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Absolutely continuous curves in Wasserstein spaces with applications to continuity equation and to nonlinear diffusion equations

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Contents

Introduction

In this PhD Thesis we deal with some aspects of the applications of the theory of optimal mass transportation (and the related Kantorowitch-Rubinstein-Wasserstein distance) to evolution partial differential equations.

Essentially, the Thesis is made up of three parts.

The first part (Chapters 2 and 3) is devoted to the link between the measure valued solutions of the continuity equation, their "probabilistic" representation formula, and the absolutely continuous curves with values in the Wasserstein space $\mathscr{P}_p(X)$. In particular, our main results (see [Lis05]) extend to an arbitrary separable and complete metric space X the previous contributions of [AGS05] (Hilbert spaces) and [LV05a] (locally compact metric spaces), and provide a characterization of the solutions of the continuity equation for a large class of Banach spaces.

In the second part (Chapter 4) we apply the results obtained in the first part and the theory of gradient flows in metric spaces to the study of evolution equations of diffusion type with variable coefficients. We follow the interpretation of these equations, originally suggested by [JKO98], as "gradient flows" of suitable energy functionals in Wasserstein spaces.

The third part (Chapter 5) is devoted to the problem of stability of flows associated to a sequence of non regular vector fields. We prove, in collaboration with Luigi Ambrosio and Giuseppe Savaré [ALS05], a general theorem of stability and we apply it to show the convergence of the iterated composition of optimal transport maps arising from the so called minimizing movements approximation scheme.

Let us now explain in greater detail the main results of the present Thesis, referring to the single sections for more precise definitions, statements, and proofs of the results, as well as for further bibliographical notes.

Continuity equation and absolutely continuous curves in $\mathscr{P}_p(X)$.

Continuity equation in $\mathbb{R}^n$. In order to illustrate the motivation that led us to the study of the absolutely continuous curves in Wasserstein spaces, we start with the principal example: the continuity equation on $\mathbb{R}^n$

$$\partial_t \mu_t + \operatorname{div}(\boldsymbol{v}_t \mu_t) = 0, \qquad \text{in } (0,T)\times\mathbb{R}^n. \tag{1}$$

Here μ_t, $t \in [0,T]$, is a family in $\mathscr{P}_p(\mathbb{R}^n)$, the set of Borel probability measures on $\mathbb{R}^n$ with finite p-moment, i.e. $\int_{\mathbb{R}^n} |x|^p \, d\mu(x) < +\infty$, and $\boldsymbol{v}$ is a Borel velocity vector field $\boldsymbol{v} : (0,T)\times\mathbb{R}^n \to \mathbb{R}^n$ (we always use the notation $\boldsymbol{v}_t(x) := \boldsymbol{v}(t,x)$) satisfying, for $p > 1$, the following integrability condition

$$\int_0^T \int_{\mathbb{R}^n} \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x)\, dt < +\infty, \tag{2}$$

and the equation (1) has to be intended in the sense of distributions. We consider a continuous (in the sense of narrow topology, see Subsection 1.4) time dependent family of elements of $\mathscr{P}_p(\mathbb{R}^n)$, $p > 1$, μ_t, $t \in [0,T]$, which is a solution of (1). We assume, for a moment, that the vector field $\boldsymbol{v}$ is sufficiently regular, in such a way that, for every $x \in \mathbb{R}^n$, there exists a unique global solution of the Cauchy problem

$$\dot{X}_t(x) = \boldsymbol{v}_t(X_t(x)), \qquad X_0(x) = x, \qquad t \in [0,T]. \tag{3}$$

In this case, it is well known that the solution of equation (1) is representable by the formula

$$\mu_t = (X_t)_\# \mu_0. \tag{4}$$

The expression $(X_t)_\#\mu_0$ denotes the push forward of the initial measure μ_0 through the map $X_t : \mathbb{R}^n \to \mathbb{R}^n$, which is defined by $(X_t)_\#\mu_0(B) := \mu_0((X_t)^{-1}(B))$ for every Borel set B of $\mathbb{R}^n$. Taking into account (3) and (2), and using Hölder's inequality, we obtain that for every $s,t \in [0,T]$, with $s < t$,

$$\int_{\mathbb{R}^n} \|X_s(x) - X_t(x)\|^p \, d\mu_0(x) \le (t-s)^{p-1} \int_s^t \int_{\mathbb{R}^n} \|\boldsymbol{v}_r(x)\|^p \, d\mu_r(x)\, dr.$$

Recalling the definition of the p-Wasserstein distance between μ, $\nu \in \mathscr{P}_p(\mathbb{R}^n)$,

$$W_p(\mu,\nu) := \left(\min\left\{ \int_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^p \, d\gamma(x,y) : \gamma \in \Gamma(\mu,\nu) \right\} \right)^{\frac{1}{p}},$$

where $\Gamma(\mu,\nu)$ is the set of Borel probability measures on the product space $\mathbb{R}^n \times \mathbb{R}^n$ having first marginal equal to μ and second marginal equal to ν, and using the admissible measure $\gamma = (X_s(\cdot), X_t(\cdot))_\#\mu_0 \in \Gamma(\mu_s,\mu_t)$ we obtain the inequality

$$W_p^p(\mu_s,\mu_t) \le (t-s)^{p-1} \int_s^t \int_{\mathbb{R}^n} \|\boldsymbol{v}_r(x)\|^p \, d\mu_r(x)\, dr. \tag{5}$$

This last inequality and (2) imply that the curve $t \mapsto \mu_t$ is absolutely continuous in $\mathscr{P}_p(\mathbb{R}^n)$. By Lebesgue differentiation Theorem, we have that

$$|\mu'|^p(t) := \lim_{s\to t} \frac{W_p^p(\mu_s,\mu_t)}{|t-s|^p} \le \int_{\mathbb{R}^n} \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x) \qquad \text{for a.e. } t \in (0,T), \tag{6}$$

and, because of the assumption (2), we obtain that the curve μ_t belongs to the space $AC^p([0,T];\mathscr{P}_p(\mathbb{R}^n))$, i.e. the space of the absolutely continuous curves $\mu:[0,T]\to\mathscr{P}_p(\mathbb{R}^n)$ such that $|\mu'|\in L^p(0,T)$.

In the applications, it is important to work with non regular vector fields $\boldsymbol{v}$, satisfying only (2). In this case, the flow X_t associated to $\boldsymbol{v}$ is not defined, in general, and the representation (4) does not make sense. Nevertheless, another type of representation, strictly linked to the previous one, is possible (see Theorem 3.1): every continuous time dependent Borel probability solution $t\mapsto\mu_t$ of the continuity equation (1) with the vector field $\boldsymbol{v}$ satisfying (2) is representable by means of a Borel probability measure η on the space of continuous functions $C([0,T];\mathbb{R}^n)$. The measure η is concentrated on the set of the curves

$$\{t\mapsto X_t : X_\cdot \text{ is an integral solution of (3) and } \dot{X}_\cdot\in L^p(0,T;\mathbb{R}^n)\}.$$

Now the relation between η and μ_t is given by

$$(e_t)_\#\eta=\mu_t \qquad \forall t\in[0,T], \tag{7}$$

where $e_t : C([0,T];\mathbb{R}^n)\to\mathbb{R}^n$ denotes the evaluation map, defined by $e_t(u(\cdot)):=u(t)$, and the push forward is defined by $(e_t)_\#\eta(B):=\eta(\{u\in C([0,T];\mathbb{R}^n) : u(t)\in B\})$ for every Borel set B of $\mathbb{R}^n$. When (7) holds, we say that η represents the curve μ_t.
Starting from this representation of the solution μ_t of (1) and taking into account the set where η is concentrated, it is not difficult to show that the curve μ_t belongs to the space $AC^p([0,T];\mathscr{P}_p(\mathbb{R}^n))$ and the estimate (6) still holds.

Since, for a given curve $t\in[0,T]\mapsto\mu_t$, solution of (1), there are many Borel vector fields $\boldsymbol{v}$ satisfying (2) such that (1) holds, a natural question arises: does a Borel vector field $\tilde{\boldsymbol{v}}$ exist such that the continuity equation (1) holds, and $\tilde{\boldsymbol{v}}$ is a minimizer of the L^p norm $\int_0^T\int_{\mathbb{R}^n}\|\boldsymbol{v}_t(x)\|^p\,d\mu_t(x)\,dt$? For $p=2$ this problem has a natural physical interpretation: since the norm $\int_{\mathbb{R}^n}\|\boldsymbol{v}_t(x)\|^2\,d\mu_t(x)$ is the kinetic energy at the time t, we are looking for a vector field $\boldsymbol{v}$ which minimizes the action $\int_0^T\int_{\mathbb{R}^n}\|\boldsymbol{v}_t(x)\|^2\,d\mu_t(x)\,dt$. The answer is positive and the minimizers are characterized by the fact that the equality holds in (6). Moreover this minimizing vector field is unique and, thanks to the equality in (6), it plays the role of tangent vector to the curve $t\mapsto\mu_t$.

This result holds also for separable Hilbert spaces ([AGS05] Theorem 8.3.1). In this Thesis we extend it to a suitable class of separable Banach spaces, and to dual of separable Banach spaces, even considering solutions without finite p-moment; both these results are, in fact, a direct consequence of a more general property which holds in arbitrary separable and complete metric spaces and which is interesting by itself. Indeed, recently many papers have appeared dealing with various aspects of measure metric spaces or, in particular, of Riemannian manifolds, strictly connected to Kantorovitch-Rubinstein-Wasserstein distance (see e.g. [LV05a], [LV05b], [Stu05b], [Stu05c], for metric measure spaces, and [WO05], [OV00], [Stu05a], [vRS05], [CEMS01] for Riemannian manifolds).

The case of an arbitrary separable and complete metric space. In a separable and complete metric space X, without additional structure, the notions of velocity vector field and of continuity equation do not make sense, but we still have the notion of (scalar) metric velocity of absolutely continuous curves, which is sufficient to obtain quantitative estimates like (6) as well as for the opposite one.

We recall that, when $u : [0,T] \to X$ is an absolutely continuous curve, for almost every $t \in [0,T]$, the following limit

$$|u'|(t) := \lim_{h \to 0} \frac{d(u(t+h), u(t))}{|h|}$$

exists, and $|u'| \in L^1([0,T])$. We denote by $AC^p([0,T];X)$ the class of absolutely continuous curves, defined on $[0,T]$ with values in X, such that $|u'| \in L^p(0,T)$.

We also recall that on $\mathscr{P}_p(X)$, the set of Borel probability measures on X with finite moment of order p, the p-Wasserstein distance (more properly the p-Kantorowitch-Rubinstein-Wasserstein distance) is defined by

$$W_p(\mu,\nu) := \left(\min \left\{ \int_{X \times X} d^p(x,y)\, d\gamma(x,y) : \gamma \in \Gamma(\mu,\nu) \right\} \right)^{\frac{1}{p}},$$

where $\Gamma(\mu,\nu)$ is the set of Borel probability measures on the product space $X \times X$ having first marginal equal to μ and second marginal equal to ν. $\mathscr{P}_p(X)$ is a separable, complete metric space as well.

Let us try to give a brief account of the metric point of view, describing the main results.

In **Theorem 2.2** we prove that, given a curve $t \mapsto \mu_t$ belonging to $AC^p([0,T];\mathscr{P}_p(X))$, for $p > 1$, there exists a Borel probability measure $\tilde{\eta}$ on the space $C([0,T];X)$ of the continuous curves in X, which is concentrated on the set $AC^p([0,T];X)$. This measure $\tilde{\eta}$ represents the curve μ through the relation

$$(e_t)_{\#}\tilde{\eta} = \mu_t \qquad \forall t \in [0,T]. \tag{8}$$

Moreover, and most importantly, this measure plays the role of a minimal vector field. Indeed, the following inequality holds

$$\int_{C([0,T];X)} |u'|^p(t)\, d\tilde{\eta}(u) \le |\mu'|^p(t) \qquad \text{for a.e. } t \in (0,T). \tag{9}$$

On the other hand, in **Theorem 2.1** we prove that if η is a Borel probability measure on the space $C([0,T];X)$, concentrated on the set $AC^p([0,T];X)$ such that

$$\int_{C([0,T];X)} \int_0^T |u'|^p(t)\, dt\, d\eta(u) < +\infty, \tag{10}$$

then the curve defined by $\mu_t := (e_t)_{\#}\eta$ belongs to $AC^p([0,T];\mathscr{P}_p(X))$ and the opposite inequality holds

$$|\mu'|^p(t) \le \int_{C([0,T];X)} |u'|^p(t)\, d\eta(u) \qquad \text{for a.e. } t \in (0,T).$$

Then the equality holds in (9) and, consequently, the measure $\tilde{\eta}$ satisfies a kind of minimality property. We observe that the condition (10) is the metric counterpart of the integrability condition (2). Notice that Theorem 2.2 is a theorem of representation of curves in $AC^p([0,T];\mathscr{P}_p(X))$ as superposition of curves of the same kind, $AC^p([0,T];X)$, in the space X.

In order to cover the various cases, which are important for the applications, we will also extend Theorem 2.2 in two directions. The first extension is given by **Corollary 2.3** and deals with the case of a pseudo metric space (where the distance d can assume the value $+\infty$, see Remark 1.2). An important example is provided by $\mathscr{P}(X)$, the space of Borel probability measures on X without assumptions on the finiteness of the p-moments, endowed with the p-Wasserstein (pseudo)distance: it is a pseudo metric space when X is not bounded. This extension to pseudo metric spaces is possible since all the results of Theorems 2.1 and 2.2 are expressed in terms of the metric derivative, and this concept involves only the infinitesimal behaviour of the distance along the curve. Then we can substitute the pseudo distance with a topologically equivalent bounded distance and with the same infinitesimal behaviour. It is also interesting to note that if two (topologically equivalent) distances on the same space X induce the same class of absolutely continuous curves and the same metric velocity, then the corresponding Wasserstein pseudo distances on $\mathscr{P}(X)$ enjoy the same property (Corollary 2.4).

The second easy extension, **Corollary 2.6**, concerns a possible application to the case of non separable metric spaces. Considering the set of tight measures (see Remark 1.11) it is not difficult to show that if all the measures μ_t of a curve belonging to $AC^p([0,T];\mathscr{P}(X))$ are tight, then they are supported in a separable closed subspace of X.

Geodesics in $\mathscr{P}_p(X)$. A first application of Theorems 2.1 and 2.2 is given in Section 2.2 and provides a characterization of the geodesics of the metric space $\mathscr{P}_p(X)$ as superposition of geodesics of the metric space X, under the hypothesis that X is a length space (i.e., for every couple x, y of points of X, the distance between x and y is the infimum of the lengths of absolutely continuous curves joining x to y). First of all we show that if X is a length space then $\mathscr{P}_p(X)$ is a length space as well (**Proposition 2.7**). In **Theorem 2.9** we prove the characterization of geodesics of $\mathscr{P}_p(X)$ by a straightforward application of Theorem 2.2, since the geodesics are a particular class of absolutely continuous curves. A similar result was obtained in [LV05a] with the further assumption that the space X is locally compact

(see also [Vil06]).
We point out that a characterization of the geodesics of the space $\mathscr{P}_p(X)$ is also useful in order to prove convexity along geodesics of functionals defined on $\mathscr{P}_p(X)$ (on this important subject see e.g. [McC97] where the convex functionals in $\mathscr{P}_2(\mathbb{R}^n)$ were studied for the first time, and [Stu05a], [LV05a] where the convexity is used in order to give a definition of Ricci curvature bounds in metric measure spaces).

Continuity equation in Banach spaces. Chapter 3 is devoted to the deep link between solutions of the continuity equation with vector fields satisfying a suitable condition of L^p integrability and curves of $AC^p([0,T];\mathscr{P}_p(X))$. In Section 3.1 we study the continuity equation in Banach spaces. The continuity equation in a Banach space X

$$\partial_t \mu_t + \mathrm{div}(\boldsymbol{v}_t \mu_t) = 0 \tag{11}$$

where μ_t, $t \in [0,T]$, is a time dependent continuous family of probability measures in the space X and $\boldsymbol{v} : (0,T) \times X \to X$ is a vector field satisfying the L^p integrability condition

$$\int_0^T \int_X \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x)\, dt < +\infty, \tag{12}$$

is imposed in the duality with smooth functions with bounded (Frechét) differential (we refer to Section 3.1 for the precise notion of solution).
In **Theorem 3.2** we prove that in a separable Banach space X satisfying the Radon-Nicodým property (see Subsection 1.6.2) and in the dual of a separable Banach space, for any curve $\mu \in AC^p([0,T];\mathscr{P}_p(X))$, $p > 1$, there exists a vector field $\tilde{\boldsymbol{v}}$, satisfying (12) and the continuity equation, such that the following inequality (analogous of (9)) holds

$$\int_X \|\tilde{\boldsymbol{v}}_t(x)\|^p \, d\mu_t(x) \le |\mu'|^p(t) \qquad \text{for a.e. } t \in [0,T]. \tag{13}$$

The proof of the existence of a minimal vector field is based on the representation Theorem 2.2 for curves $AC^p([0,T];\mathscr{P}_p(X))$, and on the differentiability (in a strong vector sense) almost everywhere for absolutely continuous curves in Banach spaces having the Radon-Nicodým property.
Thanks to these results, by disintegrating the measure $\tilde{\eta}$, representing the curve μ, with respect to e_t (see Subsection 1.4.3 for the disintegration Theorem) and denoting the disintegrated measures by $\tilde{\eta}_{t,x}$, in Banach spaces having the Radon-Nicodým property it is possible to construct a vector field

$$\tilde{\boldsymbol{v}}_t(x) := \int_{\{u \in C([0,T];X) : u(t) = x\}} \dot{u}(t)\, d\tilde{\eta}_{t,x}(u) \tag{14}$$

satisfying the continuity equation and the inequality (13). The integral in (14) is a Bochner integral and we refer to the proof for the discussion about the good definition and the measurability of the vector field $\tilde{\boldsymbol{v}}$.

With some modifications the proof works also for the dual of a separable Banach space. In this case the absolutely continuous curves, in general, are not differentiable almost everywhere with respect to the strong topology, but are differentiable almost everywhere with respect to the weak-* topology (see Theorem 1.16). However, this weak-* differentiability, together with the metric differentiability, is sufficient to carry out the proof, where now the measurability of a vector field has to be intended in the weak-* sense and the integral (14) is the weak-* integral (see Subsection 1.6.1).

The next relevant question concerns the possibility to prove if equality holds in (13). As in the case of $\mathbb{R}^n$, if every continuous solution μ_t of the continuity equation, with the vector field $\boldsymbol{v}$ satisfying (12), belongs to $AC^p([0,T],\mathscr{P}_p(X))$ and

$$|\mu'|^p(t) \le \int_X \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x) \qquad \text{for a.e. } t \in [0,T],$$

then we can conclude that the equality holds in (13). In **Theorem 3.6** we adopt the strategy of reducing the statement to the finite dimensional case by projecting X on spaces of finite dimension. A crucial assumption here is a property of approximation for the Banach space X, precisely the standard Bounded Approximation Property (see Definition 3.3), which can be suitably modified when X is the dual of a separable Banach space (see Definition 3.4). In Remark 3.5 we give some examples of Banach spaces satisfying these properties. Here we point out that we can apply our results to the space $X = l^\infty = (l^1)^*$, which is particularly relevant since it contains an isometric copy of any separable metric space (see Remark 3.5). The final main result of Section 3.1 is then **Theorem 3.7** which states that in a separable Banach space having Radon-Nicodým property and Bounded Approximation Property (or in the dual of a separable Banach space satisfying a weak-* Bounded Approximation Property) the vector field $\tilde{\boldsymbol{v}}$ defined in (14) realizes the equality in (13) and is of minimal norm among all others vector fields such that the continuity equation and the integrability condition (12) hold. If the norm of X is strictly convex, then the minimal vector field is uniquely determined for μ_t-a.e. $x \in X$ and a.e. $t \in [0,T]$.

As a final observation, we point out that under the assumptions that the separable Banach space X satisfies the Radon-Nicodým property and the Bounded Approximation Property (or X is the dual of a separable Banach space satisfying the weak-* Bounded Approximation Property), using the fact that $\mathscr{P}_p(X)$ is a geodesic space (Proposition 2.7), Theorem 3.6 and the existence of a minimal vector field $\tilde{\boldsymbol{v}}$, we recover the Benamou-Brenier formula ([BB00])

$$W_p^p(\mu,\nu) = \min\left\{ \int_0^1 \int_X \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x)\, dt : (\mu_t, \boldsymbol{v}_t) \in \mathscr{A}(\mu,\nu) \right\}, \tag{15}$$

where $\mathscr{A}(\mu,\nu)$ is the set of the couples $(\mu_t, \boldsymbol{v}_t)$ such that $t \mapsto \mu_t$ is (narrowly) continuous, $\mu_0 = \mu$, $\mu_1 = \nu$, $\boldsymbol{v}_t$ satisfies (12) with $T = 1$, and the continuity equation (11). In Corollary 2.10 we give also a metric version of this formula in geodesic metric spaces.

All the results of Chapter 2 and of Section 3.1 are contained in the paper [Lis05].

The continuity equation in $\mathbb{R}^n$ with a non smooth Riemannian distance. Theorem 3.10 extends the validity of Theorem 3.7 to the case of a (non smooth) Riemannian distance on $\mathbb{R}^n$. Even if we assume that the distance is equivalent to the euclidean one, this is not a consequence of the previous result, since the Wasserstein distance depends on the metric of the space. More precisely, we consider the Riemannian distance

$$d(x,y) = \inf\left\{\int_0^1 \sqrt{\langle G(\gamma(t))\dot\gamma(t),\dot\gamma(t)\rangle}\,dt : \gamma\in AC([0,1];X),\, \gamma(0)=x,\, \gamma(1)=y\right\},$$

induced by a given metric tensor, represented by a symmetric matrix valued function $G : \mathbb{R}^n \to \mathbb{M}_n$ satisfying a uniform ellipticity condition

$$\lambda|\xi|^2 \le \langle G(x)\xi,\xi\rangle \le \Lambda|\xi|^2 \qquad \forall x\in\mathbb{R}^n \quad \forall \xi\in\mathbb{R}^n \tag{16}$$

for some constants $\Lambda, \lambda > 0$, and the following regularity property

$$x \mapsto \langle G(x)\xi,\xi\rangle \qquad \text{is lower semi continuous } \forall\xi\in\mathbb{R}^n. \tag{17}$$

Again, the proof is based on the metric Theorem 2.2 and the property of almost-everywhere differentiability of absolutely continuous curves. We observe that (17) has been assumed only to prove the equality (see Proposition 3.8)

$$|u'|(t) = \sqrt{\langle G(u(t))\dot u(t),\dot u(t)\rangle} \qquad \text{for a.e. } t\in[0,T], \tag{18}$$

for any absolutely continuous curve $u : [0,T]\to\mathbb{R}^n$, when $\mathbb{R}^n$ is endowed with the distance d.

Diffusion equations with variable coefficients.

A suitable Riemannian metric on $\mathbb{R}^n$ arises naturally when we try to interpret the diffusion equations with variable coefficients as “gradient flows” of a suitable energy functional with respect to a Wasserstein distance. Chapter 4 is entirely devoted to this issue.

In the case of the linear Fokker-Planck equation and of the porous media equation, the gradient flow structure with respect to the 2- Wasserstein distance has been pointed out by [JKO98] and [Ott01]. In [JKO98], the seminal paper on this subject, the authors prove that the solutions of the linear Fokker-Planck equation

$$\partial_t u - \mathrm{div}(\nabla u + u\nabla V) = 0, \qquad \text{in } (0,+\infty)\times\mathbb{R}^n, \tag{19}$$

with initial datum $u_0 \ge 0$, $\|u_0\|_{L^1(\mathbb{R}^n)} = 1$, $\int_{\mathbb{R}^n}|x|^2u_0(x)\,dx < +\infty$, can be constructed as limits of the (variational formulation of the) Euler implicit time discretization of the gradient flow induced by the functional

$$\phi(u) := \int_{\mathbb{R}^n} u(x)\log(u(x))\,dx + \int_{\mathbb{R}^n} V(x)u(x)\,dx \tag{20}$$

on the set $\{u \in L^1(\mathbb{R}^n) : u \geq 0, \|u\|_1 = 1, \int_{\mathbb{R}^n} |x|^2 u(x)\,dx < +\infty\}$ (here we identify the probability measures with their densities with respect to the Lebesgue measure $\mathscr{L}^n$) with respect to the 2-Wasserstein distance. Namely, given a time step $\tau > 0$ and an initial datum u_0, they consider the sequence (u^k) obtained by the recursive minimization of

$$u \mapsto \frac{1}{2\tau} W_2^2(u, u^{k-1}) + \phi(u), \quad k = 1, 2, \cdots, \tag{21}$$

with the initial condition $u^0 = u_0$, and the related piecewise constant functions $\overline{U}_{\tau,t} := u^{[t/\tau]}$, $t > 0$, ($[t/\tau]$ denoting the integer part of t/τ) interpolating the values u^k on a uniform grid $\{0, \tau, 2\tau, \cdots, k\tau, \cdots\}$ of step size τ. The authors prove the convergence for $\tau \downarrow 0$ of $\overline{U}_\tau$ to a solution of (19), when V is a smooth non negative function with the gradient satisfying a suitable condition of growth.

This approximation scheme is the minimizing movements approximation scheme introduced in the general setting of metric spaces by De Giorgi [DG93].

For a larger class of equations, the problem of the convergence to the solution of the method described above has attracted a lot of attention in recent years. Related results are treated in [Ott96], [Agu05], [CG03], [CG04]; a comprehensive convergence scheme at the PDE level which encompasses the Otto's method is illustrated in §11.1.3 of [AGS05]. Another interesting paper is [CMV06] where the interpretation of a class of evolution equation as gradient flow allows to deduce new properties of the solutions.

Here we consider the class of nonlinear diffusion equations, with a drift term, of the type

$$\partial_t u - \operatorname{div}(\nabla f(u) + u\nabla V) = 0, \qquad \text{in } (0, +\infty) \times \mathbb{R}^n, \tag{22}$$

with initial datum $u_0 \geq 0$, $u_0 \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} |x|^2 u_0(x)\,dx < +\infty$, where f is a suitable non decreasing function and V a sufficiently regular function. The equation (22) is the gradient flow (see e.g. [AGS05]) of the energy functional

$$\phi(u) := \int_{\mathbb{R}^n} F(u(x))\,dx + \int_{\mathbb{R}^n} V(x)u(x)\,dx \tag{23}$$

defined on $\{u \in L^1(\mathbb{R}^n) : u \geq 0, \|u\|_1 = 1, \int_{\mathbb{R}^n} |x|^2 u(x)\,dx < +\infty\}$, where F is a regular convex function, linked to f by the relation $F'(u)u - F(u) = f(u)$.

In this Thesis we consider the variable coefficients nonlinear diffusion equation with drift

$$\partial_t u_t(x) - \operatorname{div}(A(x)(\nabla f(u_t(x)) + u_t(x)\nabla V(x)) = 0, \tag{24}$$

where A is a symmetric positive definite matrix valued function, depending on the spatial variable $x \in \mathbb{R}^n$ (we use always the notation $u_t(x) := u(t, x)$).

The equation (24) describes linear or nonlinear diffusion with drift in nonhomogeneous and anisotropic material. There are many references for diffusion equations in the fields of mathematical analysis, mathematical physics and of the applications. For instance we can see [Cra75] for diffusion equations, [Ris84] and [Fra05] for linear and non linear Fokker-Planck equations. Diffusion equations with variable coefficients arises also in models of economy [BM00] and in approximation of kinetic models [PT05], or in the linearization of fast diffusion equation [CLMT02].

Under the assumptions that the matrix function A is sufficiently regular (for instance of class C^2), that V is smooth, and $f(u) = u$, the proof of [JKO98] can be modified in order to show that (24) is the gradient flow of the functional (23) with respect to the 2-Wasserstein distance on $\mathbb{R}^n$, where $\mathbb{R}^n$ is endowed with a suitable Riemannian distance d. Precisely the distance induced by the metric tensor $G = A^{-1}$. When A is less regular, for instance only continuous, this kind of proof fails, even if the algorithm (21) is still meaningful. Our purpose is to show, also in the case of low regularity of A, that the minimizing movements algorithm for the functional (23) still converges to a solution of the equation (24). In order to show this, we follow the approach of the theory of curves of maximal slope in metric spaces ([DGMT80], [DMT85], [Amb95], [AGS05]). Combining that theory with our previous result (Theorem 3.10) on the existence of the minimal vector field, we obtain our main result about the diffusion equation with variable coefficients: **Theorem 4.4**. Here $A : \mathbb{R}^n \to \mathbb{M}_n$ is a symmetric matrix valued function, Borel measurable, satisfying a uniform ellipticity condition

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \forall x \in \mathbb{R}^n \quad \forall \xi \in \mathbb{R}^n, \tag{25}$$

such that the inverse matrix function $G := A^{-1}$ satisfies (17), $V : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex lower semi continuous function, bounded from below, whose domain has non empty interior, and $F : [0, +\infty) \to (-\infty, +\infty]$ is a regular convex function satisfying suitable conditions, listed in Chapter 4. The assumptions on F and V ensure that the functional ϕ is convex along geodesics of $\mathscr{P}_2(\mathbb{R}^n)$ with the standard Wasserstein distance induced by the euclidean distance (notice that, in general ϕ is not convex along geodesics of $\mathscr{P}_2(\mathbb{R}^n)$ with the Wasserstein distance induced by the Riemannian distance d). Under these assumptions, the algorithm (21), starting from u_0 in the domain of ϕ, converges (up to sub sequences) to a distributional solution u of the nonlinear diffusion equation (24).

Given a bounded convex open set $\Omega \subset \mathbb{R}^n$, we observe that the function

$$V(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \\ +\infty & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

is an admissible potential. Hence our theorem provides a solution of (24) in $(0, +\infty) \times \Omega$ satisfying (a weak formulation of) the homogeneous Neumann boundary condition $A\nabla f(u_t(x)) \cdot \boldsymbol{n}(x) = 0$ in $(0, +\infty) \times \partial\Omega$.

A related issue is the long time asymptotic behavior for the equation (24). It is important to have conditions on the functional ϕ which ensure the convergence of the solution, as

$t \to \infty$, to the stationary state, when it exists and it is unique. The stationary state, corresponding to a minimum of the functional ϕ, when it exists and it is unique, is the same for the equation (24) and for the same equation with $A = I$. Applying a result of [CMV03] we can prove significant results of asymptotic behaviour for our equation with variable coefficients. When the potential function V is regular, non negative and strictly uniformly convex, i.e. satisfying

$$\nabla^2 V(x) \geq \alpha I \qquad \text{for } \alpha > 0,$$

then there exists a unique minimizer u_∞ for the functional ϕ, which turns out to be a stationary state for the equation (24). In **Theorem 4.5** we prove that there is an exponential rate of convergence for $t \to \infty$ in "relative entropy", more precisely if u_t is a solution of the equation, with initial datum u_0 in the domain of ϕ, given by Theorem 4.4 then

$$\phi(u_t) - \phi(u_\infty) \leq e^{-2\lambda\alpha t}(\phi(u_0) - \phi(u_\infty)) \qquad \forall t \in (0, +\infty), \tag{26}$$

where λ is the ellipticity constant in (25). Moreover there is also exponential rate of convergence in Wasserstein distance

$$W_{2,G}(u_t, u_\infty) \leq e^{-\lambda\alpha t}\sqrt{\frac{2}{\lambda\alpha}(\phi(u_0) - \phi(u_\infty))} \qquad \forall t \in (0, +\infty). \tag{27}$$

In the particular case of linear diffusion, where $F(u) = u \log u$, thanks to Csiszár-Kullback inequality

$$\|u - \tilde{u}\|^2_{L^1(\mathbb{R}^n)} \leq 2(\phi(u) - \phi(\tilde{u})) \tag{28}$$

(see [Csi63], [Kul59] and [UAMT00] for generalized version), we also have convergence in $L^1(\mathbb{R}^n)$ with exponential rate of decay

$$\|u_t - u_\infty\|_{L^1(\mathbb{R}^n)} \leq e^{-\lambda\alpha t}\sqrt{2(\phi(u_0) - \phi(u_\infty))} \qquad \forall t \in (0, +\infty). \tag{29}$$

Rate of convergence in various relative entropies, for the linear Fokker-Planck equation with variable coefficients, are studied in the interesting paper [AMTU01].

Another problem is that of the contractivity of the Wasserstein distance along the solutions of the equation. In recent years, for several classes of evolution equations, this issue has attracted a lot of attention, see e.g. [CMV06], [LT04], [vRS05] in Riemannian manifolds, and [AGS05] also in the abstract metric context.

The problem can be described as follows: given two initial data u_0^1 and u_0^2 in the domain of the functional ϕ, and u^1, u^2 the corresponding solutions of the equation, does a constant $\alpha \in \mathbb{R}$ exist such that

$$W_2(u_t^1, u_t^2) \leq e^{\alpha t} W_2(u_0^1, u_0^2) \qquad \forall t \in (0, +\infty)? \tag{30}$$

Clearly the most interesting case is $\alpha < 0$.

About this problem, for variable coefficients equation, we have only a partial result. In **Theorem 4.6** we prove the contraction inequality (30) only for the linear diffusion and a scalar matrix $A(x) = a(x)I$, under a suitable hypothesis on the coefficients and the potential (see the assumption (4.64)).

All the results of this Chapter are contained in the paper [Lis06].

Convergence of iterated transport maps.

In Chapter 5 (in collaboration with Luigi Ambrosio and Giuseppe Savaré, see [ALS05]) we deal with the problem of the convergence of iterated transport maps arising from the discretization method (21).

In order to illustrating the problem, let us assume that we are given a functional ϕ defined on $\mathscr{P}_2(\mathbb{R}^n)$, whose domain is contained in the set of measures absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^n$ (for example, the internal energy functional (23)). Let us consider the minimizing movement approximation scheme for ϕ, with respect to the 2-Wasserstein distance W_2, introduced in (21). Under quite general assumptions on ϕ, as illustrated in Theorem 5.8, it is possible to prove the existence of the limit $u_t\mathscr{L}^n = \lim_{\tau\downarrow 0} \overline{U}_{\tau,t}\mathscr{L}^n$ in $\mathscr{P}_2(\mathbb{R}^n)$ with respect to W_2, for every time $t \geq 0$. The limit curve $\mu_t = u_t\mathscr{L}^n$ is a solution of the continuity equation for a vector field $\boldsymbol{v}_t \in L^2(\mu_t;\mathbb{R}^n)$ linked to μ_t itself through a nonlinear relation depending on the particular form of the functional ϕ. When a suitable "upper gradient inequality" is satisfied by μ_t, it turns out that $\boldsymbol{v}_t$ is the minimal vector field as in Chapter 3.

Denoting by $\boldsymbol{t}^k$ the unique optimal transport map between $\mu^k = u^k\mathscr{L}^n$ and $\mu^{k+1} = u^{k+1}\mathscr{L}^n$, (see Section 1.5), we consider the *iterated transport map*

$$\boldsymbol{T}^k := \boldsymbol{t}^{k-1} \circ \boldsymbol{t}^{k-2} \circ \cdots \circ \boldsymbol{t}^1 \circ \boldsymbol{t}^0,$$

mapping $\bar{\mu} = \mu^0$ to μ^k. We want to study the convergence of the maps $\boldsymbol{T}_{\tau,t} := \boldsymbol{T}^{[t/\tau]}$ as $\tau \downarrow 0$. A simple formal argument shows that their limit should be $\boldsymbol{X}_t$, where $\boldsymbol{X}_t$ is the flow associated to the minimal vectorfield $\boldsymbol{v}_t$, i.e. a map defined for $\bar{\mu}$-a.e. $x \in \mathbb{R}^n$, as the solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{X}_t(x) = \boldsymbol{v}_t\left(\boldsymbol{X}_t(x)\right), \\ \boldsymbol{X}_0(x) = x. \end{cases} \tag{31}$$

In order to make this intuition precise, and to show the convergence result, we use several auxiliary results, all of them with an independent interest: the first one, proved in Section 5.1, is a general stability result for flows associated to vector fields in the same spirit of the results proved in [DL89], [Amb04b] and based in particular on the Young measure technique in the space of (absolutely) continuous maps adopted in [Amb04b] (see also the Lecture

Notes [Amb04a]). The main new feature here, with respect to the previous results, is that we use the information that the limit vector field $\boldsymbol{v}$ is the minimal vector field, while no regularity is required either on the approximating vector fields or on the approximating flows. More precisely the main stability result for flows, **Theorem 5.1** can be illustrated as follows. Let $(\mu_t^n, \boldsymbol{v}_t^n)$ be a sequence of solutions of the continuity equation and let $(\mu_t, \boldsymbol{v}_t)$ be a solution of the continuity equation, such that the sequence μ_t^n narrowly converges to μ_t and the sequence of the L^p norms of $\boldsymbol{v}_t^n$ converges to the one of $\boldsymbol{v}_t$. We prove that the flows associated to the fields $\boldsymbol{v}_t^n$ converge (in a kind of L^p sense) to the flow associated to $\boldsymbol{v}_t$, under the assumption that $\boldsymbol{v}_t$ is the minimal vector field associated to μ_t and the Cauchy problem (31) admits at most one solution for μ_0-a.e. $x \in \mathbb{R}^n$. Notice that we have not assumed any regularity on the approximating vector fields ensuring the uniqueness of the approximating flows, but only the existence of the flows. In Subsection 5.1.2 we collect some sufficient conditions on the vector field $\boldsymbol{v}$ ensuring that the hypothesis of the stability Theorem are satisfied.

In Section 5.2 we recall from [AGS05] some definitions about the subdifferential formulation of gradient flows of a functional ϕ in $\mathscr{P}_2(\mathbb{R}^n)$ and we prove the general Theorem 5.8 relative to the convergence of the discrete solutions $\overline{M}_\tau$ to the continuous one μ. In Proposition 5.9 we show that the convergence scheme works for the gradient flow of the *internal energy* functional

$$\phi(\mu) := \int_\Omega F(u)\, dx, \quad \text{with} \quad \mu = u\mathscr{L}^d \llcorner \Omega, \tag{32}$$

under the assumption of boundedness of the initial density u_0. We point out that we have not required conditions ensuring the geodesic convexity of ϕ. Namely we have not required the convexity of Ω and the McCann displacement convexity condition on F. The crucial role in the proof is played by an "upper gradient inequality". In this case the gradient flow of ϕ corresponds to a weak formulation of a suitable nonlinear diffusion equation with homogeneous Neumann boundary conditions.

In the final Section 5.3, in **Theorem 5.13**, we prove the convergence, as $\tau \to 0$, of the iterated transport map $\boldsymbol{T}_{\tau,t}$, previously defined, to the flow $\boldsymbol{X}$ associated to $\boldsymbol{v}$ when the Cauchy problem (31), admits at most one solution for μ_0-a.e. $x \in \mathbb{R}^n$. The proof is an application of the stability result in Theorem 5.1.

Finally, in order to the apply Theorem 5.13 to the internal energy functional (the main concrete example) we discuss some sufficient conditions, depending on the regularity of the initial data. This discussion is summarized in **Corollary 5.15**.

In the paper [ALS05] the results about the convergence of the iterated transport map $\boldsymbol{T}_\tau$ are applied to the study of the gradient flow in $L^2(\Omega, \mathbb{R}^n)$ of a particular class of policonvex functionals, solving an open problem of convergence raised in the beautiful paper [ESG05].

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Chapter 1

Preliminaries

In this Chapter we define the main objects we are dealing with and we state some well known results (in general without proof) which are important in the development of the Thesis.
In all this Chapter we denote by I the bounded, closed interval $[0, T]$.

1.1 Absolutely continuous curves in metric spaces and metric derivative

In this Section we give the definition of the class of absolutely continuous curves with values in a general metric space, and we recall a basic result about metric differentiability.
Let (Y, d) be a metric space. We say that a curve $u : I \to Y$ belongs to $AC^p(I; Y)$, $p \geq 1$, if there exists $m \in L^p(I)$ such that

$$d(u(s), u(t)) \leq \int_s^t m(r)\, dr \qquad \forall s, t \in I \quad s \leq t. \tag{1.1}$$

A curve $u \in AC^1(I; Y)$ is called absolutely continuous in Y, and a curve $u \in AC^p(I; Y)$, for $p > 1$, is called absolutely continuous with finite p-energy.
The elements of $AC^p(I; Y)$ satisfy the nice property of a.e. metric differentiability. Precisely we have the following Theorem (see [AGS05] for the proof).

Theorem 1.1. *If $u \in AC^p(I; Y)$, $p \geq 1$, then for $\mathscr{L}^1$-a.e. $t \in I$ there exists the limit*

$$\lim_{h \to 0} \frac{d(u(t+h), u(t))}{|h|}. \tag{1.2}$$

We denote the value of this limit by $|u'|(t)$ and we call it metric derivative *of u at the point t. The function $t \mapsto |u'|(t)$ belongs to $L^p(I)$ and*

$$d(u(s), u(t)) \leq \int_s^t |u'|(r)\, dr \qquad \forall s, t \in I \quad s \leq t.$$

Moreover $|u'|(t) \leq m(t)$ for $\mathscr{L}^1$-a.e. $t \in I$, for each m such that (1.1) holds.

Remark 1.2 (pseudo distance)**.** If $d : Y \times Y \to [0, +\infty]$ satisfies all the usual axioms of the distance but can also assume the value $+\infty$, we call it *pseudo distance* [1] and the space (Y, d) *pseudo metric space*. A pseudo distance induces on Y a topology (the topology generated by the open balls) exactly as a distance and, defining

$$\tilde{d}(x, y) := d(x, y) \wedge 1,$$

the space $\tilde{Y} := (Y, \tilde{d})$ is a bounded metric space, topologically equivalent to Y (see [Bou58]). We observe explicitly that $C(I; Y) = C(I; \tilde{Y})$, the definition of absolutely continuous curves in Y makes sense and Theorem 1.1 holds. Moreover, if $u \in AC^p(I; Y)$ then $d(u(t), u(s)) < +\infty$ for every $s, t \in I$, and the metric derivative of u with respect to $\tilde{d}$ coincides with the metric derivative of u with respect to d. Then it follows that $AC^p(I; Y) = AC^p(I; \tilde{Y})$. □

1.2 $L^p(I; X)$ spaces

In this Section we introduce the metric valued $L^p(I; X)$ spaces and we recall an important compactness criterion in $L^p(I; X)$, proved in [RS03], which is a key tool in the proof of our main result in Chapter 2.

