $\mathbb{G}$ we have*

$$\int_E |D_H f|_g \, d\mu_g = \alpha_{Q-1} \int_{\mathbb{R}} \mathcal{S}^{Q-1}\left(E \cap f^{-1}(t)\right) \, dt \,. \tag{19}$$

PROOF. By virtue of Theorem 2.6 we have

$$\int_E |D_H f|_g \, d\mu_g = \int_{\mathbb{R}} \Big(\int_{E \cap f^{-1}(t)} \theta^g_{Q-1}(\nu_H(x)) \, d\mathcal{S}^{Q-1}(x) \Big) \, dt \,.$$

In view of Proposition 2.18 we get $\theta^g_{Q-1}(\nu_H(x)) = \alpha_{Q-1}$, so the proof is complete. □

Remark 2.23 In this remark we present some applications of (19).

1. *Classical Coarea formula.* Taking into account Definition 1.3, by classical results on rectifiable sets in Euclidean spaces we have $\omega_{n-1}\mathcal{S}^{n-1} = \omega_{n-1}\mathcal{H}^{n-1} = \sigma$, where σ is the $n-1$ surface measure in $\mathbb{E}^n$. In view of these facts, formula (19) and calculation in Example 2.20 give us the classical coarea formula.

2. *Coarea formula in $\mathbb{H}^{2n+1}$ with respect to the Carnot-Carathéodory distance.* Observing that the Carnot-Carathéodory distance in $\mathbb{H}^{2n+1}$ is $\mathcal{R}$-invariant (Example 2.16), formula (19) yields the coarea formula proved in [26].

3. *Coarea formula in $\mathbb{H}^{2n+1}$ with respect to the maximum distance.* The maximum distance in Example 2.21 gives us the coarea formula (19) with $\alpha_{Q-1} = 2\omega_{2n-1}$.

3 Characteristic sets

In this section we study the size of the characteristic set for $C^{1,1}$ smooth hypersurfaces. The following variant of the Blow-up Theorem provides estimates for the $Q-2$ densities of $C^{1,1}$-hypersurfaces in 2-step graded groups.

Theorem 3.1 (Blow-up estimates) *Let $\mathbb{G}$ be a 2-step graded group with grading $\mathcal{G} = V_1 \oplus V_2$ and let Σ be a $C^{1,1}$-hypersurface with a characteristic point $x \in \Sigma$. Then there exist two constants $c_1, c_2 > 0$, such that*

$$c_2 \geq \limsup_{r \to 0} \frac{\sigma(\Sigma \cap xB_r)}{r^{Q-2}} \geq \liminf_{r \to 0} \frac{\sigma(\Sigma \cap xB_r)}{r^{Q-2}} \geq c_1 \,, \tag{20}$$

where c_1 depends on the Lipschitz constant of the normal field on Σ.

Proof. Let $(X_1, \ldots, X_m)$ be an orthonormal frame of V_1 and let $(Z_1, \ldots, Z_s)$ be an orthonormal basis of V_2. We represent Σ in a neighbourhood of x with the set $A \cap f^{-1}(t) \subset \Sigma$, where A is an open subset of $\mathbb{G}$, $f \in C^{1,1}(A, \mathbb{R})$ and $f(x) = t$.

Since $x \in \Sigma$ is a characteristic point, the horizontal gradient $Xf(x)$ is vanishing, so $\sum_{l=1}^s Z_l f(x)\, Z_l \neq 0$ and we can define the unit vector

$$W_1(x) = \frac{\sum_{l=1}^s Z_l f(x)\, Z_l}{|\sum_{l=1}^s Z_l f(x)\, Z_l|_g} \,.$$

