On the Heat flow on metric measure spaces: existence, uniqueness and stability

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Abstract

We prove existence and uniqueness of the gradient flow of the Entropy functional under the only assumption that the functional is λ-geodesically convex for some $\lambda \in \mathbb{R}$. Also, we prove a general stability result for gradient flows of geodesically convex functionals which $\Gamma-$converge to some limit functional. The stability result applies directly to the case of the Entropy functionals on compact spaces.

1 Introduction

In [12] and [9] Sturm and Lott-Villani proposed a definition of bound from below of the Ricci curvature of metric-measure spaces, i.e. for metric spaces equipped with a reference positive measure. The main features of their approach are:
Compatibility. If the metric-measure space is a Riemannian manifold equipped with the volume measure, then the bound provided by the abstract definition coincides with the lower bound on the Ricci curvature of the manifold.
Intrinsicness. The definition is based on intrinsic properties of the space and it does not refer to extrinsic properties (e.g. approximability by smoother spaces).
Stability. Curvature bounds are stable w.r.t. the natural passage to the limit of the objects used to define them (in this case, measured-Gromov-Hausdorff convergence).
Interest. From a bound on the Ricci curvature they can derive geometrical and analytical consequences on the space itself.

In this setting, a natural - rather general - question is: which properties/constructions valid for Riemannian manifolds with a uniform bound from below on the Ricci curvature can be generalized to general metric measure spaces satisfying a curvature bound?

Among others, a pretty basic one (also recently posed in Villani's monograph [14]) concerns the Heat flow: indeed, we know that on a Riemannian manifold we can define a mass-preserving Heat flow if and only if the Ricci curvature is uniformly bounded from below.

Let us recall the basic facts of the theory. The weak lower bound on Ricci curvature for metric measure spaces (X, d, m) proposed by Sturm and Lott-Villani is: the Ricci curvature is bounded from below by λ provided the Entropy functional relative to the reference measure m is λ-geodesically convex on the space $(\mathscr{P}_2(X), W_2)$ (they also propose a definition of weak curvature bound plus bound from above on the dimension of the space, but this is outside the analysis of this paper).

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Also, on a Riemannian manifold with Ricci curvature bounded from below, the Heat flow is given by the (unique) curve of maximal slope of the Entropy functional relative to the volume measure.

Therefore it is quite natural to *define* the Heat flow on a metric-measure space satisfying a weak Ricci curvature bound as the curve of maximal slope of the Entropy relative to the reference measure. However, this definition arises a problem: neither existence, nor uniqueness of these curves are ensured in general.

Let us recall what the general theory of curves of maximal slope - as developed in [1] - provides concerning existence and uniqueness for λ-geodesically convex and lower semicontinuous functionals under additional technical assumptions:

Existence. Existence is guaranteed provided the functional is finite for the initial datum and the metric space is locally compact. Nothing in general is known without this assumption. This means that if we want to apply this result to the Entropy functional on $(\mathscr{P}_2(X), W_2)$, we need to assume X to be compact, because otherwise $\mathscr{P}_2(X)$ is not locally compact.

Uniqueness. At this level of generality, nothing is known. This is not particularly surprising if one observes that in $\mathbb{R}^2$ endowed with the sup-norm the functional $(x, y) \mapsto x$ is convex and has uncountably many curves of maximal slope for any given initial datum. Clearly this is a limit example, because neither the norm nor the functional are strictly convex. But, at least this shows that uniqueness is problematic at a very general level.

The aim of this paper is twofold. On one side we show that for the case of the Entropy functional the λ-geodesic convexity hypothesis is enough to ensure uniqueness and - for boundedly compact spaces - existence, as soon as the initial datum has finite Entropy. On the other side, we prove a general stability result - valid in a compact setting - which roughly said ensures that Gromov-Hausdorff convergence of metric spaces and Γ-convergence of λ-geodesically convex functionals imply convergence of the corresponding curves of maximal slope.

Existence and uniqueness are based on a property of the slope of the Entropy which we believe of an independent interest: its square is convex w.r.t. (usual) linear combination of measures. We think that this is an interesting link between geodesic convexity in $\mathscr{P}_2(X)$ and linear convexity: indeed, from the only assumption that Entropy is λ-geodesically convex for some λ we can deduce that the squared slope is convex w.r.t. linear interpolation. Once this convexity is proven, uniqueness follows by a 1-line argument (see the proof of Theorem 15). Also, once we have convexity, it is mainly a technical difficulty to prove that there is weak lower semicontinuity, and once the latter is proven, existence for the boundedly compact case follows with the techniques described in [1] (see the proof of Theorem 17).