Let (X, d) be a separable, complete metric space. We say that a function $u : I \to X$ belongs to $\mathcal{L}^p(I; X)$, $p \in [1, +\infty)$ if u is Lebesgue measurable and

$$\int_0^T d^p(u(t), \bar{x})\, dt < +\infty$$

for some (and thus for every) $\bar{x} \in X$.
The metric space $L^p(I; X)$ is the space of equivalence class (with respect to the equality a.e.) of functions in $\mathcal{L}^p(I; X)$, endowed with the distance

$$d_p(u, v) := \left(\int_0^T d^p(u(t), v(t))\, dt \right)^{\frac{1}{p}}.$$

Since X is separable and complete, the space $L^p(I; X)$ is separable and complete too.
We recall a useful compactness criterion in $L^p(I; X)$ (it follows from Theorem 2, Proposition 1.10, Remark 1.11 of [RS03], since the last two can be extended to $L^p(I; X)$).

Theorem 1.3. *A family $\mathscr{A} \subset L^p(I; X)$ is precompact if $\mathscr{A}$ is bounded,*

$$\lim_{h \downarrow 0} \sup_{u \in \mathscr{A}} \int_0^{T-h} d^p(u(t+h), u(t))\, dt = 0,$$

[1] Often pseudo distance means a $d : Y \times Y \to [0, +\infty]$ which satisfies all the usual axioms of the distance, except the property $d(x, y) = 0 \Leftrightarrow y = x$. In place of this property, it satisfies only $d(x, x) = 0$. We do not consider this variant. In [Bou58] the application d is called écart.

and there exists a function $\psi : X \to [0,+\infty]$ whose sublevels $\lambda_c(\psi) := \{x \in X : \psi(x) \le c\}$ are compact for every $c \ge 0$, such that

$$\sup_{u\in\mathscr{A}} \int_0^T \psi(u(t))\,dt < +\infty. \tag{1.3}$$

1.3 Metric Sobolev spaces $W^{1,p}(I;X)$

In this Section we give the definition of $W^{1,p}(I;X)$ Sobolev spaces with values in the metric space X.

In the finite dimensional case $X = \mathbb{R}^n$ it is well known that the Sobolev spaces $W^{1,p}(I;\mathbb{R}^n)$, for $p > 1$ can be characterized by

$$\left\{ u \in L^p(I;\mathbb{R}^n) : \sup_{0<h<T} \int_0^{T-h} (\Delta_h u(t))^p\,dt < +\infty \right\},$$

where $\Delta_h u$, for $h \in (0,T)$, denotes the differential quotient

$$\Delta_h u(t) := \frac{|u(t+h) - u(t)|}{h}, \qquad t \in [0, T-h].$$

When X is a separable, complete metric space and $p > 1$, still denoting by $\Delta_h u$, for $h \in (0,T)$, the differential quotient

$$\Delta_h u(t) := \frac{d(u(t+h), u(t))}{h}, \qquad t \in [0, T-h],$$

we can define

$$W^{1,p}(I;X) := \left\{ u \in L^p(I;X) : \sup_{0<h<T} \int_0^{T-h} (\Delta_h u(t))^p\,dt < +\infty \right\}. \tag{1.4}$$

The following Lemma shows that the spaces $AC^p(I;X)$ are strictly linked to the Sobolev spaces $W^{1,p}(I;X)$, as in the well known case $X = \mathbb{R}$.

Lemma 1.4. *Let $p > 1$. If $u \in AC^p(I;X)$ then (the equivalence class of) $u \in W^{1,p}(I;X)$. If $u \in W^{1,p}(I;X)$ then there exists a unique continuous representative $\tilde{u} \in C(I;X)$ (in particular $\tilde{u}(t) = u(t)$ for $\mathscr{L}^1$-a.e. $t \in I$). Moreover $\tilde{u} \in AC^p(I;X)$ and the application $T : W^{1,p}(I;X) \to C(I;X)$ defined by $Tu = \tilde{u}$ is a Borel map.*

Proof. The proof of first assertion can be carried out exactly as in the case $X = \mathbb{R}$ (see for example [Bre83] Proposition VIII.3), by using

$$(\Delta_h u(t))^p \le \frac{1}{h} \int_t^{t+h} |u'|^p(r)\,dr.$$

Now we assume that $u \in W^{1,p}(I;X)$ and we consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ dense in X. Defining $u_n(t) := d(u(t), y_n)$, the triangular inequality implies

$$|u_n(t+h) - u_n(t)| \le d(u(t+h), u(t)). \tag{1.5}$$

The fact that $u \in W^{1,p}(I;X)$ and $p > 1$ implies that $u_n \in W^{1,p}(I)$ again for [Bre83] Proposition VIII.3. Hence there exist $\tilde{u}_n$ absolutely continuous such that $\tilde{u}_n = u_n$ a.e. and $\tilde{u}_n$ is a.e. differentiable.

We introduce the negligible set

$$N = \bigcup_{n \in \mathbb{N}} (\{t \in I : \tilde{u}_n(t) \neq u_n(t)\} \cup \{t \in I : \tilde{u}'_n(t) \text{ do not exists}\}),$$

and we define $m(t) := \sup_n |\tilde{u}'_n(t)|$ for all $t \in I \setminus N$. Clearly, by the density of $\{y_n\}$, we have for all $t, s \in I \setminus N$, with $s < t$,

$$d(u(t), u(s)) = \sup_n |\tilde{u}_n(t) - \tilde{u}_n(s)| \leq \sup_n \int_s^t |\tilde{u}'_n(r)|\, dr \leq \int_s^t m(r)\, dr. \tag{1.6}$$

We show that $m \in L^p(I)$. Actually, by (1.5), if $t \in I \setminus N$ then

$$|\tilde{u}'_n(t)| = \lim_{h \to 0} \frac{|\tilde{u}_n(t+h) - \tilde{u}_n(t)|}{|h|} \leq \liminf_{h \to 0} |\Delta_h u(t)|, \tag{1.7}$$

which implies $m(t) \leq \liminf_{h \to 0} |\Delta_h u(t)|$. By Fatou's Lemma and $u \in W^{1,p}(I;X)$ we obtain

$$\int_0^T |m(t)|^p\, dt \leq C. \tag{1.8}$$

(1.6) and Hölder's inequality show that $u : I \setminus N \to X$ is uniformly continuous, thus, by the completeness of X, it admits a unique continuous extension $\tilde{u} : I \to X$ which also satisfies

$$d(\tilde{u}(t), \tilde{u}(s)) \leq \int_s^t m(r)\, dr \qquad \forall t, s \in I, \quad s < t, \tag{1.9}$$

and then, for (1.8), $\tilde{u} \in AC^p(I;X)$.

We have thus proved that $u \in AC^p(I;X)$ if and only if $u \in C(I;X)$ and $\sup_{0<h<T} \int_0^{1-T} (\Delta_h u(t))^p\, dt < +\infty$.

In order to prove that T is a Borel map, we observe that $W^{1,p}(I;X)$ and $AC^p(I;X)$ are Borel subsets of $L^p(I;X)$ and $C(I;X)$ respectively, since the map

$$u \mapsto \sup_{0<h<T} \int_0^{T-h} \frac{d^p(u(t+h), u(t))}{h^p}\, dt$$

is lower semi continuous from $L^p(I;X)$ to $[0, +\infty]$ and from $C(I;X)$ to $[0, +\infty]$.

Moreover T is an isometry from $(W^{1,p}(I;X), d_p)$ to $(C(I;X), d_p)$ and the thesis follows by observing that the Borel sets of $(C(I;X), d_\infty)$ coincides with the Borel sets of $(C(I;X), d_p)$.

This last assertion is a general fact: if Y is a separable and complete metric space (Polish space) and Y_w is the same space with an Hausdorff topology weaker than the original, then the Borel sets of Y coincides with the Y_w ones (see for instance [Sch73] Corollary 2, pag 101). □

1.4 Borel probability measures, narrow topology and tightness

This Section contains the definition of narrow convergence for sequences of Borel probability measures in a separable metric space and the important Prokhorov criterion of compactness.

Given a separable metric space Y, we denote with $\mathscr{P}(Y)$ the set of Borel probability measures on Y. We say that a sequence $\mu_n \in \mathscr{P}(Y)$ *narrowly converges* to $\mu \in \mathscr{P}(Y)$ if

$$\lim_{n\to+\infty} \int_Y \varphi(y)\, d\mu_n(y) = \int_Y \varphi(y)\, d\mu(y) \qquad \forall \varphi \in C_b(Y), \tag{1.10}$$

where $C_b(Y)$ is the space of continuous bounded real functions defined on Y.

It is well known that the narrow convergence is induced by a distance on $\mathscr{P}(Y)$ (see [AGS05]) and we call *narrow topology* the topology induced by this distance. In particular the compact subsets of $\mathscr{P}(Y)$ coincides with sequentially compact subsets of $\mathscr{P}(Y)$.
We also recall that if $\mu_n \in \mathscr{P}(Y)$ narrowly converges to $\mu \in \mathscr{P}(Y)$ and $\varphi : Y \to (-\infty, +\infty]$ is a lower semi continuous function bounded from below, then

$$\liminf_{n\to+\infty} \int_Y \varphi(y)\, d\mu_n(y) \geq \int_Y \varphi(y)\, d\mu(y). \tag{1.11}$$

Remark 1.5. As showed in [AGS05], (1.10) can be checked only on bounded Lipschitz functions.

A subset $\mathscr{T} \subset \mathscr{P}(Y)$ is said to be tight if

$$\forall \varepsilon > 0 \quad \exists K_\varepsilon \subset Y \text{ compact} : \mu(Y \setminus K_\varepsilon) < \varepsilon \quad \forall \mu \in \mathscr{T}, \tag{1.12}$$

or, equivalently, if there exists a function $\varphi : Y \to [0, +\infty]$ with compact sublevels $\lambda_c(\varphi) := \{y \in Y : \varphi(y) \leq c\}$, such that

$$\sup_{\mu\in\mathscr{T}} \int_Y \varphi(y)\, d\mu(y) < +\infty. \tag{1.13}$$

The importance of tight sets is due to the following Theorem:

Theorem 1.6 (Prokhorov)**.** *Let Y be a separable and complete metric space. $\mathscr{T} \subset \mathscr{P}(Y)$ is tight if and only if it is relatively compact in $\mathscr{P}(Y)$.*

1.4.1 Push forward of measures

In this Subsection we introduce the push forward operator for probability measures.

If Y, Z are separable metric spaces, $\mu \in \mathscr{P}(Y)$ and $F : Y \to Z$ is a Borel map, the *push forward of μ through F*, denoted by $F_\# \mu \in \mathscr{P}(Z)$, is defined as follows:

$$F_\# \mu(B) := \mu(F^{-1}(B)) \qquad \forall B \in \mathscr{B}(Z), \tag{1.14}$$

where $\mathscr{B}(Z)$ is the family of Borel subsets of Z. It is not difficult to check that this definition is equivalent to

$$\int_Z \varphi(z)\, d(F_\# \mu)(z) = \int_Y \varphi(F(y))\, d\mu(y) \tag{1.15}$$

for every bounded Borel function $\varphi : Z \to \mathbb{R}$. More generally (1.15) holds for every $F_\#\mu$-integrable function $\varphi : Z \to \mathbb{R}$. We will often use this fact.
We recall the following composition rule:

$$(G \circ F)_\# \mu = G_\#(F_\# \mu) \quad \forall \mu \in \mathscr{P}(Y), \quad \forall F : Y \to Z, \quad G : Z \to W \quad \text{Borel functions,} \tag{1.16}$$

and the continuity property:

$$F : Y \to Z \quad \text{is continuous} \quad \Longrightarrow \quad F_\# : \mathscr{P}(Y) \to \mathscr{P}(Z) \quad \text{is narrowly continuous.} \tag{1.17}$$

1.4.2 Convergence in measure

This Subsection is devoted to the link between convergence in measure for maps and narrow convergence of the measures associated to the maps, as in the theory of Young measures. This link is a useful tool in the proof of a stability Theorem in Section 5.1.

Let X, Y be metric spaces and $\mu \in \mathscr{P}(X)$. A sequence of Borel maps $f_n : X \to Y$ is said to converge in μ-measure to $f : X \to Y$ if

$$\lim_{n\to\infty} \mu\left(\{x \in X : d_Y(f_n(x), f(x)) > \delta\}\right) = 0 \qquad \forall \delta > 0.$$

This is equivalent to the $L^1(\mu)$ convergence to 0 of the maps $1 \wedge d_Y(f_n, f)$. It is also well known that if $Y = \mathbb{R}$ and $|f_n|^p$ is equi-integrable, then $f_n \to f$ in μ-measure if and only if $f_n \to f$ in $L^p(\mu)$.
The following Lemma will be used in the proof of Theorem 5.1.

Lemma 1.7. *Let $f_n, f : X \to Y$ be Borel maps and let $\mu \in \mathscr{P}(X)$. Then $f_n \to f$ in μ-measure if and only if $(\boldsymbol{i}, f_n)_\# \mu$ converges to $(\boldsymbol{i}, f)_\# \mu$ narrowly in $\mathscr{P}(X \times Y)$.*

Proof. If $f_n \to f$ in μ-measure then $\varphi(x, f_n(x))$ converges in $L^1(\mu)$ to $\varphi(x, f(x))$, and therefore thanks to (1.15) we immediately obtain the convergence of the push-forward.

Conversely, let $\delta > 0$ and, for any $\varepsilon > 0$, let $w \in C_b(X; Y)$ such that $\mu(\{f \neq w\}) \leq \varepsilon$. We define

$$\varphi(x, y) := 1 \wedge \frac{d_Y(y, w(x))}{\delta} \in C_b(X \times Y)$$

and notice that

$$\int_{X\times Y} \varphi \, d(\boldsymbol{i}, f_n)_\# \mu \geq \mu(\{d_Y(w, f_n) > \delta\}), \quad \int_{X\times Y} \varphi \, d(\boldsymbol{i}, f)_\# \mu \leq \mu(\{w \neq f\}).$$

Taking into account the narrow convergence of the push-forward we obtain that

$$\limsup_{n\to\infty} \mu(\{d_Y(f,f_n) > \delta\}) \le \limsup_{n\to\infty} \mu(\{d_Y(w,f_n) > \delta\}) + \mu(\{w \ne f\}) \le 2\mu(\{w \ne f\}) \le 2\varepsilon$$

and since ε is arbitrary the proof is achieved. □

Lemma 1.8. *Let $f : X \to Y$ be a Borel map, $\mu \in \mathscr{P}(X)$, and let $\boldsymbol{v} \in L^p(\mu;\mathbb{R}^d)$ for some $p \in (1,+\infty)$. Then, setting $\nu = f_\#\mu$, we have $f_\#(\boldsymbol{v}\mu) = \boldsymbol{w}\nu$ for some $\boldsymbol{w} \in L^p(\nu;\mathbb{R}^d)$ with*

$$\|\boldsymbol{w}\|_{L^p(\nu;\mathbb{R}^d)} \le \|\boldsymbol{v}\|_{L^p(\mu;\mathbb{R}^d)}. \tag{1.18}$$

In case of equality we have $\boldsymbol{v} = \boldsymbol{w} \circ f$ μ-a.e. in X.

Proof. Let q be the dual exponent of p, $\boldsymbol{\nu} := f_\#(\boldsymbol{v}\mu)$, and $\varphi \in L^\infty(Y;\mathbb{R}^d)$; denoting by ν^α, $\alpha = 1, \cdots, d$, the components of $\boldsymbol{\nu}$ we have

$$\left|\sum_{\alpha=1}^d \int_Y \varphi^\alpha \, d\nu^\alpha\right| = \left|\sum_{\alpha=1}^d \int_X (\varphi^\alpha \circ f)\, \boldsymbol{v}^\alpha \, d\mu\right| \le \|\varphi \circ f\|_{L^q(\mu;\mathbb{R}^d)} \|\boldsymbol{v}\|_{L^p(\mu;\mathbb{R}^d)} = \|\varphi\|_{L^q(\nu;\mathbb{R}^d)} \|\boldsymbol{v}\|_{L^p(\mu;\mathbb{R}^d)}.$$

Since φ is arbitrary this proves (1.18) and, as a consequence, the same identities above hold when $\varphi \in L^q(\nu;\mathbb{R}^d)$. In case of equality it suffices to choose $\varphi = |\boldsymbol{w}|^{p-2}\boldsymbol{w}$ to obtain that $\boldsymbol{v}$ coincides with $|\varphi \circ f|^{q-2}(\varphi \circ f) = \boldsymbol{w} \circ f$ μ-a.e. in X. □

1.4.3 Disintegration theorem

In this Subsection we recall the so-called disintegration theorem for probability measures. It is a classical result in probability theory. The proof can be found, for instance, in [DM78] and in [AFP00]. This Theorem is an important tool in Chapters 3 and 4.

Let X, Y be separable metric spaces. A measure valued map $y \in Y \mapsto \mu_y \in \mathscr{P}(X)$ is said to be a Borel map if $y \mapsto \mu_y(B)$ is a Borel map for any Borel set $B \in \mathscr{B}(X)$. Moreover a monotone class argument implies that

$$y \in Y \mapsto \int_X f(x)\, d\mu_y(x)$$

is a Borel map for every Borel map $f : X \to [0,+\infty]$.

Theorem 1.9. *Let X, Y be Polish spaces, $\mu \in \mathscr{P}(X)$, let $\pi : X \to Y$ be a Borel map and $\nu := \pi_\#\mu \in \mathscr{P}(Y)$. Then there exists a ν-a.e. uniquely determined Borel family of probability measures $\{\mu_y\}_{y\in Y} \subset \mathscr{P}(X)$ such that μ_y is concentrated on $\pi^{-1}(y)$ for ν-a.e. $y \in Y$, and*

$$\int_X f(x)\, d\mu(x) = \int_Y \left(\int_X f(x) d\mu_y(x)\right) d\nu(y) \tag{1.19}$$

for any Borel map $f : X \to [0,+\infty]$.

1.5 Kantorovitch-Rubinstein-Wasserstein distance

In this Section we give the definition of Wasserstein distance (more precisely Kantorovitch-Rubinstein-Wasserstein distance) between Borel probability measures on a metric space.

Let (X, d) be a separable and complete metric space. We fix $p \geq 1$ and denote by $\mathscr{P}_p(X)$ the space of Borel probability measures having finite p-moment, i.e.

$$\mathscr{P}_p(X) = \left\{ \mu \in \mathscr{P}(X) : \int_X d^p(x, x_0)\, d\mu(x) < +\infty \right\}, \tag{1.20}$$

where $x_0 \in X$ is an arbitrary point of X (clearly this definition does not depend on the choice of x_0). Notice also that this condition is always satisfied if the diameter of X is finite; in this case $\mathscr{P}_p(X) = \mathscr{P}(X)$.

Given $\mu, \nu \in \mathscr{P}(X)$ we define the set of admissible plans $\Gamma(\mu, \nu)$ as follows:

$$\Gamma(\mu, \nu) := \{\gamma \in \mathscr{P}(X \times X) : \pi^1_\# \gamma = \mu,\ \pi^2_\# \gamma = \nu\},$$

where $\pi^1(x, y) := x$ and $\pi^2(x, y) := y$ are the projections on the first and the second component respectively.

The p-Kantorovitch-Rubinstein-Wasserstein distance between $\mu, \nu \in \mathscr{P}_p(X)$ is defined by

$$W_p(\mu, \nu) := \left(\min \left\{ \int_{X \times X} d^p(x, y)\, d\gamma(x, y) : \gamma \in \Gamma(\mu, \nu) \right\} \right)^{\frac{1}{p}}. \tag{1.21}$$

Since $\Gamma(\mu, \nu)$ is tight and $\mu \otimes \nu \in \Gamma(\mu, \nu)$ satisfies $\int_{X \times X} d^p(x, y)\, d\mu \otimes \nu(x, y) < +\infty$, the existence of the minimum, in the above definition, is a consequence of standard Direct Methods in Calculus of Variations.

We denote by

$$\Gamma_o(\mu, \nu) := \left\{ \gamma \in \Gamma(\mu, \nu) : \int_{X \times X} d^p(x, y)\, d\gamma(x, y) = W_p^p(\mu, \nu) \right\}$$

the set of optimal plans.

Being X separable and complete, $\mathscr{P}_p(X)$, endowed with the distance W_p, is a separable and complete metric space too. We refer to [Vil03] and [AGS05] for the proof and for a characterization of convergence and compactness in $\mathscr{P}_p(X)$. Here we only recall that, given a sequence $\mu_n \in \mathscr{P}_p(X)$ and $\mu \in \mathscr{P}_p(X)$,

$$\lim_{n \to \infty} W_p(\mu_n, \mu) = 0 \iff \begin{cases} \mu_n \text{ narrowly converge to } \mu, \\ \displaystyle\lim_{n \to \infty} \int_X d^p(x, x_0)\, d\mu_n(x) = \int_X d^p(x, x_0)\, d\mu(x). \end{cases} \tag{1.22}$$

We conclude this section by recalling a few basic facts from the theory of optimal transportation (see for instance [GM96], [Vil03], [Eva99], [AGS05] for much more on this subject).

In the case when $\mu \in \mathscr{P}_p^r(\mathbb{R}^n)$, the subset of $\mathscr{P}_p(\mathbb{R}^n)$ made of absolutely continuous measures with respect to Lebesgue measure $\mathscr{L}^n$, it can be shown [Bre91, GM96] that the minimum problem (1.21) has a unique solution γ, and γ is induced by a transport map $\boldsymbol{t}:\mathbb{R}^n \to \mathbb{R}^n$:

$$\gamma = (\boldsymbol{i}, \boldsymbol{t})_{\#}\mu.$$

In particular $\boldsymbol{t}$ is the unique solution of Monge's optimal transport problem

$$\min\left\{\int_{\mathbb{R}^n} |\boldsymbol{r} - \boldsymbol{i}|^p \, d\mu : \ \boldsymbol{r}_{\#}\mu = \nu\right\},$$

of which (1.21) is a relaxed version. Finally, we recall that if also $\nu \in \mathscr{P}^r(\mathbb{R}^n)$, then

$$\boldsymbol{s} \circ \boldsymbol{t} = \boldsymbol{i} \quad \mu\text{-a.e.} \quad \text{and} \quad \boldsymbol{t} \circ \boldsymbol{s} = \boldsymbol{i} \quad \nu\text{-a.e.},$$

where $\boldsymbol{s}$ is the optimal transport map between ν and μ.

The following Remarks are important for generalizations of our main Theorem of Chapter 2.

Remark 1.10 (W_p is a pseudo distance on $\mathscr{P}(X)$)**.** If we take $\mu, \nu \in \mathscr{P}(X)$, without hypothesis on the moments, when X is unbounded, $W_p(\mu, \nu)$ can be equal to $+\infty$ and W_p is a pseudo distance on $\mathscr{P}(X)$, according to Remark 1.2. More generally, taking a pseudo distance d on X, the space $\mathscr{P}(X)$, endowed with the pseudo distance W_p, is a pseudo metric space. □

Remark 1.11 (Non separable spaces)**.** It is possible to work with complete metric spaces, not necessarily separable, considering the set of tight measures. Let X be a complete metric space, we define the set of tight probability measures (see e.g. [Par67])

$$\mathscr{P}^\tau(X) := \{\mu \in \mathscr{P}(X) : \mu \text{ is tight}\}.$$

We have that $\mu \in \mathscr{P}^\tau(X)$ if and only if $\operatorname{supp}\mu$ is separable. Indeed if μ is tight there exists a sequence of compacts K_n such that $\mu(X \setminus K_n) < \frac{1}{n}$ from which it follows that $\operatorname{supp}\mu \subset \overline{\cup_{n\in\mathbb{N}} K_n}$ and then is separable. The other implication follows by Prokorov's Theorem.
On the other hand if $\mu_n \in \mathscr{P}^\tau(X)$ narrowly converges in $\mathscr{P}(X)$ to μ then $\mu \in \mathscr{P}^\tau(X)$ because $\operatorname{supp}\mu \subset \overline{\cup_{n\in\mathbb{N}} \operatorname{supp}\mu_n}$ (see e.g. Proposition 5.1.8 of [AGS05]).
In the space $\mathscr{P}^\tau(X)$, W_p is a complete pseudo distance. The completeness can be proved as in [AGS05] by observing that a Cauchy sequence μ_n with respect to W_p in $\mathscr{P}^\tau(X)$ can be considered in $\mathscr{P}(\overline{\cup_{n\in\mathbb{N}} \operatorname{supp}\mu_n})$ and $\overline{\cup_{n\in\mathbb{N}} \operatorname{supp}\mu_n}$ is a complete separable space. □

1.6 Absolutely continuous curves in Banach Spaces

This Section is devoted to the particular case of absolutely continuous curves in Banach spaces.

1.6.1 Bochner and weak-* integral

We assume that the reader is familiar with the concept of measurable vector valued function (see e.g. [Yos95] or [DU77]). We recall the following useful Theorem.

Theorem 1.12 (Pettis)**.** *Let X be a separable Banach space, (Ω, Σ, μ) be a measure space, and $\phi : \Omega \to X$ a function. ϕ is μ measurable if and only if for every $f \in X^*$ the functions $\varphi_f : \Omega \to \mathbb{R}$ defined by $\varphi_f(\omega) := \langle f, \phi(\omega)\rangle$, are μ measurable.*

We say that a μ measurable function $\phi : \Omega \to X$ is Bochner integrable if there exists a sequence of simple functions ϕ_n such that $\lim_n \int_\Omega \|\phi_n - \phi\| \, d\mu = 0$ and we define

$$\int_A \phi \, d\mu := \lim_n \int_A \phi \, d\mu \qquad \forall A \in \Sigma,$$

where the value of the integral does not depend on the approximating sequence. We recall that a μ measurable function $\phi : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|\phi\| \, d\mu < +\infty$.

We say that $\phi \in L^p(\mu; X)$ if ϕ is μ Bochner integrable and $\int_\Omega \|\phi\|^p \, d\mu < +\infty$.

Among all the basic properties of Bochner integral we recall the useful

Theorem 1.13. *Let ϕ be Bochner integrable on $I = [0, T]$ with respect to Lebesgue measure. Then for almost every $s \in I$ one has*

$$\lim_{h \to 0} \frac{1}{h} \int_s^{s+h} \|\phi(t) - \phi(s)\| \, dt = 0,$$

and consequently for almost every $s \in I$,

$$\lim_{h \to 0} \frac{1}{h} \int_s^{s+h} \phi(t) \, dt = \phi(s).$$

Let $X = E^*$ the dual of a separable Banach space E. A function $\phi : \Omega \to X$ is said to be weakly-* μ-measurable if for every $f \in E$ the maps $\varphi_f(\omega) := \langle \phi(\omega), f\rangle$ are μ-measurable. If $\varphi_f \in L^1(\mu)$ for every $f \in E$ then ϕ is said to be weakly-* μ-integrable (or Gelfand integrable) and, by a simple application of closed graph theorem, for every $A \in \Sigma$ we define $\int_A \phi \, d\mu \in E^*$ by

$$\langle \int_A \phi \, d\mu, f\rangle = \int_A \langle \phi, f\rangle \, d\mu.$$

We say that $\phi \in L^p_{w*}(\mu; X)$ if ϕ is weakly-* μ integrable and $\int_\Omega \|\phi\|^p \, d\mu < +\infty$.

1.6.2 Radon-Nicodým property

Definition 1.14. Let Σ be a σ-algebra of the set Ω. A vector measure with values in a Banach space X is a countably additive application $F : \Sigma \to X$. A vector measure F is said to be absolutely continuous with respect to a nonnegative finite real valued measure μ on Σ if $\lim_{\mu(B)\to 0} F(B) = 0$. The variation of a vector measure is defined by

$$|F|(B) = \sup\{\sum_{i=1}^{n} \|E_i\| : \{E_i\} \subset \Sigma \text{ finite partition of } B\}.$$

The vector measure is said to be of bounded variation if $|F|(\Omega) < +\infty$.

Definition 1.15. A Banach space X has the Radon-Nicodým property with respect to a measure space (Ω, Σ, μ) if for every vector measure $F : \Sigma \to X$ of bounded variation, absolutely continuous with respect to μ there exists $\phi \in L^1(\mu; X)$ such that

$$F(B) = \int_B \phi \, d\mu \qquad \forall B \in \Sigma.$$

A Banach space X has the Radon-Nicodým property if X has the Radon-Nicodým property with respect to every finite measure space.

We recall that if X has the Radon-Nicodým property with respect to a Lebesgue measure on $[0, 1]$ then X has the Radon-Nicodým property. Other equivalent formulations of the Radon-Nicodým property and many examples of spaces that have or don't have Radon-Nicodým property can be founded in [DU77]. Here we recall only that reflexive Banach spaces, separable dual spaces, l^1, $L^p(\mu; X)$ for $p \in (1, +\infty)$ when X has the Radon-Nicodým property, have the Radon-Nikodým property. The spaces $L^1(\mu)$, with μ not purely atomic, l^∞, $L^\infty([0, 1])$, $C(\overline{\Omega})$ where Ω is a bounded open subset of $\mathbb{R}^n$, do not have the Radon-Nicodým property.

1.6.3 Characterization of absolutely continuous curves in Banach spaces

It is well known that a real valued curve $u : I \to \mathbb{R}$ is absolutely continuous if and only if u is $\mathscr{L}^1$-a.e. differentiable in I, $\dot{u} \in L^1(I)$ and $u(t) - u(s) = \int_s^t \dot{u}(r) \, dr$ for every $s, t \in I$, $s \leq t$. For curves with values in a Banach space, in general this characterizations is not true (see Example 1.17 below). If the Banach space have the Radon-Nikodým property then this characterization is still true. A similar result holds in the dual space of a separable Banach space, with the difference that the absolutely continuous curves are only weakly-* differentiable and the integrals are weak-* integrals.

Theorem 1.16. *Let X be a Banach space satisfying the Radon-Nikodým property (resp. X is the dual of a separable Banach space) and $p \geq 1$. A curve $u \in AC^p(I; X)$ if and only if*

(i) u is differentiable for a.e. $t \in I$, (resp. u is weakly-$*$ differentiable for a.e. $t \in I$)

(ii) $\dot{u}(t) := \lim_{h\to 0} \frac{u(t+h)-u(t)}{h} \in L^p(I; X)$,
(resp. $\dot{u}(t) := w^* - \lim_{h\to 0} \frac{u(t+h)-u(t)}{h} \in L^p_{w*}(I; X)$)

(iii) $u(t) - u(s) = \int_s^t \dot{u}(r)\,dr \qquad \forall s, t \in I, \quad s \le t.$

Moreover we have

$$\|\dot{u}(t)\| = |u'|(t) \qquad \text{for a.e. } t \in I. \tag{1.23}$$

Proof. We consider the case of X satisfying the Radon-Nikodým property. Assume that $u \in AC^p(I;X)$. Given an interval $[a,b] \subset I$ we set

$$F([a,b]) := u(b) - u(a)$$

and we extend F to a vector measure with bounded total variation defined in $\mathscr{B}(I)$ absolutely continuous with respect to the Lebesgue measure. We denote the extension still by F. By the Radon-Nikodým property there exists a Bochner integrable function $\phi : I \to X$ such that

$$F(B) = \int_B \phi(t)\,dt, \qquad \forall B \in \mathscr{B}(I),$$

in particular $u(t) - u(s) = \int_s^t \phi(r)\,dr$. Then, recalling Theorem 1.13, (i) holds and $\dot{u}(t) = \phi(t)$ for a.e. $t \in I$, and consequently (iii) holds. On the other hand, for every point $t \in I$ where $\dot{u}(t)$ exists, by the continuity of the norm we have

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}\right\| = \left\|\lim_{h\to 0}\frac{u(t+h)-u(t)}{h}\right\|,$$

and (1.23) holds. Now (ii) follows from (1.23), since $|u'|(\cdot) \in L^p(I)$.

Now we consider the case of $X = E^*$. For every $f \in E$ we define $\varphi_f : I \to \mathbb{R}$ by

$$\varphi_f(t) := \langle u(t), f\rangle.$$

Clearly, $\varphi_f \in AC^p(I;\mathbb{R})$. Taking a countable dense set $\{f_n\}_{n\in\mathbb{N}} \subset E$, we can see, without difficult, that for $\mathscr{L}^1$-a.e. $t \in I$ there exists $\dot{\varphi}_{f_n}(t)$ and $|u'|(t) = \lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}\right\| < +\infty$. For every t as above we define the continuous linear functional $L_t : E \to \mathbb{R}$ by continuous extension of

$$L_t f_n := \dot{\varphi}_{f_n}(t),$$

since $|L_t f_n| \le |u'|(t)\,\|f_n\|$. Then $\|L_t\| \le |u'|(t)$.

We show that $L_t = \dot{u}(t)$. Indeed for every $\varepsilon > 0$ there exists f_n such that $\|f_n - f\| < \varepsilon$, and

$$\begin{aligned}
\left|\left\langle \frac{u(t+h)-u(t)}{h} - L_t, f\right\rangle\right| &\le \left|\left\langle \frac{u(t+h)-u(t)}{h} - L_t, f - f_n\right\rangle\right| + \\
&\qquad + \left|\left\langle \frac{u(t+h)-u(t)}{h} - L_t, f_n\right\rangle\right| \\
&\le \left(\left\|\frac{u(t+h)-u(t)}{h}\right\| + |u'|(t)\right)\|f_n - f\| + \\
&\qquad + \left|\left\langle \frac{u(t+h)-u(t)}{h} - L_t, f_n\right\rangle\right|,
\end{aligned}$$

from which it follows that

$$\limsup_{h\to 0}\left|\left\langle \frac{u(t+h)-u(t)}{h} - L_t, f\right\rangle\right| \le 2|u'|(t)\varepsilon.$$

Then (i) is proved and

$$\|\dot u(t)\| \le |u'|(t) \qquad \text{for a.e. } t\in I. \tag{1.24}$$

The function $\dot u : I \to X$ is $\mathscr{L}^1$ w-* measurable, since for every $f \in E$, $\langle \dot u(t), f\rangle = \lim_{h\to 0}\left\langle \frac{u(t+h)-u(t)}{h}, f\right\rangle$ which is limit of continuous functions. Now (ii) follows from (1.24). We prove (iii). By the definition of the weak-* integral and the absolute continuity of φ_f we have

$$\begin{aligned}\langle u(t)-u(s), f_n\rangle &= \varphi_{f_n}(t)-\varphi_{f_n}(s) = \int_s^t \dot\varphi_{f_n}(r)\,dr\\ &= \int_s^t \langle \dot u(r), f_n\rangle\,dr = \langle \int_s^t \dot u(r)\,dr, f_n\rangle.\end{aligned}$$

Finally, observing that $\left\|\frac{u(t+h)-u(t)}{h}\right\| \le \left|\frac{1}{h}\int_t^{t+h}\|\dot u(r)\|\,dr\right|$, then for every Lebesgue point $t\in I$ of $\|\dot u(\cdot)\|$ and every $t \in I$ such that $|u'|(t)$ exists, we have

$$|u'|(t) \le \|\dot u(t)\|$$

and (1.23) holds by (1.24).

Conversely, in both cases, if (i), (ii), (iii) hold then is obvious that $u \in AC^p(I;X)$. □

Example 1.17. In $X = L^1(0,1)$ we consider the curve $u : [0,1] \to X$ defined by

$$u(t) = \chi_{(0,t)}.$$

Clearly $\|u(t)-u(s)\|_{L^1} \le |t-s|$ but the differential quotients do not converge in X. This is a further proof that $L^1(0,1)$ do not have the Radon-Nikodým property.

On the other hand we consider the same sequence in $C([0,1])^*$, identifying $L^1(0,1)$ with a subset of $C([0,1])^*$. The curve is then defined by

$$\langle u(t), \varphi\rangle = \int_0^1 \chi_{(0,t)}(x)\varphi(x)\,dx \qquad \forall\,\varphi\in C([0,1]).$$

Also in this case, we have that $\|u(t)-u(s)\|_{C([0,1])^*} \le |t-s|$, and

$$\left\langle \frac{u(t+h)-u(t)}{h}, \varphi\right\rangle \to \varphi(t) = \langle \delta_t, \varphi\rangle$$

but the differential quotients do not strongly converge. □

1.7 Curves of maximal slope in metric spaces

In this Section we give a brief account of the theory of curves of maximal slope for functionals in metric spaces and the related minimizing movements approximation scheme. We recall

a general theorem of existence for curves of maximal slope proved in [AGS05] following the ideas of [DGMT80], [DMT85] and [Amb95]. We will apply this theorem in Chapter 4, in order to prove existence and approximation of solutions for a large class of variable coefficients diffusion equations. These solutions are interpreted as curve of maximal slope for a suitable energy functional in the metric space of the probability measures $\mathscr{P}_2(\mathbb{R}^n)$, where the metric on $\mathbb{R}^n$ is a suitable Riemannian distance which depends on the coefficients of the equation.
In this Section we follow the presentation of [AGS05].

Let X be a complete metric space and $\phi : X \to (-\infty, +\infty]$ be an extended real functional. We denote by

$$D(\phi) := \{x \in X : \phi(x) \in \mathbb{R}\}$$

the effective domain of ϕ. We say that the functional ϕ is proper if $D(\phi)$ is not empty (i.e. ϕ is not identically $+\infty$).

The following definition is a kind of "modulus of the gradient" for functionals defined in metric spaces.

Definition 1.18 (Upper gradient)**.** A Borel function $g : X \to [0, +\infty]$ is called a *strong upper gradient* for the proper function ϕ if, for every $u \in AC^2_{\text{loc}}(I; X)$ such that $g(u)|u'| \in L^1_{\text{loc}}(I)$ we have

$$|\phi(u(t)) - \phi(u(s))| \le \int_s^t g(u(r))|u'|(r)\, dr \qquad \forall s, t \in I, \quad s < t. \tag{1.25}$$

In particular, if $g(u)|u'| \in L^1_{\text{loc}}(I)$ then $\phi \circ u$ is locally absolutely continuous and

$$\left| \frac{d}{dt} \phi(u(t)) \right| \le g(u(t))|u'|(t) \qquad \text{for a.e. } t \in I. \tag{1.26}$$

As showed in [AGS05], if ϕ is lower semi continuous, an important example of upper gradient is given by the *global slope* defined as follows:

$$S_\phi(x) := \sup_{y \neq x} \frac{(\phi(x) - \phi(y))^+}{d(x, y)} \qquad \forall x \in D(\phi). \tag{1.27}$$

For our purposes, it is important to work with an other concept of slope, the *local slope* defined by:

$$|\partial \phi|(x) := \limsup_{y \to x} \frac{(\phi(x) - \phi(y))^+}{d(x, y)} \qquad \forall x \in D(\phi). \tag{1.28}$$

In general the local slope is not an upper gradient, according to Definition 1.18, but it satisfies a weaker definition of upper gradient as showed in [AGS05]. However we do not need this definition.

The notions of metric derivative and upper gradient allow to define the concept of curve of maximal slope in metric spaces.

Definition 1.19 (Curve of maximal slope)**.** We say that $u : [0, +\infty) \to X$ is a curve of maximal slope for ϕ, with respect to the strong upper gradient g, if $u \in AC^2_{\mathrm{loc}}((0, +\infty); X)$ and

$$\frac{1}{2}\int_s^t g^2(u(r))\,dr + \frac{1}{2}\int_s^t |u'|^2(r)\,dr = \phi(u(s)) - \phi(u(t)) \qquad \forall s, t \in [0, +\infty), \quad s < t. \tag{1.29}$$

The equality (1.29) is called *energy identity*. We observe that (1.29) shows that $g \circ u \in L^2_{\mathrm{loc}}(0, +\infty)$. Moreover, with our definition of upper gradient, it is sufficient to prove the inequality $\leq$ in (1.29) in order to show that u is a curve of maximal slope for ϕ.

1.7.1 The minimizing movements approximation scheme

The existence of curves of maximal slope can be proved by means of the recursive algorithm of *minimizing movements*. This algorithm, introduced in the general setting of metric spaces by De Giorgi [DG93], yields one way to construct approximate solutions (see [Amb95] and [AGS05]).

Given a time step $\tau > 0$ and $u^0 \in D(\phi)$ we define recursively a sequence in X setting

$$U^0_\tau = u^0; \qquad U^n_\tau \text{ is a minimizer in } X \text{ of the function } V \mapsto \frac{1}{2\tau} d^2(V, U^{n-1}_\tau) + \phi(V). \tag{1.30}$$

The definition of the sequence U^n_τ is based on the existence of minimizers of the function $v \mapsto \frac{1}{2\tau} d^2(v, u) + \phi(v)$ for a fixed $u \in X$. In order to ensuring this existence, the following set of general hypothesis is sufficient.

(H1) On the separable complete metric space X we assume that there exists an Hausdorff topology σ on X, weaker than the topology induced by d, such that

$$x_n \rightharpoonup x, \quad y_n \rightharpoonup y \quad \Rightarrow \quad \liminf_n d(x_n, y_n) \geq d(x, y), \tag{1.31}$$

where we indicated by $\rightharpoonup$ the convergence with respect to σ.

(H2) $\phi : X \to (-\infty, +\infty]$ is proper and σ lower semi continuous on bounded sequences:

$$\sup_{n,m} d(x_n, x_m) < +\infty, \quad x_n \rightharpoonup x \quad \Rightarrow \quad \liminf_n \phi(x_n) \geq \phi(x). \tag{1.32}$$

(H3) there exist $\bar{\tau} > 0$ and $\bar{x} \in X$ such that

$$\inf_{y \in X} \{\phi(y) + \frac{1}{2\bar{\tau}} d^2(\bar{x}, y)\} > -\infty. \tag{1.33}$$

(H4) every bounded set contained in a sublevel set of ϕ is σ sequentially precompact, i.e. if $x_n \in X$ is a sequence such that $\sup_{n,m} d(x_n, x_m) < +\infty$ and $\sup_n \phi(x_n) < +\infty$, then there exists a subsequence of x_n which is σ-convergent.