Hence we can build an orthonormal basis $(W_1, \ldots, W_s)$ of V_2. Now, we consider an open neighbourhood of the origin $V \subset \mathbb{R}^q$ and the function $F : V \longrightarrow \mathbb{R}$ defined as

$$F(y, z) = f\Big(x \, \exp\Big(\sum_{l=1}^m y_l X_l + \sum_{l=1}^s z_l W_l\Big)\Big) \,,$$

so (y, z) are weighted coordinates. We have $\partial_{z_1} F(0) = W_1 f(x) \neq 0$ and $\partial_{z_l} F(0) = 0$ for any $l = 2, \ldots, s$. Hence, by the Implicit Function Theorem there exists a hyperplane $\mathcal{Q} = \{(y, z') \mid y \in \mathbb{R}^m,\, z' = (z_2, \ldots, z_s) \in \mathbb{R}^{s-1}\}$, an open neighbourhood of the origin $U \subset V \cap \mathcal{Q}$ and a map $\phi \in C^{1,1}(U, V)$ such that $\phi(0) = 0$ and $F(\phi(u)) = t$ for any $u \in U$. We can represent a neighbourhood of x in Σ as $x \exp \phi(U) = x\Sigma_0$ and for any suitable small $r > 0$ we get

$$\sigma(\Sigma \cap xB_r) = \sigma(x\Sigma_0 \cap xB_r) = \int_{\phi^{-1}(\tilde{B}_r)} \sqrt{\det\left(g_{ij}(x\phi(u))\right)}\, du \,,$$

where $\tilde{B}_r = \ln B_r$. As coordinates y have weight 1 and coordinates z' have weight 2, the representation of the restriction of δ_r to $\mathcal{Q}$ is as follows

$$\delta_r(y, z') = (r y_i, r^2 z_j) \,.$$

Then the jacobian of $\delta_{r|\mathcal{Q}}$ is r^{Q-2}. Now, we make a change of variable $u = \delta_r u'$, obtaining

$$\sigma(\Sigma \cap xB_r) = r^{Q-2} \int_{\delta_{1/r}\phi^{-1}(\tilde{B}_r)} \sqrt{\det\left(g_{ij}(x\phi(\delta_r u'))\right)}\, du' \,. \tag{21}$$

Next, we study the shape of the domain $\delta_{1/r}\phi^{-1}(\tilde{B}_r)$ as $r \to 0$. By the Implicit Function Theorem there exists a map $\varphi \in C^{1,1}(U, \mathbb{R})$ such that $\phi(y, z') = (y, \varphi(y, z'), z')$. Thus, we can represent the set $\delta_{1/r}\phi^{-1}(\tilde{B}_r)$ as follows

$$\delta_{1/r}\phi^{-1}\delta_r(\tilde{B}_1) = \{(y, z') \in \mathcal{Q} \mid \left(y, \varphi\left(\delta_r(y, z')\right) r^{-2}, z'\right) \in \tilde{B}_1\} . \tag{22}$$

By our choice of coordinates and the fact that $Xf(x) = 0$ we have

$$\partial_{y_k}\varphi(0) = -\frac{\partial_{y_k}F(0)}{\partial_{z_1}F(0)} = 0 \qquad \partial_{z_l}\varphi(0) = -\frac{\partial_{z_l}F(0)}{\partial_{z_1}F(0)} = 0$$

for any $k = 1, \ldots, m$ and $l = 2, \ldots, s$. Hence, we have proved that $D_y\varphi(0) = 0$ and $D_{z'}\varphi(0) = 0$. By Taylor formula for $C^{1,1}$ smooth functions we get

$$\varphi\left(\delta_r(y, z')\right) = \theta\left((ry, r^2z')\right) \, |(ry, r^2z')|^2 \tag{23}$$

where $|\cdot|$ is a norm on the space $\mathcal{Q}$ and θ is a map which is bounded by the Lipschitz constant of $D\varphi$. Let $C \subset \mathbb{R}^q$ be an open Euclidean ball contained in $\tilde{B}_1$ and let us define the set

$$E = \{(y, z') \in \mathcal{Q} \mid \ (y, L|y|^2, z') \in C\} ,$$

where $L = 2\|\theta\|_\infty$. Now, we aim to prove that for any $(y, z') \in E$

$$\mathbf{1}_{\delta_{1/r}\phi^{-1}(B_r)}\left((y, z')\right) \longrightarrow 1 \quad \text{as} \quad r \to 0 . \tag{24}$$