Finally, we collect some problems concerning the Heat flow which remain open:

General initial datum. All our results require the initial datum to have finite Entropy. What can be said for general ones, e.g. Delta's?

Stability for the locally compact case. The stability result that we proved applies to λ-geodesically convex functionals on compact spaces. Can this be generalized? A variant of Theorem 21 valid for boundedly compact spaces is not hard to obtain, but since if (X, d) is boundedly compact, $(\mathscr{P}_2(X), W_2)$ is not boundedly compact anymore, such generalization cannot be applied to the study of the Entropy on non compact spaces.

Non normalized spaces. Among the results proven here, which ones can be gener-

alized to σ-finite metric-measure spaces?
Heat flow as GF of the Dirichlet energy. In [13] Sturm and Ohta proved that in a Finsler manifold the gradient flow of the Entropy w.r.t. W_2 and the gradient flow of the Dirichlet energy w.r.t. $L^2(\mathrm{m})$ coincide. The fact that these two approaches coincide on such a rather general case, raises the question on whether the same can be proven in abstract or not. Observe that here it is part of the problem the definition of the Dirichlet energy itself.
Heat equation. Is there any 'PDE' solved by the Heat flow in this generality? This is of course a very loose question, as before even trying to write down an equation we should understand better the structure of spaces with bounds from below on the Ricci curvature. In particular, a first step should be the proof of existence of the tangent space.

I would like to thank Giuseppe Savaré for an important contribution he gave to the development of this paper. Specifically, while I was trying to prove the convexity of the squared slope via Propositions 11, 12 and 13, Savaré showed me some calculations, still unpublished. It turned out that his calculations are closely related to part 1 of Proposition 11, and only after having spoken with him I became convinced of the validity of the proposition.
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2 Preliminaries

All the metric spaces (X, d) we will consider are complete, separable and geodesic.

We assume the reader to be familiar with the definition of the space $(\mathscr{P}_2(X), W_2)$, which is complete, separable and geodesic as well, due to our assumptionS on the base space. We will denote the cost of a plan $\boldsymbol{\gamma}$ by $C(\boldsymbol{\gamma})$:

$$C(\boldsymbol{\gamma}) := \int d^2(x, y) d\boldsymbol{\gamma}(x, y).$$

Given two measures $\mu, \nu \in \mathscr{P}_2(X)$ the set of admissible plans from μ to ν is denoted by $\textsc{Adm}(\mu, \nu)$, i.e. $\boldsymbol{\gamma} \in \textsc{Adm}(\mu, \nu)$ if and only if $\pi^1_\# \boldsymbol{\gamma} = \mu$ and $\pi^2_\# \boldsymbol{\gamma} = \nu$. The set of optimal plans $\textsc{Opt}(\mu, \nu) \subset \textsc{Adm}(\mu, \nu)$ is the set of those plans with minimal cost among those in $\textsc{Adm}(\mu, \nu)$, i.e. $\boldsymbol{\gamma} \in \textsc{Opt}(\mu, \nu)$ if and only if $C(\boldsymbol{\gamma}) = W_2^2(\mu, \nu)$.

A functional $E : X \to \mathbb{R} \cup \{+\infty\}$ is said to be λ-geodesically convex (in the literature sometimes also called weakly λ-geodesically convex) provided for any $x, y \in X$ there exists a constant speed geodesic $t \mapsto \gamma(t) \in X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$E(\gamma(t)) \leq (1 - t)E(\gamma(0)) + tE(\gamma(1)) - \frac{\lambda}{2} d^2(x, y), \qquad \forall t \in [0, 1].$$

The *slope* of a functional $E : X \to \mathbb{R} \cup \{+\infty\}$ is the map $|\nabla E| : \{E < \infty\} \to \mathbb{R} \cup \{+\infty\}$ defined as

$$|\nabla E|(x) := \varlimsup_{y \to x} \frac{\big(E(x) - E(y)\big)^+}{d(x, y)}.$$

If E is λ-geodesically convex the slope admits the representation

$$|\nabla E|(x) := \sup_{y \neq x} \left(\frac{E(x) - E(y)}{d(x, y)} - \frac{\lambda^-}{2} d(x, y) \right)^+,$$

(this formula is just a small variant of equation 2.4.14 of [1]), where $\lambda^- := \max\{0, -\lambda\}$.