We observe that (1.33) is always satisfied if ϕ is bounded from below.
We define the piecewise constant function $\bar{U}_\tau : [0, +\infty) \to X$ as follows:

$$\bar{U}_\tau(t) := U_\tau^k \quad \text{if} \quad t \in ((k-1)\tau, k\tau]. \tag{1.34}$$

Given $u^0 \in D(\phi)$, under our assumptions (H1), (H2), (H3), (H4), it is possible to prove that there exists a sequence $\tau_n \to 0$ and a curve $u \in AC^2_{\text{loc}}([0, +\infty); X)$ such that

$$\bar{U}_{\tau_n}(t) \rightharpoonup u(t) \qquad \forall t \in [0, +\infty). \tag{1.35}$$

Every limit curve u of this kind is called *generalized minimizing movement* for ϕ starting from u^0.
We define the *relaxed slope* $|\partial^- \phi|$ by

$$|\partial^- \phi|(x) := \inf \left\{ \liminf_{n \to +\infty} |\partial \phi|(x_n) : x_n \rightharpoonup x, \, \sup_n d(x_n, x) < +\infty, \, \sup_n \phi(x_n) < +\infty \right\}. \tag{1.36}$$

We observe that, by definition, $|\partial^- \phi|$ is sequentially lower semi continuous with respect to the weak topology σ in d-bounded sublevel sets of ϕ. Moreover, as a direct consequence of the definition of $|\partial^- \phi|$, we have that

$$|\partial^- \phi|(x) \leq |\partial \phi|(x) \qquad \forall x \in D(\phi).$$

Moreover, if $g(x) \leq |\partial \phi|(x)$, for every $x \in D(\phi)$, and g is sequentially lower semi continuous with respect to the weak topology σ in d-bounded sublevel sets of ϕ, then $g(x) \leq |\partial^- \phi|(x)$ for every $x \in D(\phi)$.

A very general existence Theorem for curves of maximal slope (which is a simple variant of Theorem 2.3.3 of [AGS05]) is the following

Theorem 1.20. *Assume that the hypothesis (H1), (H2), (H3), (H4) are satisfied. If g is a strong upper gradient for ϕ such that*

$$g(x) \leq |\partial^- \phi|(x) \qquad \forall x \in X, \tag{1.37}$$

and u is a generalized minimizing movement for ϕ starting from $u^0 \in D(\phi)$, then u is a curve of maximal slope for ϕ with respect to g. In particular the energy identity (1.29) holds. Moreover, if $\bar{U}_{\tau_n}(t)$ is a sequence of discrete solutions satisfying (1.35), we have

$$\lim_{n \to \infty} \phi(\bar{U}_{\tau_n}(t)) = \phi(u(t)) \qquad \forall t \in [0, +\infty), \tag{1.38}$$

$$\lim_{n \to \infty} |\partial \phi|(\bar{U}_{\tau_n}(t)) = |\partial^- \phi|(u(t)) \qquad \text{in } L^2_{\text{loc}}([0, +\infty)), \tag{1.39}$$

and, defining the piecewise constant function,

$$|U'_\tau|(t) = \frac{d(U_\tau^n, U_\tau^{n-1})}{\tau} \qquad \text{if } t \in ((n-1)\tau, n\tau),$$

we have

$$\lim_{n\to\infty} |U'_{\tau_n}| = |u'| \qquad \text{in } L^2_{\mathrm{loc}}([0,+\infty)). \tag{1.40}$$

Moreover

$$|u'|(t) = g(u(t)) = |\partial^- \phi|(u(t)) \qquad \text{for a.e. } t \in [0,+\infty), \tag{1.41}$$

the function $t \mapsto \phi(u(t))$ *is locally absolutely continuous and*

$$\frac{d}{dt}\phi(u(t)) = -g(u(t))^2 \qquad \text{for a.e. } t \in [0,+\infty). \tag{1.42}$$

Chapter 2

Absolutely continuous curves in Wasserstein spaces and geodesics

In this Chapter we state and prove our main results about the characterization of absolutely continuous curves with finite p-energy in the Wasserstein space $\mathscr{P}_p(X)$. We recall that $\mathscr{P}_p(X)$ denotes the space of Borel probability measures with finite moment of order p on the metric space X. $\mathscr{P}_p(X)$ is endowed with the p-Wasserstein distance. The motivation for the study of this kind of curves and a brief description of the results is illustrated in the introduction. Here we recall only that our main result is a theorem of representation of absolutely continuous curves with finite p-energy in the Wasserstein space $\mathscr{P}_p(X)$ as superposition of curves of the same kind (absolutely continuous with finite p-energy) in the complete and separable metric space X. The superposition is described by a Borel probability measure on the space of continuous curves concentrated on the absolutely continuous curves with finite p-energy. In Section 2.1 we give precise statements of the results together with some important corollaries and detailed proofs. In Section 2.2 we give a first application of our representation theorem to the study of geodesics in $\mathscr{P}_p(X)$, when X is a length space. We obtain a theorem of representation of geodesics in $\mathscr{P}_p(X)$ as superposition of geodesics in the space X without assumption of local compactness on the space X. This result generalizes a similar result of Lott-Villani on locally compact length spaces (see [LV05a] and [Vil06]). The second main application of our representation theorem will be given in Chapter 3, where we study the continuity equation in Banach spaces and in $\mathbb{R}^n$ with a non smooth Riemannian metric.

The results of this Chapter are contained in the paper [Lis05].

2.1 Characterization of the class of curves $AC^p(I;\mathscr{P}_p(X))$

Let X be a complete and separable metric space with metric $d : X \times X \to [0,+\infty)$.
We denote by I the closed interval $[0,T]$ of $\mathbb{R}$ and by $\Gamma := C(I;X)$ the separable and complete metric space of continuous curves in X, endowed with the metric of the uniform convergence induced by d:

$$d_\infty(u,\tilde{u}) = \sup_{t\in I} d(u(t),\tilde{u}(t)). \tag{2.1}$$

We also define the *evaluation map* $e_t : \Gamma \to X$ in this way

$$e_t(u) = u(t) \tag{2.2}$$

and notice that e_t is continuous.

The following result shows that if we have a superposition of absolutely continuous curves in X with finite p-energy, described by a probability measure η on the space of continuous curves Γ, then the curve $\mu_t := (e_t)_\#\eta$ is absolutely continuous in $\mathscr{P}_p(X)$ with finite p-energy.

Theorem 2.1. *If $\eta \in \mathscr{P}(\Gamma)$ is concentrated on $AC^p(I;X)$, i.e. $\eta(\Gamma \setminus AC^p(I;X)) = 0$, with $p \in [1,+\infty)$, such that*

$$\mu_0 := (e_0)_\#\eta \in \mathscr{P}_p(X) \tag{2.3}$$

and

$$\int_\Gamma \int_0^T |u'|^p(t)\,dt\,d\eta(u) < +\infty, \tag{2.4}$$

then the curve $t \mapsto \mu_t := (e_t)_\#\eta$ belongs to $AC^p(I;\mathscr{P}_p(X))$. Moreover for a.e. $t \in I$, $|u'|(t)$ exists for $\eta-$a.e. $u \in \Gamma$ and

$$|\mu'|^p(t) \le \int_\Gamma |u'|^p(t)\,d\eta(u) \qquad \textit{for a.e. } t \in I. \tag{2.5}$$

Proof. First of all we check that for a.e. $t \in I$, $|u'|(t)$ exists for $\eta-$a.e. $u \in \Gamma$.
We set $\Lambda := \{(t,u) \in I \times \Gamma : |u'|(t) \text{ do not exists}\}$ and we observe that Λ is a Borel subset of $I \times \Gamma$ since the maps $(t,u) \mapsto \frac{d(u(t+h),u(t))}{|h|}$ are continuous from $I \times \Gamma$ to $\mathbb{R}$ for every $h \neq 0$. Since η is concentrated on $AC^p(I;X)$, we have that for η-a.e. $u \in \Gamma$, $\mathscr{L}^1(\{t \in I : (t,u) \in \Lambda\}) = 0$ and then Fubini's Theorem implies that for a.e. $t \in I$, $\eta(\{u \in \Gamma : (t,u) \in \Lambda\}) = 0$.
Now we prove that $\mu_t = (e_t)_\#\eta$ has finite p-moment for every $t \in I$. Given a point $\bar{x} \in X$, we have

$$\begin{aligned}
\int_X d^p(x,\bar{x})\,d\mu_t(x) &= \int_\Gamma d^p(u(t),\bar{x})\,d\eta(u) \\
&\le 2^{p-1}\int_\Gamma (d^p(u(0),\bar{x}) + d^p(u(0),u(t)))\,d\eta(u) \\
&\le 2^{p-1}\int_\Gamma \left(d^p(u(0),\bar{x}) + \left(\int_0^t |u'|(r)\,dr\right)^p \right) d\eta(u) \\
&\le 2^{p-1}\int_\Gamma \left(d^p(u(0),\bar{x}) + \int_0^T |u'|^p(r)\,dr \right) d\eta(u)
\end{aligned}$$

and this is finite by (2.3) and (2.4).
Now we take $s,t \in I$ with $s<t$ and $\gamma_{s,t} := (e_s, e_t)_\# \eta$. Since $\gamma_{s,t} \in \Gamma(\mu_s,\mu_t)$, by the definition of W_p, the fact that η is concentrated on $AC^p(I;X)$, and Hölder's inequality, we have

$$\begin{aligned} W_p^p(\mu_s,\mu_t) &\le \int_{X\times X} d^p(x,y)\,d\gamma_{s,t}(x,y) = \int_\Gamma d^p(e_s(u),e_t(u))\,d\eta(u) \\ &\le \int_\Gamma \left(\int_s^t |u'|(r)\,dr\right)^p d\eta(u) \le \int_\Gamma |s-t|^{p-1}\int_s^t |u'|^p(r)\,dr\,d\eta(u) \qquad (2.6)\\ &= |s-t|^{p-1}\int_s^t\int_\Gamma |u'|^p(r)\,d\eta(u)\,dr, \end{aligned}$$

where the last equality follows by (2.4) and Fubini-Tonelli Theorem.
The estimates (2.6) and (2.4) imply that μ_t is absolutely continuous. The thesis is now a simple consequence of (2.6) and Lebesgue differentiation Theorem. □

The most important result is the converse of the above theorem. Precisely, every absolutely continuous curve with finite p-energy in the space $\mathscr{P}_p(X)$ is a superposition of curves of the same kind on X. The superposition is described by a measure $\tilde\eta$ which represents μ_t through the relation $\mu_t = (e_t)_\#\tilde\eta$. Moreover, for this measure $\tilde\eta$ the equality holds in (2.5).

Theorem 2.2 (Representation of absolutely continuous curves in Wasserstein spaces)**.** *If μ_t is an absolutely continuous curve in $\mathscr{P}_p(X)$ with finite p-energy, $p>1$,*
i.e. $\mu_t \in AC^p(I;\mathscr{P}_p(X))$, then there exists $\tilde\eta \in \mathscr{P}(\Gamma)$ such that

(i) $\tilde\eta$ is concentrated on $AC^p(I;X)$,

(ii) $(e_t)_\#\tilde\eta = \mu_t \qquad \forall t\in I$,

(iii)

$$|\mu'|^p(t) = \int_\Gamma |u'|^p(t)\,d\tilde\eta(u) \qquad \textit{for a.e. } t\in I.$$

Before proving Theorem 2.2 we state and prove the extension of Theorems 2.1 and 2.2 to *pseudo metric spaces* and to $\mathscr{P}(X)$, according to Remarks 1.2 and 1.10. This extension to pseudo metric spaces is possible since all the results of Theorems 2.1 and 2.2 are expressed in terms of the metric derivative. Since this concept of metric derivative involves only the infinitesimal behaviour of the distance along the curve, we can substitute the pseudo distance with a topologically equivalent bounded distance and with the same infinitesimal behaviour.

Corollary 2.3. *Let (X,d) be a pseudo metric space, separable and d-complete.*
If $\eta \in \mathscr{P}(\Gamma)$ is concentrated on $AC^p(I;X)$ with $p\in[1,+\infty)$, such that

$$\int_\Gamma\int_0^T |u'|^p(t)\,dt\,d\eta(u) < +\infty,$$

then the curve $t\mapsto \mu_t := (e_t)_\#\eta$ belongs to $AC^p(I;(\mathscr{P}(X),W_p))$. Moreover for a.e. $t\in I$, $|u'|(t)$ exists for $\eta-$a.e. u and

$$|\mu'|^p(t) \le \int_\Gamma |u'|^p(t)\,d\eta(u) \qquad \textit{for a.e. } t\in I.$$

If the curve $t \mapsto \mu_t$ *belongs to* $AC^p(I;(\mathscr{P}(X), W_p))$*, with* $p > 1$*, then there exists* $\tilde{\eta} \in \mathscr{P}(\Gamma)$ *such that*

(i) $\tilde{\eta}$ *is concentrated on* $AC^p(I;X)$*,*

(ii) $(e_t)_\# \tilde{\eta} = \mu_t \qquad \forall t \in I$*,*

(iii)

$$|\mu'|^p(t) = \int_\Gamma |u'|^p(t)\, d\tilde{\eta}(u) \qquad \text{for a.e. } t \in I.$$

Proof.[Proof of Corollary 2.3] The proof of the first part is exactly the proof of Theorem 2.1 since the hypothesis (2.3) was only needed in order to ensure that the p-moment of μ_t is finite.
In order to prove the second part we take into account the Remark 1.2. We define

$$\tilde{d}(x,y) := d(x,y) \wedge 1,$$

and we denote by $\tilde{X}$ the bounded metric space $(X, \tilde{d})$, topologically equivalent to X. Then $\mathscr{P}(X) = \mathscr{P}(\tilde{X}) = \mathscr{P}_p(\tilde{X})$ and $AC^p(I;X) = AC^p(I;\tilde{X})$, since the metric derivative with respect to d coincides with the metric derivative with respect to $\tilde{d}$. Denoting by $\tilde{W}_p$ the Wasserstein distance with respect to $\tilde{d}$, we have that $\tilde{W}_p(\mu,\nu) \le W_p(\mu,\nu)$.
If $\mu_t \in AC^p(I;\mathscr{P}(X))$, then $\mu_t \in AC^p(I;\mathscr{P}_p(\tilde{X}))$ and, applying Theorem 2.2, we can find $\tilde{\eta} \in \mathscr{P}(\Gamma)$ concentrated on $AC^p(I;\tilde{X}) = AC^p(I;X)$ such that

$$\int_\Gamma |u'|^p(t)\, d\tilde{\eta}(u) = |\mu'|^p_{\tilde{W}_p}(t) \quad \text{for a.e. } t \in I$$

where $|\mu'|_{\tilde{W}_p}(t)$ denotes the metric derivative with respect to the distance $\tilde{W}_p$.
Applying the first part of this Corollary to $\tilde{\eta}$ we obtain

$$|\mu'|^p(t) \le \int_\Gamma |u'|^p(t)\, d\tilde{\eta}(u) = |\mu'|^p_{\tilde{W}_p}(t) \le |\mu'|^p(t) \tag{2.7}$$

for a.e. $t \in I$, which complete the proof. □

Reasoning as in the proof of the above corollary we can prove the following corollary, interesting by itself.

Corollary 2.4. *If* $X_1 := (X, d_1)$*,* $X_2 := (X, d_2)$ *are two topologically equivalent, separable and complete pseudo metric spaces, such that* $AC^p(I;X_1) = AC^p(I;X_2)$ *and for every* $u \in AC^p(I;X_i)$*,* $i = 1,2$ *it holds*

$$|u'|_{d_1}(t) = |u'|_{d_2}(t) \qquad \text{for a.e. } t \in I,$$

then $AC^p(I;(\mathscr{P}(X), W_{p,d_1})) = AC^p(I;(\mathscr{P}(X), W_{p,d_2}))$ *and for every* $\mu \in AC^p(I;(\mathscr{P}(X), W_{p,d_i}))$*,* $i = 1,2$ *it holds*

$$|\mu'|_{W_{p,d_1}}(t) = |\mu'|_{W_{p,d_2}}(t) \qquad \text{for a.e. } t \in I.$$

Remark 2.5. By using Corollary 2, (pag. 101) of [Sch73], the Corollary 2.4 can be proved even when the topology induced by one of the distances is weaker than the other one. □

In the case of a complete but non separable metric space, Theorem 2.2 can be stated considering the set of tight measures as in Remark 1.11.

Corollary 2.6. *Let (X,d) be a complete metric space (in general non separable). If the curve $t \mapsto \mu_t$ belongs to $AC^p(I;(\mathscr{P}^\tau(X),W_p))$, with $p>1$, then there exist a closed separable subspace $X_0 \subset X$ and a measure $\tilde\eta \in \mathscr{P}(C(I;X_0))$ such that* $\operatorname{supp}\mu_t \subset X_0$ *for every $t \in I$ and (i), (ii), (iii) of Theorem 2.2 hold.*

Proof. It is sufficient to apply the second part of Corollary 2.3 in the separable complete metric space

$$X_0 := \overline{\cup_{s\in\mathbb{Q}\cap I}\operatorname{supp}\mu_s},$$

by observing that $\operatorname{supp}\mu_t \subset X_0$ for every $t \in I$. Indeed for $t \in I$ we take a sequence $t_n \in \mathbb{Q}\cap I$ convergent to t, by the narrow continuity of the curve μ, μ_{t_n} narrowly converges to μ_t and then, recalling Remark 1.11, we have $\operatorname{supp}\mu_t \subset X_0$. □

Now we give the proof of Theorem 2.2.

Proof.[Proof of Theorem 2.2] We prove the theorem in the particular case $T=1$, i.e. $I=[0,1]$. Obviously it is not restrictive.

For any integer $N \geq 1$, we divide the unitary interval I in 2^N equal parts, and we denote the nodal points by

$$t^i := \frac{i}{2^N} \qquad i=0,1,\dots,2^N.$$

We also denote by X_i, with $i=0,1,\dots,2^N$, 2^N+1 copies of the same space X and define the product space

$$\boldsymbol{X}_N := X_0 \times X_1 \times \ldots \times X_{2^N}.$$

Choosing optimal plans

$$\gamma_N^i \in \Gamma_o(\mu_{t^i},\mu_{t^{i+1}}) \qquad i=0,1,\dots,2^N-1,$$

there exists (see, for example, Lemma 5.3.2 and Remark 5.3.3 of [AGS05]) $\gamma_N \in \mathscr{P}(\boldsymbol{X}_N)$ such that

$$\pi^i_\#\gamma_N = \mu_{t^i} \qquad \text{and} \qquad \pi^{i,i+1}_\#\gamma_N = \gamma_N^i,$$

where we denoted by $\pi^i : \boldsymbol{X}_N \to X_i$ the projection on the i-th component and by $\pi^{i,j} : \boldsymbol{X}_N \to X_i \times X_j$ the projection on the i,j-th components.

We define $\sigma : \boldsymbol{x} = (x_0,\dots,x_{2^N}) \in \boldsymbol{X}_N \to \sigma_{\boldsymbol{x}} \in L^p(I;X)$ by

$$\sigma_{\boldsymbol{x}}(t) := x_i \qquad \text{if} \quad t \in [t^i,t^{i+1}),$$

and we also set

$$\eta_N := \sigma_\# \gamma_N \in \mathscr{P}(L^p(I;X)).$$

Step 1. (Tightness of $\{\eta_N\}_{N\in\mathbb{N}}$) In order to obtain the existence of a narrow limit point η of the sequence $\{\eta_N\}_{N\in\mathbb{N}}$, by Prokhorov's theorem, it is sufficient to prove its tightness by exhibiting a function $\Phi : L^p(I;X) \to [0,+\infty]$ whose sublevels $\lambda_c(\Phi) := \{u \in L^p(I;X) : \Phi(u) \le c\}$ are compact in $L^p(I;X)$ for any $c \in \mathbb{R}_+$, and

$$\sup_{N\in\mathbb{N}} \int_{L^p(I;X)} \Phi(u)\, d\eta_N(u) < +\infty. \tag{2.8}$$

First of all we observe that $\mathscr{A} := \{\mu_t : t \in I\}$ is compact (because it is a continuous image of a compact) in $\mathscr{P}_p(X)$ and consequently in $\mathscr{P}(X)$. In particular $\mathscr{A}$ is bounded in $\mathscr{P}_p(X)$ and then there exists C_1 such that

$$\int_X d^p(x,\bar{x})\, d\mu_t(x) = W_p^p(\mu_t, \delta_{\bar{x}}) \le C_1 \qquad \forall t \in I. \tag{2.9}$$

Since, by Prokhorov's Theorem, $\mathscr{A}$ is tight there exists $\psi : X \to [0,+\infty]$ whose sublevels $\lambda_c(\psi) := \{x \in X : \psi(x) \le c\}$ are compact in X for any $c \in \mathbb{R}_+$, such that

$$C_2 := \sup_{t\in I} \int_X \psi(x)\, d\mu_t(x) < +\infty. \tag{2.10}$$

We define $\Phi : L^p(I;X) \to [0,+\infty]$ as follows

$$\Phi(u) := \int_0^1 d^p(u(t),\bar{x})\, dt + \int_0^1 \psi(u(t))\, dt + \sup_{0<h<1} \int_0^{1-h} \frac{d^p(u(t+h),u(t))}{h}\, dt,$$

where $\bar{x}$ is a given point of X.

The compactness of $\lambda_c(\Phi)$ in $L^p(I;X)$ is immediate since the hypothesis of Theorem 1.3 are satisfied, and Φ is lower semi continuous by a simple application of Fatou's Lemma.

The proof of (2.8) requires some computations.

As a first step we show that

$$\sup_{N\in\mathbb{N}} \int_{L^p(I;X)} \int_0^1 (d^p(u(t),\bar{x}) + \psi(u(t)))\, dt\, d\eta_N(u) < +\infty.$$

By (2.9) and (2.10)

$$\begin{aligned}
\int_{L^p(I;X)} \int_0^1 d^p(u(t),\bar{x}) + \psi(u(t))\, dt\, d\eta_N(u) &= \int_0^1 \int_{\boldsymbol{X}_N} d^p(\sigma\boldsymbol{x}(t),\bar{x}) + \psi(\sigma\boldsymbol{x}(t))\, d\gamma_N(\boldsymbol{x})\, dt \\
&= \sum_{i=0}^{2^N-1} \int_{t^i}^{t^{i+1}} \int_X d^p(x,\bar{x}) + \psi(x)\, d\mu_{t^i}(x)\, dt \\
&= \frac{1}{2^N} \sum_{i=0}^{2^N-1} \int_X d^p(x,\bar{x}) + \psi(x)\, d\mu_{t^i}(x) \\
&\le \frac{1}{2^N} \sum_{i=0}^{2^N-1} (C_1 + C_2) = C_1 + C_2.
\end{aligned}$$

As a second step we show that

$$\sup_{N\in\mathbb{N}}\int_{L^p(I;X)}\sup_{0<h<1}\int_0^{1-h}\frac{d^p(u(t+h),u(t))}{h}\,dt\,d\eta_N(u)<+\infty. \tag{2.11}$$

In order to prove (2.11), we show that if $\boldsymbol{x}\in\boldsymbol{X}_N$ then

$$\sup_{0<h<1}\int_0^{1-h}\frac{d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))}{h}\,dt\le\left(2^p+\frac{(2^N)^p}{2^N-1}\right)\sum_{i=0}^{2^N-1}d^p(x_i,x_{i+1}). \tag{2.12}$$

If $h<1/2^N$ then

$$\int_0^{1-h}d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt=\sum_{i=0}^{2^N-2}\int_{t^i}^{t^{i+1}}d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt=h\sum_{i=0}^{2^N-2}d^p(x_i,x_{i+1})$$

since $\sigma\boldsymbol{x}(t+h)=\sigma\boldsymbol{x}(t)$ if $t\in[t^i,t^{i+1}-h)$.

If $1/2^N\le h<1$ we take the integer $k\ge1$ such that

$$\frac{k}{2^N}\le h<\frac{k+1}{2^N}, \tag{2.13}$$

so that the triangular inequality yields

$$d(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\le\sum_{i=0}^{k}d(\sigma\boldsymbol{x}(t+t^{i+1}),\sigma\boldsymbol{x}(t+t^i)),$$

and, using Holder's discrete inequality,

$$d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\le(k+1)^{p-1}\sum_{i=0}^{k}d^p(\sigma\boldsymbol{x}(t+t^{i+1}),\sigma\boldsymbol{x}(t+t^i)).$$

Then

$$\begin{aligned}\int_0^{1-h}d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt &\le \int_0^{1-t^k}d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt\\ &\le \int_0^{1-t^k}(k+1)^{p-1}\sum_{i=0}^{k}d^p(\sigma\boldsymbol{x}(t+t^{i+1}),\sigma\boldsymbol{x}(t+t^i))\,dt\\ &= (k+1)^{p-1}\sum_{i=0}^{k}\frac{1}{2^N}\sum_{j=0}^{2^N-k-1}d^p(x_{i+j+1},x_{i+j}) \end{aligned}\tag{2.14}$$

and, observing that, in (2.14), $d^p(x_{j+1},x_j)$ is counted at most $k+1$ times, we have

$$\int_0^{1-h}d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt\le\frac{(k+1)^p}{2^N}\sum_{j=0}^{2^N-1}d^p(x_{j+1},x_j). \tag{2.15}$$

Using (2.13) we obtain

$$\int_0^{1-h}d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt\le h\frac{(k+1)^p}{k}\sum_{j=0}^{2^N-1}d^p(x_{j+1},x_j).$$

Since the function $\lambda\mapsto\frac{(\lambda+1)^p}{\lambda}$ is increasing in the interval $(1/(p-1),+\infty)$, decreasing in $(0,1/(p-1)$ and $1\le k\le 2^N-1$ we have

$$\frac{(k+1)^p}{k}\le\begin{cases}2^p & \text{if } k\le\frac{1}{p-1}\\ \frac{(2^N)^p}{2^N-1} & \text{if } k>\frac{1}{p-1}\end{cases}$$

and (2.12) is proved.

Since

$$\begin{aligned}\int_{\boldsymbol{X}_N}\sum_{i=0}^{2^N-1} d^p(x_i,x_{i+1})\,d\gamma_N(\boldsymbol{x}) &= \sum_{i=0}^{2^N-1} W_p^p(\mu_{t^i},\mu_{t^{i+1}}) \le \sum_{i=0}^{2^N-1}\left(\int_{t^i}^{t^{i+1}}|\mu'|(t)\,dt\right)^p \qquad (2.16)\\ &\le \frac{1}{(2^N)^{p-1}}\sum_{i=0}^{2^N-1}\int_{t^i}^{t^{i+1}}|\mu'|^p(t)\,dt = \frac{1}{(2^N)^{p-1}}\int_0^1|\mu'|^p(t)\,dt\end{aligned}$$

we have, taking into account (2.12),

$$\begin{aligned}\int_{L^p(I;X)}\sup_{0<h<1}\int_0^{1-h}\frac{d^p(u(t+h),u(t))}{h}\,dt\,d\eta_N(u) &\le \left(C_p+\frac{(2^N)^p}{2^N-1}\right)\frac{1}{(2^N)^{p-1}}\int_0^1|\mu'|^p(t)\,dt\\ &\le (C_p+2)\int_0^1|\mu'|^p(t)\,dt\end{aligned}$$

which is finite because $|\mu'|\in L^p(I)$.

Then there exists $\eta\in\mathscr{P}(L^p(I;X))$ and a subsequence N_k such that $\eta_{N_k}\to\eta$ narrowly in $\mathscr{P}(L^p(I;X))$ if $k\to+\infty$.

Step 2. (η is concentrated on $W^{1,p}(I;X)$) Let us define the sequence of lower semi continuous functions $f_N : L^p(I;X)\to[0,+\infty]$ as follows

$$f_N(u) := \sup_{1/2^N\le h<1}\int_0^{1-h}\frac{d^p(u(t+h),u(t))}{h^p}\,dt;$$

clearly they satisfy the monotonicity property

$$f_N(u)\le f_{N+1}(u) \qquad \forall\, u\in L^p(I;X). \qquad (2.17)$$

We prove that

$$\int_{L^p(I;X)} f_N(u)\,d\eta_N(u)\le C. \qquad (2.18)$$

We fix $1/2^N\le h<1$ and take the integer $k\ge 1$ as in (2.13). From (2.15) and

$$\frac{1}{2^N}\le (2^N)^{p-1}\frac{h^p}{k^p} \qquad (2.19)$$

which is the first inequality of (2.13) rewritten, it follows that

$$\int_0^{1-h} d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt\le h^p\frac{(k+1)^p}{k^p}(2^N)^{p-1}\sum_{j=0}^{2^N-1}d^p(x_{j+1},x_j)$$

which implies

$$\sup_{1/2^N\le h<1}\int_0^{1-h}\frac{d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))}{h^p}\,dt\le 2^p(2^N)^{p-1}\sum_{j=0}^{2^N-1}d^p(x_{j+1},x_j).$$

Integrating and using (2.16) we obtain (2.18) with $C:=2^p\int_0^1|\mu'|^p(t)\,dt$.

By (2.17) and (2.18) we have that

$$\int_{L^p(I;X)} f_N(u)\,d\eta_{N_k}(u)\le C \qquad (2.20)$$

for every k such that $N_k \geq N$.

The lower semi continuity of f_N, (1.11) and the bound (2.20) yield

$$\int_{L^p(I;X)} f_N(u)\,d\eta(u) \leq C \qquad \forall\, N \in \mathbb{N},$$

and consequently, by monotone convergence Theorem, we have that

$$\int_{L^p(I;X)} \sup_{N\in\mathbb{N}} f_N(u)\,d\eta(u) \leq C, \qquad \text{and} \qquad \sup_{N\in\mathbb{N}} f_N(u) < +\infty \quad \text{for } \eta-\text{a.e. } u \in L^p(I;X). \tag{2.21}$$

Since

$$\sup_{N\in\mathbb{N}} f_N(u) = \sup_{0<h<1} \int_0^{1-h} \frac{d^p(u(t+h),u(t))}{h^p}\,dt,$$

(2.21) shows that η is concentrated on $W^{1,p}(I;X)$ by it's very definition (1.4).

Recalling that $W^{1,p}(I;X)$ is a Borel subset of $L^p(I;X)$, the measure η can be considered as a Borel measure on $W^{1,p}(I;X)$, i.e. $\eta \in \mathscr{P}(W^{1,p}(I;X))$.

Thanks to Lemma 1.4 we can define

$$\tilde{\eta} := T_{\#}\eta \in \mathscr{P}(\Gamma)$$

which is concentrated, by definition, on $AC^p(I;X)$.

Step 3. (Proof of (ii) and (iii)) In order to prove (iii), we show preliminarily that for all $s_1, s_2 \in I$, $s_1 < s_2$, we have

$$\int_{L^p(I;X)} \int_{s_1}^{s_2} \frac{d^p(u(t+h),u(t))}{h^p}\,dt\,d\eta(u) \leq \int_{s_1}^{s_2+h} |\mu'|^p(t)\,dt \tag{2.22}$$

for every $h \in (0, 1-s_2)$.

We fix $h \in (0,1-s_2)$ and for every $N \in \mathbb{N}$ such that $\frac{1}{2^N} \leq h$ we take $k \geq 1$ such that (2.13) holds. Setting

$$s_1^N := \frac{\bar{j}}{2^N} := \max\left\{\frac{j}{2^N} : \frac{j}{2^N} \leq s_1\right\}, \qquad s_2^N := \frac{\bar{i}}{2^N} := \min\left\{\frac{j}{2^N} : \frac{j}{2^N} \leq s_1\right\},$$

and reasoning as in the proof of (2.15) we obtain

$$\int_{s_1}^{s_2} d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt \leq \frac{(k+1)^p}{2^N} \sum_{j=\bar{j}}^{\bar{i}+k} d^p(x_{j+1},x_j)$$

and by (2.19)

$$\int_{s_1}^{s_2} d^p(\sigma\boldsymbol{x}(t+h),\sigma\boldsymbol{x}(t))\,dt \leq h^p \frac{(k+1)^p}{k^p} (2^N)^{p-1} \sum_{j=\bar{j}}^{\bar{i}+k} d^p(x_{j+1},x_j).$$

Integrating we obtain

$$\int_{L^p(I;X)} \int_{s_1}^{s_2} \frac{d^p(u(t+h),u(t))}{h^p}\,dt\,d\eta_N(u) \leq \left(\frac{k+1}{k}\right)^p \int_{s_1^N}^{s_2^N+h} |\mu'|^p(t)\,dt$$

from which (2.22) follows passing to the limit for $N \to +\infty$.

Clearly we have

$$\begin{aligned}\int_{L^p(I;X)} \int_{s_1}^{s_2} \frac{d^p(u(t+h),u(t))}{h^p}\,dt\,d\eta(u) &= \int_{W^{1,p}(I;X)} \int_{s_1}^{s_2} \frac{d^p(Tu(t+h),Tu(t))}{h^p}\,dt\,d\eta(u)\\ &= \int_\Gamma \int_{s_1}^{s_2} \frac{d^p(\tilde u(t+h),\tilde u(t))}{h^p}\,dt\,d\tilde\eta(\tilde u).\end{aligned}$$

By this last relation and (2.22), by Fatou's Lemma and the fact that $\tilde\eta$ is concentrated on $AC^p(I;X)$, letting h going to 0, we obtain

$$\int_\Gamma \int_{s_1}^{s_2} |u'|^p(t)\,dt\,d\tilde\eta(u) \le \int_{s_1}^{s_2} |\mu'|^p(t)\,dt \tag{2.23}$$

for every $s_1, s_2 \in I$ such that $s_1 < s_2$. By Fubini and Lebesgue differentiation Theorems, (2.23) yields

$$\int_\Gamma |u'|(t)^p\,d\tilde\eta(u) \le |\mu'|^p(t) \qquad \text{for a.e. } t \in I. \tag{2.24}$$

In order to show (ii) we prove that for every $t \in I$,

$$\int_\Gamma \varphi(u(t))\,d\tilde\eta(u) = \int_X \varphi(x)\,d\mu_t(x) \qquad \forall \varphi \in C_b(X). \tag{2.25}$$

We fix $\varphi \in C_b(X)$ and we introduce the sequence of piecewise constant function

$$g_N(t) := \int_X \varphi(x)\,d\mu_{t_i}(x) \qquad \text{if } t \in [t^i, t^{i+1}),$$

which converges uniformly in I when $N \to +\infty$ to the function

$$g(t) := \int_X \varphi(x)\,d\mu_t(x);$$

in particular, for every test function $\zeta \in C_b(I)$, we have that

$$\lim_{N\to+\infty} \int_0^1 \zeta(t) g_N(t)\,dt = \int_0^1 \zeta(t) g(t)\,dt. \tag{2.26}$$

On the other hand

$$\begin{aligned}\int_0^1 \zeta(t) g_N(t)\,dt &= \int_0^1 \zeta(t) \int_{L^p(I;X)} \varphi(u(t))\,d\eta_N(u)\,dt\\ &= \int_{L^p(I;X)} \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta_N(u).\end{aligned}$$

Since the map

$$u \mapsto \int_0^1 \zeta(t)\varphi(u(t))\,dt$$

is continuous and bounded from $L^p(I;X)$ to $\mathbb{R}$, then by the narrow convergence of η_{N_k} we have

$$\lim_{k\to+\infty} \int_{L^p(I;X)} \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta_{N_k}(u) = \int_{L^p(I;X)} \int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta(u).$$

By Fubini's Theorem and the definition of $\tilde\eta$,

$$\begin{aligned}\int_{L^p(I;X)}\int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\eta(u) &= \int_\Gamma\int_0^1 \zeta(t)\varphi(u(t))\,dt\,d\tilde\eta(u)\\ &= \int_0^1 \zeta(t)\int_\Gamma \varphi(u(t))\,d\tilde\eta(u)\,dt.\end{aligned}$$

By the uniqueness of the limit then

$$\int_0^1 \zeta(t)\int_\Gamma \varphi(u(t))\,d\tilde\eta(u)\,dt = \int_0^1 \zeta(t)\int_X \varphi(x)\,d\mu_t(x)\,dt \qquad \forall\zeta\in C_b(I),$$

from which

$$\int_\Gamma \varphi(u(t))\,d\tilde\eta(u) = \int_X \varphi(x)\,d\mu_t(x) \qquad \text{for a.e. } t\in I. \tag{2.27}$$

The applications $t\mapsto\int_X\varphi(x)\,d\mu_t(x)$ and $t\mapsto\int_\Gamma\varphi(u(t))\,d\tilde\eta(u)$ are continuous because the applications $t\in I\mapsto\mu_t\in\mathscr{P}(X)$ and $t\in I\mapsto(e_t)_\#\tilde\eta\in\mathscr{P}(X)$ are narrowly continuous (recall that $e_t:\Gamma\to X$ is continuous). Then (2.27) is true for every $t\in I$ and (2.25) is proved.

Finally, by (2.24) and (2.5) of Theorem 2.1 applied to $\tilde\eta$ we obtain (iii). □

2.2 Wasserstein geodesics

In this Section we apply Theorem 2.2 in order to give a characterization of the geodesics of the metric space $(\mathscr{P}_p(X),W_p)$ in terms of the geodesics of the space (X,d), under the further assumption that (X,d) is a length space.

In all this section I denotes the unitary interval $[0,1]$ and (X,d) is a separable complete metric space.

2.2.1 Length and geodesics spaces

In this Subsection we give the definition of length space and geodesic space, (see [AT04] and [BBI01] for an exhaustive treatment of this subject) and we prove that the space $\mathscr{P}_p(X)$ is a length space when X is a length space. Analogously $\mathscr{P}_p(X)$ is a geodesic space when X is a geodesic space.

Recalling that the length of $u\in AC(I;X)$ is defined by $L(u):=\int_0^1|u'|(t)\,dt$, we say that X is a *length space* if for every $x,y\in X$,

$$d(x,y)=\inf\{L(u): u\in AC(I;X),\ u(0)=x,\ u(1)=y\} \tag{2.28}$$

and X is a *geodesic space* if for every $x,y\in X$, the inf in (2.28) is a minimum, i.e. there exists $u\in AC(I;X)$ such that

$$u(0)=x,\quad u(1)=y,\quad d(x,y)=L(u). \tag{2.29}$$

We call u satisfying (2.29) *minimizing geodesic* of X. A curve $u : I \to X$ satisfying

$$d(u(t), u(s)) = |t - s| d(u(0), u(1)) \qquad \forall s, t \in I \tag{2.30}$$

is called *constant speed minimizing geodesic.*
If u is a constant speed minimizing geodesic then it is a minimizing geodesic, since $|u'|(t) = d(u(0), u(1))$ for every $t \in I$. Conversely every minimizing geodesic can be reparametrized in such a way (2.30) holds.
We define the set

$$G := \{u : I \to X : u \text{ is a constant speed minimizing geodesics of } X\}$$

and we observe that it is immediate to check that G is a closed subset of Γ.
The following property will be useful: $u \in G$, for $p > 1$, if and only if

$$\int_0^1 |u'|^p(t)\, dt \leq d^p(u(0), u(1)). \tag{2.31}$$

Indeed if $u \in G$ then the equality holds in (2.31). Conversely if $u \notin G$ then $|u'|$ is not constant and the strict convexity of $\alpha \mapsto |\alpha|^p$ yields $\int_0^1 |u'|^p(t)\, dt > d^p(u(0), u(1))$. In other words, the elements of G are the unique minimizers of the p-energy.

Proposition 2.7. *If X is a length (resp. geodesic) space then $\mathscr{P}_p(X)$ is a length (resp. geodesic) space too.*

Proof. First of all we define the Lipschitz constant $\mathrm{Lip} : \Gamma \to [0, +\infty]$ as

$$\mathrm{Lip}(u) = \sup_{s,t \in I, s \neq t} \frac{d(u(s)), d(u(t))}{|s - t|}$$

and we observe that Lip is a lower semi continuous function. Moreover for every $u \in AC(I; X)$ we have that $|u'|(t) \leq \mathrm{Lip}(u)$ for a.e. $t \in I$. We say that u is a Lipschitz curve if $\mathrm{Lip}(u) < +\infty$.

We recall that (see for instance [AGS05]) every $u \in AC(I; X)$ of length $L(u)$ can be reparametrized in such a way that $\mathrm{Lip}(u) = L(u) = |u'|(t)$ for a.e. $t \in I$.

We take $\mu, \nu \in \mathscr{P}_p(X)$ and an optimal plan $\gamma \in \Gamma_o(\mu, \nu)$.
We fix $\varepsilon > 0$ and we define the multi-valued application $\Sigma_\varepsilon : X \times X \to 2^{C(I;X)}$ as follows:

$$\Sigma_\varepsilon(x, y) := \{u \in AC(I; X) : u(0) = x,\ u(1) = y,\ d(x, y) + \varepsilon \geq \mathrm{Lip}(u)\}.$$

Since X is a length space $\Sigma_\varepsilon(x, y)$ is not empty. Indeed taking $u_\varepsilon \in AC(I; X)$ such that $u_\varepsilon(0) = x$, $u_\varepsilon(1) = y$ and $d(x, y) + \varepsilon \geq L(u_\varepsilon)$, then the reparametrization of u_ε satisfying $\mathrm{Lip}(u) = L(u)$ belongs to $\Sigma_\varepsilon(x, y)$. In order to obtain a γ-measurable selection of the multi-function Σ_ε we can apply Aumann's selection Theorem (see Theorem III.22 of [CV77]). For this application it is sufficient to show that the graph of Σ_ε, defined by

$$\mathscr{G}(\Sigma_\varepsilon) := \{(x, y, u) \in X \times X \times C(I; X) : u \in \Sigma_\varepsilon(x, y)\}$$

is Borel measurable. This is true since $\mathscr{G}(\Sigma_\varepsilon)$ is closed. Indeed, taking $(x_n, y_n, u_n) \in \mathscr{G}(\Sigma_\varepsilon)$ convergent to (x, y, u), $d(x_n, x) \to 0$, $d(y_n, y) \to 0$ and $d_\infty(u_n, u) \to 0$ by the uniqueness of the limit we obtain that $u(0) = x$ and $u(1) = y$. Moreover, using the lower semi continuity of the Lipschitz constant, we can pass to the limit in $d(x_n, y_n) + \varepsilon \geq \mathrm{Lip}(u_n)$ obtaining that $d(x, y) + \varepsilon \geq \mathrm{Lip}(u)$, which yields that u is a Lipschitz curve and then $u \in AC(I; X)$.