Consider $(y, z') \in E$ and choose $r_0 > 0$ such that for any $r \in (0, r_0)$ we have $|(y, rz')| \leq \sqrt{2}|y|$. Then, by equation (23) for any $r \in (0, r_0)$ we get

$$r^{-2}|\varphi\left(\delta_r(y, z')\right)| = |\theta\left((ry, r^2z')\right)| \, |(y, rz')|^2 \leq 2\|\theta\|_\infty \, |y|^2 = L \, |y|^2 .$$

Since C is convex and $\delta_{1/r}\phi^{-1}\delta_r(\tilde{B}_1)$ has representation (22) it follows that (y, z') belongs to $\delta_{1/r}\phi^{-1}\delta_r(\tilde{B}_1)$ for any $r \in (0, r_0)$, so the limit (24) is proved. In view of Fatou Theorem and (24) we obtain

$$\liminf_{r \to 0} \int_{\delta_{1/r}\phi^{-1}(\tilde{B}_r)} \sqrt{\det\left(g_{ij}(x\phi(\delta_r u'))\right)} \, du' \geq \int_E \sqrt{\det\left(g_{ij}(x)\right)} \, du .$$

Since the metric is left invariant and g_{ij} is computed on a left invariant orthonormal basis, the factor $\det\left(g_{ij}(x)\right)$ is a constant C independent of x. Now, we observe that the set E is an open set, whose size depends on the Lipschitz constant of the normal field $x \longrightarrow (D\varphi(x), 1)$. Then, in view of formula (21) the positive constant $c_1 = C\,\mathcal{L}^{q-1}(E)$ satisfies our claim. To get the upper estimate we observe directly from the representation (22) that there exists a bounded set F which contains $\delta_{1/r}\phi^{-1}(\tilde{B}_r)$ for any $r > 0$. Thus, we can choose $c_2 = C\mathcal{L}^{q-1}(F)$. $\square$

Before stating the next theorem, we recall the definition of Hausdorff dimension of a subset E in a metric space (X, d):

$$\mathcal{H}_d\text{–}\dim(E) = \inf\{\alpha > 0 \mid \mathcal{H}_d^\alpha(E) = 0\} .$$

Theorem 3.2 *Let $\mathbb{G}$ be a 2-step graded group and let $\Sigma \subset \mathbb{G}$ be a $C^{1,1}$-hypersurface. Then there exist two constants c_1, c_2 as in Theorem 3.1 such that*

$$c_2\, \mathcal{S}^{Q-2}\left(C(\Sigma)\right) \geq \sigma(C(\Sigma)) \geq\ c_1\, \mathcal{S}^{Q-2}\left(C(\Sigma)\right)\ , \tag{25}$$

moreover we have

$$\mathcal{H}_{d_C}\text{–dim}\left(C(\Sigma)\right) \leq Q-2\,. \tag{26}$$

Proof. We adopt the notation of Theorem 2.10.18 in [9], where $V = \Sigma$, $\mu = \sigma \llcorner \Sigma$ and F is the family of balls with respect to the homogeneous metric d and $\zeta(B_{x,r}) = r^\alpha$ for any $x \in \mathbb{G}$ and $r > 0$. By virtue of the estimates (20) and Theorems 2.10.17(2), 2.10.18(1) of [9] we get our claim

$$c_1\, \mathcal{S}^{Q-2}\left(C(\Sigma)\right) \leq \sigma\left(C(\Sigma)\right) \leq\ c_2\, \mathcal{S}^{Q-2}\left(C(\Sigma)\right)\ .$$

Now, let us fix $\alpha \in (Q-2, +\infty)$ and observe that by (20) we have

$$\limsup_{r\to 0} \frac{\sigma\llcorner\Sigma(B_{x,r})}{r^\alpha} \geq t$$

for any $t > 0$ and any $x \in C(\Sigma)$. Thus, again Theorem 2.10.18 of [9] implies that for each $t > 0$ we have

$$t\, \mathcal{S}^\alpha(C(\Sigma)) \leq \sigma(C(\Sigma)) \leq \sigma(\Sigma)\,.$$