If E is λ-geodesically convex and lower semicontinuous, the slope is a strong upper gradient (in the terminology of [1]). This means that for any locally absolutely continuous curve $(0, +\infty) \ni t \mapsto x(t) \in X$ it holds

$$|E(x(s)) - E(x(t))| \le \int_t^s |\dot{x}(r)||\nabla E|(x(r))dr, \qquad 0 < t \le s$$

where $|\dot{x}(t)|$ is the metric derivative of $t \mapsto x(t)$. From the above inequality it follows that for any locally absolutely continuous curve $(0, +\infty) \ni t \mapsto x(t) \in X$ such that $E(x(t)) < \infty$ for $t > 0$ it holds

$$E(x(t)) \le E(x(s)) + \frac{1}{2}\int_t^s |\dot{x}(r)|^2 dr + \frac{1}{2}\int_t^s |\nabla E|^2(x(r))dr \qquad 0 < t \le s. \qquad (1)$$

A curve $(0, +\infty) \ni t \mapsto x(t) \in X$ is called of maximal slope for E starting from $\overline{x}$ provided $E(x(t)) < \infty$ for $t > 0$, $x(t) \to \overline{x}$ as $t \to 0$ and the opposite inequality holds:

$$E(x(t)) \ge E(x(s)) + \frac{1}{2}\int_t^s |\dot{x}(r)|^2 dr + \frac{1}{2}\int_t^s |\nabla E|^2(x(r))dr \qquad 0 < t \le s, \qquad (2)$$

or, which is the same, if equality holds. If $E(\overline{x}) < \infty$ then the above (in)equality is required to hold also for $t = 0$.

A metric space (X, d) equipped with a reference Borel probability measure $\mathrm{m} \in \mathscr{P}(X)$ is called normalized metric measure space.

On a normalized metric measure space (X, d, m), the Entropy functional $\mathrm{Ent}_{\mathrm{m}}(\cdot) : \mathscr{P}(X) \to \mathbb{R} \cup \{+\infty\}$ is defined as:

$$\mathrm{Ent}_{\mathrm{m}}(\mu) := \begin{cases} \displaystyle\int e(\rho(x))d\mathrm{m}(x) & \text{if } \mu = \rho\mathrm{m}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $e : [0, +\infty) \to [0, +\infty)$ is given by

$$e(z) := z \log z.$$

In the sequel, we will often use the fact that densities in the sublevels of the Entropy are equi-integrable. This is a well known consequence of inequality

$$\int e(\rho)d\mathrm{m} + 1 \ge \int_A e(\rho)d\mathrm{m} = \mathrm{m}(A)\left(\frac{1}{\mathrm{m}(A)}\int_A e(\rho)d\mathrm{m}\right) \ge \mathrm{m}(A)e\left(\frac{\int_A \rho d\mathrm{m}}{\mathrm{m}(A)}\right),$$

and the superlinearity of e.

Throughout the whole paper we will assume that the normalized metric measure spaces we are dealing with satisfy the following:

Assumption 1 (Bound on the Ricci curvature) *The space (X, d, m) is such that the Entropy functional is λ-geodesically convex on $(\mathscr{P}_2(X), W_2)$ for some $\lambda \in \mathbb{R}$.*

We pass now to the description of convergence of metric spaces and metric measure spaces.

The *distortion* of a map f from a metric space (X,d) to a metric space (X',d') is defined as

$$\mathrm{Dis}(f) := \sup_{x,y\in X} \big|d(x,y) - d'(f(x),f(y))\big|.$$

A map between two metric spaces (X,d) and (X',d') is called ε-isometry provided:

$$\mathrm{Dis}(f) \le \varepsilon, \quad \sup_{x'\in X'} d(x', f(X)) \le \varepsilon.$$

Definition 2 (Gromov-Hausdorff convergence) *Let (X_n,d_n), $n\in\mathbb{N}$, and (X,d) be compact metric spaces. We say that the sequence (X_n,d_n) converges to (X,d) in the Gromov-Hausdorff sense if for every n there is a map $f_n : X \to X_n$ which is an ε_n-isometry, with $\varepsilon_n \to 0$ as $n\to\infty$.*