Denoting by $S_\varepsilon : X \times X \to \Gamma$ this measurable selection, we observe that it is well defined the measure

$$\eta := (S_\varepsilon)_\# \gamma \in \mathscr{P}(\Gamma).$$

The curve

$$\mu_t := (e_t)_\# \eta, \qquad t \in I,$$

satisfies $\mu_0 = \mu$, $\mu_1 = \nu$ and

$$W_p(\mu, \nu) + \varepsilon \geq \int_0^1 |\mu'|(t)\, dt.$$

Indeed, applying Theorem 2.1 to η we obtain

$$\begin{aligned} \int_0^1 |\mu'|(t)\, dt &\leq \int_0^1 \int_\Gamma |u'|(t)\, d\eta(u)\, dt \leq \int_0^1 \int_\Gamma \mathrm{Lip}(u)\, d\eta(u)\, dt \\ &= \int_\Gamma \mathrm{Lip}(u)\, d\eta(u) \leq \int_{X \times X} (d(x, y) + \varepsilon)\, d\gamma(x, y) \\ &\leq \left(\int_{X \times X} d^p(x, y)\, d\gamma(x, y) \right)^{\frac{1}{p}} + \varepsilon \end{aligned}$$

which concludes the proof in the case of a length space X. When X is a geodesic space the proof works with $\varepsilon = 0$. □

Remark 2.8. A sufficient condition on the complete space X, ensuring that a length space X is a geodesic space, is that all the closed balls of X are compact and any two points of X can be connected by an absolutely continuous curve (see e.g. [AT04]).
A more general condition, which replaces the compactness of the closed balls, is that there exists another topology τ on X, weaker than the topology induced by the metric, such that d is lower semi continuous with respect to τ and all d-bounded subsets of X are precompact in the topology τ. The sufficiency of this condition can be proved using the same arguments of the proof of Proposition 3.3.1 of [AGS05].

2.2.2 Characterization of geodesics of $\mathscr{P}_p(X)$

An important consequence of Theorem 2.2 is the following theorem of representation of geodesics of the space $\mathscr{P}_p(X)$ as superposition of geodesics of the space X.

Theorem 2.9 (Representation of geodesics)**.** *Let X be a separable and complete length space.*

A curve μ_t is a constant speed minimizing geodesic of $\mathscr{P}_p(X)$ if and only if there exists $\eta \in \mathscr{P}(\Gamma)$ such that

(i) η is concentrated on G,

(ii) $(e_t)_\# \eta = \mu_t \qquad \forall t \in I$,

(iii) $W_p^p(\mu_0, \mu_1) = \int_\Gamma d^p(u(0), u(1))\, d\eta(u)$.

We observe explicitly that if (i), (ii), (iii) hold then

$$W_p^p(\mu_s, \mu_t) = \int_\Gamma d^p(u(s), u(t))\, d\eta(u), \quad \forall\, s, t \in I,$$

that is $\gamma_{t,s} := (e_t, e_s)_\# \eta \in \Gamma_o(\mu_t, \mu_s)$.

Proof. Let μ_t be a constant speed minimizing geodesic of $\mathscr{P}_p(X)$ and let η be given by Theorem 2.2 applied to μ_t. We have only to check (i) and (iii).
By (iii) of Theorem 2.2 and the fact that μ_t is a constant speed geodesic, we have

$$\begin{aligned}\int_\Gamma d^p(u(0), u(1))\, d\eta(u) \quad &\geq \quad W_p^p(\mu_0, \mu_1) = \int_0^1 |\mu'|^p(t)\, dt = \int_0^1 \int_\Gamma |u'|^p(t)\, d\eta(u)\, dt \\ &= \quad \int_\Gamma \int_0^1 |u'|^p(t)\, dt\, d\eta(u) \geq \int_\Gamma d^p(u(0), u(1))\, d\eta(u)\end{aligned}$$

which shows that η is concentrated on G, by (2.31), and (iii) holds.

Conversely, we assume that (i), (ii), (iii) hold. Setting $\gamma_{t,s} := (e_t, e_s)_\# \eta \in \Gamma(\mu_t, \mu_s)$, for $t, s \in I$, and using the fact that η is concentrated on G, we obtain

$$\begin{aligned}W_p^p(\mu_t, \mu_s) \quad &\leq \quad \int_{X\times X} d^p(x, y)\, d\gamma_{t,s}(x, y) = \int_\Gamma d^p(u(t), u(s))\, d\eta(u) \\ &= \quad |t-s|^p \int_\Gamma d^p(u(0), u(1))\, d\eta(u) = |t-s|^p \int_{X\times X} d^p(x, y)\, d\gamma(x, y) \\ &= \quad |t-s|^p W_p^p(\mu_0, \mu_1).\end{aligned}$$

The triangular inequality shows that the above inequality cannot be strict. □

The following Corollary is a metric generalization of the Benamou-Brenier formula (see [BB00] and (15) in the introduction).

Corollary 2.10. *Let X be a separable and complete geodesic space. Then for every μ, $\nu \in \mathscr{P}_p(X)$ we have*

$$W_p^p(\mu, \nu) = \min\left\{ \int_0^1 \int_\Gamma |u'|^p(t)\, d\eta(u)\, dt : \eta \in \mathscr{A}(\mu, \nu) \right\},$$

where the set of admissible measures is $\mathscr{A}(\mu, \nu) := \{\eta \in \mathscr{P}(\Gamma) : \eta(\Gamma \setminus AC^p(I; X)) = 0,\ (e_0)_\# \eta = \mu,\ (e_1)_\# \eta = \nu,\ \int_\Gamma \mathcal{E}_p(u)\, d\eta(u) < +\infty\}$.

Proof. Corollary 2.3 shows that

$$W_p^p(\mu, \nu) \leq \int_0^1 \int_\Gamma |u'|^p(t)\, d\eta(u)\, dt$$

for every $\eta \in \mathscr{A}(\mu, \nu)$. Since, by Proposition 2.7, $\mathscr{P}_p(X)$ is a geodesic space, taking μ_t a constant speed minimizing geodesic such that $\mu_0 = \mu$ and $\mu_1 = \nu$, the measure η given by Theorem 2.9 is admissible and realizes the equality. □

Chapter 3

The continuity equation

This Chapter is entirely devoted to the relation between solutions of the continuity equation with a vector field satisfying a suitable condition of L^p integrability, for $p > 1$, and the class of absolutely continuous curves $AC^p([0,T]; \mathscr{P}_p(X))$ as illustrated in detail in the introduction. We consider the continuity equation in Banach spaces and in $\mathbb{R}^n$ with a non smooth Riemannian distance.

The results of this Chapter are contained in the paper [Lis05].

3.1 The continuity equation in Banach spaces

In the euclidean space $\mathbb{R}^n$ the notion of distributional solution of the continuity equation is well known. In a general Banach space it is necessary to give a precise definition of solution, choosing carefully the family of test functions. Moreover, since we treat simultaneously the case of a separable Banach space and the case of the dual space (in general not separable) of a separable Banach space, it is also necessary to specify the notion of measurability and of the integral of vector fields.

We consider a separable Banach space $(X, \|\cdot\|)$ satisfying the Radon-Nicodým property (resp. the dual $X = E^*$ of a separable Banach space E). Through this section we treat the separable case pointing out, inside brackets, the necessary changes for the dual case. The norm $\|\cdot\|$ denotes the norm in X in both cases and the norm $\|\cdot\|_*$ denotes the norm of X^* in the separable case and the norm of E in the dual case. According to Remark 1.11 we work in the space $\mathscr{P}^\tau(X)$, observing that in the separable case $\mathscr{P}^\tau(X) = \mathscr{P}(X)$.

In this section we denote by I the closed interval $[0,T]$ and we fix the exponent $p > 1$. Given a narrowly continuous curve $\mu : I \to \mathscr{P}^\tau(X)$, $t \mapsto \mu_t$, we can associate to it the probability measure $\bar{\mu} \in \mathscr{P}^\tau(I \times X)$ defined by

$$\int_{I\times X} \varphi(t,x)\, d\bar{\mu}(t,x) := \frac{1}{T}\int_0^T \int_X \varphi(t,x)\, d\mu_t(x)\, dt \tag{3.1}$$

for every bounded Borel function $\varphi : I \times X \to \mathbb{R}$. We say that a time dependent vector field $\boldsymbol{v} : I \times X \to X$ belongs to $L^p(\bar{\mu}; X)$ (resp. $L^p_{w^*}(\bar{\mu}; X)$) if $\boldsymbol{v}$ is $\bar{\mu}$ Bochner integrable (resp. if $\boldsymbol{v}$ is weakly-* $\bar{\mu}$ measurable, i.e. for every $f \in E$ the maps $\varphi_f(t,x) := \langle \boldsymbol{v}_t(x), f\rangle$ are $\bar{\mu}$ measurable) and $\int_{I\times X} \|\boldsymbol{v}_t(x)\|^p \, d\bar{\mu}(t,x)$ is finite (see e.g. [DU77] for the definition and the properties of the Bochner integral and of the weak-* integral (or Gelfand integral), here we recall only that the weak-* integral of $\boldsymbol{v}$ is defined by

$$\left\langle \int_{I\times X} \boldsymbol{v}_t(x)\, d\bar{\mu}(t,x), f \right\rangle = \int_{I\times X} \langle \boldsymbol{v}_t(x), f\rangle \, d\bar{\mu}(t,x) \qquad \forall f \in E).$$

We say that $(\mu, \boldsymbol{v})$, where μ is narrowly continuous and $\boldsymbol{v} \in L^p(\bar{\mu}; X)$ satisfies the *continuity equation*

$$\partial_t \mu_t + \operatorname{div}(\boldsymbol{v}_t \mu_t) = 0 \tag{3.2}$$

if the relation

$$\frac{d}{dt}\int_X \phi \, d\mu_t = \int_X \langle D\phi, \boldsymbol{v}_t\rangle \, d\mu_t \qquad \forall \phi \in C^1_b(X) \quad (\text{resp. } \forall \phi \in C^1_{b*}(X)) \tag{3.3}$$

holds in the sense of distributions in $(0,T)$; here $C^1_b(X)$ denotes the space of functions $\phi : X \to \mathbb{R}$ such that ϕ is bounded, Fréchet differentiable, and the application $D\phi : X \to X^*$ is continuous and bounded (in all this paragraph D denotes the Fréchet differential), whereas $C^1_{b*}(X)$ denotes the space of functions $\phi : X \to \mathbb{R}$ such that ϕ is bounded, Fréchet differentiable and the application $D\phi : X \to X^*$ is continuous, bounded and, for every $x \in X$, $D\phi(x) \in E \subset E^{**}$. Here E is identified with the image of the canonical injection of E in E^{**}.

In order to simplify the statements of the Theorems of this Section, we define the following set,

$$\begin{aligned} EC_p(X) := \big\{ &(\mu, \boldsymbol{v}) : \mu : I \to \mathscr{P}^\tau(X) \text{ is narrowly continuous, } \boldsymbol{v} \in L^p(\bar{\mu}; X), \\ &(\text{resp. } \boldsymbol{v} \in L^p_{w^*}(\bar{\mu}; X)),\ (\mu, \boldsymbol{v}) \text{ satisfies the continuity equation} \big\}. \end{aligned}$$

In the space $\mathscr{P}^\tau(X)$ we always use the pseudo distance W_p.

3.1.1 A probabilistic representation of solutions of the continuity equation on $\mathbb{R}^n$.

We recall a useful Theorem of representation of the solutions of the continuity equation in $\mathbb{R}^n$ (see Theorem 8.2.1 in [AGS05] and also the lecture notes of the CIME course [Amb05] for the proof).

Theorem 3.1. *If $\mu : [0,T] \to \mathscr{P}(\mathbb{R}^n)$ and a Borel vector field $\boldsymbol{v}$ such that, for $p > 1$,*

$$\int_0^T \int_{\mathbb{R}^n} \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x)\, dt < +\infty,$$

satisfy the continuity equation in $\mathbb{R}^n$, then there exists $\eta \in \mathscr{P}(\Gamma)$ concentrated on the set

$$\{u \in AC^p([0,T];\mathbb{R}^n) : u \text{ is an integral solution of } \dot{u}(t) = \boldsymbol{v}_t(u(t))\}$$

such that

$$\mu_t = (e_t)_\# \eta.$$

3.1.2 The continuity equation in Banach spaces: results

This Subsection contains the main application of our representation Theorem 2.2. As illustrated in detail in the introduction, we show that given an absolutely continuous (with respect to Wassertsein distance) curve of probability measures with finite p-energy, we can construct a minimal vector field belonging to L^p, in such a way that the continuity equation holds. This minimal vector field is defined by means of the measure $\tilde{\eta}$ which represents the curve. Moreover the $L^p(\mu_t)$ norm of the minimal vector field is equal to the metric derivative of the curve for almost every time t. The precise statement is given in Theorem 3.7, and it is a collection of the two next Theorems. In Theorem 3.2 we prove the existence of the vector field and the inequality (3.4). In Theorem 3.6 we prove that every solution of the continuity equation with the vector field in L^p is an absolutely continuous curve with finite p-energy, and the opposite inequality of (3.4) holds. For the proof of Theorem 3.6 we need a property of approximation for the Banach space given in Definitions 3.3 and 3.4.

Theorem 3.2. *If $\mu \in AC^p(I;\mathscr{P}^\tau(X))$ then there exists a vector field $\boldsymbol{w} : I \times X \to X$ such that $(\mu, \boldsymbol{w}) \in EC_p(X)$ and*

$$\|\boldsymbol{w}_t\|_{L^p(\mu_t;X)} \leq |\mu'|(t) \qquad \text{for a.e. } t \in I. \tag{3.4}$$

Proof. In order to carry out the proof for both cases we put $X_0 = X$ and $\Gamma_0 = \Gamma$ in the separable case, whereas in the dual case we take X_0 a separable Banach space $X_0 \subset X$ containing the support of every μ_t, and $\Gamma_0 := C(I;X_0)$. Let $\eta := \tilde{\eta} \in \mathscr{P}(\Gamma_0)$ be given by Corollary 2.3 (resp. $\eta := \tilde{\eta} \in \mathscr{P}(\Gamma_0)$ given by Corollary 2.6). We denote by $\bar{\eta} \in \mathscr{P}(I \times \Gamma_0)$ the measure $\bar{\eta} := \frac{1}{T}\mathscr{L}^1_{|I} \otimes \eta$. Defining the evaluation map $e : I \times \Gamma_0 \to I \times X_0$ by $e(t,u) = (t, e_t(u))$, it is immediate to check that $e_\# \bar{\eta} = \bar{\mu}$.

The disintegration of $\bar{\eta}$ with respect to e yields a Borel family of probability measures $\bar{\eta}_{t,x}$ on Γ_0 concentrated on $\{u : e_t(u) = x\}$ such that for every $\varphi : I \times \Gamma_0 \to \mathbb{R}$, $\varphi \in L^1(\bar{\eta})$, we have

$$u \mapsto \varphi(t,u) \in L^1(\bar{\eta}_{t,x}) \text{ for } \bar{\mu}\text{-a.e. } (t,x) \in I \times X_0, \tag{3.5}$$

$$(t,x) \mapsto \int_{\Gamma_0} \varphi(t,u)\, d\bar{\eta}_{t,x}(u) \in L^1(\bar{\mu}), \tag{3.6}$$

$$\int_{I\times\Gamma_0} \varphi(t,u)\, d\bar{\eta}(t,u) = \int_{I\times X_0}\int_{\Gamma_0} \varphi(t,u)\, d\bar{\eta}_{t,x}(u)\, d\bar{\mu}(t,x)$$

and the measures $\bar{\eta}_{t,x}$ are uniquely determined for $\bar{\mu}$-a.e. $(t,x) \in I \times X_0$.

In the following of the proof, we take into account Theorem 1.16. We denote by $\dot{u}(t) := \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}$ the pointwise derivative of the curve u (resp. $\dot{u}(t) := w^* - \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}$ the weak-* derivative of the curve u, and we observe explicitly that, in general, X_0 is not weakly-* closed and then $\dot{u}(t) \in X$), which is defined $\mathscr{L}^1$- almost everywhere when u is absolutely continuous.
We observe that the set

$$A := \{(t,u) \in I \times \Gamma_0 : \dot{u}(t) \text{ exists}\}$$

$$(\text{resp. } A := \{(t,u) \in I \times \Gamma_0 : \dot{u}(t) \text{ exists}, |u'|(t) \text{ exists}, \ \|\dot{u}(t)\| = |u'|(t)\})$$

is a Borel set and $\bar{\eta}(A^c) = 0$. Indeed, defining, for $h \neq 0$, the continuous functions $g_h : I \times \Gamma \to X_0$, by $g_h(t,u) = \frac{u(t+h)-u(t)}{h}$, (in the definition of g_h we extend the functions u outside of I by $u(s) = u(0)$ for $s < 0$ and $u(s) = u(T)$ for $s > T$), the completeness of X_0 yields that

$$A^c = \{(t,u) : \limsup_{(h,k)\to(0,0)} \|g_h(t,u) - g_k(t,u)\| > 0\},$$

and then A is a Borel set because of the continuity of the functions $(t,u) \mapsto \|g_h(t,u) - g_k(t,u)\|$ for every $h \neq 0$ and $k \neq 0$. (in the dual case A is a Borel set since, for a dense subset $\{f_n\}_{n\in\mathbb{N}} \subset E$ we have

$$A := \Big\{(t,u) \in I \times \Gamma_0 : \lim_{h\to 0} \left\langle \frac{u(t+h)-u(t)}{h}, f_n \right\rangle \text{ exists}, |u'|(t) \text{ exists},$$
$$|u'|(t) = \sup_{n\in\mathbb{N}} \lim_{h\to 0} \left\langle \frac{u(t+h)-u(t)}{h}, \frac{f_n}{\|f_n\|_E} \right\rangle \}).$$

Since $\bar{\eta}$ is concentrated on $I \times AC^p(I, X_0)$, Fubini's Theorem implies that $\bar{\eta}(A^c) = 0$.
Then the map $\psi : I \times \Gamma_0 \to X$ defined by

$$\psi(t,u) = \dot{u}(t)$$

is well defined for $\bar{\eta}$-a.e. $(t,u) \in I \times \Gamma_0$.
We prove that ψ is $\bar{\eta}$ Bochner (resp. $\bar{\eta}$ weak-*) integrable and $\|\psi\|^p \in L^1(\bar{\eta})$. Taking $f \in X^*$ (resp. $f \in E$), we define for $(t,u) \in A$, $\psi_f(t,u) = \langle f, \dot{u}(t)\rangle$. Since ψ_f is limit of continuous functions is a Borel function in A and then $\bar{\eta}$ measurable. Since X_0 is separable, by Pettis Theorem ψ is a $\bar{\eta}$ measurable function (resp. By definition ψ is a $\bar{\eta}$ weak-* measurable function) and, recalling (iii) of Theorem 2.2,

$$\int_{I\times\Gamma_0} \|\dot{u}(t)\|^p \, d\bar{\eta}(t,u) = \int_{I\times\Gamma_0} |u'|^p(t)\, d\bar{\eta}(u) = \frac{1}{T}\int_0^T |\mu'|^p(t)\, dt < +\infty. \tag{3.7}$$

Since for $\bar{\mu}$-a.e. $(t,x) \in I \times X_0$, we have $\bar{\eta}_{t,x}(\{u : (t,u) \in A^c\}) = 0$, the map $\psi(t,\cdot)$ is well defined for $\bar{\eta}_{t,x}$-a.e. $u \in \Gamma_0$ and $\psi(t,\cdot)$ is $\bar{\eta}_{t,x}$ Bochner integrable (resp. $\psi(t,\cdot)$ is $\bar{\eta}_{t,x}$ weak-* integrable). Indeed for every $f \in X^*$ (resp. $f \in E$),

$$\int_{I\times\Gamma_0} |\psi_f(t,u)|\, d\bar{\eta}(t,u) \quad \leq \quad \|f\|_* \int_{I\times\Gamma_0} \|\dot{u}(t)\|\ d\bar{\eta}(t,u)$$

i.e. $\psi_f \in L^1(\bar{\eta})$, and by (3.5) $\psi_f(t,\cdot) \in L^1(\bar{\eta}_{t,x})$, then Pettis Theorem yields that $\psi(t,\cdot)$ is $\bar{\eta}_{t,x}$ measurable (resp. then by definition $\psi(t,\cdot)$ is $\bar{\eta}_{t,x}$ is weak-* measurable) and $\|\psi(t,\cdot)\| \in L^1(\bar{\eta}_{t,x})$ by (3.5) and (3.7).

Consequently it is well defined the vector field

$$\boldsymbol{w}_t(x) := \int_{\Gamma_0} \dot{u}(t)\, d\bar{\eta}_{t,x}(u) \qquad \text{for } \bar{\mu}\text{-a.e. } (t,x) \in I \times X_0, \tag{3.8}$$

and $\boldsymbol{w} \in L^p(\bar{\mu}, X)$ (resp. $\boldsymbol{w} \in L^p_{w*}(\bar{\mu}, X)$). Indeed for every $f \in X^*$ (resp. $f \in E$),

$$\langle f, \boldsymbol{w}_t(x)\rangle = \int_{\Gamma_0} \psi_f(t,u)\, d\bar{\eta}_{t,x}(u)$$

and by (3.6) the maps $(t,x) \mapsto \langle f, \boldsymbol{w}_t(x)\rangle \in L^1(\bar{\mu})$. Pettis Theorem implies that $\boldsymbol{w}$ is $\bar{\mu}$ measurable (resp. By definition $\boldsymbol{w}$ is $\bar{\mu}$ weak-* measurable) and moreover

$$\begin{aligned}
\int_{I\times X_0} \|\boldsymbol{w}_t(x)\|^p\ d\bar{\mu}(t,x) \quad &\leq \quad \int_{I\times X_0}\int_{\Gamma_0} \|\dot{u}(t)\|^p\ d\bar{\eta}_{t,x}(u)\, d\bar{\mu}(t,x) \\
&= \quad \int_{I\times\Gamma_0} \|\dot{u}(t)\|^p\ d\bar{\eta}(t,u) < +\infty.
\end{aligned}$$

The inequality (3.4) follows from the definition of $\boldsymbol{w}$, Jensen's inequality and (iii) of Theorem 2.2. Indeed for every $[a,b] \subset I$,

$$\begin{aligned}
\int_a^b \int_{X_0} \|\boldsymbol{w}_t(x)\|^p\ d\mu_t(x) dt \quad &= \quad \int_{I\times X_0} T\chi_{[a,b]}(t)\, \|\boldsymbol{w}_t(x)\|^p\ d\bar{\mu}(t,x) \\
&\leq \quad \int_{I\times X_0} T\chi_{[a,b]}(t) \int_{\Gamma_0} \|\dot{u}(t)\|^p\ d\bar{\eta}_{t,x}(u)\, d\bar{\mu}(t,x) \\
&= \quad \int_{I\times\Gamma_0} T\chi_{[a,b]}(t)\, \|\dot{u}(t)\|^p\ d\bar{\eta}(t,u) = \int_a^b |\mu'|(t)\, dt.
\end{aligned}$$

Now we prove that (3.3) holds. Taking $\phi \in C^1_b(X)$ (resp $\phi \in C^1_{b*}(X)$), as a consequence of the absolute continuity of μ_t in $(\mathscr{P}(X_0), W_p)$, the application $t \mapsto \int_{X_0} \phi\, d\mu_t$ is absolutely continuous. Indeed for every $s,t \in I$, taking $\gamma_{s,t} \in \Gamma_o(\mu_s,\mu_t)$, we have

$$\begin{aligned}
\left|\int_{X_0} \phi\, d\mu_t - \int_{X_0} \phi\, d\mu_s\right| \quad &\leq \quad \int_{X_0\times X_0} |\phi(y) - \phi(x)|\, d\gamma_{s,t}(x,y) \\
&\leq \quad \sup_{x\in X_0} \|D\phi(x)\|_* \int_{X_0\times X_0} \|x-y\|\ d\gamma_{s,t}(x,y) \\
&\leq \quad \sup_{x\in X_0} \|D\phi(x)\|_*\, W_p(\mu_s,\mu_t).
\end{aligned}$$

Moreover, by (iii) of Theorem 1.16 and the differentiability of test functions,

$$\begin{aligned}\int_{X_0} \phi\, d\mu_t - \int_{X_0} \phi\, d\mu_s &= \int_{\Gamma_0} \phi(u(t)) - \phi(u(s))\, d\eta(u) \\ &= \int_{\Gamma_0} \langle D\phi(u(s)), u(t) - u(s)\rangle\, d\eta(u) \\ &\quad + \int_{\Gamma_0} \|u(t) - u(s)\|\, \omega_{u(s)}(u(t))\, d\eta(u) \\ &= \int_{\Gamma_0} \langle D\phi(u(s)), \int_s^t \dot u(r)\, dr\rangle\, d\eta(u) \\ &\quad + \int_{\Gamma_0} \|u(t) - u(s)\|\, \omega_{u(s)}(u(t))\, d\eta(u)\end{aligned}$$

where

$$\omega_x(y) = \frac{\phi(y) - \phi(x) - \langle D\phi(x), y - x\rangle}{\|y - x\|}.$$

Dividing by $t - s$ and passing to the limit for $t \to s$, for a.e. $s \in I$ we have

$$\frac{d}{ds}\int_{X_0} \phi\, d\mu_s = \int_{\Gamma_0} \langle D\phi(u(s)), \dot u(s)\rangle\, d\eta(u).$$

Indeed, for a.e. $s \in I$, a simple application of Lebesgue Theorem yields

$$\lim_{t\to s} \frac{1}{t - s}\int_{\Gamma_0} \langle D\phi(u(s)), \int_s^t \dot u(r)\, dr\rangle\, d\eta(u) = \int_{\Gamma_0} \langle D\phi(u(s)), \dot u(s)\rangle\, d\eta(u)$$

and we must only prove that

$$\lim_{t\to s} \frac{1}{t - s}\int_{\Gamma_0} \|u(t) - u(s)\|\, \omega_{u(s)}(u(t))\, d\eta(u) = 0. \tag{3.9}$$

We take a Lebesgue point s of the application $t \mapsto |\mu'|^p(t)$ such that $\dot u(s)$ exists for η-a.e. $u \in \Gamma_0$ and a sequence $t_n \to s$. We define the sequence $f_n : \Gamma_0 \to \mathbb{R}$ by $f_n(u) := \left\|\frac{u(t_n)-u(s)}{t_n-s}\right\| \omega_{u(s)}(u(t_n))$. Since $\sup_{x\in X_0} \|D\phi(x)\|_* < +\infty$, in particular ϕ is Lipschitz and

$$|\omega_x(y)| \le \frac{|\phi(y) - \phi(x)|}{\|y - x\|} + \frac{|\langle D\phi(x), y - x\rangle|}{\|y - x\|} \le C$$

with C independent on $x \in X$ and $y \in X$. Clearly $f_n(u) \to 0$ for η-a.e. $u \in \Gamma_0$ and,

$$|f_n(u)| \le g_n(u) := C\left\|\frac{u(t_n) - u(s)}{t_n - s}\right\| \qquad \text{for } \eta\text{-a.e. } u \in \Gamma_0.$$

The sequence g_n is η-equiintegrable since,

$$\begin{aligned}\int_{\Gamma_0} g_n(u)^p\, d\eta(u) &\le C\int_{\Gamma_0} \frac{1}{|t_n - s|}\left|\int_s^{t_n} |u'|(r)^p\, dr\right| d\eta(u) \\ &= \frac{C}{|t_n - s|}\left|\int_s^{t_n} |\mu'|(r)^p\, dr\right| \to C|\mu'|^p(s) < +\infty,\end{aligned}$$

which implies that $\int_{\Gamma_0} g_n(u)^p\, d\eta(u) \le C_0$ for every $n \in \mathbb{N}$. By Vitali's convergence Theorem, and since the sequence t_n is arbitrary, we obtain (3.9).

Finally, taking into account the definition of $\boldsymbol{w}_t$ we easily obtain

$$\frac{d}{dt}\int_{X_0} \phi\, d\mu_t = \int_{X_0} \langle D\phi, \boldsymbol{w}_t\rangle\, d\mu_t, \qquad \text{for a.e. } t \in I.$$

Since this pointwise derivative is also a distributional derivative, we can conclude. □

Before stating the next Theorem we give two definitions.

Definition 3.3. A separable Banach space X satisfies the Bounded Approximation Property (BAP) if there exists a sequence of finite rank linear operators $T_n : X \to X$ such that

$$\lim_{n\to\infty} \|T_n x - x\| = 0 \qquad \forall\, x \in X. \tag{3.10}$$

We observe explicitly that if (3.10) holds, then, by Banach-Steinhaus Theorem, there exists $M \geq 1$ such that

$$\|T_n x\| \leq M \, \|x\| \qquad \forall\, x \in X. \tag{3.11}$$

Definition 3.4. The dual X of a separable Banach space satisfies the Weak* Bounded Approximation Property (wBAP) if there exists a sequence of finite rank linear operators $T_n : X \to X$ such that

$$w^* - \lim_{n\to\infty} T_n x = x \qquad \forall\, x \in X, \tag{3.12}$$

there exists $M \geq 1$ such that

$$\|T_n x\| \leq M \, \|x\| \qquad \forall\, x \in X, \tag{3.13}$$

$$\limsup_{n\to+\infty} \|T_n x\| \leq \|x\| \qquad \forall\, x \in X, \tag{3.14}$$

and

$$T_n \text{ are weakly-* continuous.} \tag{3.15}$$

Remark 3.5. We observe that, being X separable, Definition 3.3 is equivalent to the following one, valid also for non separable Banach spaces: there exists $M \geq 1$ such that for every $\varepsilon > 0$ and for every compact $K \subset X$ there exists a finite rank linear operator $T_{\varepsilon,K} : X \to X$ such that $\|T_{\varepsilon,K}\| \leq M$ and $\sup_{x\in K} \|T_{\varepsilon,K} x - x\| < \varepsilon$ (see [LT77]).
Clearly a Banach space with a Schauder basis has the (BAP). Then, for instance, the property is satisfied by the Sobolev spaces $W^{k,p}(\Omega)$ for $p \in [1, +\infty)$ and $k \geq 0$, where Ω is a smooth bounded open subset of $\mathbb{R}^n$. Even if M is a compact, smooth n-dimensional manifold with or without boundary, the spaces of continuous functions $C^k(M)$ and the Sobolev spaces $W^{k,p}(M)$, for $p \in [1, +\infty)$ and $k \geq 0$, satisfy the property (see [FW01]).
We observe that l^∞ satisfies the (wBAP) as we can see taking the operators $T_n x := (x_1, \ldots, x_n, 0, \ldots)$. The space l^∞ is of fundamental importance since every separable metric space can be isometrically embedded in l^∞. An isometric embedding $J : Y \to l^\infty$ of a separable metric space (Y, d) in l^∞, can be defined by taking a countable dense set $\{y_n\}_{n\in\mathbb{N}} \subset Y$ and defining

$$J(y)_n := d(y, y_n) - d(y_n, y_1)$$

(see e.g. [AT04] and [AK00]).

Theorem 3.6. *Assume that X satisfies the bounded approximation property (BAP) (resp. the property (wBAP)).*
If $(\mu, \boldsymbol{v}) \in EC_p(X)$ then $\mu \in AC^p(I; \mathscr{P}^\tau(X))$ and

$$|\mu'|(t) \le \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \textit{for a.e. } t \in I. \tag{3.16}$$

Proof. For all $n \in \mathbb{N}$ we set $X_n := T_n(X)$ and $m := \dim(T_n(X))$. Let $\{e_i\}_{i=1,\dots,m}$ be a basis of X_n. There exist $f_1^n, \dots, f_m^n \in X^*$ (resp. $f_1^n, \dots, f_m^n \in E$, thanks to (3.15)) such that $T_n x = \sum_{i=1}^m \langle f_i^n, x\rangle e_i$.
Denoting by $P_n : X_n \to \mathbb{R}^m$ the linear isomorphism given by $P_n(\sum_{i=1}^m a_i e_i) = (a_1, \dots, a_m)$, we define the projection $\pi_n : X \to \mathbb{R}^m$, as

$$\pi_n = P_n \circ T_n$$

and the relevement $\tilde{\pi}_n : \mathbb{R}^m \to X$, as

$$\tilde{\pi}_n(a_1, \dots, a_m) = \sum_{i=1}^m a_i e_i.$$

We observe that

$$\pi_n \circ \tilde{\pi}_n = id_{\mathbb{R}^m} \qquad \text{and} \qquad \tilde{\pi}_n \circ \pi_n = T_n. \tag{3.17}$$

We define

$$\mu_t^n := \pi_{n\#}\mu_t \qquad \text{and} \qquad \boldsymbol{v}_t^n(y) := \int_X \pi_n(\boldsymbol{v}_t)(x))\, d\mu_{ty}(x)$$

where μ_{ty} is the disintegration of μ_t with respect to π_n which is concentrated on $\{x : \pi_n x = y\}$. Using a test function of the form $\phi = \psi \circ \pi_n$ with $\psi \in C_c^\infty(\mathbb{R}^m)$, it is easy to check that $(\mu_t^n, \boldsymbol{v}_t^n)$ satisfies the continuity equation in the distribution sense on $\mathbb{R}^m$.
For our purposes it's useful to consider on $\mathbb{R}^m$ the norm

$$|||y|||_{\mathbb{R}^m} := \|\tilde{\pi}_n y\|, \tag{3.18}$$

satisfying, by (3.17),

$$|||\pi_n x|||_{\mathbb{R}^m} = \|T_n x\| \qquad \forall x \in X, \tag{3.19}$$

and

$$\limsup_{n\to+\infty} \|\boldsymbol{v}_t^n\|_{L^p(\mu_t^n;\mathbb{R}^m)} \le \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \text{for a.e. } t \in I. \tag{3.20}$$

In fact, using the property of disintegration, (3.19) and Jensen's inequality,

$$\begin{aligned}
\|\boldsymbol{v}_t^n\|^p_{L^p(\mu_t^n;\mathbb{R}^m)} &= \int_{\mathbb{R}^m} |||\boldsymbol{v}_t^n(y)|||^p_{\mathbb{R}^m} \, d\mu_t^n(y) \\
&= \int_{\mathbb{R}^m} \left|\left|\left| \pi_n \int_{\{x:\pi_n x = y\}} \boldsymbol{v}_t(x)\, d\mu_{ty}(x) \right|\right|\right|^p_{\mathbb{R}^m} d\mu_t^n(y) \\
&= \int_{\mathbb{R}^m} \left\| T_n \int_{\{x:\pi_n x = y\}} \boldsymbol{v}_t(x)\, d\mu_{ty}(x) \right\|^p d\mu_t^n(y) \\
&= \int_{\mathbb{R}^m} \left\| \int_{\{x:\pi_n x = y\}} T_n \boldsymbol{v}_t(x)\, d\mu_{ty}(x) \right\|^p d\mu_t^n(y) \\
&\le \int_{\mathbb{R}^m} \int_{\{x:\pi_n x = y\}} \|T_n \boldsymbol{v}_t(x)\|^p \, d\mu_{ty}(x)\, d\mu_t^n(y) \\
&= \int_X \|T_n \boldsymbol{v}_t(x)\|^p \, d\mu_t(x).
\end{aligned}$$

In the separable case, by (3.11) and (3.10), Lebesgue dominated convergence Theorem implies

$$\lim_{n\to\infty} \int_X \|T_n \boldsymbol{v}_t(x)\|^p \, d\mu_t(x) = \int_X \|\boldsymbol{v}_t(x)\|^p \, d\mu_t(x)$$

from which (3.20) follows. In the dual case, (3.20) follows by using (3.14) instead of Lebesgue Theorem.

Theorem 3.1 states that there exists $\eta^n \in \mathscr{P}(\Gamma_{\mathbb{R}^m})$ such that $(e_t)_\# \eta^n = \mu_t^n$ and η^n is concentrated on the set $\{u \in AC^p(I;\mathbb{R}^m) : u \text{ is an integral solution of } \dot{u}(t) = \boldsymbol{v}_t^n(u(t))\}$. By the first part of Corollary 2.3 and (1.23) we conclude that $\mu_t^n \in AC^p(I;\mathscr{P}_p(\mathbb{R}^m))$ and

$$|(\mu^n)'|(t) \le \|\boldsymbol{v}_t^n\|_{L^p(\mu_t;\mathbb{R}^m)} \qquad \text{for a.e. } t \in I \tag{3.21}$$

where the Wasserstein pseudo distance, and consequently the metric derivative, is made with respect to the distance induced by the norm (3.18).

Defining

$$\tilde{\mu}_t^n := (T_n)_\# \mu_t = (\tilde{\pi}_n)_\# \mu_t^n,$$

in the separable case, (3.10) implies that $\tilde{\mu}_t^n \to \mu_t$ narrowly and hence we have

$$W_p(\mu_t,\mu_s) \le \liminf_{n\to\infty} W_p(\tilde{\mu}_t^n, \tilde{\mu}_s^n) = \liminf_{n\to\infty} W_p(\mu_t^n, \mu_s^n), \tag{3.22}$$

by the lower semicontinuity of the Wasserstein distance and the isometric embedding (3.18).

In the dual case, the inequality (3.22) can be obtained by introducing the distance

$$d_w(x,y) := \sum_{n=1}^{+\infty} \frac{1}{2^n} |\langle x-y, f_n\rangle|, \tag{3.23}$$

where $\{f_n\}_{n\ge 1}$ is a countable dense subset of $\{f \in E : \|f\|_E \le 1\}$. d_w is a distance on X which metrizes the weak-* topology on bounded sets (see e.g. [Bre83]) and, in general, the topology induced by d_w is weaker than the weak-* topology. We can assume that all the supports of $(T_n)_\# \mu_t$ and μ_t, for all $t \in I$, are contained in a separable subspace X_0, since T_n are of finite rank $(T_n)_\# \mu_t \in \mathscr{P}^\tau(X)$, and $t \mapsto \mu_t$ is narrowly continuous. Taking $\varphi \in C_b((X_0, d_w)) \subset C_b((X_0, w*))$ by (3.12) $\lim_{n\to\infty} \varphi(T_n x) = \varphi(x)$ for every $x \in X_0$ and Lebesgue dominated convergence Theorem implies that $(T_n)_\# \mu_t$ narrowly converges to μ_t in $\mathscr{P}((X_0, d_w))$. Consequently a sequence of optimal plans $\gamma_n \in \Gamma_o((T_n)_\# \mu_t, (T_n)_\# \mu_s)$ is tight in $\mathscr{P}((X_0, d_w) \times (X_0, d_w))$. Let γ be a limit point of γ_n in $\mathscr{P}((X_0, d_w) \times (X_0, d_w))$; since Borel sets of X_0 coincides with Borel sets of (X_0, d_w) (see [Sch73]), $\gamma \in \mathscr{P}(X_0 \times X_0)$. By the lower semi continuity of the application $(x, y) \to \|x - y\|^p$ with respect to the product topology induced by d_w in $X_0 \times X_0$ we obtain

$$\liminf_{n\to\infty} \int_{X_0\times X_0} \|x - y\|^p \, d\gamma_n(x, y) \geq \int_{X_0\times X_0} \|x - y\|^p \, d\gamma(x, y),$$

and then (3.22) follows.

Finally, by (3.21), (3.20) and (3.22) we have

$$W_p(\mu_t, \mu_s) \leq \int_s^t \|\boldsymbol{v}_r\|_{L^p(\mu_r;X)} \, dr,$$

which shows that $\mu_t \in AC^p(I; \mathscr{P}^\tau(X))$ and the inequality (3.16) holds. □

Using the two previous Theorems, and the fact that the strict convexity of the norm implies the strict convexity of the norm of $L^p(\bar{\mu}; X)$, it is immediate to prove the following Theorem.

Theorem 3.7 (Existence and uniqueness of the minimal vector field)**.** *Assume that X is a separable Banach space satisfying the Radon-Nicodým property and the bounded approximation property (BAP) (resp. X is the dual of a separable Banach space and satisfies the property (wBAP)).*

If $\mu \in AC^p(I; \mathscr{P}^\tau(X))$ then there exists a vector field $\boldsymbol{v} : I \times X \to X$ such that $(\mu, \boldsymbol{v}) \in EC_p(X)$ and

$$|\mu'|(t) = \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \textit{for a.e. } t \in I. \tag{3.24}$$

Moreover $\boldsymbol{v}_t$ is minimal, since for any $\boldsymbol{w} \in L^p(\bar{\mu}; X)$ (resp. $\boldsymbol{w} \in L^p_{w^*}(\bar{\mu}; X)$) such that $(\mu_t, \boldsymbol{w}_t) \in EC_p(X)$ we have

$$\|\boldsymbol{w}_t\|_{L^p(\mu_t;X)} \geq \|\boldsymbol{v}_t\|_{L^p(\mu_t;X)} \qquad \textit{for a.e. } t \in I. \tag{3.25}$$

If the norm of X is strictly convex, then $\boldsymbol{v}$ is uniquely determined in $L^p(\bar{\mu}; X)$.

3.2 The continuity equation in $\mathbb{R}^n$ with a Riemannian metric

In view of the study of evolution equations with variable coefficients as gradient flows in Wasserstein spaces, it is important to consider $\mathbb{R}^n$ with a suitable Riemannian distance.

Indeed (see Section 4.2), the solutions of the equation

$$\partial_t u_t(x) - \mathrm{div}(A(x)(\nabla u_t(x) + u_t(x)\nabla V(x))) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{R}^n, \tag{3.26}$$

where A is a symmetric matrix valued function, depending on the spatial variable $x \in \mathbb{R}^n$, satisfying a uniform ellipticity condition, can be interpreted as gradient flows of the energy functional

$$\phi(u) := \int_{\mathbb{R}^n} u(x)\log(u(x))\,dx + \int_{\mathbb{R}^n} V(x)u(x)\,dx$$

in the Wasserstein space $\mathscr{P}_2(\mathbb{R}^n)$, only if the Wasserstein distance on $\mathscr{P}_2(\mathbb{R}^n)$ is induced by the Riemannian distance on $\mathbb{R}^n$ related to the metric tensor $G = A^{-1}$.
Since we are interested to weak assumptions of regularity on the coefficients of A, we consider a non smooth metric tensor, just satisfying a suitable lower semi continuity property.

3.2.1 The Riemannian metric on $\mathbb{R}^n$

Let us denote by $\mathbb{M}_n$ the space of square $n \times n$ real matrices and consider a matrix-valued function $G : \mathbb{R}^n \to \mathbb{M}_n$.
We assume that $G(x)$ is symmetric for every $x \in \mathbb{R}^n$ and satisfies the following uniform ellipticity condition:

$$\lambda|\xi|^2 \le \langle G(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \qquad \forall x \in \mathbb{R}^n \quad \forall \xi \in \mathbb{R}^n \tag{3.27}$$

for some constants $\Lambda, \lambda > 0$.
We assume that for every $\xi \in \mathbb{R}^n$ the application

$$x \mapsto \langle G(x)\xi, \xi\rangle \qquad \text{is lower semi continuous.} \tag{3.28}$$

We denote by d the Riemannian distance on $\mathbb{R}^n$ induced by the metric tensor G. More precisely d is defined as follows:

$$d(x,y) = \inf\left\{\int_0^1 \sqrt{\langle G(\gamma(t))\dot\gamma(t), \dot\gamma(t)\rangle}\,dt : \gamma \in AC([0,1];\mathbb{R}^n),\ \gamma(0) = x,\ \gamma(1) = y\right\}. \tag{3.29}$$

Clearly the condition (3.27) implies that the distance d is equivalent to the euclidean one. We denote by $\mathbb{R}^n_G$ the metric space $(\mathbb{R}^n, d)$. From the equivalence of the distances it follows that the class of continuous functions is the same in the two topologies and $AC^p(I;\mathbb{R}^n) = AC^p(I;\mathbb{R}^n_G)$ for every $p \ge 1$.