Since Σ can be realized as a countable union of relatively compact hypersurfaces we can assume that Σ is relatively compact. Then $\sigma(\Sigma)$ is finite, so letting $t \to \infty$ we get $\mathcal{S}^\alpha(C(\Sigma)) = 0$. This ends the proof. □

Remark 3.3 In the assumptions of Theorem 3.2, if $\sigma(\Sigma) < \infty$ it follows

$$c_1\, \mathcal{H}^{Q-2}(C(\Sigma)) \leq \sigma(C(\Sigma)) \leq \sigma(\Sigma) < \infty\,,$$

so the characteristic set has finite $Q-2$ Hausdorff measure.

We observe that the Carnot-Carathéodory distance d_C is always greater than the Riemannian distance ρ, when both of them are built with the same left invariant metric. Hence, for any set $E \subset \mathbb{G}$ and $\alpha > 0$ we have $\mathcal{H}^\alpha_\rho(E) \leq \mathcal{H}^\alpha_{d_C}(E)$. So the following inequality holds

$$\mathcal{H}_{d_C}\text{–dim}(E) \geq \mathcal{H}_\rho\text{–dim}(E)\,. \tag{27}$$

Now, by Theorem 1.4(1) of [2], for any $\alpha > 0$ there exists a $C^{1,1}$-hypersurface Σ_α in the Heisenberg group $\mathbb{H}^n$ such that $\mathcal{H}_{|\cdot|}\text{–dim}(C(\Sigma_\alpha)) \geq 2n - \alpha$, where $|\cdot|$ is the Euclidean norm in $\mathbb{H}^n$, viewed as a vector space. It is clear that $\mathcal{H}_\rho\text{–dim}(C(\Sigma)) = \mathcal{H}_{|\cdot|}\text{–dim}(C(\Sigma))$, so by (27) we get

$$\mathcal{H}_{d_C}\text{–dim}(C(\Sigma_\alpha)) \geq 2n - \alpha = Q - 2 - \alpha\,, \tag{28}$$

where $Q = 2n + 2$ is the Hausdorff dimension of $\mathbb{H}^n$ with respect to a homogeneous distance. Thus, by virtue of Theorem 3.2 we get

$$Q - 2 - \alpha \leq \mathcal{H}_{d_C}\text{–dim}(C(\Sigma_\alpha)) \leq Q - 2\,,$$

it follows that the estimate (26) is optimal.

Aknowledgements. I am deeply grateful to Paolo Tilli for his suggestion about a technical point of the work. I thank Fulvio Ricci for his comments which clarified the geometrical content of some important definitions. I also thank Hans Martin Reimann for his usefull remarks.

References

[1] L.Ambrosio, B.Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318**, 527-555, (2000)

[2] Z.M.Balogh, *Size of characteristic sets and functions with prescribed gradients*, preprint (2000)

[3] A.Bellaïche, *The Tangent space in sub-Riemannian geometry*, in *Subriemannian Geometry*, Progress in Mathematics, **144**. ed. by A.Bellaiche and J.Risler, Birkhäuser Verlag, Basel, 1996.

[4] Y.D.Burago, V.A.Zalgaller, *Geometric inequalities*, Grundlehren Math. Springer, Berlin.

[5] L.Capogna, D.Danielli, N.Garofalo, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*, Comm. Anal. Geom. **2**, n.2, 203-215, (1994)

[6] D.Danielli, N.Garofalo, D.M. Nhieu, *Traces Inequalities for Carnot-Carathéodory Spaces and Applications*, Ann. Scuola. Norm. Sup., **27**, 195-252, (1998)

[7] M.Derridj, *Sur un théorém de traces*, Ann. Inst. Fourier, Grenoble, **22**, 2, 73-83, (1972)

[8] M.Gromov, *Carnot-Carathéodory spaces seen from within*, in *Subriemannian Geometry*, Progress in Mathematics, **144**. ed. by A.Bellaiche and J.Risler, Birkhauser Verlag, Basel, 1996.