A sequence of ε_n-isometries with $\varepsilon_n \to 0$ is called a sequence of *approximate isometries.* Recall that for any such sequence there exists a sequence of *approximate inverses*, i.e. maps $g_n : X_n \to X$ such that

$$\sup_{x\in X} d\big(x, g_n(f_n(x))\big) \to 0, \quad \sup_{x_n\in X_n} d_n\big(x_n, f_n(g_n(x_n))\big) \to 0,$$

as $n\to\infty$. It is immediate to verify that for any choice of such g_n's it holds

$$\mathrm{Dis}(g_n) \to 0, \quad \sup_{x\in X} d(x, g_n(X_n)) \to 0,$$

as $n\to\infty$.

Two sequences of approximate isometries $(f_n),(f'_n)$ are said *equivalent* provided

$$\sup_{x\in X} d_n\big(f_n(x), f'_n(x)\big) \to 0$$

as $n\to\infty$. Similarly for sequences of approximate inverses $(g_n),(g'_n)$. It is immediate to verify that being approximate inverses is a statement which depends only on the equivalence class of the approximate isometries, i.e. (g_n) is a sequence of approximate inverses of (f_n) if and only if (g'_n) is a sequence of approximate inverses of (f'_n) whenever (f'_n) and (g'_n) are equivalent to (f_n) and (g_n) respectively.

Definition 3 (Convergence of points) *Let (X_n,d_n), $n\in\mathbb{N}$, (X,d) be compact metric spaces, $f_n : X\to X_n$ approximate isometries and $g_n : X_n \to X$ be approximate inverses of the f_n's. We say that $n\mapsto x_n\in X_n$ converges to $x\in X$* via (f_n) *provided $g_n(x_n)$ converges to x in (X,d).*

It is immediate to verify that this notion of convergence depends only on the equivalence class of the approximate isometries.

Definition 4 (De Giorgi's $\Gamma-$convergence of functionals) *Let (X_n,d_n), $n\in\mathbb{N}$, be compact metric spaces which converge to the space (X,d) in the GH topology. Let (f_n) be a family of approximate isometries ensuring the GH convergence of X_n to X, as in the definition above. Also, let $\mathscr{E}_n : X_n \to \mathbb{R}\cup\{\pm\infty\}$, $\mathscr{E} : X\to\mathbb{R}\cup\{\pm\infty\}$ be given functionals. We say that the sequence $n\mapsto\mathscr{E}_n$ $\Gamma-$converges to $\mathscr{E}$* via the approximate isometries f_n *if $\mathscr{E}_n\circ f_n : X\to\mathbb{R}\cup\{\pm\infty\}$ $\Gamma-$converges to $\mathscr{E}$.*

Using the above definition of convergence of points, the one of Γ-convergence may be written in the more familiar way (see [3]) as follows: $\mathscr{E}_n$ is Γ-converging to $\mathscr{E}$ if

$$\inf_{(x_n)} \varliminf_{n\to\infty} \mathscr{E}_n(x_n) \geq \mathscr{E}(x),$$
$$\inf_{(x_n)} \varlimsup_{n\to\infty} \mathscr{E}_n(x_n) \leq \mathscr{E}(x),$$

where the infima are taken among all sequences $n \mapsto x_n \in X_n$ converging to x via (f_n).

Remark 5 (Importance of the chosen approximate isometries) *It is important to explicitly refer to some chosen approximate isometries when defining* $\Gamma-$*convergence: it is needed to avoid self isometries of the limit space. In other words, we need to know which sequences* $n \mapsto x_n \in X_n$ *are converging to some limit* $x \in X$ *and to identify the limit.*

To be more explicit, consider the case $X_n \equiv X$ *where* X *is a space admitting an isometry* $\mathrm{I} : X \to X$ *different from the identity. Now let* $\mathscr{E} : X \to \mathbb{R}$ *be a lower semicontinuous functional satisfying* $\mathscr{E} \neq \mathscr{E} \circ \mathrm{I}$ *and let* $\mathscr{E}_n = \mathscr{E}$ *for every* n*. In this situation we would like to say that the sequence of functionals* $(\mathscr{E}_n)$ $\Gamma-$*converges to* $\mathscr{E}$*. However, it is unclear whether this is the case or not if we don't fix a sequence of approximate isometries: for instance, if we take as isometry from* X *to* X_n *the map* I*, in general we won't have that* $\mathscr{E} \circ \mathrm{I}$ $\Gamma-$*converges to* $\mathscr{E}$*.*