The following Proposition shows that for absolutely continuous curves the metric derivative coincides with the norm (on the tangent space) of the pointwise derivative.

Proposition 3.8. *Let I be an open interval. If $u \in AC(I;\mathbb{R}^n_G)$ then $\dot u(t) := \lim_{h\to 0}\frac{u(t+h)-u(t)}{h}$ exists for a.e. $t \in I$ and*

$$|u'|(t) = \sqrt{\langle G(u(t))\dot u(t), \dot u(t)\rangle} \qquad \textit{for a.e. } t \in I. \tag{3.30}$$

Proof. The first assertion is classic and is proved in the more general setting of Banach spaces in Theorem 1.16.

In order to prove (3.30) we choose $t \in I$ such that $|u'|(t)$ and $\dot{u}(t)$ exist.
Using the curve $s \in [0,1] \mapsto u(t+sh)$ which connect $u(t)$ to $u(t+h)$ we can estimate

$$\begin{aligned} d(u(t+h),u(t)) &\leq \int_0^1 \sqrt{\langle G(u(t+sh))\dot{u}(t+sh)h, \dot{u}(t+sh)h\rangle}\, ds \\ &= \left| \int_t^{t+h} \sqrt{\langle G(u(\tau))\dot{u}(\tau), \dot{u}(\tau)\rangle}\, d\tau \right|. \end{aligned}$$

Then it follows that

$$|u'|(t) \leq \sqrt{\langle G(u(t))\dot{u}(t), \dot{u}(t)\rangle}$$

for any Lebesgue point of $\tau \mapsto \sqrt{\langle G(u(\tau))\dot{u}(\tau), \dot{u}(\tau)\rangle}$.
Conversely, by the assumption of lower semi continuity (3.28), for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\langle G(x)\xi, \xi\rangle \geq \langle G(u(t))\xi, \xi\rangle - \varepsilon|\xi|^2 \qquad \forall x \in B_{\delta_\varepsilon}(u(t)) \tag{3.31}$$

By the continuity of u there exists $h_\varepsilon > 0$ such that for every h, $|h| < h_\varepsilon$ we have $u(t+h) \in B_{\delta_\varepsilon}(u(t))$. By the ellipticity assumption (3.27), the symmetric matrix $G(u(t)) - \varepsilon I$ is positive definite when $\varepsilon < \lambda$. Since the Riemannian distance induced by G in $B_{\delta_\varepsilon}(u(t))$ coincides with d and $G(u(t)) - \varepsilon I$ is a constant metric tensor in $B_{\delta_\varepsilon}(u(t))$ (the geodesics in this last case are the segments), it follows that

$$d(u(t+h),u(t)) \geq \sqrt{\langle (G(u(t)) - \varepsilon I)(u(t+h) - u(t)), u(t+h) - u(t)\rangle}.$$

Dividing by $|h|$ and passing to the limit for $h \to 0$ we have that

$$|u'|(t) \geq \sqrt{\langle (G(u(t)) - \varepsilon I)\dot{u}(t), \dot{u}(t)\rangle}.$$

Being ε arbitrary we conclude. □

Remark 3.9. We observe that the property of lower semi continuity (3.28) has been assumed only to prove the inequality

$$|u'|(t) \geq \sqrt{\langle G(u(t))\dot{u}(t), \dot{u}(t)\rangle}.$$

The validity of Proposition 3.8 under the only assumption of uniform ellipticity and Borel measurability of the coefficients of G is an open problem.

3.2.2 The continuity equation in $\mathbb{R}^n_G$.

In view of the application to the study of evolution equations with variable coefficients (Section 4.2), we study the continuity equation in $\mathbb{R}^n$ endowed with the Riemannian distance induced by a metric tensor G, satisfying the properties described in the previous Subsection.

Since the Euclidean distance and the Riemannian distance d induced by G are equivalent by (3.27), the Wasserstein pseudo distances $W_{p,G}$, induced by the Riemannian distance d, and $W_{p,I}$, induced by the Euclidean distance, are equivalent in $\mathscr{P}(X)$. Therefore the class of absolutely continuous curves of probability measures coincide: $AC^p(I;(\mathscr{P}(\mathbb{R}^n), W_{p,G})) = AC^p(I;(\mathscr{P}(\mathbb{R}^n), W_{p,I}))$. Clearly the notion of solution of the continuity equation does not depend on the metric, whereas the minimal velocity vector field given by Theorem 3.7 is intimately linked to the metric.

Fixing $I = [0,T]$ and the summability exponent $p > 1$, we consider the set $EC_p(\mathbb{R}^n)$ exactly as in Section 3.1.

We observe that, in this case $(\mu_t, \boldsymbol{v}_t)$ is a solution of the continuity equation if and only if

$$\partial_t \mu_t + \operatorname{div}(\boldsymbol{v}_t \mu_t) = 0 \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{R}^n),$$

i.e. in the sense of distribution on $(0,T)\times\mathbb{R}^n$.

For a Borel vector field $\boldsymbol{v} : I\times\mathbb{R}^n \to \mathbb{R}^n$ and the measure $\bar\mu \in \mathscr{P}(I\times\mathbb{R}^n)$ associated to μ_t as in (3.1), we define

$$\|\boldsymbol{v}\|^p_{L^p_G(\bar\mu;\mathbb{R}^n)} = \int_0^T \int_{\mathbb{R}^n} \langle G(x)\boldsymbol{v}_t(x), \boldsymbol{v}_t(x)\rangle^{p/2}\, d\mu_t(x)\, dt.$$

Clearly $L^p_G(\bar\mu;\mathbb{R}^n)$ is a Banach space and the norm of $L^p_G(\bar\mu;\mathbb{R}^n)$ is equivalent to the norm of $L^p(\bar\mu;\mathbb{R}^n)$.

We denote by $|\mu'|_G$ the metric derivative of the absolutely continuous curve μ_t with respect to the distance $W_{p,G}$.

The following Theorem of existence and uniqueness of the minimal velocity field holds.

Theorem 3.10. *If $\mu \in AC^p(I;(\mathscr{P}(\mathbb{R}^n), W_{p,G}))$ then there exists a vector field $\tilde{\boldsymbol{v}} : I\times\mathbb{R}^n \to \mathbb{R}^n$ such that $(\mu, \tilde{\boldsymbol{v}}) \in EC_p(\mathbb{R}^n)$ and*

$$|\mu'|_G(t) = \|\tilde{\boldsymbol{v}}_t\|_{L^p_G(\mu_t;\mathbb{R}^n)} \qquad \textit{for a.e. } t \in I. \tag{3.32}$$

Proof. The proof can be carried out similarly to the proof for Banach spaces, by using Proposition 3.8 instead of Theorem 1.16. We give only a sketch of the proof.

We start with $\eta = \tilde\eta \in \mathscr{P}(\Gamma)$ given by Theorem 2.2 in $\mathbb{R}^n_G$ and we continue as in the proof of Theorem 3.2. In this finite dimensional case the problem of measurability of the map $\psi : I\times\Gamma \to \mathbb{R}^n$ defined by

$$\psi(t,u) = \dot u(t)$$

is much more simple. The minimal vector field is defined by

$$\tilde{\boldsymbol{v}}_t(x) := \int_\Gamma \dot u(t)\, d\bar\eta_{t,x}(u) \qquad \text{for } \bar\mu\text{-a.e. } (t,x) \in I\times\mathbb{R}^n, \tag{3.33}$$

and $\tilde{\boldsymbol{v}} \in L^p(\bar{\mu};\mathbb{R}^n)$. All the estimates of the proof can be carried out by using the equality (3.30) and observing that, for every $x \in \mathbb{R}^n$, the function $y \mapsto \langle G(x)y, y\rangle^{p/2} = |y|_x^p$ is convex. Then we easily obtain the inequality

$$|\mu'|_G(t) \geq \|\tilde{\boldsymbol{v}}_t\|_{L^p_G(\mu_t;\mathbb{R}^n)} \qquad \text{for a.e. } t \in I.$$

The assertion that $(\mu_t, \tilde{\boldsymbol{v}}_t)$ satisfies the continuity equation is proved as in the proof of Theorem 3.2. In order to prove the equality (3.32), since $(\mu_t, \tilde{\boldsymbol{v}}_t)$ satisfies the continuity equation and $\tilde{\boldsymbol{v}} \in L^p(\bar{\mu};\mathbb{R}^n)$, by Theorem 3.1 there exists $\eta \in \mathscr{P}(C([0,T];\mathbb{R}^n))$ such that $(e_t)_\#\eta = \mu_t$ and η is concentrated on the set
$\{u \in AC^p([0,T];\mathbb{R}^n) : u \text{ is an integral solution of } \dot{u}(t) = \tilde{\boldsymbol{v}}_t(u(t))\}$.
Taking $s, t \in I$ with $s < t$ and $\gamma_{s,t} := (e_s, e_t)_\#\eta \in \Gamma(\mu_s, \mu_t)$, we have

$$\begin{aligned} W^p_{p,G}(\mu_s,\mu_t) &\leq \int_{\mathbb{R}^n\times\mathbb{R}^n} d^p(x,y)\, d\gamma_{s,t}(x,y) = \int_\Gamma d^p(e_s(u), e_t(u))\, d\eta(u) \\ &\leq \int_\Gamma (t-s)^{p-1} \int_s^t \langle G(u(r))\dot{u}(r), \dot{u}(r)\rangle^{p/2}\, dr\, d\eta(u) \\ &= \int_\Gamma (t-s)^{p-1} \int_s^t \langle G(u(r))\tilde{\boldsymbol{v}}_r(u(r)), \tilde{\boldsymbol{v}}_r(u(r))\rangle^{p/2}\, dr\, d\eta(u) \\ &= (t-s)^{p-1} \int_s^t \int_{\mathbb{R}^n} \langle G(x)\tilde{\boldsymbol{v}}_r(x), \tilde{\boldsymbol{v}}_r(x)\rangle^{p/2}\, d\mu_r(x)\, dr. \end{aligned}$$

Lebesgue differentiation Theorem implies that

$$|\mu'|_G(t) \leq \|\tilde{\boldsymbol{v}}_t\|_{L^p_G(\mu_t;\mathbb{R}^n)} \qquad \text{for a.e. } t \in I.$$

The uniqueness of $\tilde{\boldsymbol{v}}$ is a consequence of the linearity of the continuity equation with respect to the vector field, the strict convexity of the norm in $L^p_G(\mu_t;\mathbb{R}^n)$ and the minimal property.
□

Chapter 4

Variable coefficients diffusion equations and gradient flows in Wasserstein spaces

In this Chapter we study a class of nonlinear diffusion equations of the type

$$\partial_t u_t(x) - \operatorname{div}(A(x)(\nabla f(u_t(x)) + u_t(x)\nabla V(x)) = 0, \qquad \text{in } (0,+\infty)\times\mathbb{R}^n \tag{4.1}$$

where A is a symmetric matrix valued function, depending on the spatial variable $x \in \mathbb{R}^n$, satisfying a uniform ellipticity condition, f is a suitable non decreasing function and V is a convex function. We interpret this equation as gradient flow of the energy functional

$$\phi(u) := \int_{\mathbb{R}^n} F(u(x))\,dx + \int_{\mathbb{R}^n} V(x)u(x)\,dx \tag{4.2}$$

in the Wasserstein space $\mathscr{P}_2(\mathbb{R}^n_G)$, where $\mathbb{R}^n_G$ denotes the space $\mathbb{R}^n$ endowed with the Riemannian distance induced by the metric tensor $G = A^{-1}$ (the uniform ellipticity condition implies that A^{-1} exists). Here F is a suitable convex function linked to f by the relation $f(u) = F'(u)u - F(u)$.

Referring to the Introduction for bibliographical references on this subject, here we recall that the gradient flow structure, with respect to the 2-Wasserstein distance, for the linear Fokker Planck equation

$$\partial_t u_t(x) - \operatorname{div}(\nabla u_t(x) + u_t(x)\nabla V(x)) = 0, \qquad \text{in } (0,+\infty)\times\mathbb{R}^n \tag{4.3}$$

has been pointed out by [JKO98].
The authors prove that for an initial datum $u_0 \geq 0$, $\|u_0\|_{L^1(\mathbb{R}^n)} = 1$, $\int_{\mathbb{R}^n} |x|^2 u_0(x)\,dx < +\infty$, a solution of (4.3) can be constructed as limits of the (variational formulation of the) Euler

implicit time discretization of the gradient flow induced by the functional

$$\phi(u) := \int_{\mathbb{R}^n} u(x)\log(u(x))\,dx + \int_{\mathbb{R}^n} V(x)u(x)\,dx \tag{4.4}$$

on the set $\{u \in L^1(\mathbb{R}^n) : u \geq 0,\ \|u\|_1 = 1,\ \int_{\mathbb{R}^n} |x|^2 u(x)\,dx < +\infty\}$ (here we identify the probability measures with their densities with respect to the Lebesgue measure $\mathscr{L}^n$) with respect to the 2-Wasserstein distance. Namely, given a time step $\tau > 0$ and an initial datum u_0, they consider the sequence (u^k) obtained by the recursive minimization of

$$u \mapsto \frac{1}{2\tau} W_2^2(u, u^{k-1}) + \phi(u), \quad k = 1, 2, \cdots, \tag{4.5}$$

with the initial condition $u^0 = u_0$, and the related piecewise constant functions $\overline{U}_{\tau,t} := u^{[t/\tau]}$, $t > 0$, ($[t/\tau]$ denoting the integer part of t/τ) interpolating the values u^k on a uniform grid $\{0, \tau, 2\tau, \cdots, k\tau, \cdots\}$ of step size τ. The authors prove the convergence for $\tau \downarrow 0$ of $\overline{U}_\tau$ to a solution of (4.3), when V is a smooth non negative function with the gradient satisfying a suitable condition of growth.

Under the assumption that the matrix function A is sufficiently regular (for instance of class C^2), V is smooth and $f(u) = u$, the proof of [JKO98] can be modified in order to show that (4.1) is the gradient flow of the functional (4.2) with respect to the 2-Wasserstein distance on $\mathbb{R}^n_G$, where $F(u) = u \log u$. When A is less regular than C^2, this kind of proof fails, even if the algorithm (4.5) is still meaningful.

Our purpose is to prove the convergence, to a solution of the equation, of the piecewise constant functions constructed with the variational algorithm of minimizing movements, under minimal assumption of regularity on A and for a larger class of functions F. This is possible by combining the theory of curves of maximal slope in metric spaces, recalled in Section 1.7, and our Theorem 3.10 of existence and uniqueness of the minimal vector field.

Moreover we study the problem of the asymptotic behaviour of the solutions when the stationary state exists and is unique. When the potential V is uniformly convex, we prove that the solution converges with exponential decay, as $t \to \infty$, to the stationary state. We have convergence in entropy and in Wasserstein distance. In the case of linear diffusion we also have convergence in L^1 norm.

Finally, in a particular case, we study the problem of the contractivity of the Wasserstein distance along the solutions.

The results of this Chapter are contained in the paper [Lis06].

4.1 Internal and Potential energy functionals

In this Section we introduce the internal energy and potential energy functionals.
Let $F : [0, +\infty) \to \mathbb{R}$ be a proper, lower semi continuous convex function. We assume that

$$F \in C^1(0, +\infty), \tag{4.6}$$

$$F(0) = 0, \qquad \lim_{z \downarrow 0} \frac{F(z)}{z^\alpha} > -\infty. \tag{4.7}$$

for some $\alpha > \frac{n}{n+2}$, and that F has superlinear growth at infinity, i.e.

$$\lim_{z \to +\infty} \frac{F(z)}{z} = +\infty. \tag{4.8}$$

Let us define the internal energy functional $\mathscr{F} : \mathscr{P}_2(\mathbb{R}^n) \to (-\infty, +\infty]$ as follows:

$$\mathscr{F}(\mu) = \begin{cases} \int_{\mathbb{R}^n} F(u(x))\,dx & \text{if } \mu = u\mathscr{L}^n \in \mathscr{P}_2^r(\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.9}$$

The condition (4.7) ensures that F^-, the negative part of F, is integrable on $\mathbb{R}^n$. By the superlinear growth condition (4.8), the functional $\mathscr{F}$ is lower semi continuous with respect to the narrow convergence on W_2-bounded sequences, in particular is lower semi continuous in $\mathscr{P}_2(\mathbb{R}^n)$ (see e.g. [AFP00]).

Let $V : \mathbb{R}^n \to (-\infty, +\infty]$ be a function such that

$$V \text{ is bounded from below and lower semi continuous.} \tag{4.10}$$

We define the potential energy functional $\mathscr{V} : \mathscr{P}_2(\mathbb{R}^n) \to (-\infty, +\infty]$ as follows:

$$\mathscr{V}(\mu) = \int_{\mathbb{R}^n} V(x)\,d\mu(x). \tag{4.11}$$

By the assumption (4.10) and (1.11), the functional $\mathscr{V}$ is lower semi continuous with respect to the narrow convergence and, consequently, with respect to the Wasserstein distance.

Denoting by $\mathrm{Int}(A)$ the internal part of the set A, we assume that

$$\Omega := \mathrm{Int}(D(V)) \neq \emptyset, \qquad \mathscr{L}^n(\partial\Omega) = 0. \tag{4.12}$$

We consider the sum of internal energy and potential energy functional $\phi : \mathscr{P}_2(\mathbb{R}^n) \to (-\infty, +\infty]$ defined by

$$\phi(\mu) := \mathscr{F}(\mu) + \mathscr{V}(\mu). \tag{4.13}$$

We observe that by the definitions of $\mathscr{F}$ and $\mathscr{V}$, if $\mu \in D(\phi)$, then $\operatorname{supp}\mu \subset \overline{\Omega}$, $\mu(\partial\Omega) = 0$ and $\mu \in \mathscr{P}_2^r(\mathbb{R}^n)$. It follows that the functional ϕ can be thought in the space $\mathscr{P}_2(\Omega)$ and all the integrals on the whole $\mathbb{R}^n$ with respect to the measure μ are in effect integrals on Ω.

4.2 Diffusion equations with variable coefficients

Let us consider a Borel measurable matrix-valued function $A : \mathbb{R}^n \to \mathbb{M}^{n\times n}$.
We assume that $A(x)$ is symmetric for every $x \in \mathbb{R}^n$ and satisfies the following uniform ellipticity condition:

$$\lambda|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \qquad \forall x \in \mathbb{R}^n \quad \forall \xi \in \mathbb{R}^n \tag{4.14}$$

for some constants $\Lambda, \lambda > 0$.
We observe that $G := A^{-1}$ is well defined (thanks to (4.14)), it is symmetric, and continuous. Moreover it satisfies a uniform ellipticity condition:

$$\Lambda^{-1}|\xi|^2 \le \langle G(x)\xi, \xi\rangle \le \lambda^{-1}|\xi|^2 \qquad \forall x \in \mathbb{R}^n \quad \forall \xi \in \mathbb{R}^n. \tag{4.15}$$

We assume that A is such that G satisfies the lower semi continuity property (3.28).

Given $F : [0, +\infty) \to \mathbb{R}$ satisfying (4.6), (4.7), (4.8), and , $V : \mathbb{R}^n \to (-\infty, +\infty]$ satisfying (4.10), (4.12), we also assume that

$$V \quad \text{is convex,} \tag{4.16}$$

$$\text{the map} \quad z \mapsto z^n F(z^{-n}) \qquad \text{is convex and not increasing,} \tag{4.17}$$

and F is doubling: i.e. there exists a constant $C > 0$ such that

$$F(z + \tilde{z}) \le C(1 + F(z) + F(\tilde{z})) \qquad \forall z, \tilde{z} \in [0, +\infty). \tag{4.18}$$

We also define the function $L_F : [0, +\infty) \to \mathbb{R}$ as follows:

$$L_F(z) := zF'(z) - F(z).$$

We observe that the convexity of F implies that $L_F \ge 0$ and L_F is not decreasing.
We also point out that the convexity of V implies that V is locally Lipschitz in Ω and consequently $\mathscr{L}^n$-a.e. differentiable in Ω (for these classical results see e.g. [EG92]). We denote by ∇V its gradient defined $\mathscr{L}^n$-a.e..

We will study partial differential equations of diffusion type given by the continuity equation

$$\partial_t u + \operatorname{div}(\boldsymbol{v}u) = 0 \qquad \text{in } \Omega \times (0, +\infty) \tag{4.19}$$

with the velocity vector field of the form

$$\boldsymbol{v}u = -A\nabla L_F(u) - A\nabla V u. \tag{4.20}$$

Example 4.1. The main examples of functions F satisfying the above assumptions are:

$$F(z) = z \log z \tag{4.21}$$

and

$$F(z) = \frac{1}{\alpha - 1} z^\alpha \qquad \text{for } \alpha > 1. \tag{4.22}$$

The partial differential equation corresponding to (4.21) and to (4.22), respectively, are the heat equation with variable coefficients

$$\partial_t u - \operatorname{div}(A \nabla u) = 0, \tag{4.23}$$

and the porous medium equation with variable coefficients

$$\partial_t u - \operatorname{div}(A \nabla u^\alpha) = 0. \tag{4.24}$$

These equations describe a diffusion in a non isotropic and non homogeneous material. □

The case of the matrix function A equal to the identity matrix I is deeply studied in [AGS05]. Here we only recall that, in the case $A = I$, the assumptions (4.16), (4.17) together to all the assumptions of Subsection 4.1 ensure that the functional ϕ is convex along geodesics in the space $(\mathscr{P}_2(\mathbb{R}^n), W_{2,I})$.

In the case $A = I$, the local slope defined (see (1.28) in) by

$$|\partial\phi|_I(\mu) := \limsup_{W_{2,I}(\mu,\nu)\to 0} \frac{(\phi(\mu) - \phi(\nu))^+}{W_{2,I}(\mu,\nu)}$$

is (see [AGS05] Theorem 10.4.6)

$$|\partial\phi|_I(\mu) = \begin{cases} \left\| \frac{\nabla L_F(u)}{u} + \nabla V \right\|_{L^2_I(\mu;\mathbb{R}^n)} & \text{if } \mu \in D(|\partial\phi|_I) \\ +\infty & \text{otherwise} \end{cases} \tag{4.25}$$

where

$$D(|\partial\phi|_I) = \{\mu = u\mathscr{L}^n \in D(\phi) : L_F(u) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n), \frac{\nabla L_F(u)}{u} + \nabla V \in L^2(\mu;\mathbb{R}^n)\}.$$

Imitating this known case, we can define the functional

$$g(\mu) := \begin{cases} \left\| A\frac{\nabla L_F(u)}{u} + A\nabla V \right\|_{L^2_G(\mu;\mathbb{R}^n)} & \text{if } \mu \in D(g) \\ +\infty & \text{otherwise} \end{cases} \tag{4.26}$$

where $D(g) := D(|\partial\phi|_I)$.

It is not known if the function g is the local slope of ϕ with respect to the distance $W_{2,G}$,

$$|\partial\phi|_G(\mu) := \limsup_{W_{2,G}(\mu,\nu)\to 0} \frac{(\phi(\mu) - \phi(\nu))^+}{W_{2,G}(\mu,\nu)}$$

but, in view of an application of the metric theory illustrated in Section 1.7, in particular Theorem 1.20, it is sufficient to show that g is an upper gradient for ϕ in the space $(\mathscr{P}_2(\mathbb{R}^n), W_{2,G})$, according to Definition 1.18, and satisfies the inequality $g(\mu) \le |\partial^-\phi|_G(\mu)$ for every $\mu \in D(g)$, where the relaxed slope $|\partial^-\phi|_G$, defined in (1.36) is

$$|\partial^-\phi|_G(\mu) := \inf\left\{ \liminf_{n\to+\infty} |\partial\phi|_G(\mu_n) : \mu_n \rightharpoonup \mu \text{ narrowly}, \sup_n W_{2,G}(\mu_n,\mu) < +\infty, \sup_n \phi(\mu_n) < +\infty \right\}.$$

We start proving two crucial lemmata. The first, Lemma 4.2, is a sort of chain rule in Wasserstein space. The proof uses the similar result in the "flat" space $\mathbb{R}^n$, which holds since the functional ϕ is geodesically convex in the space $(\mathscr{P}_2(\mathbb{R}^n), W_{2,I})$. The second, Lemma 4.3, shows that g is a strong upper gradient, in the metric space $(\mathscr{P}_2(\mathbb{R}^n), W_{2,G})$, and the inequality $g(\mu) \leq |\partial^- \phi|_G(\mu)$ holds.

Lemma 4.2. *Let $\mu \in AC^2_{\mathrm{loc}}(I; \mathscr{P}_2(\mathbb{R}^n_G))$ and $\tilde{\boldsymbol{v}}_t$ the vector field given by Theorem 3.10. If*

$$t \mapsto g(\mu_t)|\mu'|_G(t) \in L^1_{\mathrm{loc}}(I) \tag{4.27}$$

then $\phi \circ \mu_t$ is locally absolutely continuous and the chain rule holds:

$$\frac{d}{dt}\phi(\mu_t) = \int_{\mathbb{R}^n} \left\langle \frac{\nabla L_F(u_t(x))}{u_t(x)} + \nabla V(x), \tilde{\boldsymbol{v}}_t(x) \right\rangle d\mu_t(x) \qquad \text{for a.e. } t \in I, \tag{4.28}$$

where $\mu_t = u_t \mathscr{L}^n$.

Proof. By the equivalence of the distances W_G and W_I, (4.27) yields that $\int_J |\partial\phi|_I(\mu_t)|\mu'|_I(t)\,dt < +\infty$ for every bounded interval $J \subset (0, +\infty)$. Since ϕ is convex along geodesics in $(\mathscr{P}_2(\mathbb{R}^n), W_I)$ we have that (see Section 10.1.2 of [AGS05]) $\phi \circ \mu_t$ is absolutely continuous and the Wasserstein chain rule holds:

$$\frac{d}{dt}\phi(\mu_t) = \int_{\mathbb{R}^n} \left\langle \frac{\nabla L_F(u_t(x))}{u_t(x)} + \nabla V(x), \boldsymbol{v}_t(x) \right\rangle d\mu_t(x) \qquad \text{for a.e. } t \in I \tag{4.29}$$

where $\boldsymbol{v}_t$ is the vector field associated to the curve μ_t given by Theorem 3.10 in the case of the identity matrix $G = I$.

Moreover (see [AGS05] Corollary 10.3.15) the vector field $\frac{\nabla L_F(u_t(x))}{u_t(x)}$ belongs to the closure of $\{\nabla\varphi : \varphi \in C^\infty_c(\mathbb{R}^n)\}$ in the topology of $L^2(\mu_t; \mathbb{R}^n)$. Since Theorem 3.10 implies that $\mathrm{div}((\boldsymbol{v}_t - \tilde{\boldsymbol{v}}_t)\mu_t) = 0$, we have

$$\int_{\mathbb{R}^n} \left\langle \frac{\nabla L_F(u_t(x))}{u_t(x)} + \nabla V(x), \boldsymbol{v}_t(x) \right\rangle d\mu_t(x) = \int_{\mathbb{R}^n} \left\langle \frac{\nabla L_F(u_t(x))}{u_t(x)} + \nabla V(x), \tilde{\boldsymbol{v}}_t(x) \right\rangle d\mu_t(x).$$

□

Lemma 4.3. *The function g defined in (4.26) is a strong upper gradient for ϕ, lower semi continuous with respect to the narrow convergence in bounded sublevel sets of ϕ, more precisely*

$$\mu_n \to \mu \quad \text{narrowly}, \quad \sup_{m,n} W_{2,G}(\mu_m, \mu_n) < +\infty, \quad \sup_n \phi(\mu_n) < +\infty \quad \Longrightarrow \quad \liminf_{n\to\infty} g(\mu_n) \geq g(\mu). \tag{4.30}$$

Moreover the following inequality holds

$$g(\mu) \leq |\partial^- \phi|_G(\mu) \qquad \forall \mu \in D(g). \tag{4.31}$$

Proof. First of all we prove that g is an upper gradient.

Let $\mu \in AC^2_{\rm loc}(I; \mathscr{P}_2(\mathbb{R}^n))$ such that $t \mapsto g(\mu_t)|\mu'|_G(t) \in L^1_{\rm loc}(I)$. By Lemma 4.2, Cauchy-Schwartz inequality and Theorem 3.10, we have

$$\begin{aligned}\left|\frac{d}{dt}\phi(\mu_t)\right| &= \left|\int_{\mathbb{R}^n} \left\langle \frac{\nabla L_F(u_t(x))}{u_t(x)} + \nabla V(x), \tilde{\boldsymbol{v}}_t(x)\right\rangle d\mu_t(x)\right| \\ &= \left|\int_{\mathbb{R}^n} \langle A(x)\left(\frac{\nabla L_F(u_t(x))}{u_t(x)} + \nabla V(x)\right), \tilde{\boldsymbol{v}}_t(x)\rangle_G \, d\mu_t(x)\right| \\ &\le \left\| A\frac{\nabla L_F(u_t)}{u_t} + A\nabla V\right\|_{L^2_G(\mu_t;\mathbb{R}^n)} \|\tilde{\boldsymbol{v}}_t\|_{L^2_G(\mu_t;\mathbb{R}^n)} = g(\mu_t)|\mu'|_G(t)\end{aligned}$$

which states that g is an upper gradient.

Now we prove that

$$g(\mu) \le |\partial\phi|_G(\mu) \qquad \forall \mu \in D(g). \tag{4.32}$$

Let $\boldsymbol{\xi} \in C^\infty_c(\Omega;\mathbb{R}^n)$ be a regular vector field and we denote by $X(t,x)$ the flow associated to $\boldsymbol{\xi}$: i.e. the unique maximal solution of the Cauchy problem

$$\dot{X}(t,x) = \boldsymbol{\xi}(X(t,x)), \quad X(0,x) = x. \tag{4.33}$$

Fixing $\mu \in D(g)$ we consider the curve $\mu_t := X(t,\cdot)_\#\mu$ and we observe that for t sufficiently small $\mu_t \in D(\phi)$.

By the definition of W_G, using the admissible plan $(\boldsymbol{i}(\cdot), X(t,\cdot))_\#\mu$, and the definition of Riemannian distance d we have

$$\begin{aligned} W^2_G(\mu,\mu_t) &\le \int_{\mathbb{R}^n} d^2(x, X(t,x))\, d\mu(x) \le \int_{\mathbb{R}^n} t\int_0^t \langle G(X(s,x))\dot{X}(s,x), \dot{X}(s,x)\rangle\, ds\, d\mu(x) \\ &= t\int_{\mathbb{R}^n}\int_0^t \langle G(X(s,x))\boldsymbol{\xi}(X(s,x)), \boldsymbol{\xi}(X(s,x))\rangle\, ds\, d\mu(x) \\ &= t\int_0^t \|\boldsymbol{\xi}\|^2_{L^2_G(\mu_s;\mathbb{R}^n)}\, ds. \end{aligned} \tag{4.34}$$

On the other hand, changing variable in the integral, by the regularity of the flow $X(t,\cdot)$, for t sufficiently small we obtain,

$$\begin{aligned}\mathscr{F}(\mu_t) - \mathscr{F}(\mu) &= \int_{\mathbb{R}^n} F\left(\frac{u(x)}{\det(\nabla(X(t,x)))}\right)\det(\nabla(X(t,x))) - F(u(x))\, dx \\ &= \int_{\mathbb{R}^n} \Psi(u(x), \det(\nabla(X(t,x))) - \Psi(u(x),1)\, dx\end{aligned}$$

where $\Psi(z,s) := sF(\frac{z}{s})$ is defined for $z \in [0,+\infty)$ and $s \in (0,+\infty)$.

An elementary computation shows that $\frac{\partial}{\partial s}\Psi(s,z) = -L_F(\frac{z}{s})$. Therefore

$$\begin{aligned}\lim_{t\downarrow 0}\frac{\mathscr{F}(\mu_t) - \mathscr{F}(\mu)}{t} &= \int_{\mathbb{R}^n} \frac{d}{dt}\Psi(u(x), \det(\nabla(X(t,x)))_{|t=0}\, dx \\ &= -\int_{\mathbb{R}^n} L_F(u(x))\frac{d}{dt}\det(\nabla(X(t,x)))_{|t=0}\, dx = -\int_{\mathbb{R}^n} L_F(u(x))\,\mathrm{div}\,\boldsymbol{\xi}(x)\, dx \\ &= \int_{\mathbb{R}^n} \langle\nabla L_F(u(x)), \boldsymbol{\xi}(x)\rangle\, dx. \end{aligned} \tag{4.35}$$

Moreover, since V is locally Lipschitz in Ω and $\boldsymbol{\xi}$ has compact support in Ω, we have

$$\begin{aligned}\lim_{t\downarrow 0}\frac{\mathscr{V}(\mu_t)-\mathscr{V}(\mu)}{t} &= \lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^n}(V(X(t,x))-V(x))u(x)\,dx\\ &= \int_{\mathbb{R}^n}\langle\nabla V(x),\boldsymbol{\xi}(x)\rangle u(x)\,dx. \end{aligned}\tag{4.36}$$

We observe that, since $u_t \rightharpoonup u$ weakly in $L^1(\Omega)$ when $t\to 0$ by the regularity of $\boldsymbol{\xi}$ and the ellipticity condition (4.14) the application $t\mapsto \|\boldsymbol{\xi}\|^2_{L^2_G(\mu_t;\mathbb{R}^n)}$ is continuous in 0. Consequently we have

$$\lim_{t\downarrow 0}\left(\frac{1}{t}\int_0^t \|\boldsymbol{\xi}\|^2_{L^2_G(\mu_s;\mathbb{R}^n)}\,ds\right)^{\frac{1}{2}} = \|\boldsymbol{\xi}\|_{L^2_G(\mu;\mathbb{R}^n)}\,. \tag{4.37}$$

Finally, (4.34), (4.35), (4.36), (4.37) yield

$$\begin{aligned}|\partial\phi|_G(\mu) &\geq \lim_{t\downarrow 0}\frac{\phi(\mu_t)-\phi(\mu)}{W_G(\mu_t,\mu)} \geq \lim_{t\downarrow 0}\frac{\phi(\mu_t)-\phi(\mu)}{t\left(\frac{1}{t}\int_0^t\|\boldsymbol{\xi}\|^2_{L^2_G(\mu_s;\mathbb{R}^n)}\,ds\right)^{\frac{1}{2}}}\\ &= \left(\lim_{t\downarrow 0}\frac{\mathscr{F}(\mu_t)-\mathscr{F}(\mu)}{t}+\lim_{t\downarrow 0}\frac{\mathscr{V}(\mu_t)-\mathscr{V}(\mu)}{t}\right)\lim_{t\downarrow 0}\frac{1}{\left(\frac{1}{t}\int_0^t\|\boldsymbol{\xi}\|^2_{L^2_G(\mu_s;\mathbb{R}^n)}\,ds\right)^{\frac{1}{2}}}\\ &= \frac{1}{\|\boldsymbol{\xi}\|_{L^2_G(\mu;\mathbb{R}^n)}}\int_{\mathbb{R}^n}\langle\nabla L_F(u(x))+\nabla V(x)u(x),\boldsymbol{\xi}(x)\rangle\,dx\\ &= \frac{1}{\|\boldsymbol{\xi}\|_{L^2_G(\mu;\mathbb{R}^n)}}\int_{\mathbb{R}^n}\langle A(x)\frac{\nabla L_F(u(x))}{u(x)}+A(x)\nabla V(x),\boldsymbol{\xi}(x)\rangle_G\,d\mu(x).\end{aligned}$$

By density and duality formula for the norm in $L^2_G(\mu;\mathbb{R}^n)$ we obtain (4.32).

Now we prove (4.30). Setting

$$l := \liminf_n g(\mu_n),$$

it is not restrictive to assume (eventually extracting a subsequence) that

$$\sup_n g(\mu_n) < +\infty, \qquad \lim_n g(\mu_n) = l < +\infty, \tag{4.38}$$

since the case $l=+\infty$ is obvious.

Denoting by $\boldsymbol{w}_n u_n := A^{\frac{1}{2}}(\nabla L_F(u_n)+\nabla V u_n)$ we can write $g(\mu_n)=\int_{\mathbb{R}^n}|\boldsymbol{w}_n|^2\,d\mu_n$. By (4.38) and the narrow convergence of μ_n to μ, Theorem 5.4.4 of [AGS05] states that there exists a vector field $\boldsymbol{w}\in L^2(\mu;\mathbb{R}^n)$ such that, up to extracting a subsequence,

$$\lim_n\int_{\mathbb{R}^n}\langle\varphi,\boldsymbol{w}_n\rangle\,d\mu_n = \int_{\mathbb{R}^n}\langle\varphi,\boldsymbol{w}\rangle\,d\mu \qquad \forall\varphi\in C^0_c(\mathbb{R}^n;\mathbb{R}^n), \tag{4.39}$$

and

$$\liminf_n\int_{\mathbb{R}^n}|\boldsymbol{w}_n|^2\,d\mu_n \geq \int_{\mathbb{R}^n}|\boldsymbol{w}|^2\,d\mu. \tag{4.40}$$

We must only prove that

$$A^{\frac{1}{2}}(x)(\nabla L_F(u(x))+\nabla V(x)u(x)) = \boldsymbol{w}(x)u(x) \qquad \text{for } \mathscr{L}^n\text{-a.e. } x\in\mathbb{R}^n. \tag{4.41}$$

Since $\sup_n \mathscr{F}(\mu_n) < +\infty$, $\sup_n W_2(\mu_n, \mu) < +\infty$ and F is superlinear, by weak compactness in $L^1(\Omega)$ we have that $u_n \rightharpoonup u$ weakly in $L^1(\Omega)$. Since ∇V is locally bounded in Ω we have

$$\lim_n \int_{\mathbb{R}^n} \langle \varphi, A^{\frac{1}{2}} \nabla V(x) \rangle u_n(x)\, dx = \int_{\mathbb{R}^n} \langle \varphi, A^{\frac{1}{2}} \nabla V(x) \rangle u(x)\, dx \qquad \forall \varphi \in C_c^0(\Omega; \mathbb{R}^n). \tag{4.42}$$

Now we show that the sequence $L_F(u_n)$ is bounded in $BV_{\rm loc}(\mathbb{R}^n)$, i.e., for every $\tilde{\Omega} \subset\subset \mathbb{R}^n$ the sequence $L_F(u_n)_{|\tilde{\Omega}}$ is bounded in $BV(\tilde{\Omega})$.
The convexity of F yields $L_F(u) \le F(2u) - 2F(u)$, from which, by the doubling condition (4.18), we obtain that $L_F(u_n)$ is bounded in $L^1_{\rm loc}(\mathbb{R}^n)$. Moreover

$$\int_{\mathbb{R}^n} |\nabla L_F(u_n(x))|\, dx = \int_{\mathbb{R}^n} \left| \frac{\nabla L_F(u_n(x))}{u_n(x)} \right| u_n(x)\, dx \le \left(\int_{\mathbb{R}^n} \left| \frac{\nabla L_F(u_n(x))}{u_n(x)} \right|^2 u_n(x)\, dx \right)^{\frac{1}{2}} \tag{4.43}$$

which is bounded by (4.38) and (4.14).
By compactness in $BV_{\rm loc}(\mathbb{R}^n)$ (see Theorem 3.23 of [AFP00]) there exists a function $L \in BV_{\rm loc}(\mathbb{R}^n)$ such that, up to considering a subsequence, $L_F(u_n) \to L$ in $L^1_{\rm loc}(\mathbb{R}^n)$. Again, up to considering a subsequence, we can suppose that $L_F(u_n(x)) \to L(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. By the monotonicity of the application $z \mapsto L_F(z)$ and a truncation argument we can prove that $L(x) = L_F(u(x))$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. Then for every $\varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^n)$ we have (see for instance Proposition 3.13 of [AFP00])

$$\lim_n \int_{\mathbb{R}^n} \langle \varphi, \nabla L_F(u_n) \rangle\, dx = \int_{\mathbb{R}^n} \langle \varphi, \nabla L_F(u) \rangle\, dx. \tag{4.44}$$

Since for every measurable subset $B \subset \mathbb{R}^n$ we have

$$\begin{aligned} \int_B |\nabla L_F(u_n(x))|\, dx &= \int_B \left| \frac{\nabla L_F(u_n(x))}{\sqrt{u_n(x)}} \right| \sqrt{u_n(x)}\, dx \\ &\le \left(\int_B \frac{|\nabla L_F(u_n(x))|^2}{u_n(x)}\, dx \right)^{\frac{1}{2}} \left(\int_B u_n(x)\, dx \right)^{\frac{1}{2}} \end{aligned}$$

the equintegrability of $\{u_n\}$ (recall that u_n is L^1 weakly convergent) and the bound (4.38) imply the equintegrability of $\{\nabla L_F(u_n)\}$. Then the convergence of $\nabla L_F(u_n)$ is also weak in L^1, and the symmetry of A implies that

$$\lim_n \int_{\mathbb{R}^n} \langle \varphi, A^{\frac{1}{2}} \nabla L_F(u_n) \rangle\, dx = \int_{\mathbb{R}^n} \langle \varphi, A^{\frac{1}{2}} \nabla L_F(u) \rangle\, dx \qquad \forall \varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^n). \tag{4.45}$$

Taking into account (4.39), (4.42) and (4.45) we deduce (4.41).