[9] H.Federer, *Geometric Measure Theory*, Springer, 1969

[10] G.B. Folland, E.M. Stein, *Hardy Spaces on Homogeneous groups*, Princeton University Press, 1982

[11] B.Franchi, S.Gallot, R.L.Wheeden, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann. **300**, 557-571 (1994)

[12] B.Franchi, R.L.Wheeden, *Compensation couples and isoperimetric estimates for vector fields*, Colloq. Math. **74**, 9-27, (1997).

[13] B.Franchi, R.Serapioni, F.Serra Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston Jour. Math., **22**, 859-889, (1996)

[14] B.Franchi, R.Serapioni, F.Serra Cassano, *Rectifiability and Perimeter in the Heisenberg group*, to appear in Math. Ann.

[15] B.Franchi, R.Serapioni, F.Serra Cassano, *Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups*, to appear in Comm. Anal. Geom.

[16] B.Franchi, R.Serapioni, F.Serra Cassano, *Rectifiability and Perimeter in Carnot groups of step 2*, forthcoming

[17] N.Garofalo, D.M.Nhieu, *Isoperimetric and Sobolev Inequalities for Carnot-Carathéodory Spaces and the Existence of Minimal Surfaces*, Comm. Pure Appl. Math. **49**, 1081-1144 (1996)

[18] J.Heinonen, *Calculus on Carnot groups*, Ber. Univ. Jyväskylä Math. Inst., **68**, 1-31, (1995).

[19] A.Kaplan, *Lie groups of Heisenberg type*, Rend. Sem. Mat. Univ. Politec. Torino, (1983), Special Issue 117-130 (1984)

[20] A.Korányi, H.M.Reimann, *Foundation for the Theory of Quasiconformal Mappings on the Heisenberg Group*, Adv. Math., **111**, 1-87, (1995).

[21] V.Magnani, *Differentiability and Area formula on stratified Lie groups*, Houston Jour. Math., **27**, n.2, (2001)

[22] V.Magnani, *On a general coarea inequality and applications*, to appear in Ann. Acad. Sci. Fenn. Math.

[23] G.A.Margulis, G.D.Mostow, *Some remarks on the definition of tangent cones in a Carnot-Carathéodory space*, J. Analyse Math. **80**, 299-317 (2000)

[24] R.Monti, *Some properties of Carnot-Carathéodory balls in the Heisenberg group*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Mem. s.9 vol.11, 155-167 (2000)

[25] R.Monti, F.Serra Cassano, *Surface measures in Carnot-Carathédory spaces*, to appear in Calc. Var. and P.D.E.

[26] P.Pansu, *Une inégalité isoperimetrique sur le groupe de Heisenberg*, C.R. Acad. Sc. Paris, **295**, Série I, 127-130, (1982)

[27] P.Pansu, *Métriques de Carnot-Carathéodory quasiisométries des espaces symétriques de rang un*, Ann. Math., **129**, 1-60 (1989)

[28] C.Riehm, *The automorphism group of a composition of quadratic forms*, Trans. A.M.S. **269**, 403-414, (1982)

[29] C.Riehm, *Explicit spin representations and Lie algebras of Heisenberg type*, J. London Math. Soc. (2), **29**, 49-62, (1984)

[30] S.D.Pauls, *A notion of rectifiability modelled on Carnot groups*, preprint (2000)

[31] S.D.Pauls, *The Large Scale Geometry of Nilpotent Lie groups*, to appear in Comm. Anal. Geom.

[32] N.Th.Varopoulos, L.Saloff-Coste, T.Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, 1992.

[33] S.K.Vodop'yanov, *$\mathcal{P}$-differentiability on Carnot groups in different topologies and related topics*, Proc.on Analysis and Geometry,603–670, Sobolev Institute Press, Novosibirsk, 2000.