Still, observe that the definition of Γ*-limit depends only on the equivalence class of approximate isometries chosen.*

Definition 6 (Measured Gromov-Hausdorff convergence) *Let* (X_n, d_n, m_n)*,* $n \in \mathbb{N}$*,* (X, d, m) *be normalized compact metric measure spaces. We say that* (X_n, d_n, m_n) *converge to* (X, d, m) *in the measured Gromov-Hausdorff topology (MGH in the following) if* (X_n, d_n) *is converging to* (X, d) *in the GH sense and there exist approximate isometries* $f_n : X \to X_n$ *with approximate inverses* $g_n : X_n \to X$ *such that* $(g_n)_\#\mathrm{m}_n$ *weakly converge to* m *as* $n \to \infty$*.*

In the above definition and in the rest of the work, weak convergence of measures stands for convergence tested against continuous and bounded functions. For more details on the Measured-Gromov-Hausdorff convergence, see [7] and [6]. In [9] the definition was used in order to study the stability properties of Ricci curvature bounds, while in [12] the approach is slightly different.

3 Existence and uniqueness

3.1 γ-variation of the Entropy

Let $\mathscr{P}_2^{ac}(X) \subset \mathscr{P}_2(X)$ be the set of measures absolutely continuous w.r.t. $\mathrm{m} \in \mathscr{P}(X)$.

Definition 7 ('Good plans') *The set* $\mathrm{GP} \subset \mathscr{P}(X^2)$ *is the set of plans* $\boldsymbol{\gamma}$ *such that:*

i) $\pi^1_\#\boldsymbol{\gamma}, \pi^2_\#\boldsymbol{\gamma}$ *are absolutely continuous with density uniformly bounded away from 0 and* ∞*,*

ii) $\sup_{(x,y)\in\mathrm{Supp}(\boldsymbol{\gamma})} d(x,y) < \infty.$

Given $\boldsymbol{\gamma} \in \mathrm{GP}$ and $\mu = f\mathrm{m} \in \mathscr{P}_2^{ac}(X)$, we define the plan $\boldsymbol{\gamma}_\mu \in \mathscr{P}(X^2)$ as:

$$d\boldsymbol{\gamma}_\mu(x,y) := \frac{d\mu(x)}{d\pi^1_\#\boldsymbol{\gamma}(x)} d\boldsymbol{\gamma}(x,y),$$

and the measure $\nu_{\boldsymbol{\gamma},\mu}$ as $\nu_{\boldsymbol{\gamma},\mu} := \pi^2_\#\boldsymbol{\gamma}_\mu$. Observe that since $\boldsymbol{\gamma}_\mu \ll \boldsymbol{\gamma}$, we have $\nu_{\boldsymbol{\gamma},\mu} \ll \pi^2_\#\boldsymbol{\gamma} \ll \mathrm{m}$. We will denote the density of $\nu_{\boldsymbol{\gamma},\mu}$ with $g_{\boldsymbol{\gamma},f}$.

Notice that from (ii) of the definition of GP we have that the cost of a plan $\boldsymbol{\gamma} \in \mathrm{GP}$ is always finite (even if $\boldsymbol{\gamma} \notin \mathscr{P}_2(X^2)$) and that $\mu \in \mathscr{P}_2(X)$ implies $\nu_{\boldsymbol{\gamma},\mu} \in \mathscr{P}_2(X)$ as well.

Letting $\pi^1_\#\boldsymbol{\gamma} = \overline{f}\mathrm{m}$ and $\pi^2_\#\boldsymbol{\gamma} = \overline{g}\mathrm{m}$, the function $g_{\boldsymbol{\gamma},f}$ is given by

$$g_{\boldsymbol{\gamma},f}(y) = \overline{g}(y) \int \frac{f(x)}{\overline{f}(x)} d\boldsymbol{\gamma}_y(x),$$

where $\{\boldsymbol{\gamma}_y\}$ is the disintegration of $\boldsymbol{\gamma}$ w.r.t. its second marginal.

It is not part of the definition of GP the requirement that the plans are optimal. Still, observe that if an optimal plan $\boldsymbol{\gamma}$ belongs to GP, then $\boldsymbol{\gamma}_\mu$ is optimal as well (because $\mathrm{Supp}(\boldsymbol{\gamma}_\mu) \subset \mathrm{Supp}(\boldsymbol{\gamma})$).