The inequality (4.31) is a consequence of the definition (1.36) of $|\partial^- \phi|_G$, the property (4.30) and the inequality (4.32). □

Minimizing movements approximation scheme and convergence to a solution of the equation. Now, thanks to the two previous Lemmata, we can apply the general

metric theory of Section 1.7. The convergence of the minimizing movement approximation scheme introduced in Subsection 1.7.1 shows both existence and approximation for solutions of the equation (4.1).
Given a time step $\tau > 0$ and $\mu_0 = u_0\mathscr{L}^n \in D(\phi)$ we define recursively a sequence in $\mathscr{P}_2(\mathbb{R}^n)$ setting

$$M_\tau^0 = \mu_0; \qquad M_\tau^n \text{ is a minimizer in } \mathscr{P}_2(\mathbb{R}^n) \text{ of the function } \nu \mapsto \frac{1}{2\tau}W_{2,G}^2(\nu, M_\tau^{n-1}) + \phi(\nu). \tag{4.46}$$

Since the hypothesis (H1), (H2), (H3), (H4) of Section 1.7 are satisfied when the topology σ is the narrow topology on $\mathscr{P}(\mathbb{R}^n)$, defining the piecewise constant function $\overline{M}_\tau : [0, +\infty) \to \mathscr{P}_2(\mathbb{R}^n)$ as follows

$$\overline{M}_\tau(t) := M_\tau^k \quad \text{if} \quad t \in ((k-1)\tau, k\tau], \tag{4.47}$$

there exists a sequence $\tau_n \to 0$ and a curve $\mu \in AC^2_{\mathrm{loc}}([0, +\infty); \mathscr{P}_2(\mathbb{R}^n))$ such that

$$\overline{M}_{\tau_n}(t) \quad \text{narrowly converges to} \quad \mu_t \qquad \forall t \in [0, +\infty). \tag{4.48}$$

Every limit curve μ of this kind is called *generalized minimizing movement* for ϕ starting from μ_0.

The following Theorem is the main result of this Chapter.

Theorem 4.4 (Existence and convergence)**.** *If $\mu_0 = u_0\mathscr{L}^n \in D(\phi)$ then a generalized minimizing movement μ_t starting from μ_0 is a curve of maximal slope for ϕ with respect to the upper gradient g defined in (4.26) and the energy identity holds*

$$\frac{1}{2}\int_s^t |\mu'|_G^2(r)\,dr + \frac{1}{2}\int_s^t g^2(\mu_r)\,dr = \phi(\mu_s) - \phi(\mu_t) \qquad \forall s, t \in [0, +\infty), \quad s < t. \tag{4.49}$$

Moreover, if $\overline{M}_{\tau_n}(t)$ is a sequence of discrete solutions satisfying (4.48), we have

$$\lim_{n\to\infty} \phi(\overline{M}_{\tau_n}(t)) = \phi(\mu_t) \qquad \forall t \in [0, +\infty), \tag{4.50}$$

$$\lim_{n\to\infty} |\partial\phi|_G(\overline{M}_{\tau_n}(t)) = g(\mu_t) \qquad \text{in } L^2_{\mathrm{loc}}([0, +\infty)), \tag{4.51}$$

Moreover

$$|\mu'|(t) = g(\mu_t) = |\partial^-\phi|_G(\mu_t) \qquad \text{for a.e. } t \in [0, +\infty), \tag{4.52}$$

the function $t \mapsto \phi(\mu_t)$ is locally absolutely continuous and

$$\frac{d}{dt}\phi(\mu_t) = -g(\mu_t)^2 \qquad \text{for a.e. } t \in [0, +\infty). \tag{4.53}$$

The function u defined by the curve $\mu_t = u_t\mathscr{L}^n$ is a nonnegative distributional solution of the variable coefficients nonlinear diffusion equation

$$\partial_t u - \mathrm{div}(A\nabla(L_F(u)) + A\nabla V u) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}^n \tag{4.54}$$

satisfying $L_F(u) \in L^1_{\mathrm{loc}}((0, +\infty); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n))$ and $t \mapsto g^2(\mu_t) \in L^1_{\mathrm{loc}}(0, +\infty)$.

The notion of solution of the equation (4.54) is

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi(x)u(t,x)\,dx=\int_{\mathbb{R}^n}\langle A(x)\nabla(L_F(u(t,x)))+A(x)\nabla V(x)u(t,x),\nabla\varphi(x)\rangle\,dx \qquad \forall\varphi\in C^1_b(\mathbb{R}^n), \tag{4.55}$$

where the equality is understood in the sense of distributions in $(0,+\infty)$ and $C^1_b(\mathbb{R}^n)$ is the space of differentiable bounded functions $\varphi:\mathbb{R}^n\to\mathbb{R}$ with $\nabla\varphi$ bounded.

Proof. The first part of the assertion is an application of Theorem 1.20, since the hypothesis are satisfied in the metric space $(\mathscr{P}_2(\mathbb{R}^n),W_{2,G})$ where the weak topology σ is the narrow topology.

We show that $\mu_t=u_t\mathscr{L}^n$ is a solution of the equation (4.54).

Let $\tilde{\boldsymbol{v}}$ be the vector field associated to μ given by Theorem 3.10. By Lemma 4.2 $\phi\circ\mu_t$ is absolutely continuous. Then from (4.49) we have

$$-\frac{d}{dt}\phi(\mu_t)=\frac{1}{2}|\mu'|^2_G(t)+\frac{1}{2}g^2(\mu_t) \qquad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,+\infty)$$

which, by (3.32) and the definition of g, can be rewritten as

$$-\frac{d}{dt}\phi(\mu_t)=\frac{1}{2}\,\|\tilde{\boldsymbol{v}}_t\|^2_{L^2_G(\mu_t;\mathbb{R}^n)}+\frac{1}{2}\left\|A\frac{\nabla L_F(u_t)}{u_t}+A\nabla V\right\|^2_{L^2_G(\mu_t;\mathbb{R}^n)} \qquad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,+\infty).$$

On the other hand (4.28) implies that

$$\frac{d}{dt}\phi(\mu_t)=\int_{\mathbb{R}^n}\langle G(x)A(x)(\frac{\nabla L_F(u_t(x))}{u_t(x)}+\nabla V(x)),\tilde{\boldsymbol{v}}_t(x)\rangle\,d\mu_t(x) \qquad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,+\infty).$$

Then the equality holds in Cauchy-Schwartz inequality in the Hilbert space $L^2_G(\mu_t;\mathbb{R}^n)$, and this implies that

$$\tilde{\boldsymbol{v}}_t(x)=-A(x)\frac{\nabla L_F(u_t(x))}{u_t(x)}-A(x)\nabla V(x) \qquad \text{for } \mu_t\text{-a.e. } x\in\mathbb{R}^n.$$

□

The following Theorem shows that if the potential V is uniformly convex, then we have exponential convergence to the stationary state, as in the case $A=I$ (see [CMV03]). Notice that we have not required any regularity on the matrix A but only the ellipticity condition (4.14).

Theorem 4.5 (Asymptotic behaviour)**.** *Assume that the potential $V\in C^2(\mathbb{R}^n)$ satisfies $V\geq 0$ and*

$$\nabla^2V(x)\geq\alpha I \qquad \textit{for } \alpha>0.$$

Then there exists a unique minimizer $\mu_\infty=u_\infty\mathscr{L}^n$ for the functional ϕ, (μ_∞ is called stationary state).

Let λ be the ellipticity constant in (4.14) for the matrix A. Then we have

$$g(\mu)^2\geq 2\lambda\alpha(\phi(\mu)-\phi(\mu_\infty)) \qquad \forall\mu\in D(\phi). \tag{4.56}$$

Moreover, for every $\mu_0 \in D(\phi)$, denoting by μ_t the corresponding solution given by Theorem 4.4, we have

$$\phi(\mu_t) - \phi(\mu_\infty) \leq e^{-2\lambda\alpha t}(\phi(\mu_0) - \phi(\mu_\infty)) \qquad \forall t \in (0, +\infty) \tag{4.57}$$

and

$$W_{2,G}(\mu_t, \mu_\infty) \leq e^{-\lambda\alpha t}\sqrt{\frac{2}{\lambda\alpha}(\phi(\mu_0) - \phi(\mu_\infty))} \qquad \forall t \in (0, +\infty). \tag{4.58}$$

In the particular case of linear diffusion, $F(u) = u \log u$, we also have

$$\|u_t - u_\infty\|_{L^1(\mathbb{R}^n)} \leq e^{-\lambda\alpha t}\sqrt{2(\phi(\mu_0) - \phi(\mu_\infty))} \qquad \forall t \in (0, +\infty). \tag{4.59}$$

Proof. The proof follows by the analogous result for the case $A = I$ Theorem 2.1 of [CMV03] and the ellipticity condition (4.14) on A. The existence and uniqueness of the stationary state is exactly the same for the case $A = I$ and follows by the geodesically strict convexity of ϕ on $(\mathscr{P}_2(\mathbb{R}^n), W_{2,I})$ (see e.g. [McC97] and [CMV03]).
By the ellipticity of A and the definitions of the local slope for ϕ and the upper gradient g,

$$g(\mu)^2 = \int_{\mathbb{R}^n} \langle A\left(\frac{\nabla L_F(u)}{u} + \nabla V\right), \frac{\nabla L_F(u)}{u} + \nabla V\rangle\, d\mu \geq \lambda \int_{\mathbb{R}^n} \left|\frac{\nabla L_F(u)}{u} + \nabla V\right| d\mu = \lambda |\partial\phi|_I^2(\mu). \tag{4.60}$$

Since by Theorem 2.1 of [CMV03] we have

$$|\partial\phi|_I^2(\mu) \geq 2\alpha(\phi(\mu) - \phi(\mu_\infty)) \qquad \forall \mu \in D(\phi)$$

then (4.56) follows.
Now (4.57) is a consequence of (4.56), (4.53) and Gronwall's Lemma.
Again for Theorem 2.1 of [CMV03], we have that

$$W_{2,I}(\mu, \mu_\infty) \leq \sqrt{\frac{2}{\alpha}(\phi(\mu) - \phi(\mu_\infty))} \qquad \forall \mu \in D(\phi)$$

and (4.58) follows by (4.57) and the inequality

$$W_{2,G}(\mu, \nu) \leq \sqrt{\lambda^{-1}}W_{2,I}(\mu, \nu).$$

Finally, in the case of linear diffusion, (4.59) is a consequence of (4.57) and the Csizár-Kullback inequality

$$\|u_t - u_\infty\|^2_{L^1(\mathbb{R}^n)} \leq 2(\phi(\mu_t) - \phi(\mu_\infty)).$$

□

4.3 Wasserstein contraction

In this section we consider the particular case of the variable coefficients linear Fokker-Planck where the matrix of the coefficients is of the form $A(x) = a(x)I$ and a is a scalar function.

Under suitable assumptions we can prove a uniqueness result and a contraction estimate in Wasserstein distance for gradient flows of the energy functional

$$\phi(\mu) = \begin{cases} \int_{\mathbb{R}^n} u(x)\log(u(x))\,dx + \int_{\mathbb{R}^n} V(x)u(x)\,dx & \text{if } \mu = u\mathscr{L}^n \in \mathscr{P}_2^r(\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.61}$$

given by Theorem 4.4. In this case the equation satisfied by the gradient flow is

$$\partial_t u - \operatorname{div}(a\nabla u + a\nabla V u) = 0 \qquad \text{in } [0,+\infty)\times\mathbb{R}^n. \tag{4.62}$$

The uniform ellipticity condition (4.14) can be obviously rewritten as

$$0 < \lambda \le a(x) \le \Lambda \qquad \forall x \in \mathbb{R}^n. \tag{4.63}$$

In the proof of the following Theorem we use known results about transport maps and Wasserstein distance in the case of $\mathbb{R}^n$ with the euclidean distance. We denote by W_2 (respectively $W_{2,G}$) the Wasserstein distance in $\mathscr{P}_2(\mathbb{R}^n)$ where $\mathbb{R}^n$ is endowed with the euclidean distance (respectively the Riemannian distance induced by $G(x) = \frac{1}{a(x)}I$).

Theorem 4.6. *Assume that $a \in C^1(\mathbb{R}^n)$ and the condition (4.63) holds.*
Let $\mu_0^1 = u_0^1\mathscr{L}^n \in D(\phi)$, $\mu_0^2 = u_0^2\mathscr{L}^n \in D(\phi)$ and $\mu_t^1 = u_t^1\mathscr{L}^n$, $\mu_t^2 = u_t^2\mathscr{L}^n$ solutions of the equation (4.62) given by Theorem 4.4. If there exists $\alpha \in \mathbb{R}$ such that

$$\langle \nabla a(x) - a(x)\nabla V(x) - \nabla a(y) + a(y)\nabla V(y), x-y\rangle + n|\sqrt{a(x)} - \sqrt{a(y)}|^2 \le \alpha|x-y|^2 \quad \forall x,y \in \mathbb{R}, \tag{4.64}$$

then

$$W_2(\mu_t^1,\mu_t^2) \le e^{\alpha t} W_2(\mu_0^1,\mu_0^2) \qquad \forall t \in (0,+\infty). \tag{4.65}$$

Before to prove Theorem 4.6, we observe that the most interesting case is $\alpha < 0$, where we have exponential decay. Moreover, by the equivalence of the Wasserstein distances

$$\sqrt{\Lambda^{-1}}W_2(\mu,\nu) \le W_{2,G}(\mu,\nu) \le \sqrt{\lambda^{-1}}W_2(\mu,\nu),$$

we obtain immediately

$$W_{2,G}(\mu_t^1,\mu_t^2) \le \sqrt{\Lambda/\lambda}\; e^{\alpha t} W_{2,G}(\mu_0^1,\mu_0^2) \qquad \forall t \in (0,+\infty). \tag{4.66}$$

Proof. Applying Lemma 4.3.4 of [AGS05] we obtain that the map $t \mapsto W_2(\mu_t^1,\mu_t^2)$ is absolutely continuous and

$$\frac{d}{dt}W_2^2(\mu_t^1,\mu_t^2) \le \frac{d}{ds}W_2^2(\mu_s^1,\mu_t^2)_{|s=t} + \frac{d}{dt}W_2^2(\mu_t^1,\mu_s^2)_{|s=t} \tag{4.67}$$

for $\mathscr{L}^1$-a.e. $t \in (0,+\infty)$.
Moreover we recall that (see Proposition 8.4.7 and Remark 8.4.8 of [AGS05]) for any absolutely continuous curve μ_t in $\mathscr{P}_2(\mathbb{R}^n)$ and any measure $\sigma \in \mathscr{P}_2(\mathbb{R}^n)$, and for a Borel

vector field $\boldsymbol{v}_t$ such that $\int_0^T \int_{\mathbb{R}^n} |\boldsymbol{v}_t(x)|^2 \, d\mu_t(x) \, dt < +\infty$ and $(\mu_t, \boldsymbol{v}_t)$ satisfies the continuity equation, we have that

$$\frac{d}{dt} W_2^2(\mu_t, \sigma) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \langle x - y, \boldsymbol{v}_t(x) \rangle \, d\gamma(x, y) \qquad \forall \gamma \in \Gamma_o(\mu_t, \sigma) \tag{4.68}$$

for $\mathscr{L}^1$-a.e. $t \in (0, T)$.

Denoting by $r : \mathbb{R}^n \to \mathbb{R}^n$ the optimal transport map for the euclidean quadratic cost $|x-y|^2$ from μ_t^1 to μ_t^2 we recall that r is μ_t^1-a.e. differentiable and $\det(\nabla r(x)) > 0$ for μ_t^1-a.e. $x \in \mathbb{R}^n$ and there exists a μ_t^1 negligible set N such that r is strictly monotone on $\mathbb{R}^n \setminus N$. We also recall that r is essentially injective and the optimal transport map from μ_t^2 to μ_t^1 is the inverse function r^{-1}.

Since u^1 and u^2 are solutions of the equation (4.62), by Theorem 4.4 $u_t^i \in W^{1,1}(\mathbb{R}^n)$, $i = 1, 2$ and

$$\int_0^T \int_{\mathbb{R}^n} \left(\left| \frac{\nabla u_t^i(x)}{u_t^i(x)} \right|^2 + |\nabla V(x)|^2 \right) u_t^i(x) \, dx \, dt < +\infty$$

for every $T > 0$, $i = 1, 2$, we can apply (4.68) obtaining that

$$\frac{d}{ds} W_2^2(\mu_s^1, \mu_t^2)_{|s=t} = -2 \int_{\mathbb{R}^n} \langle x - r(x), a(x) \frac{\nabla u_t^1(x)}{u_t^1(x)} + a(x) \nabla V(x) \rangle u_t^1(x) \, dx \tag{4.69}$$

and

$$\frac{d}{ds} W_2^2(\mu_t^1, \mu_s^2)_{|s=t} = -2 \int_{\mathbb{R}^n} \langle y - r^{-1}(y), a(y) \frac{\nabla u_t^2(y)}{u_t^2(y)} + a(y) \nabla V(y) \rangle u_t^2(y) \, dy. \tag{4.70}$$

Then by (4.67)

$$\begin{aligned} \frac{d}{dt} W_2^2(\mu_t^1, \mu_t^2) \leq & -2 \int_{\mathbb{R}^n} \langle x - r(x), a(x) \nabla u_t^1(x) \rangle \, dx - 2 \int_{\mathbb{R}^n} \langle y - r^{-1}(y), a(y) \nabla u_t^2(y) \rangle \, dy \\ & - 2 \int_{\mathbb{R}^n} \langle x - r(x), a(x) \nabla V(x) \rangle u_t^1(x) \, dx - 2 \int_{\mathbb{R}^n} \langle y - r^{-1}(y), a(y) \nabla V(y) \rangle u_t^2(y) \, dy \end{aligned} \tag{4.71}$$

By definition of transport we have that

$$\begin{aligned} & - 2 \int_{\mathbb{R}^n} \langle x - r(x), a(x) \nabla V(x) \rangle u_t^1(x) \, dx - 2 \int_{\mathbb{R}^n} \langle y - r^{-1}(y), a(y) \nabla V(y) \rangle u_t^2(y) \, dy = \\ & = -2 \int_{\mathbb{R}^n} \langle x - r(x), a(x) \nabla V(x) - a(r(x)) \nabla V(r(x)) \rangle u_t^1(x) \, dx \end{aligned} \tag{4.72}$$

On the other hand, since u_t^1 and u_t^2 belong to $W^{1,1}(\mathbb{R}^n)$ and r and r^{-1} are monotone and $a \in C^1(\mathbb{R})$, using a similar argument of the proof of Lemma 10.4.5 of [AGS05], and a truncation argument, the following weak formula of integration by parts holds ($\operatorname{tr} \nabla$ is the absolutely continuous part of the distributional divergence)

$$\int_{\mathbb{R}^n} a(x) \langle r(x) - x, \nabla u_t^1(x) \rangle \, dx \leq - \int_{\mathbb{R}^n} \operatorname{tr} \nabla (a(x)(r(x) - x)) u_t^1(x) \, dx, \tag{4.73}$$

and similarly

$$\int_{\mathbb{R}^n} a(y) \langle r^{-1}(y) - y, \nabla u_t^2(y) \rangle \, dy \leq - \int_{\mathbb{R}^n} \operatorname{tr} \nabla (a(y)(r^{-1}(y) - y)) u_t^2(y) \, dy. \tag{4.74}$$

Then we have

$$\begin{aligned}
&-2\int_{\mathbb{R}^n}\langle x-r(x),a(x)\nabla u_t^1(x)\rangle\,dx-2\int_{\mathbb{R}^n}\langle y-r^{-1}(y),a(y)\nabla u_t^2(y)\,dy\\
&\le 2\int_{\mathbb{R}^n}(\langle x-r(x),\nabla a(x)\rangle+(n-\operatorname{tr}\nabla r(x))a(x))u_t^1(x)\,dx\\
&\quad+2\int_{\mathbb{R}^n}(\langle y-r^{-1}(y),\nabla a(y)\rangle+(n-\operatorname{tr}(\nabla r(r^{-1}(y)))^{-1})a(y))u_t^2(y)\,dy\\
&=2\int_{\mathbb{R}^n}(\langle x-r(x),\nabla a(x)\rangle+(n-\operatorname{tr}\nabla r(x))a(x))u_t^1(x)\,dx \qquad (4.75)\\
&\quad+2\int_{\mathbb{R}^n}(\langle r(x)-x,\nabla a(r(x))\rangle+(n-\operatorname{tr}(\nabla r(x))^{-1})a(r(x)))u_t^1(x)\,dx\\
&=2\int_{\mathbb{R}^n}\langle x-r(x),\nabla a(x)-\nabla a(r(x))\rangle u_t^1(x)\,dx\\
&\quad+2\int_{\mathbb{R}^n}(na(x)+na(r(x))-a(x)\operatorname{tr}\nabla r(x)-a(r(x))\operatorname{tr}(\nabla r(x))^{-1})u_t^1(x)\,dx.
\end{aligned}$$

Since all the eigenvalues of $\nabla r(x)$ are strictly positive we easily obtain

$$-a(x)\operatorname{tr}\nabla r(x)-a(r(x))\operatorname{tr}(\nabla r(x)^{-1})\le -2n\sqrt{a(x)}\sqrt{a(r(x))}$$

and then

$$\begin{aligned}
&-2\int_{\mathbb{R}^n}\langle x-r(x),a(x)\nabla u_t^1(x)\rangle\,dx-2\int_{\mathbb{R}^n}\langle y-r^{-1}(y),a(y)\nabla u_t^2(y)\rangle\,dy\\
&\le 2\int_{\mathbb{R}^n}\langle x-r(x),\nabla a(x)-\nabla a(r(x))\rangle+n\big(\sqrt{a(x)}-\sqrt{a(r(x))}\big)^2u_t^1(x)\,dx
\end{aligned} \tag{4.76}$$

Combining (4.71) with (4.72) and (4.76) and using the assumption (4.64) we obtain

$$\frac{d}{dt}W_2^2(\mu_t^1,\mu_t^2)\le 2\alpha\int_{\mathbb{R}^n}|x-r(x)|^2u_t^1(x)\,dx=2\alpha W_2^2(\mu_t^1,\mu_t^2) \tag{4.77}$$

and Gronwall's Lemma shows that

$$W_2(\mu_t^1,\mu_t^2)\le e^{\alpha t}W_2(\mu_0^1,\mu_0^2)\qquad \forall t\in(0,+\infty). \tag{4.78}$$

□

Remark 4.7 (The one dimensional case). In the one dimensional case another simple result of contraction in Wasserstein distance for variable coefficients Fokker-Planck equation is available.

We consider the equation

$$\partial_t u_t(x)=\partial_x(a(x)\partial_x u_t(x))+\partial_x(b(x)u_t(x))\qquad \text{in } (0,+\infty)\times\mathbb{R} \tag{4.79}$$

with $b:\mathbb{R}\to\mathbb{R}$ and $a:\mathbb{R}\to\mathbb{R}$ continuous, satisfying

$$0<a(x)<\Lambda. \tag{4.80}$$

Defining $g(x) := \frac{1}{a(x)}$ we can consider the change of variable $f(x) = \int_0^x \sqrt{g(s)}\, ds$ and we observe that, by (4.80), f is surjective on $\mathbb{R}$ and $f'(x) > 0$, i.e. f is a diffeomorphism. Then we can define the new density v implicitly by

$$u(x) = \sqrt{g(x)}v(f(x))$$

which yields

$$v(y) = \sqrt{a(f^{-1}(y))}u(f^{-1}(y)).$$

In the new variable y the equation is

$$\partial_t v = \partial^2_{yy} v + \partial_y \left(\left(\frac{a \circ f^{-1}}{2} \partial_y (g \circ f^{-1}) + b \circ f^{-1} \sqrt{g \circ f^{-1}} \right) v \right), \tag{4.81}$$

which can be written in the form

$$\partial_t v = \partial^2_y v + \partial_y (V' v), \tag{4.82}$$

where the potential is obviously defined by

$$V(y) = \int_0^y \frac{a \circ f^{-1}}{2} \partial_y (g \circ f^{-1}) + b \circ f^{-1} \sqrt{g \circ f^{-1}}.$$

The equation (4.82) is the gradient flow with respect to the Wasserstein distance W_2 in $\mathbb{R}$ with the euclidean metric, of the functional

$$\phi(\mu) = \begin{cases} \int_{\mathbb{R}} u(x) \log(u(x))\, dx + \int_{\mathbb{R}} V(x) u(x)\, dx & \text{if } \mu = u\mathscr{L}^1 \in \mathscr{P}_2^r(\mathbb{R}) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.83}$$

Assume that there exists $\alpha \in \mathbb{R}$ such that

$$V''(x) \geq \alpha \qquad \forall x \in \mathbb{R}.$$

Applying the well known result for convex functionals on $(\mathscr{P}_2(\mathbb{R}), W_2)$ (see [AGS05] Theorem 11.1.4), it is immediate to show that if $\mu_0^1 = v_0^1 \mathscr{L}^1 \in D(\phi)$, $\mu_0^2 = v_0^2 \mathscr{L}^1 \in D(\phi)$ and $\mu_t^1 = v_t^1 \mathscr{L}^1$, $\mu_t^2 = v_t^2 \mathscr{L}^1$ are solutions of the equation (4.82) given by Theorem 4.4, then the following 2-Wasserstein contraction holds

$$W_2(\mu_t^1, \mu_t^2) \leq e^{-\alpha t} W(\mu_0^1, \mu_0^2) \qquad \forall t \in (0, +\infty). \tag{4.84}$$

□

Chapter 5

Stability of flows and convergence of iterated transport maps

In this final Chapter we study the convergence of the iterated composition of optimal transport maps arising from the minimizing movements approximation scheme as illustrated in the introduction. The convergence result for iterated transport maps is a consequence of a general theorem of stability for flows associated to a sequence of vector fields.
All the results of this Chapter are contained in the paper [ALS05].

5.1 Stability of flows associated to a minimal vector field

We fix a positive time $T > 0$ and $I = [0, T]$, a summability exponent $p \in (1, +\infty)$ and an open set $\Omega \subset \mathbb{R}^d$. Here, and in the following, we identify a probability measure $\mu \in \mathscr{P}(\Omega)$ with a probability measure $\mu \in \mathscr{P}(\mathbb{R}^d)$, still denoted by the same name, simply considering the trivial extension of μ to all $\mathbb{R}^d$, i.e. $\mu(B) = \mu(B \cap \Omega)$ for every Borel set of $\mathbb{R}^d$.

We denote by $EC_p(\Omega)$ the set of the couples $(\mu_t, \boldsymbol{v}_t) \in EC_p(\mathbb{R}^d)$ such that $\mu_t \in \mathscr{P}_p(\Omega)$, $\forall t \in [0, T]$, recalling the definitions in Section 3.1.

Notice that the continuity equation is imposed in $\mathbb{R}^d$ and $\mu_t(\mathbb{R}^d \setminus \Omega) = 0$ for every $t \in I$. Moreover, in this case of a finite dimensional space, $(\mu_t, \boldsymbol{v}_t)$ satisfies the continuity equation if and only if

$$\partial_t \mu_t + \operatorname{div}(\boldsymbol{v}_t \mu_t) = 0 \qquad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d). \tag{5.1}$$

Moreover a simple approximation argument shows that if $(\mu_t, \boldsymbol{v}_t) \in EC_p(\mathbb{R}^d)$, then

$$\mu_t \in \mathscr{P}(\Omega), \quad \mu_0 \in \mathscr{P}_p(\Omega) \qquad \Longrightarrow \qquad \mu_t \in \mathscr{P}_p(\Omega) \ \ \forall t \in [0, T].$$

So, the assumption that $\mu_t \in \mathscr{P}_p(\Omega)$ could be replaced by $\mu_0 \in \mathscr{P}_p(\Omega)$.

5.1.1 Flows and their stability

Given a reference time $s \in [0,T]$, a measure $\bar\mu_s \in \mathscr{P}_p(\Omega)$, and a Borel field $\boldsymbol{v} : (t,x) \in (0,T)\times\Omega \to \boldsymbol{v}_t(x) \in \mathbb{R}^d$, a *flow* associated to $\boldsymbol{v}_t$, is a map $\boldsymbol{X}(t,s,x) : [0,T]\times[0,T]\times\Omega \to \Omega$ such that $t \mapsto \boldsymbol{X}(t,s,x)$ is absolutely continuous in $[0,T]$ and it is an integral solution of the ODE

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{X}(t,s,x) = \boldsymbol{v}_t\left(\boldsymbol{X}(t,s,x)\right) \\ \boldsymbol{X}(s,s,x) = x \end{cases} \tag{5.2}$$

for $\bar\mu_s$-a.e. x, satisfying

$$\int_\Omega \int_0^T \Big|\frac{d}{dt}\boldsymbol{X}(t,s,x)\Big|^p \, dt \, d\bar\mu_s(x) < +\infty. \tag{5.3}$$

When $s = 0$ we will speak of *forward flows* and we will often omit to indicate the explicit occurrence of s in $\boldsymbol{X}(t,0,x)$, writing either $\boldsymbol{X}(t,x)$ or $\boldsymbol{X}_t(x)$; analogously, the case $s = T$ corresponds to *backward flows.*

If $\bar\mu_s \in \mathscr{P}_p(\Omega)$, defining the narrowly continuous family of measures

$$\mu_t = \boldsymbol{X}(t,s,\cdot)_{\#}\bar\mu_s, \tag{5.4}$$

it is immediate to check that $(\mu_t, \boldsymbol{v}_t)$ belongs to $EC_p(\Omega)$ with the condition $\mu_s = \bar\mu_s$; in this case we say that $\boldsymbol{X}$ is a flow associated with $(\mu_t, \boldsymbol{v}_t)$.

A forward flow associated to $(\mu_t, \boldsymbol{v}_t) \in EC_p(\Omega)$ is a solution of the ODE (5.2) associated to the vector field $\boldsymbol{v}_t$, satisfying the transport condition (5.4) for $s = 0$, i.e. $\mu_t = \boldsymbol{X}(t,\cdot)_{\#}\mu_0$.

Without assuming any regularity on $\boldsymbol{v}_t$ we do not know anything about the existence and the uniqueness of a flow associated to $(\mu_t, \boldsymbol{v}_t)$. We postpone a detailed discussion of this aspect to the next section .

We are mainly interested in the *stability properties* of flows: supposing that $\boldsymbol{X}^n$ and $\boldsymbol{X}$ are *forward* flows associated with $(\mu^n, \boldsymbol{v}^n)$ and $(\mu, \boldsymbol{v})$ respectively. In the next Theorem we give sufficient conditions on the measures and on their velocity vector fields for the convergence of $\boldsymbol{X}^n$ to $\boldsymbol{X}$. Notice that we do not assume uniqueness of the solutions of the ODE (5.2) associated to the approximant vector fields $\boldsymbol{v}^n$ but we only assume uniqueness for μ_0- a.e. $x \in \Omega$ of the solutions of the ODE (5.2) associated to the limit vector field $\boldsymbol{v}$.

Its proof uses, in the same spirit of [Amb04b], [Amb04a], narrow convergence in the space of continuous maps as a technical tool for proving convergence in measure of the flows.

Theorem 5.1 (Stability of flows)**.** *We are given $(\mu_t^n, \boldsymbol{v}_t^n)$, $(\mu_t, \boldsymbol{v}_t) \in EC_p(\Omega)$ such that μ_t^n narrowly converges to μ_t,*

$$\limsup_{n\to\infty} \int_0^T \int_\Omega |\boldsymbol{v}_t^n|^p \, d\mu_t^n \, dt \le \int_0^T \int_\Omega |\boldsymbol{v}_t|^p \, d\mu_t \, dt. \tag{5.5}$$

We assume that there exist functions $\boldsymbol{d}^n : \Omega \to \Omega$ *such that*

$$\mu_0^n = \boldsymbol{d}^n_{\#}\mu_0 \quad \textit{with} \quad \lim_{n\to\infty} \int_\Omega |\boldsymbol{d}^n - \boldsymbol{i}|^p \, d\mu_0 = 0, \tag{5.6}$$

and there exist forward flows $\boldsymbol{X}^n$ *associated to* $(\mu_t^n, \boldsymbol{v}_t^n)$.

If $\boldsymbol{v}_t$ *is the unique minimal velocity field associated to* μ_t *given by Corollary 3.7 and*

$$\textit{the ODE (5.2) associated to } \boldsymbol{v} \textit{ admits } \text{at most one } \overline{\Omega}\text{-valued solution } \textit{for } \mu_0\textit{-a.e. } x \in \Omega, \tag{5.7}$$

then there exists a unique forward flow $\boldsymbol{X} : [0,T] \times \Omega \to \overline{\Omega}$ *associated to* $(\mu_t, \boldsymbol{v}_t)$. *Moreover the flows* $\boldsymbol{X}^n$ *converge in* L^p *to* $\boldsymbol{X}$, *in the following sense*

$$\lim_{n\to\infty} \int_\Omega \sup_{t\in[0,T]} \Big| \boldsymbol{X}^n(t, \boldsymbol{d}^n(x)) - \boldsymbol{X}(t,x) \Big|^p \, d\mu_0(x) = 0. \tag{5.8}$$

Proof. We denote by $\Gamma = C^0([0,T];\overline{\Omega})$ the complete and separable metric space of continuous maps from $[0,T]$ to $\overline{\Omega}$, whose generic element will be denoted by γ, and we denote by $\boldsymbol{e}_t : \Gamma \to \overline{\Omega}$ the evaluation maps $\boldsymbol{e}_t(\gamma) = \gamma(t)$.

Since $\boldsymbol{X}^n(t,x) \in \Omega$ for every $x \in \Omega$, we can define probability measures $\boldsymbol{\eta}^n$ in Γ by

$$\boldsymbol{\eta}^n := (\boldsymbol{X}^n(\cdot,x))_{\#}\, \mu_0^n, \tag{5.9}$$

where $x \mapsto \boldsymbol{X}^n(\cdot,x)$ is the natural map from Ω to Γ. Since $\boldsymbol{e}_t \circ (\boldsymbol{X}^n(\cdot,x)) = \boldsymbol{X}^n(t,x)$ we immediately obtain that

$$(\boldsymbol{e}_t)_{\#}\boldsymbol{\eta}^n = \boldsymbol{X}^n(t,\cdot)_{\#}\mu_0^n = \tilde{\mu}_t^n, \tag{5.10}$$

where we denoted by $\tilde{\mu}_t^n$ the trivial extension of μ_t^n to $\overline{\Omega}$ obtained by setting $\tilde{\mu}_t^n(\partial\Omega) = 0$.

Step 1. (Tightness of $\boldsymbol{\eta}^n$) We claim that the family $\{\boldsymbol{\eta}^n\}$ is tight as $n \to \infty$, that any limit point $\boldsymbol{\eta}$ is concentrated on $\overline{\Omega}$*-valued and absolutely continuous* maps in $[0,T]$ and that

$$(\boldsymbol{e}_t)_{\#}\boldsymbol{\eta} = \tilde{\mu}_t, \tag{5.11}$$

$$\int_\Gamma \int_0^T |\dot{\gamma}(t)|^p \, dt \, d\boldsymbol{\eta}(\gamma) \le \int_0^T \int_\Omega |\boldsymbol{v}_t|^p \, d\mu_t \, dt, \tag{5.12}$$

$$\gamma(t) \in \Omega \quad \mathscr{L}^1\text{-a.e. in } [0,T], \text{ for } \boldsymbol{\eta}\text{-a.e. } \gamma. \tag{5.13}$$

Indeed, by Ascoli-Arzelà theorem, the functional

$$\Psi(\gamma) := \begin{cases} |\gamma(0)|^p + \int_0^T |\dot{\gamma}(t)|^p \, dt & \text{if } \gamma \text{ is absolutely continuous} \\ +\infty & \text{otherwise} \end{cases}$$

is coercive in Γ, and since $\dot{\gamma} = \boldsymbol{v}^n(\gamma)$ for $\boldsymbol{\eta}^n$-a.e. $\gamma \in \Gamma$ and the p-moment of μ_0^n converges to the p-moment of μ_0 by (5.6), we get

$$\begin{aligned} \limsup_{n\to\infty} \int_\Gamma \Psi(\gamma) \, d\boldsymbol{\eta}^n &= \limsup_{n\to\infty} \int_\Gamma |\gamma(0)|^p \, d\boldsymbol{\eta}^n + \int_0^T \int_\Gamma |\boldsymbol{v}_t^n(\gamma(t))|^p \, d\boldsymbol{\eta}^n \, dt \\ &\le \limsup_{n\to\infty} \int_\Omega |x|^p \, d\mu_0^n(x) + \int_0^T \int_\Omega |\boldsymbol{v}_t^n|^p \, d\mu_t^n \, dt < +\infty, \end{aligned}$$

thus proving the tightness of the family. Moreover, the lower semicontinuity of Ψ gives that $\int \Psi\, d\boldsymbol{\eta}$ is finite, so that $\boldsymbol{\eta}$ is concentrated on the absolutely continuous maps. An analogous argument proves (5.12), while (5.11) can be achieved passing to the limit in (5.10).

Finally, by introducing the upper semicontinuous function

$$j_\Omega(x) := \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \in \partial\Omega, \end{cases}$$

(5.11) and Fubini's theorem give

$$0 = \int_0^T \int_\Omega j_\Omega(x)\, d\tilde{\mu}_t(x)\, dt = \int_0^T \int_\Gamma j_\Omega(\gamma(t))\, d\boldsymbol{\eta}(\gamma)\, dt = \int_\Gamma \int_0^T j_\Omega(\gamma(t))\, dt\, d\boldsymbol{\eta}(\gamma) \tag{5.14}$$

which is equivalent to (5.13).

Step 2. (Representation of $\boldsymbol{v}_t$) Let $\boldsymbol{\eta}$ be a limit point as in Step 1, along some sequence $n_i \to \infty$. By the previous claim we know that for $\boldsymbol{\eta}$-a.e. $\gamma \in \Gamma$ the vector field $\boldsymbol{z}_t(\gamma) := \dot{\gamma}(t)$ is well defined up to a $\mathscr{L}^1$-negligible subset of $(0,T)$. By Fubini's theorem and (5.12), it coincides with a Borel vector field $\boldsymbol{z} \in L^p(\mathscr{L}^1 \times \boldsymbol{\eta}; \mathbb{R}^d)$ with

$$\int_0^T \int_\Gamma |\boldsymbol{z}_t(\gamma)|^p\, d\boldsymbol{\eta}(\gamma) dt = \int_\Gamma \int_0^T |\dot{\gamma}(t)|^p\, dt\, d\boldsymbol{\eta}(\gamma) \le \int_0^T \int_\Omega |\boldsymbol{v}_t|^p\, d\mu_t\, dt. \tag{5.15}$$

in particular there exists a Borel set $\mathcal{T} \subset (0,T)$ of full measure such that $\boldsymbol{z}_t \in L^p(\boldsymbol{\eta})$ for every $t \in \mathcal{T}$.

Define now $\boldsymbol{\nu}_t := (\boldsymbol{e}_t)_\#(\boldsymbol{z}_t \cdot \boldsymbol{\eta})$, and notice that Lemma 1.8 gives $\boldsymbol{\nu}_t$ is well defined and absolutely continuous with respect to $\mu_t = (\boldsymbol{e}_t)_\#\boldsymbol{\eta}$ for any $t \in \mathcal{T}$. Then, writing $\nu_t = \boldsymbol{w}_t \mu_t$, Lemma 1.8 again and (5.15) give

$$\int_0^T \int_\Omega |\boldsymbol{w}_t|^p\, d\mu_t\, dt \le \int_0^T \int_\Gamma |\boldsymbol{z}_t(\gamma)|^p\, d\boldsymbol{\eta}\, dt \le \int_0^T \int_\Omega |\boldsymbol{v}_t|^p\, d\mu_t\, dt. \tag{5.16}$$

Now, let us show that $\boldsymbol{w}_t$ is an admissible velocity field relative to μ_t: indeed, for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega \varphi\, d\mu_t = \frac{d}{dt} \int_\Gamma \varphi(\gamma(t))\, d\boldsymbol{\eta} &= \int_\Gamma \langle \nabla\varphi(\gamma(t)), \dot{\gamma}(t) \rangle\, d\boldsymbol{\eta} \\ = \int_\Gamma \langle \nabla\varphi(\gamma(t)), \boldsymbol{z}_t(\gamma) \rangle\, d\boldsymbol{\eta} &= \int_\Omega \langle \nabla\varphi, \boldsymbol{w}_t \rangle\, d\mu_t. \end{aligned} \tag{5.17}$$

As a consequence (5.16) and the minimality of $\boldsymbol{v}_t$ yield $\boldsymbol{v}_t = \boldsymbol{w}_t$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$. In addition, from the equality

$$\int_\Gamma |\dot{\gamma}(t)|^p\, d\boldsymbol{\eta} = \int_\Omega |\boldsymbol{w}_t|^p\, d(\boldsymbol{e}_t)_\#\boldsymbol{\eta}$$

and Lemma 1.8 we infer

$$\boldsymbol{v}_t(\gamma(t)) = \dot{\gamma}(t) \quad \boldsymbol{\eta}\text{-a.e. in } \Gamma \tag{5.18}$$

for $\mathscr{L}^1$-a.e. $t \in [0,T]$. Then, (5.18) and Fubini's theorem give

$$\boldsymbol{v}_t(\gamma(t)) = \dot{\gamma}(t) \quad \mathscr{L}^1\text{-a.e. in } (0,T) \tag{5.19}$$

for $\boldsymbol{\eta}$-a.e. γ. In other words, $\boldsymbol{\eta}$ is concentrated on the absolutely continuous solutions of the ODE relative to the vector field $\boldsymbol{v}_t$.

Step 3. (Conclusion) By assumption, we know that there is at most one solution $\gamma \in \Gamma$ of the ODE $\dot{\gamma} = \boldsymbol{v}_t(\gamma)$ with an initial condition $\gamma(0) = x \in \Omega$. Taking into account (5.13) we obtain that the disintegration $\{\boldsymbol{\eta}_x\}_{x\in\Omega}$ induced by the map $\boldsymbol{e}_0$ is a Dirac mass, concentrated on the unique $\overline{\Omega}$-valued solution $\boldsymbol{X}(\cdot,x)$ of the ODE starting from x at $t=0$. As a consequence

$$\boldsymbol{\eta} = (\boldsymbol{X}(\cdot,x))_{\#}\,\mu_0. \tag{5.20}$$

Since $\boldsymbol{\eta}$ does not depend on the subsequence (n_i), the tightness of $(\boldsymbol{\eta}^n)$ gives

$$(\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x)))_{\#}\,\mu_0 = (\boldsymbol{X}^n(\cdot,x))_{\#}\,\mu_0^n \;\to (\boldsymbol{X}(\cdot,x))_{\#}\,\mu_0 \text{ narrowly as } n\to\infty. \tag{5.21}$$

In order to apply Lemma 1.7 we must to check that

$$(\boldsymbol{i},\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x)))_{\#}\,\mu_0 \to (\boldsymbol{i},\boldsymbol{X}(\cdot,x))_{\#}\,\mu_0 \text{ narrowly as } n\to\infty. \tag{5.22}$$

Recalling Remark 1.5 we take a bounded Lipschitz function $\varphi : \Omega\times\Gamma \to \mathbb{R}$ and, by (5.6) we have

$$\left|\int \varphi\left(x,\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x) - \int \varphi\left(\boldsymbol{d}^n(x),\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x)\right| \le L\int |x-\boldsymbol{d}^n(x)|\,d\mu_0(x) \to 0, \tag{5.23}$$

as $n\to\infty$, and by (5.21) we have

$$\begin{aligned}\int \varphi\left(\boldsymbol{d}^n(x),\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x) &= \int \varphi\left(\boldsymbol{e}_0(\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))),\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x)\\ &\to \int \varphi\left(\boldsymbol{e}_0(\boldsymbol{X}(\cdot,x)),\boldsymbol{X}(\cdot,x)\right)\,d\mu_0(x)\end{aligned}$$

when $n\to\infty$. Then

$$\begin{aligned}\int \varphi\left(x,\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x) &= \int \varphi\left(x,\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x) - \int \varphi\left(\boldsymbol{d}^n(x),\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x)\\ &\quad + \int \varphi\left(\boldsymbol{d}^n(x),\boldsymbol{X}^n(\cdot,\boldsymbol{d}^n(x))\right)\,d\mu_0(x) \to \int \varphi\left(x,\boldsymbol{X}(\cdot,x)\right)\,d\mu_0(x)\end{aligned}$$

and (5.22) holds. Consequently Lemma 1.7 implies that the sequence of functions

$$g^n(x) := \sup_{t\in[0,T]} |\boldsymbol{X}^n(t,\boldsymbol{d}^n(x)) - \boldsymbol{X}(t,x)| \to 0 \quad \text{ in } \mu_0\text{-measure.}$$

In order to prove the L^p-convergence (5.8) we can assume with no loss of generality that $\boldsymbol{X}^n(\cdot,x) \to \boldsymbol{X}(\cdot,x)$ uniformly in $[0,T]$ for μ_0-a.e. x, and we need to show that $|g^n|^p$ is equi-integrable in $L^1(\mu_0)$. To this aim, taking into account the fact that strongly converging

sequences are equi-integrable it is sufficient to exhibit a sequence of functions $G^n \geq |g^n|^p$ such that G^n strongly converges in $L^1(\mu_0)$.
We recall a useful criterion for strong convergence in $L^1(\mu_0)$ (see for instance Exercise 1.19 of [AFP00]): if

$$\liminf_{n\to\infty} G_n(x) \geq G(x) \geq 0, \quad \text{for } \mu_0\text{-a.e. } x \in \Omega, \quad \limsup_{n\to\infty} \int_\Omega G_n \, d\mu_0 \leq \int_\Omega G \, d\mu_0 < +\infty$$

then $\lim_{n\to\infty} \int_\Omega |G_n - G| \, d\mu_0 = 0$.
We can choose

$$G^n(x) := 3^{p-1}\Big(\sup_{[0,T]} |\boldsymbol{X}(\cdot,x)|^p + |\boldsymbol{d}^n(x)|^p + T^{p-1} h^n(x)\Big),$$

where

$$h^n(x) := \int_0^T \Big|\frac{d}{dt}\boldsymbol{X}^n(t, \boldsymbol{d}^n(x))\Big|^p \, dt.$$

Since $\boldsymbol{d}^n \to \boldsymbol{i}$ in $L^p(\mu_0; \mathbb{R}^d)$ and standard lower semicontinuity result yield

$$\liminf_{n\to\infty} h^n(x) \geq h(x) := \int_0^T \Big|\frac{d}{dt}\boldsymbol{X}(t,x)\Big|^p \, dt \quad \text{for } \mu_0\text{-a.e. } x \in \Omega \tag{5.24}$$

the proof is achieved if

$$\limsup_{n\to\infty} \int_\Omega h^n(x) \, d\mu_0(x) \leq \int_\Omega h(x) \, d\mu_0(x). \tag{5.25}$$

For, we can calculate

$$\begin{aligned}\int_\Omega h^n(x) \, d\mu_0(x) &= \int_\Omega \int_0^T \Big|\frac{d}{dt}\boldsymbol{X}^n(t,x)\Big|^p \, dt \, d\mu_0^n(x) = \int_\Omega \int_0^T \big|\boldsymbol{v}_t^n(\boldsymbol{X}^n(t,x))\big|^p \, dt \, d\mu_0^n(x) \\ &= \int_0^T \int_\Omega \big|\boldsymbol{v}_t^n(y)\big|^p \, d\mu_t^n(y) \, dt\end{aligned}$$

and by (5.5) we get

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega h^n(x) \, d\mu_0(x) &\leq \int_0^T \int_\Omega \big|\boldsymbol{v}_t(y)\big|^p \, d\mu_t(y) \, dt = \int_\Omega \int_0^T \Big|\frac{d}{dt}\boldsymbol{X}(t,x)\Big|^p \, dt \, d\mu_0(x) \\ &= \int_\Omega h(x) \, d\mu_0(x).\end{aligned}$$

□

5.1.2 On the regularity of the limit vectorfield

We notice that in Theorem 5.1 no regularity is imposed on the approximating velocity fields $\boldsymbol{v}_t^n$ and that assumption (5.7) stating μ_0-a.e. uniqueness of solutions of the ODE (5.2) can be guaranteed by several assumptions on the limit vector field $\boldsymbol{v}_t$. In the following we list some of these assumptions.

Local Lipschitz condition

The simplest condition on the vector field $\boldsymbol{v}_t$ which ensures both the existence and the uniqueness of a $\overline{\Omega}$-valued solution of the ODE (5.2) for $\bar{\mu}_0$-a.e. initial datum $x \in \Omega$ is the classical local Lipschitz condition. Denoting by $\mathrm{Lip}(\boldsymbol{w}, B)$ the Lipschitz constant of a map $\boldsymbol{w}$ on B, we say that $\boldsymbol{v}_t$ satisfies the local Lipschitz condition if

$$\forall\,(t_0, x_0) \in \Big((0,T) \times \overline{\Omega}\Big) \cup \Big(\{0\} \times \Omega\Big) \quad \exists\, \varepsilon, \tau > 0 : \quad \int_{t_0}^{t_0+\tau} \mathrm{Lip}(\boldsymbol{v}_t, B_\varepsilon(x_0) \cap \overline{\Omega})\, dt < +\infty. \tag{5.26}$$

One-sided Lipschitz condition

Since we need only uniqueness of *forward* characteristics, assumption (5.7) can be guaranteed by a weaker assumptions than (5.26), a *one-sided* Lipschitz condition. The vector field $\boldsymbol{v}_t$ (or, more precisely, at least one of the functions in its equivalence class modulo μ_t-negligible sets) satisfies the one-sided Lipschitz condition if there exists $\omega \in L^1_{\mathrm{loc}}([0,T))$ such that

$$\langle \boldsymbol{v}_t(x) - \boldsymbol{v}_t(y), x - y \rangle \le \omega(t)|x-y|^2 \qquad \forall x,\, y \in \Omega. \tag{5.27}$$

See also [PR97], [BJ98], [BJM05] for other well-posedness results in this context.

Sobolev or BV regularity

As the proof clearly shows, we don't really need uniqueness of forward characteristics in a *pointwise* sense, but rather that any probability measure $\boldsymbol{\eta}$ in Γ concentrated on absolutely continuous solutions of the ODE $\dot\gamma = \boldsymbol{v}_t(\gamma)$ is representable as in (5.20) for some "natural" flow $\boldsymbol{X}$. Then assumption (5.7) could be replaced by the following more technical but also much more general one:

any probability measure $\boldsymbol{\eta}$ in $\Gamma = C^0([0,T];\overline{\Omega})$ concentrated on absolutely continuous solutions of the ODE $\quad \dot\gamma = \boldsymbol{v}_t(\gamma) \quad$ and satisfying $\quad (\boldsymbol{e}_t)_\# \boldsymbol{\eta} = \mu_t$ is representable as $\quad \boldsymbol{\eta} = (\boldsymbol{X}(\cdot, x))_\# \, \mu_0 \quad$ for some flow $\boldsymbol{X}$. (5.28)

In the following we describe several situations where this condition is satisfied.
Let us consider the case when Ω is bounded and

$$\boldsymbol{v} \in L^1_{\mathrm{loc}}\left([0,T); W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)\right), \quad [D_x \cdot \boldsymbol{v}]^- \in L^1_{\mathrm{loc}}\left([0,T); L^\infty(\Omega)\right).$$

Under these assumptions it has been proved in [DL89] (the assumption $\nabla_x \cdot \boldsymbol{v} \in L^1(L^\infty)$ made in that paper can be weakened, requiring only a bound on the negative part, arguing as in [Amb04b]) that the continuity equation (5.1) has at most one solution in the class of μ_t's of the form $\mu_t = u_t \mathscr{L}^d$ with $u \in L^\infty_{\mathrm{loc}}([0,T); L^\infty(\Omega))$, for any initial condition $\mu_0 = u_0 \mathscr{L}^d$,

$u_0 \in L^\infty(\Omega)$. As explained in [Amb04a], it is a general fact that the well-posedness of the continuity equation in the class of μ_t's above (precisely, the validity of a comparison principle) implies the validity of (5.28) for μ_t's in the same class; in this particular case $\boldsymbol{X}$ is the so-called DiPerna–Lions flow associated to $\boldsymbol{v}$ (see [DL89] and also [Amb04b] for a different characterization of it).

It was shown in [Amb04b] (the original $L^1(L^\infty)$ estimate on the negative part of the divergence has been improved to an $L^1(L^1)$ one in [Amb04a]) that also a BV regularity on $\boldsymbol{v}$ can be considered, together with the absolute continuity of the distributional divergence:

$$\boldsymbol{v} \in L^1_{\rm loc}\left([0,T); BV_{\rm loc}(\Omega;\mathbb{R}^d)\right), \quad [D_x \cdot \boldsymbol{v}]^- \in L^1_{\rm loc}\left([0,T); L^1(\Omega)\right).$$

In this case there is uniqueness of bounded solutions of (5.1) and again (5.28) holds in the class of solutions $\mu_t = u_t \mathscr{L}^d$ of (5.1) with u bounded. Other classes of vectorfields to which (5.28) applies are considered in [Hau04], [Hau03], [ACM05], [LBL04], [Ler04].

Regular densities

When we work with absolutely continuous (with respect to Lebesgue measure) measures $\mu_t = u_t \mathscr{L}^d \llcorner \Omega$ and the density u_t is a "regular" function, it is sufficient to control the local regularity of $\boldsymbol{v}_t$ in the positivity set of u and the integrability of the positive part of its divergence.

Let P_0 be an open subset of Ω with $\mu_0(\Omega \setminus P_0) = 0$. When

$$\mu_t = u_t \mathscr{L}^d \llcorner \Omega \quad \text{and the map } (t,x) \mapsto u_t(x) \text{ is continuous in } \Big((0,T)\times\overline{\Omega}\Big) \cup \Big(\{0\}\times P_0\Big) \tag{5.29}$$

we can localize condition (5.26) to the (open) positivity set of u. Thus we introduce the sets

$$P := \Big\{(t,x) \in (0,T)\times\overline{\Omega} : u_t(x) > 0\Big\} \tag{5.30}$$

and we assume that

$$u \in C^1(P), \ \boldsymbol{v}, \, D_x\boldsymbol{v} \in C^0(P), \tag{5.31}$$

$$\forall\, x_0 \in P_0 \text{ we have } u_0(x_0) > 0 \text{ and } \exists\,\varepsilon > 0 \text{ such that } \int_0^\varepsilon \sup_{x \in B_\varepsilon(x_0)} \|D_x \boldsymbol{v}_t(x)\|\, dt < +\infty, \tag{5.32}$$

so that solutions of the ODE (5.2) are locally unique in P and the Cauchy problem for $x \in P_0$ admits a unique maximal forward Ω-valued solution. If

$$\int_\varepsilon^T \int_{\{x : u_t(x) > 0\}} \big[D_x \cdot \boldsymbol{v}_t(x)\big]^+ \, d\mu_t(x)\, dt < +\infty \quad \forall\, \varepsilon > 0, \tag{5.33}$$

then the same conclusion of Theorem 5.1 hold. This fact is a direct consequence of the following "confinement" Lemma.

Lemma 5.2. *Let us suppose that* (5.29), (5.31), (5.32), (5.33) *hold; then for μ_0-a.e. initial datum $x \in \Omega$ the graph in $(0, T) \times \overline{\Omega}$ of each maximal forward solution of the ODE* (5.2) *(for $s = 0$) belongs to the open set P; in particular the maximal solution is unique by* (5.31), (5.32).

Proof. For every $x \in P_0$ let $[0, \tau(x))$ be the open domain of existence and uniqueness of the maximal solution of the system (5.2) restricted to P; classical results on perturbation of ordinary differential equations show that the map $x \mapsto \tau(x)$ is lower semicontinuous in P_0, so that the set

$$D := \Big\{(t, x) \in [0, T) \times P_0 : t < \tau(x)\Big\} \quad \text{is open in } [0, T) \times \Omega,$$

$\boldsymbol{X}$ is of class C^1 in D, and

$$r \in (0, T), \quad \limsup_{t \uparrow r} |\boldsymbol{X}_t(x)| + \frac{1}{u_t(\boldsymbol{X}_t(x))} < +\infty \qquad \Longrightarrow \qquad \tau(x) > r. \tag{5.34}$$

Moreover, by integrating $\int_0^{\tau(x)} |\frac{d}{dt} X_t(x)|\, dt$ with respect to μ_0 (see for instance [AGS05, Proposition 8.1.8] for details), one obtains that for μ_0-a.e. x the map $\boldsymbol{X}_\cdot(x)$ is bounded in $(0, \tau(x))$, so that (5.34) implies that $\tau(x)$ can be strictly less than T only if $u_t(X_t(x))$ approaches 0 as $t \uparrow \tau(x)$.

Let us denote by ϖ_t the divergence $D_x \cdot \boldsymbol{v}_t$ of the vectorfield $\boldsymbol{v}_t$; since u is of class C^1 in P it is a classical solution of the continuity equation in P and therefore a simple computation gives

$$\frac{\partial}{\partial t} \log(u_t) + \boldsymbol{v}_t \cdot D_x \log u_t = -\varpi_t \quad \text{in } P, \tag{5.35}$$

so that

$$\frac{d}{dt} \log \Big(u_t(\boldsymbol{X}(t, x))\Big) = -\varpi(t, \boldsymbol{X}(t, x)) \qquad t \in (0, \tau(x)), \tag{5.36}$$

and

$$u_t(\boldsymbol{X}_t(x)) \det D_x \boldsymbol{X}_t(x) = u_0(x), \quad \forall t \in (0, \tau(x)). \tag{5.37}$$

We introduce the decreasing family of open sets

$$E_\varepsilon := \Big\{x \in P_0 : \tau(x) > \varepsilon\Big\}$$

whose union is P_0 and for every $\varepsilon > 0$ we get

$$\begin{aligned}
&\int_{E_\varepsilon} \sup_{(\varepsilon, \tau(x))} \log \Big(u_\varepsilon(\boldsymbol{X}_\varepsilon(x))/u_t(\boldsymbol{X}_t(x))\Big)\, d\mu_0(x) \le \int_{E_\varepsilon} \int_\varepsilon^{\tau(x)} \varpi_t^+(\boldsymbol{X}_t(x))\, dt\, d\mu_0(x) \\
&\quad = \int_\varepsilon^T \int_{E_t} \varpi_t^+(\boldsymbol{X}_t(x))\, u_t(\boldsymbol{X}_t(x)) \det D_x \boldsymbol{X}_t(x)\, dx\, dt = \int_\varepsilon^T \int_{\boldsymbol{X}_t(E_t)} \varpi_t^+(y) u_t(y)\, dy\, dt \\
&\quad \le \int_\varepsilon^T \int_\Omega \varpi_t^+(x)\, d\mu_t(x)\, dt < +\infty,
\end{aligned}$$

by (5.33). It follows that $u_t(\boldsymbol{X}_t(x))$ is bounded away from 0 on $(\varepsilon, \tau(x))$ for μ_0-a.e. $x \in E_\varepsilon$ and therefore $\tau(x) = T$ for μ_0-a.e. $x \in E_\varepsilon$. Taking a sequence $\varepsilon_n \to 0$ and recalling that the union of E_{ε_n} is P_0, and that $\mu_0(\Omega \setminus P_0) = 0$, we conclude that $\tau(x) = T$ for μ_0-a.e. $x \in \Omega$. □

Remark 5.3. Recalling (5.35), condition (5.33) surely holds if

$$\int_\varepsilon^T \int_{\{x: u_t(x)>0\}} \big(\partial_t u_t\big)^- + \big(\boldsymbol{v}_t \cdot D_x u_t\big)^- \, dx \, dt < +\infty. \tag{5.38}$$

Remark 5.4. Under the same assumptions of Lemma 5.2, setting $V_\varepsilon := \big\{x \in P_0 : \tau(x) > T - \varepsilon\big\}$ we get a family of open subsets $V_\varepsilon \subset P_0$ such that

$$\mu_0(\Omega \setminus V_\varepsilon) = 0 \text{ and the restriction of } \boldsymbol{X} \text{ to } [0, T-\varepsilon] \times V_\varepsilon \text{ is of class } C^1. \tag{5.39}$$

Starting from (5.35) and arguing as in Lemma 5.2, it is easy to check that if (5.33) is replaced by the stronger condition

$$\int_\varepsilon^T \sup_{x: u_t(x)>0} [D_x \cdot \boldsymbol{v}_t(x)]^+ \, dt < +\infty \qquad \forall \varepsilon > 0, \tag{5.40}$$

then maximal solutions are unique for *every* $x_0 \in P_0$ and $\boldsymbol{X}$ is of class C^1 in $[0, T] \times P_0$.

5.2 Gradient flows of functional without geodesic convexity assumption

5.2.1 Subdifferential formulation of Gradient flows in $\mathscr{P}_2(\Omega)$

In this section we give the differential formulation of gradient flow of a functional ϕ defined on $\mathscr{P}_2(\mathbb{R}^d)$. This formulation is based on the definition of subdifferential in the space $\mathscr{P}_2(\mathbb{R}^d)$ (we refer to Chapter 10 of [AGS05] for a systematic development of the subdifferential calculus in spaces of probability measures).

We assume that Ω is an open subset of $\mathbb{R}^d$ with $\mathscr{L}^d(\partial\Omega) = 0$. We identify $\mathscr{P}_2(\Omega)$ with the set of measures $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ such that $\mu(\mathbb{R}^d \setminus \Omega) = 0$ and we consider a proper functional $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ which is lower semicontinuous with respect to the *narrow* convergence of $\mathscr{P}(\mathbb{R}^d)$ on the bounded sets of $\mathscr{P}_2(\mathbb{R}^d)$, i.e.

$$\mu_n \to \mu \quad \text{narrowly}, \quad \sup_n \int_{\mathbb{R}^d} |x|^2 \, d\mu_n(x) < +\infty \quad \Rightarrow \quad \liminf_{n\to\infty} \phi(\mu_n) \ge \phi(\mu); \tag{5.41}$$

we also assume that

$$\inf_{\mathscr{P}_2(\mathbb{R}^d)} \phi > -\infty \text{ and } \phi(\mu) = +\infty \text{ for any } \mu \in \mathscr{P}_2(\mathbb{R}^d) \setminus \mathscr{P}_2^r(\Omega). \tag{5.42}$$

$\mathrm{Dom}\,(\phi) \subset \mathscr{P}_2^r(\Omega)$ denotes the domain of finiteness of the functional. In order to define in a precise way the concept of "gradient flow" of ϕ in $\mathscr{P}_2(\Omega)$, we introduce the notion of *strong* and *limiting* subdifferential following [AGS05, 10.1.1 and 11.1.5].

Definition 5.5 (Strong and limiting subdifferentials in $\mathscr{P}_2(\Omega)$)**.** We say that $\boldsymbol{\xi} \in L^2(\mu;\mathbb{R}^d)$ belongs to the *strong subdifferential* $\partial_s\phi(\mu)$ of ϕ at μ if $\mu \in \mathrm{Dom}\,(\phi)$ and

$$\phi(\boldsymbol{t}_{\#}\mu) - \phi(\mu) \geq \int_\Omega \langle \boldsymbol{\xi}(x), \boldsymbol{t}(x) - x\rangle \, d\mu(x) + o\big(\|\boldsymbol{t} - \boldsymbol{i}\|_{L^2(\mu;\mathbb{R}^d)}\big). \tag{5.43}$$

We say that $\boldsymbol{\xi} \in L^2(\mu;\mathbb{R}^d)$ belongs to the *limiting subdifferential* $\partial_\ell\phi(\mu)$ of ϕ at μ if $\mu \in \mathrm{Dom}\,(\phi)$ and there exist sequences $\boldsymbol{\xi}_k \in \partial_s\phi(\mu_k)$ such that

$$\mu_k \to \mu \quad \text{narrowly in } \mathscr{P}(\Omega), \qquad \boldsymbol{\xi}_k\mu_k \to \boldsymbol{\xi}\mu \quad \text{in the sense of distributions in } \mathcal{D}'(\Omega), \tag{5.44}$$

$$\sup_k \Big(\phi(\mu_k), \int_\Omega \big(|x|^2 + |\boldsymbol{\xi}_k(x)|^2\big)\, d\mu_k(x)\Big) < +\infty. \tag{5.45}$$

We also set

$$|\partial_\ell\phi|(\mu) := \inf\Big\{\|\boldsymbol{\xi}\|_{L^2(\mu;\mathbb{R}^d)} : \boldsymbol{\xi} \in \partial_\ell\phi(\mu)\Big\}, \qquad \text{with the convention} \qquad \inf\emptyset = +\infty. \tag{5.46}$$

Thanks to (5.42) and (5.45), (5.44) is also equivalent to the apparently stronger condition

$$\mu_k \to \mu \quad \text{narrowly in } \mathscr{P}(\mathbb{R}^d), \qquad \boldsymbol{\xi}_k\mu_k \to \boldsymbol{\xi}\mu \quad \text{in the sense of distributions in } \mathcal{D}'(\mathbb{R}^d). \tag{5.45'}$$

In the following Lemma we compare the norm of the subdifferential with the local and relaxed slope defined in the more general setting of metric spaces in (1.28) and (1.36).

Lemma 5.6 (Comparison between subdifferentials and slope)**.** *For every* $\mu \in \mathrm{Dom}\,(\phi)$ *and* $\boldsymbol{\xi} \in \partial_s\phi(\mu)$ *we have*

$$|\partial_\ell\phi|(\mu) \leq |\partial^-\phi|(\mu) \leq |\partial\phi|(\mu) \leq \|\boldsymbol{\xi}\|_{L^2(\mu;\mathbb{R}^d)}. \tag{5.47}$$

Proof. It is obvious that the relaxed slope $|\partial^-\phi|(\mu)$ cannot be greater than $|\partial\phi|(\mu)$, which is also bounded by the norm of any element in $\partial_s\phi(\mu)$ simply by its very definition (5.43).

In order to prove the first inequality in (5.47) let us choose $\varepsilon > 0$ and measures μ_k such that

$$\mu_k \to \mu \quad \text{narrowly}, \quad \sup_k \big\{W(\mu_k,\mu), \phi(\mu_k)\big\} < +\infty, \quad \lim_{k\to\infty} |\partial\phi|(\mu_k) \leq |\partial^-\phi|(\mu) + \varepsilon.$$

Combining Lemma 10.1.2, 3.1.3, and 3.1.5 of [AGS05], we find measures $\tilde\mu_k$ and vectors $\tilde{\boldsymbol{\xi}}_k \in \partial_s\phi(\tilde\mu_k)$ such that

$$W(\mu_k, \tilde\mu_k) \leq k^{-1}, \quad \|\tilde{\boldsymbol{\xi}}_k\|_{L^2(\tilde\mu_k;\mathbb{R}^d)} \leq |\partial\phi|(\mu_k) + k^{-1}, \quad \phi(\tilde\mu_k) \leq \phi(\mu_k), \tag{5.48}$$

so that (up to extracting a subsequence such that $\boldsymbol{\xi}_k\mu_k$ is converging in the distribution sense; using the L^2 bound on $\boldsymbol{\xi}_k$ it is easy to check that the limit is representable as $\boldsymbol{\xi}\mu$) we find a limiting subdifferential $\boldsymbol{\xi}_\varepsilon \in \partial_\ell \phi(\mu)$ with $\|\boldsymbol{\xi}\| \le |\partial^- \phi|(\mu) + \varepsilon$. Being $\varepsilon > 0$ arbitrary, we get the thesis. □

Recalling the definition of $EC_2(\Omega)$ given at the beginning of Section 5.1, we are now ready to give the following

Definition 5.7 (Gradient flow)**.** We say that $\mu_t \in AC^2([0,T];\mathscr{P}_2(\Omega)$ is a gradient flow relative to ϕ with initial datum $\bar\mu \in \mathrm{Dom}\,(\phi)$ if $\phi(\mu_t) \le \phi(\bar\mu)$, $\mu_t \to \bar\mu$ in $\mathscr{P}_2(\Omega)$ as $t \downarrow 0$, and for its minimal velocity field $\boldsymbol{v}_t$ given by Corollary 3.7, we have $(\mu_t, \boldsymbol{v}_t) \in EC_p(\Omega)$ and

$$-\boldsymbol{v}_t \in \partial_\ell \phi(\mu_t) \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T). \tag{5.49}$$

One of the possible ways to show existence of gradient flows is to prove the convergence (up to subsequence) of the time discretization of (5.49) by means of a variational formulation of the implicit Euler scheme (we refer to [AGS05] for a more general discussion of this approach and an up-to-date bibliography). For the sake of completeness we illustrate the algorithm also for this particular case. Given a time step $\tau > 0$ and $\bar\mu \in \mathrm{Dom}\,(\phi)$, we recursively define a sequence of measures μ^k in such a way that $\mu^0 = \bar\mu$ and

$$\mu^k \text{ minimizes } \mu \mapsto \frac{1}{2\tau} W_2^2(\mu, \mu^{k-1}) + \phi(\mu) \tag{5.50}$$

for any integer $k \ge 1$. Then, we can define a piecewise constant discrete solution $\overline{M}_\tau$: $[0,+\infty) \to \mathscr{P}_2(\Omega)$ by

$$\overline{M}_{\tau,t} := \mu^k \qquad \text{if } t \in \big((k-1)\tau, k\tau\big]; \tag{5.51}$$

analogously, denoting by $\boldsymbol{t}^k$ the optimal transport map between μ^{k-1} and μ^k, with inverse $\boldsymbol{s}^k$ we can define a piecewise constant (with respect to time) velocity field by

$$\overline{\boldsymbol{V}}_{\tau,t} := \frac{\boldsymbol{i} - \boldsymbol{s}^k}{\tau} \in L^2(\overline{M}_{\tau,t};\mathbb{R}^d) \quad \text{with} \quad -\overline{\boldsymbol{V}}_{\tau,t} \in \partial_s \phi(\overline{M}_{\tau,t}) \quad \text{for } t \in \big((k-1)\tau, k\tau\big]. \tag{5.52}$$

With this notation the following energy convergence result holds:

Theorem 5.8 (Convergence of discrete approximations and Gradient flows)**.** *Let us assume that ϕ satisfies* (5.41) *and* (5.42) *and that*

$$\partial_\ell \phi(\mu) \quad \textit{contains at most one vector.} \tag{5.53}$$

For every $\bar\mu \in \mathrm{Dom}\,(\phi)$ there exists a vanishing subsequence of time steps $\tau_n \downarrow 0$ and a curve $\mu_t \in AC^2([0,T];\mathscr{P}_2(\Omega))$ such that the discrete solutions $\overline{M}_{\tau_n,t}$, narrowly converge to μ_t as $n \uparrow \infty$ for every $t \in [0,T]$ and $\overline{\boldsymbol{V}}_{\tau_n}\overline{M}_{\tau_n}$ converge to $\overline{\boldsymbol{v}}\mu$ in $\mathcal{D}'((0,T)\times\Omega)$, where $\overline{\boldsymbol{v}}_t$ satisfies

$$\begin{gathered}\partial_t \mu_t + D\cdot(\overline{\boldsymbol{v}}_t \mu_t) = 0 \quad \textit{in } \mathcal{D}'((0,T)\times\mathbb{R}^d), \quad -\overline{\boldsymbol{v}}_t = \partial_\ell \phi(\mu_t) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in (0,T), \\ \int_0^T \|\overline{\boldsymbol{v}}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}\,dt < +\infty.\end{gathered} \tag{5.54}$$

Moreover, if μ_t satisfies the "upper gradient inequality"

$$\phi(\mu_t) + \int_0^t |\partial^- \phi|(\mu_s) \cdot \|\boldsymbol{v}_s\|_{L^2(\mu_s;\mathbb{R}^d)}\, ds \geq \phi(\bar{\mu}) \qquad \forall\, t \in [0,T], \tag{5.55}$$

where $\boldsymbol{v}_t$ is the minimal velocity field to μ_t, then

$$\overline{\boldsymbol{v}}_t = \boldsymbol{v}_t \quad \mu_t\text{-a.e., for } \mathscr{L}^1\text{-a.e. } t \in (0,T), \tag{5.56}$$

μ_t is a gradient flow relative to ϕ according to (5.49),

$$\lim_{n\to\infty} W_2(\overline{M}_{\tau_n,t}, \mu_t) = 0, \qquad \lim_{n\to+\infty} \phi(\overline{M}_{\tau_n,t}) = \phi(\mu_t) \qquad \forall\, t \in [0,T], \tag{5.57}$$

the discrete velocity fields $\overline{\boldsymbol{V}}_{\tau_n,t}$ satisfy

$$\lim_{n\to\infty} \int_0^T \int_\Omega |\overline{\boldsymbol{V}}_{\tau_n,t}|^2 \, d\overline{M}_{\tau_n,t}\, dt = \int_0^T \int_\Omega |\boldsymbol{v}_t|^2 \, d\mu_t\, dt, \tag{5.58}$$

the map $t \mapsto \phi(\mu_t)$ is absolutely continuous, and finally

$$\frac{d}{dt}\phi(\mu_t) = -\int_\Omega |\boldsymbol{v}_t(x)|^2 \, d\mu_t(x) \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T). \tag{5.59}$$

The proof combines various *a priori* estimates and a deep variational interpolation argument due to De Giorgi: it allows to derive a discrete energy identity that gives in the limit the (standard) continuous energy identity, the absolute continuity of $t \mapsto \phi(\mu_t)$ and the convergence of all discrete quantities to their continuous counterpart.

Proof. Theorem 11.1.6 and Corollary 11.1.8 of [AGS05] yield the pointwise narrow convergence of $\overline{M}_{\tau_n}$ to μ, the distributional convergence of $\overline{\boldsymbol{V}}_{\tau_n}\overline{M}_{\tau_n}$ to $\overline{\boldsymbol{v}}\mu$ and (5.54).

If $\boldsymbol{v}_t$ is the velocity vector field associated to the curve μ_t, we have

$$|\mu'|(t) = \lim_{h\to 0} \frac{W(\mu_{t+h}, \mu_t)}{|h|} = \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)} \leq \|\overline{\boldsymbol{v}}_t\|_{L^2(\mu_t;\mathbb{R}^d)} \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T). \tag{5.60}$$

In order to prove the second part of the Theorem, we are introducing the so called "De Giorgi variational interpolants" $\tilde{M}_{\tau,t}$, which are defined as

$$\tilde{M}_{\tau,t} \quad \text{minimizes} \quad \mu \mapsto \frac{1}{2\sigma} W_2^2(\mu, \mu^{k-1}) + \phi(\mu) \quad \text{if } t = (k-1)\tau + \sigma, \quad 0 < \sigma \leq \tau, \tag{5.61}$$

(choosing when $\sigma = \tau$ $\tilde{M}_{\tau,t} = \mu^k$) together with the related optimal transport maps $\tilde{\boldsymbol{s}}_t$ which push $\tilde{M}_{\tau,t}$ on μ^{k-1} for $t = (k-1)\tau + \sigma$, and the velocities

$$\tilde{\boldsymbol{V}}_{\tau,t} := \frac{\boldsymbol{i} - \tilde{\boldsymbol{s}}_t}{\sigma} \in L^2(\tilde{M}_{\tau,t}; \mathbb{R}^d), \qquad -\tilde{\boldsymbol{V}}_{\tau,t} \in \partial_s \phi(\tilde{M}_{\tau,t}). \tag{5.62}$$

The interest of De Giorgi's interpolants relies in the following refined *discrete energy identity* (see Lemma [AGS05, 3.2.2])

$$\frac{1}{2}\int_0^t \int_\Omega |\overline{\boldsymbol{V}}_{\tau,s}|^2 \, d\overline{M}_{\tau,s}\, ds + \frac{1}{2}\int_0^t \int_\Omega |\tilde{\boldsymbol{V}}_{\tau,s}|^2 \, d\tilde{M}_{\tau,s}\, ds + \phi(\overline{M}_{\tau,t}) = \phi(\bar{\mu}) \quad \text{if } t/\tau \in \mathbb{N}. \tag{5.63}$$

Since $\tilde{M}_{\tau_n,t}$ still narrowly converge to μ_t [AGS05, Cor. 3.3.4], we pass to the limit in the above identity as $\tau_n \to 0$ by using Fatou's Lemma, the narrow lower semicontinuity of ϕ, and the very definition of relaxed slope (1.36); by (5.55) we obtain for every $t \in [0,T]$

$$\limsup_{n\to\infty} \frac{1}{2}\int_0^t \int_\Omega |\overline{\boldsymbol{V}}_{\tau_n,s}|^2 \, d\overline{M}_{\tau_n,s}\, ds + \frac{1}{2}\int_0^t |\partial^-\phi|^2(\mu_s)\, ds + \phi(\mu_t) \le \phi(\bar\mu) \tag{5.64}$$

$$\le \phi(\mu_t) + \int_0^t |\partial^-\phi|(\mu_s)\cdot \|\boldsymbol{v}_s\|_{L^2(\mu_s;\mathbb{R}^d)}\, ds, \tag{5.65}$$

i.e.

$$\limsup_{n\to\infty} \frac{1}{2}\int_0^t \int_\Omega |\overline{\boldsymbol{V}}_{\tau_n,s}|^2 \, d\overline{M}_{\tau_n,s}\, ds - \frac{1}{2}\int_0^t\int_\Omega |\boldsymbol{v}_s|^2\, d\mu_s\, ds + \frac{1}{2}\int_0^t \Big(|\partial^-\phi|(\mu_s) - \|\boldsymbol{v}_s\|_{L^2(\mu_s;\mathbb{R}^d)}\Big)^2 ds \le 0.$$

Since general lower semicontinuity results yield (see Theorem 5.4.4 in [AGS05])

$$\liminf_{n\to\infty} \frac{1}{2}\int_0^t \int_\Omega |\overline{\boldsymbol{V}}_{\tau_n,s}|^2 \, d\overline{M}_{\tau_n,s}\, ds \ge \frac{1}{2}\int_0^t\int_\Omega |\overline{\boldsymbol{v}}_s|^2\, d\mu_s\, ds \ge \frac{1}{2}\int_0^t\int_\Omega |\boldsymbol{v}_s|^2\, d\mu_s\, ds,$$

taking into account (5.47) we conclude that

$$\int_0^T \|\overline{\boldsymbol{V}}_{\tau_n,t}\|^2_{L^2(M_{\tau_n,t};\mathbb{R}^d)}\, dt \to \int_0^T \|\boldsymbol{v}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}\, dt,$$

$$\|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)} = \|\overline{\boldsymbol{v}}_t\|_{L^2(\mu_t;\mathbb{R}^d)} = |\partial^-\phi|(\mu_t) = |\partial_\ell\phi|(\mu_t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,T),$$

and, using again (5.64), $\phi(M_{\tau_n,t}) \to \phi(\mu_t)$ for all $[0,T]$ and

$$\phi(\mu_t) = \phi(\bar\mu) - \int_0^t |\partial^-\phi|(\mu_s)\cdot\|\boldsymbol{v}_s\|_{L^2(\mu_s;\mathbb{R}^d)}\, ds \quad \forall t\in[0,T].$$

Hence the map $t \mapsto \phi(\mu_t)$ is absolutely continuous and

$$\frac{d}{dt}\phi(\mu_t) = -\|\boldsymbol{v}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)} \quad \mathscr{L}^1\text{-a.e. in } (0,T).$$

By the minimality of the norm of $\boldsymbol{v}_t$ among all the possible vector fields satisfying the continuity equation (5.54) we also deduce (5.56).

Finally, in order to check the convergence of $\overline{M}_{\tau_n,t}$ to μ_t in $\mathscr{P}_2(\mathbb{R}^d)$ we apply (1.22) and we simply show the convergence of the quadratic moment of $\overline{M}_{\tau_n,t}$. Recall that, if $t \in \big((m-1)\tau, m\tau\big]$

$$\int_{\mathbb{R}^d} |x|^2\, dM_{\tau,t}(x) - \int_{\mathbb{R}^d} |x|^2\, d\bar\mu(x) = \sum_{j=1}^m \int_{\mathbb{R}^d} |x|^2\, d\mu^j(x) - \int_{\mathbb{R}^d} |x|^2\, d\mu^{j-1}(x)$$

$$\le 2\sum_{j=1}^m \int_{\mathbb{R}^d} \langle x - \boldsymbol{s}^j(x), x\rangle\, d\mu^j(x) = 2\int_0^{m\tau}\int_{\mathbb{R}^d} \langle \overline{\boldsymbol{V}}_{\tau,t}(x), x\rangle\, d\overline{M}_{\tau,t}(x),$$

whereas for the absolutely continuous curve μ_t

$$\int_{\mathbb{R}^d} |x|^2\, d\mu_t(x) - \int_{\mathbb{R}^d} |x|^2\, d\bar\mu(x) = 2\int_0^t\int_{\mathbb{R}^d} \langle \boldsymbol{v}_s(x), x\rangle\, d\mu_s(x).$$

Taking into account (5.58) and arguing as in [AGS05, Lemma 5.2.4], we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |x|^2\, dM_{\tau_n,t} \le \int_{\mathbb{R}^d} |x|^2\, d\mu_t,$$

which yields the convergence of the quadratic moments. □

5.2.2 The gradient flow of the internal energy fuctional in Ω.

In this section we apply the general convergence result Theorem 5.8 to the most important example for the applications, the internal energy functional on a domain Ω without convexity assumptions on Ω and without assumption of geodesic convexity of $\mathscr{F}$ in $\mathscr{P}_2(\mathbb{R}^d)$ (see [AGS05, Chap. 10 and 11] for other applications).

Let $F : [0,+\infty) \to (-\infty,+\infty]$ be a proper, lower semi continuous convex function satisfying (4.6), (4.7), (4.8). Let $\Omega \subset \mathbb{R}^d$ be an open set with $\mathscr{L}^d(\partial\Omega) = 0$. We consider the function $V : \mathbb{R}^d \to (-\infty,+\infty]$ simply taking

$$V(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \\ +\infty & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

which satisfies (4.10).

We consider the lower semi continuous functional $\phi : \mathscr{P}_2(\mathbb{R}^n) \to (-\infty,+\infty]$ defined by $\phi(\mu) := \mathscr{F}(\mu) + \mathscr{V}(\mu)$. Since when $\mu \in D(\phi)$, we have that $\operatorname{supp}\mu \subset \overline{\Omega}$ and $\mu \in \mathscr{P}_2^r(\mathbb{R}^n)$, then $\mu(\partial\Omega) = 0$. Consequently the functional ϕ can be thought in the space $\mathscr{P}_2(\Omega)$ and all the integrals on the whole $\mathbb{R}^n$ with respect to the measure μ are in effect integrals on Ω. Then the functional ϕ can be simply rewritten as

$$\phi(\mu) = \begin{cases} \displaystyle\int_\Omega F\left(u(x)\right)\,dx & \text{if } \mu = u \cdot \mathscr{L}^d \in \mathscr{P}^r(\Omega), \\ +\infty & \text{if } \mu \in \mathscr{P}_2(\Omega) \setminus \mathscr{P}_2^r(\Omega), \end{cases} \tag{5.66}$$

which obviously satisfies conditions (5.41) and (5.42). If

$$\mu = u \cdot \mathscr{L}^d \in \mathscr{P}_2^r(\Omega), \quad \text{with} \quad u \in L^\infty(\Omega) \tag{5.67}$$

and we set

$$L_F(u) := uF'(u) - F(u), \tag{5.68}$$

(observe that even when $\mathscr{L}^d(\Omega) = +\infty$ the inequality $0 \le L_F(u) \le uF'\big(\|u\|_\infty\big) + F(u)^-$ yields the integrability of L_F) then it is possible to show that (see [AGS05, Example 11.1.9])

$$\boldsymbol{\xi} \in \partial_\ell\phi(\mu) \quad \Rightarrow \quad L_F(u) \in W^{1,1}(\Omega) \text{ and } \nabla L_F(u) = u\boldsymbol{\xi}, \tag{5.69}$$

so that

$$\text{if} \quad \partial_\ell\phi(\mu) \neq \emptyset \quad \text{then} \quad \partial_\ell\phi(\mu) \quad \text{contains the unique element} \quad \boldsymbol{\xi} = \frac{\nabla L_F(u)}{u} \in L^2(\mu;\mathbb{R}^d). \tag{5.70}$$

The L^2-norm of $\boldsymbol{\xi}$ is a crucial quantity which we call

$$\mathscr{I}(u) := \int_\Omega \left|\frac{\nabla L_F(u)}{u}\right|^2 u\,dx = \int_\Omega \frac{|\nabla L_F(u)|^2}{u}\,dx. \tag{5.71}$$

Thus, if a curve $\mu_t = u_t\mathscr{L}^d \in \mathscr{F}_2(\Omega)$ with $u \in L^\infty((0,T)\times\Omega)$ is a gradient flow relative to ϕ starting from $\bar\mu = \bar u\mathscr{L}^d \in \mathrm{Dom}\,(\phi)$ with $\bar u \in L^\infty(\Omega)$, then

$$L_F(u) \in L^2(0,T;W^{1,2}(\Omega)), \qquad \int_0^T \mathscr{I}(u_t)\,dt < +\infty, \quad \phi(u_t) \le \phi(\bar u) < +\infty, \tag{5.72}$$

and u_t solves the nonlinear diffusion PDE

$$\partial_t u_t - \Delta L_F(u_t) = 0 \quad \text{in } (0,T)\times\Omega, \quad \partial_{\boldsymbol{n}} L_F(u_t) = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{5.73}$$

in the following weak sense

$$\frac{d}{dt}\int_\Omega \zeta(x)u_t(x)\,dx + \int_\Omega \nabla L_F(u_t(x))\cdot\nabla\zeta(x)\,dx = 0 \quad \forall\,\zeta \in C_c^\infty(\mathbb{R}^d). \tag{5.74}$$

In particular, integrating by parts against test function ζ with $\partial_{\boldsymbol{n}}\zeta = 0$ on $\partial\Omega$, we shall also see that u_t is the *unique* [BC79] weak solution of (5.74) satisfying

$$\frac{d}{dt}\int_\Omega \zeta(x)u_t(x)\,dx = \int_\Omega L_F(u_t(x))\Delta\zeta(x)\,dx \quad \forall\,\zeta \in C_c^\infty(\mathbb{R}^d), \quad \partial_{\boldsymbol{n}}\zeta = 0 \text{ on } \partial\Omega. \tag{5.75}$$

In order to apply Theorem 5.8 to the internal energy functional we saw that (5.53) and the general properties (5.41), (5.42) are satisfied. The crucial assumption of the previous theorem is in fact the "upper gradient" inequality (5.55).