Lemma 8 *Let* $\boldsymbol{\gamma} \in \mathrm{GP}$ *and* $\mu \in \mathscr{P}_2^{ac}(X)$ *such that* $\mathrm{Ent}_{\mathrm{m}}(\mu) < \infty$. *Then* $\mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma},\mu}) < \infty$.

Proof Let $\pi^1_\#\boldsymbol{\gamma} = \overline{f}\mathrm{m}$ and $\pi^2_\#\boldsymbol{\gamma} = \overline{g}\mathrm{m}$. We have

$$\begin{aligned}
\mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma},\mu}) &= \int e(g_{\boldsymbol{\gamma},f}(y))d\mathrm{m}(y) = \int e\left(\overline{g}(y)\int \frac{f(x)}{\overline{f}(x)} d\boldsymbol{\gamma}_y(x)\right) d\mathrm{m}(y) \\
&= \int e(\overline{g}(y)) \int \frac{f(x)}{\overline{f}(x)} d\boldsymbol{\gamma}_y(x) d\mathrm{m}(y) + \int \overline{g}(y) e\left(\int \frac{f(x)}{\overline{f}(x)} d\boldsymbol{\gamma}_y(x)\right) d\mathrm{m}(y) \\
&\le \frac{e\Big(\sup_y \overline{g}(y)\Big)}{\inf_y \overline{g}(y)} \int \overline{g}(y) \int \frac{f(x)}{\overline{f}(x)} d\boldsymbol{\gamma}_y(x) d\mathrm{m}(y) \\
&\qquad + \int \overline{g}(y) \int e\left(\frac{f(x)}{\overline{f}(x)}\right) d\boldsymbol{\gamma}_y(x) d\mathrm{m}(y).
\end{aligned} \tag{3}$$

Now observe that for any positive Borel function $h : X \to \mathbb{R} \cup \{+\infty\}$ it holds

$$\begin{aligned}
\int \overline{g}(y) \int \frac{h(x)}{\overline{f}(x)} d\boldsymbol{\gamma}_y(x) d\mathrm{m}(y) &= \int \frac{h(x)}{\overline{f}(x)} d\boldsymbol{\gamma}(x,y) \\
&= \int \frac{h(x)}{\overline{f}(x)} d\pi^1_\#\boldsymbol{\gamma}(x) = \int h(x) d\mathrm{m}(x),
\end{aligned} \tag{4}$$

so that from (3) we have

$$\mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma},\mu}) \le \frac{e\Big(\sup_y \overline{g}(y)\Big)}{\inf_y \overline{g}(y)} + \int e\left(\frac{f(x)}{\overline{f}(x)}\right) \overline{f}(x) d\mathrm{m}(x),$$

and the conclusion follows from the fact that $\overline{f}$ is bounded away from 0 and ∞. $\square$

Lemma 9 (Continuity) *Fix $\boldsymbol{\gamma} \in \mathrm{GP}$. The maps $\mu \mapsto \boldsymbol{\gamma}_\mu$, $\mu \mapsto \nu_{\boldsymbol{\gamma},\mu}$ are continuous on $\mathscr{P}_2^{ac}(X)$ w.r.t. the weak convergence on the sublevels of the Entropy.*

Proof It is sufficient to prove that $\mu \mapsto \boldsymbol{\gamma}_\mu$ is continuous w.r.t. the weak convergence on the sublevels of the Entropy, as then the result follows by the continuity of the projection.

Choose a sequence $n \mapsto \mu_n = f_n \mathrm{m}$ weakly converging to $\mu = f\mathrm{m}$ in $\mathscr{P}(X)$ and assume that $\sup_n \mathrm{Ent}_{\mathrm{m}}(\mu_n) < \infty$. Then we know that the sequence (f_n) is equi-integrable. Therefore (f_n) converges to f in duality with L^∞ functions (and not just with continuous and bounded ones).