With respect to the results of [AGS05, Chap.2], in Theorem 5.8 there is a slight improvement: it is sufficient to check the upper gradient inequality only on the limit curves arising from the "Minimizing Movement" scheme (instead of proving it on all the curves with finite energy). This simple remark is quite useful to show that bounded solutions to the nonlinear diffusion equation (5.73) satisfy the "upper gradient inequality" (5.55).

Proposition 5.9. *Let us assume that the function F satisfies (4.7), (4.8) and*

$$F \in C^2(0,+\infty), \qquad \text{and } F'' > 0. \tag{5.76}$$

We also suppose that the initial datum $\bar\mu = \bar u\mathscr{L}^d$ satisfies

$$\sup_{x\in\Omega} \bar u(x) < +\infty, \quad \phi(\bar\mu) = \int_\Omega F(\bar u(x))\,dx < +\infty, \quad \int_\Omega |x|^2\bar u(x)\,dx < +\infty. \tag{5.77}$$

Then the discrete solutions $M_{\tau,t}$ of the Minimizing Movement scheme converge pointwise to $\mu_t := u_t\mathscr{L}^d$ in $\mathscr{P}_2(\Omega)$ as $\tau \downarrow 0$, u_t is the unique solution in $L^\infty((0,T)\times\Omega)$ of (5.73) with the integrability conditions (5.72), and u_t satisfies the energy identity

$$\int_\Omega F(u_t)\,dx + \int_0^t\int_\Omega \frac{|\nabla L_F(u_s)|^2}{u_s}\,dx\,ds = \int_\Omega F(\bar u)\,dx \quad \forall\,t \in [0,T]. \tag{5.78}$$

Moreover, the internal energy functional ϕ introduced in (5.66) satisfies the "upper gradient inequality" along any limit curve $\mu_t = u_t\mathscr{L}^d$ and therefore all the convergence properties of Theorem 5.8 hold true.

Proof. By applying the first part of Theorem 5.8 and the discrete L^∞-estimates of [Ott01], [Agu05], we obtain that any limit curve $\mu_t := u_t \mathscr{L}^d$ of the Minimizing Movement scheme is a uniformly bounded weak solution (according to (5.75)) of (5.73) satisfying (5.72). Since bounded weak solutions are unique [BC79], we obtain the convergence of the whole sequence $M_{\tau,t}$; standard estimates on nonlinear diffusion equations show that $\|u_t\|_{L^\infty(\Omega)} \le M_\infty := \|\bar u\|_{L^\infty(\Omega)}$.

It remains to check the validity of the upper gradient inequality. Let

$$\begin{aligned} \mathscr{S} \ &:= \ \bar u + \overline{\operatorname{conv}\{u_t - \bar u : t \in [0,T]\}}^{L^2(\Omega)} \\ &\subset \ \Big\{u \in L^1(\Omega) \cap L^\infty(\Omega) : u \ge 0,\ \|u\|_{L^1(\Omega)} = 1,\ \|u\|_{L^\infty(\Omega)} \le M_\infty\Big\}, \end{aligned}$$

and the H^{-1}-like distance on $\mathscr{S}$

$$d(u_1,u_2) := \sup\Big\{\int_\Omega \zeta(x)\,(u_1-u_2)\,dx : \zeta \in C^\infty_c(\mathbb{R}^d),\quad \|\nabla\zeta\|_{L^2(\Omega;\mathbb{R}^d)} \le 1\Big\}. \tag{5.79}$$

It is not difficult to check that d is finite on $\mathscr{S}$: let us first observe that (5.74), (5.72), and a standard approximation result in $W^{1,2}(\Omega)$, yield

$$\int_\Omega \zeta\big(u_{t_1} - u_{t_0}\big)\,dx = -\int_{t_0}^{t_1}\int_\Omega \nabla L_F(u_t)\cdot\nabla\zeta\,dx\,dt \qquad \forall\,\zeta \in W^{1,2}(\Omega). \tag{5.80}$$

Choosing now a test function $\zeta \in C^\infty_c(\mathbb{R}^d)$ with $\|\nabla\zeta\|_{L^2(\Omega)} \le 1$ and $0 \le t_0 < t_1 \le T$, we get

$$\int_\Omega \zeta\big(u_{t_1} - u_{t_0}\big)\,dx \le \int_{t_0}^{t_1} \|\nabla L_F(u_t)\|_{L^2(\Omega;\mathbb{R}^d)}\,dt \le M_\infty^{1/2}\int_{t_0}^{t_1}\sqrt{\mathscr{I}(u_t)}\,dt, \tag{5.81}$$

so that by (5.72)

$$d(u_{t_1},u_{t_0}) \le M_\infty^{1/2}\int_{t_0}^{t_1}\sqrt{\mathscr{I}(u_t)}\,dt, \quad \text{and} \quad \int_0^T \mathscr{I}(u_t)\,dt < +\infty. \tag{5.82}$$

Let us now introduce the regularized convex functionals

$$\phi_\varepsilon(u) := \int_\Omega F_\varepsilon(u)\,dx \quad \forall\,u \in \mathscr{S}, \qquad F_\varepsilon(u) := \begin{cases} F(u) + \varepsilon F'(\varepsilon) - F(\varepsilon) & \text{if } u > \varepsilon, \\ F'(\varepsilon)u & \text{if } 0 \le u \le \varepsilon, \end{cases} \tag{5.83}$$

and the related "Lagrangians"

$$L_{F_\varepsilon}(u) = \begin{cases} L_F(u) - L_F(\varepsilon) & \text{if } u > \varepsilon, \\ 0 & \text{if } 0 \le u \le \varepsilon. \end{cases} \tag{5.84}$$

Observe that ϕ_ε are geodesically convex functionals on $\mathscr{S}$, since the usual segments $t \mapsto (1-t)\rho_0 + t\rho_1$, $\rho_0,\rho_1 \in \mathscr{S}$, are constant speed geodesics in $\mathscr{S}$. An upper bound for the slope of ϕ_ε

$$|\partial\phi_\varepsilon|_{\mathscr{S}}(u) := \limsup_{d(u,\rho)\to 0} \frac{\big(\phi_\varepsilon(u) - \phi_\varepsilon(\rho)\big)^+}{d(u,\rho)}$$

can be readily obtained: first of all, we recall that $\mathscr{I}(u) < +\infty$ implies $L_F(u) \in W^{1,2}(\Omega)$ and $|\nabla L_F(u)|^2/u \in L^1(\Omega)$, hence

$$\mathscr{I}(u) < +\infty \quad \Longrightarrow \quad \frac{L_{F_\varepsilon}(u)}{u} = F'_\varepsilon(u) \in W^{1,2}(\Omega), \quad \|\nabla F'_\varepsilon(u)\|^2_{L^2(\Omega;\mathbb{R}^d)} \le \varepsilon^{-1}\mathscr{I}(u); \tag{5.85}$$

therefore, assuming that $\mathscr{I}(u) < +\infty$ and using the convexity of F_ε, we get

$$\phi_\varepsilon(u) - \phi_\varepsilon(\rho) \le \int_\Omega F'_\varepsilon(u)(u-\rho)\,dx \le d(u,\rho)\|\nabla F'_\varepsilon(u)\|_{L^2(\Omega;\mathbb{R}^d)}, \tag{5.86}$$

and we can use (5.85) to obtain

$$|\partial\phi_\varepsilon|_{\mathscr{S}}(u) \le \varepsilon^{-1/2}\sqrt{\mathscr{I}(u)}. \tag{5.87}$$

Applying [AGS05, Theorem 1.2.5] and the estimate (5.82) we find that the map $t \mapsto \phi_\varepsilon(u_t)$ is absolutely continuous; since $F'_\varepsilon(u_t) \in W^{1,2}(\Omega)$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$, combining (5.86) with u_t, u_{t+h} instead of u, ρ and (5.80) with $\zeta := F'_\varepsilon(u_t)$, we get

$$\phi_\varepsilon(u_t) - \phi_\varepsilon(u_{t+h}) \le \int_\Omega F'_\varepsilon(u_t)\big(u_t - u_{t+h}\big)\,dx = \int_t^{t+h}\int_\Omega \nabla L_F(u_s)\cdot\nabla F'_\varepsilon(u_t)\,dx\,ds, \tag{5.88}$$

for $\mathscr{L}^1$-a.e. $t \in (0,T)$. Since Lebesgue differentiation Theorem for functions with values in the Hilbert space $L^2(\Omega;\mathbb{R}^d)$ yields

$$\lim_{h\to 0}\int_\Omega \Big|\frac{1}{h}\int_t^{t+h}\nabla L_F(u_s)\,ds - \nabla L_F(u_t)\Big|^2\,dx = 0 \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T),$$

dividing by $h \neq 0$ and taking the limit of (5.88) as h goes to 0 from the right and from the left, we find that the derivative of $\phi_\varepsilon \circ u$ is

$$\frac{d}{dt}\phi_\varepsilon(u_t) = -\int_\Omega \nabla F'_\varepsilon(u_t)\cdot\nabla L_F(u_t)\,dx \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T).$$

Since

$$u_t\nabla F'_\varepsilon(u_t) = \nabla L_{F_\varepsilon}(u_t) = \nabla L_F(u_t)\chi_{\varepsilon,t}(x) \quad \text{with} \quad \chi_{\varepsilon,t}(x) := \begin{cases} 1 & \text{if } u_t(x) > \varepsilon, \\ 0 & \text{if } 0 \le u_t(x) \le \varepsilon, \end{cases} \tag{5.89}$$

integrating in time we eventually find

$$\int_\Omega F_\varepsilon(u_t)\,dx + \int_0^t\int_\Omega \frac{|\nabla L_F(u_s)|^2}{u_s}\chi_{\varepsilon,s}\,dx\,ds = \int_\Omega F_\varepsilon(\bar u)\,dx. \tag{5.90}$$

Since $F_\varepsilon \downarrow F$ and it is easy to check that, e.g., $F_1(u) \in L^1(\Omega)$, we can pass to the limit as $\varepsilon \downarrow 0$ in (5.90) and we find the energy identity (5.78).

The "upper gradient inequality" (5.55) follows immediately if we show that

$$\bar{\boldsymbol{v}}_t = -\partial_\ell\phi(\mu_t) = -\frac{\nabla L_F(u_t)}{u_t}$$

is the minimal tangent vector field. Thanks to the characterization of the minimal tangent field of Theorem 8.3.1 and Proposition 8.4.5 of [AGS05], it is sufficient to show that for

$\mathscr{L}^1$-a.e. $t \in (0,T)$ there exists a family of functions $\zeta_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ such that $\nabla\zeta_\varepsilon \to \bar{\boldsymbol{v}}_t$ in $L^2(\mu_t;\mathbb{R}^d)$ as $\varepsilon \downarrow 0$. Since $u_t \in L^\infty(\Omega)$, by standard extension and approximation results, it is sufficient to find an approximating sequence $\zeta_\varepsilon \in W^{1,2}(\Omega)$; disregarding a $\mathscr{L}^1$-negligible subset of $(0,T)$ we can assume that $L_F(u_t) \in W^{1,2}(\Omega)$ and recalling (5.89) we can choose

$$\zeta_\varepsilon := F_\varepsilon'(u_t) \in W^{1,2}(\Omega) \quad \text{so that} \quad \nabla F_\varepsilon'(u_t) = \frac{\nabla L_F(u_t)}{u_t}\chi_{\varepsilon,t} \to \frac{\nabla L_F(u_t)}{u_t} \tag{5.91}$$

in $L^2(\mu_t;\mathbb{R}^d)$ as $\varepsilon \downarrow 0$. □

Remark 5.10 (The cases $0 < F(0) < +\infty$ and $F(0) = +\infty$)**.** When $0 < F(0) < +\infty$ we assume that Ω is bounded and the proof of Proposition 5.9 works simply substituting F by $F - F(0)$. When $F(0) = +\infty$, we will also assume that $\bar{u} \geq u_{\min} > 0$ $\mathscr{L}^d$-a.e. in Ω, so that the maximum principle yields $u_t \geq u_{\min}$ in $(0,T)\times\Omega$. In this case Ω has to be bounded and the calculations are even easier than in the previous Proposition: e.g. by modifying F in the interval $(0,u_{\min})$ we directly obtain (5.78) without performing any preliminary regularization of F around 0.

Remark 5.11 (The case when ϕ is displacement convex)**.** When Ω is a *convex* subset of $\mathbb{R}^d$, ϕ has compact sublevels in $\mathscr{P}_2^r(\mathbb{R}^d)$ (this is always the case if, e.g., Ω is *bounded*), and it is *displacement convex*, i.e. for any μ, $\nu \in \mathscr{P}_2^r(\Omega)$, if we denote by $\boldsymbol{t}_\mu^\nu \in L^2(\mu;\Omega)$ the optimal transport map between μ and ν relative to W_2, the map

$$t \mapsto \phi\left(((1-t)\boldsymbol{t}_\mu^\nu + t\boldsymbol{i})_\#\mu\right) \quad \text{is convex in } [0,1], \tag{5.92}$$

then the theory becomes considerably simpler: solutions to the gradient flow equation are in fact unique, the (limiting) subdifferential is characterized by the following system of variational inequalities

$$\boldsymbol{\xi} \in \partial_\ell\phi(\mu) \quad \Longleftrightarrow \quad \phi(\nu) \geq \phi(\mu) + \int_\Omega \langle \boldsymbol{\xi}, \boldsymbol{t}_\mu^\nu - \boldsymbol{i}\rangle\, d\mu \qquad \forall \nu \in \mathscr{P}^r(\Omega), \tag{5.93}$$

and inequality (5.55) is always satisfied by any absolutely continuous curve in $\mathscr{P}_2(\mathbb{R}^d)$ (see Corollary 2.4.11, §10.1.1 and Chapter 11 of [AGS05]).

Notice also that if the stronger property

$$t \mapsto \phi\left(((1-t)\boldsymbol{t}_\mu^\nu + t\boldsymbol{t}_\mu^\sigma)_\#\mu\right) \quad \text{is convex in } [0,1] \text{ for any } \mu,\ \nu,\ \sigma \in \mathscr{P}^r(\Omega)$$

holds, then even *error estimates* for the scheme can be proved, see Theorem 4.0.4 and §11.2 of [AGS05], and as a consequence one can also consider initial data that are in $\overline{\mathrm{Dom}\,(\phi)}$.

We recall that in the case of the *internal energy* functional, the assumption of displacement convexity is equivalent to McCann's condition

$$s \mapsto s^d F(s^{-d}) \text{ is convex and non increasing in } (0,+\infty), \tag{5.94}$$

which is more restrictive than convexity if the space dimension d is greater than 1.

5.3 Convergence of iterated transport maps

In this section we study the convergence as $\tau \downarrow 0$ of the iterated transport maps

$$\boldsymbol{T}^k := \boldsymbol{t}^k \circ \boldsymbol{T}^{k-1} = \boldsymbol{t}^k \circ \boldsymbol{t}^{k-1} \circ \cdots \circ \boldsymbol{t}^1, \tag{5.95}$$

associated to the Minimizing Movement scheme (5.50); recall that we denoted by $\boldsymbol{t}^k$ the (unique) optimal transport map pushing μ^{k-1} on μ^k and by $\boldsymbol{s}^k = \big(\boldsymbol{t}^k\big)^{-1}$ its inverse map, pushing μ^k to μ^{k-1}; in particular $\boldsymbol{T}^k$ maps $\bar\mu = \mu^0$ to μ^k.

We can "embed" the discrete sequence $\boldsymbol{T}^k$ into a continuous flow $\boldsymbol{T}_{\tau,t}$ such that $\boldsymbol{T}_{\tau,t} = \boldsymbol{T}^k$ if t is one of the nodes $k\tau$ of the discrete partition. To fix a simple notation, in what follows

$$\text{if } t \in \big[(k-1)\tau, k\tau\big) \quad \text{we decompose it as} \quad t = (k-1)\tau + \sigma\tau, \quad \sigma = \Big\{\frac{t}{\tau}\Big\} \in (0,1], \tag{5.96}$$

and we set

$$\boldsymbol{t}^{k-1,\sigma} := (1-\sigma)\boldsymbol{i} + \sigma\boldsymbol{t}^k, \quad \boldsymbol{t}^{\sigma,k} := \boldsymbol{t}^k \circ \big(\boldsymbol{t}^{k-1,\sigma}\big)^{-1}, \quad \text{so that} \quad \boldsymbol{t}^k = \boldsymbol{t}^{\sigma,k} \circ \boldsymbol{t}^{k-1,\sigma}, \tag{5.97}$$

$$\boldsymbol{T}_{\tau,t} := (1-\sigma)\boldsymbol{T}^{k-1} + \sigma\boldsymbol{T}^k = \boldsymbol{t}^{k-1,\sigma} \circ \boldsymbol{T}^{k-1}, \quad \text{so that} \quad \boldsymbol{T}^k = \boldsymbol{t}^{\sigma,k} \circ \boldsymbol{T}_{\tau,t}. \tag{5.98}$$

The continuous family of maps $\boldsymbol{T}_{\tau,t}$ is naturally associated to a sort of "piecewise linear" interpolant (according to McCann's displacement convexity) *continuous* interpolation $\mu_{\tau,t}$ of the sequence of measures μ^k:

$$\mu_{\tau,t} := \big(\boldsymbol{t}^{k-1,\sigma}\big)_\# \mu^{k-1} = (\boldsymbol{T}_{\tau,t})_\# \bar\mu, \quad \text{so that} \quad \mu^k = \big(\boldsymbol{t}^{\sigma,k}\big)_\# \mu_{\tau,t} \qquad \text{if } t \in \big[(k-1)\tau, k\tau\big). \tag{5.99}$$

Notice that the inverse of the map $\boldsymbol{t}^{k-1,\sigma}$ is well defined up to $\mu_{\tau,t}$-negligible sets and it coincides with the optimal transport map between $\mu_{\tau,t}$ and μ^{k-1}; moreover

$$\boldsymbol{t}^{\sigma,k} = \Big(\boldsymbol{t}^{k-1,\sigma} \circ \boldsymbol{s}^k\Big)^{-1} = \Big(\sigma\boldsymbol{i} + (1-\sigma)\boldsymbol{s}^k\Big)^{-1}. \tag{5.100}$$

We define also the velocity vector field $\boldsymbol{v}_{\tau,t}$ associated to this flow

$$\boldsymbol{v}_{\tau,t} := \overline{\boldsymbol{V}}_{\tau,t} \circ \boldsymbol{t}^{\sigma,k} = \frac{\boldsymbol{i} - \boldsymbol{s}^k}{\tau} \circ \boldsymbol{t}^{\sigma,k} \ \in L^2(\mu_{\tau,t};\mathbb{R}^d) \qquad \text{if } t \in \big[(k-1)\tau, k\tau\big). \tag{5.101}$$

Due to the uniform $C^{0,1/2}$ bound

$$\frac{1}{2}\sum_{k=0}^\infty W_2^2(\mu^{k+1},\mu^k) \le \tau \sum_{k=0}^\infty \Big(\phi(\mu^k) - \phi(\mu^{k+1})\Big) \le \tau\big(\phi(\bar\mu) - \inf\phi\big), \tag{5.102}$$

the convergence statement of Theorem 5.8 applies not only to the discrete piecewise constant solution $\overline{M}_{\tau,t}$, but also to $\mu_{\tau,t}$. The following lemma shows that $\boldsymbol{v}_{\tau,t}$ is an admissible velocity field relative to $\mu_{\tau,t}$ and provides the corresponding energy estimate.

Lemma 5.12 (Properties of the continuous interpolation)**.** *Let $\mu_{\tau,t}$, $\boldsymbol{v}_{\tau,t}$, $\boldsymbol{T}_{\tau,t}$ be defined as in* (5.99), (5.101), and (5.98) *respectively. Then $\boldsymbol{T}_{\tau,t}$ is a flow relative to $(\mu_{\tau,t}, \boldsymbol{v}_{\tau,t})$ and in particular the continuity equation*

$$\frac{d}{dt}\mu_{\tau,t} + D_x \cdot (\boldsymbol{v}_{\tau,t}\mu_{\tau,t}) = 0 \quad \textit{in } \mathcal{D}'\left((0,T)\times\Omega\right) \tag{5.103}$$

holds. Moreover we have

$$\int_0^T \int_\Omega |\boldsymbol{v}_{\tau,t}|^2 \, d\mu_{\tau,t} \, dt = \int_0^T \int_\Omega |\overline{\boldsymbol{V}}_{\tau,t}|^2 \, d\overline{M}_{\tau,t} \, dt \qquad \forall T > 0. \tag{5.104}$$

Proof. We check first that $\boldsymbol{T}_{\tau,t}$ is a flow relative to $\boldsymbol{v}_{\tau,t}$. Clearly $t \mapsto \boldsymbol{T}_{\tau,t}(x)$ is continuous and piecewise linear; in any interval $\big((k-1)\tau, k\tau\big)$, with k integer, we have that its derivative is given by

$$\frac{d}{dt}\boldsymbol{T}_{\tau,t}(x) = \frac{1}{\tau}\big(\boldsymbol{T}^k(x) - \boldsymbol{T}^{k-1}(x)\big), \tag{5.105}$$

so that (5.101) and (5.98) yield

$$\boldsymbol{v}_{\tau,t}(\boldsymbol{T}_{\tau,t}(x)) = \frac{1}{\tau}(\boldsymbol{i} - \boldsymbol{s}^k) \circ \boldsymbol{t}^{\sigma,k}(\boldsymbol{T}_{\tau,t}(x)) = \frac{1}{\tau}(\boldsymbol{i} - \boldsymbol{s}^k) \circ \boldsymbol{T}^k(x) = \frac{1}{\tau}(\boldsymbol{T}^k(x) - \boldsymbol{T}^{k-1}(x)),$$

and therefore the ODE $\frac{d}{dt}\boldsymbol{T}_{\tau,t}(x) = \boldsymbol{v}_{\tau,t}(\boldsymbol{T}_{\tau,t}(x))$ is satisfied.

Finally, by the definition of $\boldsymbol{v}_{\tau,t}$ and taking into account (5.99) we get

$$\int_\Omega |\boldsymbol{v}_{\tau,t}|^2 \, d\mu_{\tau,t} = \int_\Omega |\overline{\boldsymbol{V}}_{\tau,t}(\boldsymbol{t}^{\sigma,k}(x))|^2 \, d\mu_{\tau,t}(x) = \int_\Omega |\overline{\boldsymbol{V}}_{\tau,t}|^2 \, d\mu^k \qquad t \in \big[(k-1)\tau, k\tau\big),$$

and this immediately gives (5.104) after an integration in time. □

Theorem 5.13 (Convergence of forward iterated transport maps)**.** *Let* $\mu_{\tau,t}$, $\boldsymbol{v}_{\tau,t}$, $\boldsymbol{T}_{\tau,t}$ *be defined as in* (5.99), (5.101) *and* (5.98) *respectively starting from* $\bar\mu \in \mathrm{Dom}\,(\phi)$*. Assume that* ϕ *satisfies all the assumptions of Theorem 5.8, including the upper gradient inequality* (5.55)*, and that the minimal velocity field* $\boldsymbol{v}_t$ *relative to the gradient flow* μ_t *of* ϕ *with initial condition* $\bar\mu$ *satisfies* (5.7)*.*

Then

$$\lim_{\tau\downarrow 0} \int_\Omega \max_{t\in[0,T]} |\boldsymbol{T}_{\tau,t}(x) - \boldsymbol{X}(t,x)|^2 \, d\bar\mu(x) = 0 \qquad \forall T > 0, \tag{5.106}$$

where $\boldsymbol{X}$ *is the flow associated to* $\boldsymbol{v}_t$*.*

Proof. We apply Theorem 5.1. Notice that the convergence of $\mu_{\tau,t}$ to μ_t as $\tau \downarrow 0$ is ensured by Theorem 5.8 (that gives the convergence of $\overline{M}_{\tau,t}$) and the $C^{0,1/2}$ estimate (5.102). Moreover, (5.104) and (5.58) give

$$\limsup_{\tau\downarrow 0} \int_0^T \int_\Omega |\boldsymbol{v}_{\tau,t}|^2 \, d\mu_{\tau,t} \, dt = \limsup_{\tau\downarrow 0} \int_0^T \int_\Omega |\overline{\boldsymbol{V}}_{\tau,t}|^2 \, d\overline{M}_{\tau,t} \, dt = \int_0^T |\boldsymbol{v}_t|^2 \, d\mu_t \, dt$$

for any $T > 0$. Therefore, taking also into account that $\boldsymbol{T}_{\tau,t}(x)$ are flows relative to $\boldsymbol{v}_{t,\tau}$, all the assumptions of Theorem 5.1 are fulfilled and (5.106) is the conclusion of that theorem. □

The assumption (5.26) (and its variants considered in Subsection 5.1.2) may not be satisfied if the initial datum $\bar\mu$ is not sufficiently smooth. For this reason it is also interesting to consider the behaviour of the inverses of $\boldsymbol{T}_{\tau,t}$, reversing also the time variable. In the following theorem we focus on the case of a Sobolev regularity of $\boldsymbol{v}$ with respect to the space variable, leaving all other variants (for instance the BV ones) to the interested reader.

Recalling (5.2), given $T > 0$ we define the *backward* flow $\tilde{\boldsymbol{X}}(t,x)$ associated to $\boldsymbol{v}_t$ as $\tilde{\boldsymbol{X}}(t,x) := \boldsymbol{X}(t,T,x)$. Under the assumption

$$\boldsymbol{v} \in L^1_{\mathrm{loc}}\left((0,T]; W^{1,1}_{\mathrm{loc}}(\Omega)\right), \quad [\nabla_x \cdot \boldsymbol{v}]^+ \in L^1_{\mathrm{loc}}\left((0,T]; L^\infty(\Omega)\right), \tag{5.107}$$

considered up to a time reversal, the backward flow is well defined up to $t = 0$ and produces for $t > 0$ densities $\mu_t = \tilde{\boldsymbol{X}}(t,\cdot)_\# \bar{\mu}$ in $L^\infty(\Omega)$ for any $\bar{\mu} \in L^\infty(\Omega)$.

Analogously, we define

$$\tilde{\boldsymbol{T}}_{\tau,t} := \boldsymbol{T}_{\tau,t} \circ \boldsymbol{T}^{-1}_{\tau,T}, \tag{5.108}$$

mapping $\mu_{\tau,T}$ to $\mu_{\tau,t}$. Finally, as in Theorem 5.1, we have to take into account a correction term due to the optimal map $\boldsymbol{d}^\tau$ between μ_T and $\mu_{\tau,T}$.

Theorem 5.14 (Convergence of backward iterated transport maps)**.** *Let $\tilde{\boldsymbol{T}}_{\tau,t}$ be defined as in (5.108) starting from $\bar{\mu} \in \mathrm{Dom}(\phi)$. Assume that ϕ satisfies all the assumptions of Theorem* 5.8 *and that the tangent velocity field $\boldsymbol{v}_t$ relative to the gradient flow μ_t of ϕ with initial condition $\bar{\mu}$ satisfies (5.107). Then*

$$\lim_{\tau \downarrow 0} \int_\Omega \max_{t \in [0,T]} \left| \tilde{\boldsymbol{T}}_{\tau,t}(\boldsymbol{d}^\tau) - \tilde{\boldsymbol{X}}(t,x) \right|^2 d\mu_T = 0.$$

Proof. We have just to notice that $\tilde{\boldsymbol{T}}_{\tau,t}$ is the backward flow associated to the velocity field $\boldsymbol{v}_{\tau,t}$ defined in (5.101) and then, using the same estimates used in the proof of Theorem 5.13, apply Theorem 5.1 and Remark 5.1.2 in a time reversed situation. □

Now we conclude the discussion relative to the *internal energy* functional as in subsection 5.2.2.

We consider the internal energy functional ϕ in Ω defined in (5.66), with the convex lower semi continuous function F satisfying (4.6), (4.7), (4.8), (5.76), and $\bar{\mu} = \bar{u}\mathscr{L}^d$ satisfying (5.77). Theorem 5.8 applies, yielding a unique gradient flow $\mu_t = u_t \mathscr{L}^d$ relative to ϕ which satisfies the nonlinear PDE

$$\frac{\partial}{\partial t} u_t = D_x \cdot (\nabla L_F(u_t)) \qquad \text{in } \mathscr{D}'\left((0,T) \times \Omega\right) \tag{5.109}$$

with homogeneous Neumann boundary conditions. Since $L'_F(s) = sF''(s)$, under suitable regularity assumptions on u_t the chain rule gives the alternative formulations

$$\frac{d}{dt} u_t = D_x \cdot (u_t F''(u_t) \nabla u_t)) = D_x \cdot (u_t \nabla F'(u_t)). \tag{5.110}$$

If we want to apply to this example the results of Theorem 5.13, we should check if the tangent velocity field $\boldsymbol{v}$ given by

$$\boldsymbol{v}_t = -\frac{\nabla L_F(u_t)}{u_t} = -\nabla F'(u_t) \qquad \text{for } t > 0, \tag{5.111}$$

satisfies (5.26) or one of the other conditions discussed in Subsection 5.1.2, which are strictly related to the regularity of the solution u_t.

Besides (5.26), in the following discussion we focus our attention on the regular densities of Subsection 5.1.2 (and Remarks 5.3, 5.4). We distinguish some cases:

[(a) Ω is bounded and of class $C^{2,\alpha}$ and $\bar u \in C^\alpha$ is bounded away from 0.] When

$$\begin{cases} \Omega \text{ is bounded and of class } C^{2,\alpha} \text{ for some } \alpha > 0, \text{ and} \\ \bar u \in C^\alpha(\Omega), \quad 0 < u_{\min} \le \bar u(x) \le u_{\max} \quad \text{for } \mathscr{L}^d\text{-a.e. } x \in \Omega, \end{cases} \tag{5.112}$$

the maximum principle shows that the solution u_t satisfies the same bound

$$0 < u_{\min} \le u_t(x) \le u_{\max} \quad \forall\, x \in \Omega,\, t \in [0,T]; \tag{5.113}$$

by [LSU67, Thm. 10.1, Chap. III;Thm. 7.1, Chap. V] the variational solution of (5.109) is Hölder continuous in $[0,T] \times \overline{\Omega}$.

The smooth transformation $\rho := L_F(u)$ shows that ρ is a solution of the linear parabolic equation

$$\frac{\partial}{\partial t}\rho - a(t,x)\Delta\rho = 0 \quad \text{in } (0,T) \times \Omega$$

with homogeneous Neumann boundary conditions, where $a(t,x) = L'_F(u(t,x))$ is Hölder continuous and satisfies $0 < a_{\min} \le a \le a_{\max} < +\infty$ in $(0,T] \times \overline{\Omega}$.

Standard parabolic regularity theory [LSU67, Chap. IV] yields $D^2_x \rho \in C^{\bar\alpha}((0,T) \times \overline{\Omega})$ for some $\bar\alpha > 0$; moreover, since $\rho_0 \in C^\alpha(\Omega)$, too, then the intermediate Schauder estimates of [Lie86, Thm 6.1] yield for every $x_0 \in \Omega$ the existence of $\varepsilon, \delta > 0$ such that

$$\sup_{(t,x) \in (0,T) \times B_\varepsilon(x_0)} t^{1-\delta} \|D^2_x \rho_t(x)\| < +\infty, \tag{5.114}$$

so that $\boldsymbol{v}$ satisfies (5.26).

[(b) The Heat/Porous medium equation in $\Omega = \mathbb{R}^d$.]

Let us consider the Heat/Porous medium equation in $\mathbb{R}^d$, corresponding to the choice

$$L_F(u) = u^m, \qquad F(s) := \begin{cases} \frac{1}{m-1} s^m & \text{if } m > 1, \\ s \log s & \text{if } m = 1. \end{cases} \tag{5.115}$$

Following the approach of Paragraph *regular densities* in Subsection 5.1.2, and Remark 5.4, we assume here that an open set $P_0 \subset \mathbb{R}^d$ exists such that

$$\bar u \in C^\alpha(P_0), \quad \bar u > 0 \quad \text{in } P_0, \quad \bar u \equiv 0 \quad \text{in } \mathbb{R}^d \setminus P_0. \tag{5.116}$$

We know (see [CF79, DiB83, Zie82] and also [Fri82, Chap. 5, Thm. 3.1, 3.3]) that the solution u is Hölder continuous in $\big((0,T] \times \mathbb{R}^d\big) \cup \big(\{0\} \times P_0\big)$; still applying the local regularity theory we mentioned in point **(a)**, we obtain that $D^2_x u$, $D_x F'(u) = D_x \boldsymbol{v}$ are Hölder continuous in P, too, thus showing that (5.31) is satisfied.

The Hölder assumption on $\bar u$ shows that u is locally Hölder continuous in P up to the initial time $t = 0$, too. Arguing as before, we obtain (5.114) for every $x_0 \in P_0$, which entails (5.32).

In order to check (5.40) we will invoke the ARONSON-BENILAN estimate [AB79], [Fri82, Chap. 5, Lemma 2.1]

$$\Delta F'(u_t) \geq -\frac{k}{t}, \quad \text{with} \quad k = \frac{1}{m-1+(2/d)}, \quad \text{i.e.} \quad D_x \cdot \boldsymbol{v}_t(x) \leq \frac{k}{t}, \quad \forall\,(t,x) \in P. \tag{5.117}$$

[(c) Ω is $C^{2,\alpha}$, possibly unbounded, and $\Delta L_F(\bar u)$ is a finite measure.] The local regularity results we used in the previous points **(a), (b)** depend, in fact, only on the local behavior of L_F' around 0; let us thus assume that, in a suitable neighborhood $(0,\varepsilon_0)$, the function L_F' has a "power like" behaviour

$$c_0 u^{\kappa_0} \leq L_F'(u) \leq c_1 u^{\kappa_1} \quad \forall\, u \in (0,\varepsilon_0), \tag{5.118}$$

for given positive constants $0 < c_0 \leq c_1$, and $0 < \kappa_1 \leq \kappa_0$. Under this assumption and (5.116), the regularity results of [PV93] (see also [DiB93, Page 76]) yield the Hölder continuity of u in $\Big((0,T) \times \overline{\Omega}\Big) \cup \Big(\{0\} \times P_0\Big)$. Arguing as in the previous points we still get $D_x^2 u \in C^0(P)$, $D_x F'(u) = D_x \boldsymbol{v} \in C^0(P)$ and (5.114).

The only difference here is that we cannot invoke the regularizing effect (5.117), which seems to depend on the particular form (5.115) of L_F.

In this case, we should impose extra regularity properties on $\bar u$ which guarantee (5.38) (and therefore (5.33)): one possibility is to assume, in addition to (5.116), that

$$\Delta L_F(\bar u) \text{ is a finite measure.} \tag{5.119}$$

(5.119) and standard results for contraction semigroups in L^1 yield

$$\sup_{t>0} \int_{\{x: u_t(x)>0\}} |\partial_t u(t,x)|\, dx < +\infty; \tag{5.120}$$

for, the BREZIS-STRAUSS resolvent estimates [BS73] and CRANDALL-LIGGETT [CL71] generation Theorem show that the nonlinear operator $u \mapsto -\Delta L_F(u)$ with domain $D := \big\{u \in L^1(\Omega) : -\Delta L_F(u) \in L^1(\Omega)\big\}$ generates a contraction semigroup in L^1, whose trajectories satisfy (5.120) if (5.119) holds.

Since L_F is an increasing function, we have

$$\boldsymbol{v}_t \cdot \nabla u_t = -\frac{\nabla L_F(u_t)}{u_t} \cdot \nabla u_t \leq 0 \quad \text{in } P;$$

taking into account (5.120), in order to prove (5.38) we should show that

$$\iint_P \frac{\nabla L_F(u_t)}{u_t} \cdot \nabla u_t \, dx\, dt < +\infty. \tag{5.121}$$

We argue by approximation as in the proof of Proposition 5.9 and we consider the same kind of regularized functions ℓ_ε obtained from the entropy $\ell(u) := u\log u$ according to (5.83), which satisfy the convexity condition (5.94) and

$$\ell_\varepsilon(u) - \varepsilon \le u\log u \le \ell_\varepsilon(u) \quad \forall\, u > 0. \tag{5.122}$$

(5.72) yields $L_\varepsilon(u) := L_\varepsilon := \varepsilon\ell_\varepsilon'(u) - \ell_\varepsilon(u) \quad \in L^1(0,T;W^{1,1}(\Omega))$ with

$$\int_0^T \int_\Omega \frac{|\nabla L_\varepsilon(u)|^2}{u^2} u\,dx\,dt < +\infty \quad \forall\,\varepsilon > 0.$$

It follows from the geodesic convexity of the functional $H_\varepsilon(u) := \int_\Omega \ell_\varepsilon(u)\,dx$ and the "Wasserstein chain rule" (see §10.1.2 in [AGS05]) that the map $t \mapsto H_\varepsilon(u_t)$ is absolutely continuous and its time derivative is

$$\frac{d}{dt}H_\varepsilon(u_t) = \int_\Omega \boldsymbol{v}_t \cdot \nabla L_\varepsilon(u)\,dx = -\int_\Omega \frac{\nabla L_F(u_t)}{u_t}\frac{DL_\varepsilon(u_t)}{u_t}u_t\,dx \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0.$$

Upon an integration in time, (5.122) and $\varepsilon < e^{-1}$ yield

$$\int_\Omega u_T\log u_T\,dx + \iint_{P\cap\{u>\varepsilon\}} \frac{\nabla L_F(u_t)}{u_t}\nabla u_t\,dx\,dt \le \int_{\Omega\cap\{\bar u > 1 - e\varepsilon\}} \bar u\big([\log\bar u]^+ + \frac{\varepsilon}{1-e\varepsilon}\big)\,dx. \tag{5.123}$$

Since $\bar u$ is bounded, the right hand side of (5.123) is uniformly bounded as $\varepsilon \downarrow 0$; moreover, the finiteness of the second moment $\int_\Omega |x|^2 u_T(x)\,dx < +\infty$ and Hölder inequality yield that the entropy $\int_\Omega u_T\log u_T$ cannot take the value $-\infty$ (see for instance Remark 9.3.7 in [AGS05]); hence passing to the limit as $\varepsilon \downarrow 0$ we obtain (5.121).

We collect the above discussion and Theorem 5.13 in the following result

Corollary 5.15 (Convergence of iterated transport maps for diffusion equations)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set of class $C^{2,\alpha}$, $F : [0,+\infty) \to (-\infty,+\infty]$ be a convex lower semi continuous function satisfying (4.6), (4.7), (4.8),* (5.76)*, and let $\bar\mu = \bar u\mathscr{L}^d \in \mathscr{P}_2(\Omega)$ be satisfying* (5.77)*. We assume that there exists an open set $P_0 \subset \Omega$ such that* (5.116) *holds and that* at least one *of the following conditions is satisfied:*

$$P_0 = \Omega \text{ bounded of class } C^{2,\alpha} \quad \text{and} \quad 0 < u_{\min} \le \bar u(y) \quad \forall\, y \in \Omega, \tag{5.124a}$$

$$\Omega = \mathbb{R}^d \text{ and } L_F(u) = u^m \text{ for some } m \ge 1, \tag{5.124b}$$

$$\partial\Omega \in C^{2,\alpha},\ (5.118) \text{ holds, and } \Delta L_F(\bar u) \text{ is a finite measure in } \Omega. \tag{5.124c}$$

If $\boldsymbol{v}_t$ is the velocity vector field associated through (5.111) *to the solution u_t of the nonlinear diffusion PDE* (5.109) *with initial condition $\bar u$, then for every final time $T > 0$ there exist an open set Ω_0 with $\bar u\mathscr{L}^d(\Omega\setminus\Omega_0) = 0$ and a unique $\overline{\Omega}$-valued forward flow $\boldsymbol{X}$ which solves* (5.2) *for every $x \in \Omega_0$.*

Moreover $\boldsymbol{X} \in C^1([0,T]\times\Omega_0;\overline{\Omega})$, $\boldsymbol{X}(t,\cdot)_\#\bar u\mathscr{L}^d = u_t\mathscr{L}^d$ and the iterated transport maps

$\boldsymbol{T}_\tau$ *constructed as in* (5.97), (5.98) *from the solution of the variational algorithm* (5.50) *converge to* $\boldsymbol{X}$*:*

$$\lim_{\tau\downarrow 0}\int_\Omega \max_{[0,T]} |\boldsymbol{T}_{\tau,\cdot}(x)-\boldsymbol{X}(\cdot,x)|^2\bar u(x)\,dx=0 \qquad \forall T>0. \tag{5.125}$$

In the cases (5.124a) *and* (5.124b) *we can always choose* $\Omega_0\equiv P_0$*.*

Proof. We can apply Theorem 5.13 in a time interval $(0,T')$ with $T'>T$: conditions (4.6), (4.7), (4.8), (5.76), and (5.77) together with Proposition 5.9 ensure that we are in the "Wasserstein gradient flow" setting; each of the assumptions (5.124a,b,c) provides enough regularity on the limit vector field

$$\boldsymbol{v}_t=-\frac{\nabla L_F(u_t)}{u_t}$$

in order to check (5.26) (in the case (5.124a)), or Remark 5.4 (in the case (5.124b)), or Remark 5.3 (in the case (5.124c))

Concerning the regularity of $\boldsymbol{X}$, it follows by classical results on differential equations in the first case and by (5.40) and Remark 5.4 in the second one (together with the identification $P_0=\Omega_0$). In the third case, we can still apply (5.39) of Remark 5.4, by choosing $\varepsilon<T'-T$. □

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