Now fix $\varphi \in C_b(X^2)$. We need to show that

$$\int \varphi(x,y)\frac{f_n(x)}{\overline{f}(x)}d\boldsymbol{\gamma}(x,y) \to \int \varphi(x,y)\frac{f(x)}{\overline{f}(x)}d\boldsymbol{\gamma}(x,y),$$

as n goes to infinity, where $\overline{f}$ is the density of $\pi^1_\#\boldsymbol{\gamma}$. Observe that

$$\int \varphi(x,y)\frac{f_n(x)}{\overline{f}(x)}d\boldsymbol{\gamma}(x,y) = \int \left(\int \varphi(x,y)d\boldsymbol{\gamma}_x(y)\right) f_n(x)d\mathrm{m}(x).$$

Define $h(x) := \int \varphi(x,y)d\boldsymbol{\gamma}_x(y)$ and observe that $\sup|h| \le \sup|\varphi|$ to conclude. $\square$

Lemma 10 (Approximability in Entropy and distance) *Let $\mu,\nu \in \mathscr{P}_2^{ac}(X)$. Then there exists a sequence $(\boldsymbol{\gamma}^n) \subset \mathrm{GP}$ such that $\mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma}^n,\mu}) \to \mathrm{Ent}_{\mathrm{m}}(\nu)$ and $C(\boldsymbol{\gamma}^n) \to W_2^2(\mu,\nu)$ as $n \to \infty$.*

Proof Let $\mu = f\mathrm{m}$, $\nu = g\mathrm{m}$ and find $\nu_n = g_n\mathrm{m}$ such that $W_2(\nu_n,\nu) \to 0$, $\mathrm{Ent}_{\mathrm{m}}(\nu_n) \to \mathrm{Ent}_{\mathrm{m}}(\nu)$ as $n \to \infty$ and g_n is (essentially) bounded for every n. Fix n and choose $\boldsymbol{\gamma}^n \in \mathrm{Opt}(\mu,\nu_n)$. Choose $\varepsilon > 0$ and define

$$\boldsymbol{\gamma}^{n,\varepsilon} := (1-\varepsilon)\boldsymbol{\gamma}^n + \varepsilon(\mathrm{Id},\mathrm{Id})_\#\mathrm{m}.$$

Clearly $\pi^i_\#\boldsymbol{\gamma}^{n,\varepsilon} = (1-\varepsilon)\pi^i_\#\boldsymbol{\gamma}^n + \varepsilon\mathrm{m}$, $i = 1,2$. Now define $\overline{\boldsymbol{\gamma}}^{n,\varepsilon}$ by

$$d\overline{\boldsymbol{\gamma}}^{n,\varepsilon}(x,y) := \frac{1}{(1-\varepsilon)f(x)+\varepsilon}d\boldsymbol{\gamma}^{n,\varepsilon}(x,y).$$

Since $\frac{1}{(1-\varepsilon)f(x)+\varepsilon} \le \frac{1}{\varepsilon}$, $\pi^2_\#\overline{\boldsymbol{\gamma}}^{n,\varepsilon} \le \frac{1}{\varepsilon}\pi^2_\#\boldsymbol{\gamma}^{n,\varepsilon}$ and therefore has bounded density. Also clearly $\pi^1_\#\overline{\boldsymbol{\gamma}}^{n,\varepsilon} = \mathrm{m}$. Consider now the plan $\overline{\boldsymbol{\gamma}}^{n,\varepsilon}_\mu$ and let $\nu^{n,\varepsilon} := \pi^2_\#\overline{\boldsymbol{\gamma}}^{n,\varepsilon}_\mu = g^{n,\varepsilon}\mathrm{m}$. It is obvious that

$$\begin{aligned} C(\overline{\boldsymbol{\gamma}}^{n,\varepsilon}_\mu) &= \int d^2(x,y)\frac{f(x)}{(1-\varepsilon)f(x)+\varepsilon}d\boldsymbol{\gamma}^{n,\varepsilon}(x,y) \\ &= (1-\varepsilon)\int d^2(x,y)\frac{f(x)}{(1-\varepsilon)f(x)+\varepsilon}d\boldsymbol{\gamma}^n(x,y), \end{aligned}$$

converges to $C(\boldsymbol{\gamma}^n) = W_2^2(\mu,\nu^n)$ as $\varepsilon \downarrow 0$. Similarly, from the identity

$$g^{n,\varepsilon}(y) = \big((1-\varepsilon)g(y)+\varepsilon\big)\int \frac{f(x)}{(1-\varepsilon)f(x)+\varepsilon}d\boldsymbol{\gamma}_y(x),$$

it is immediate to verify that $\mathrm{Ent}_{\mathrm{m}}(\nu^{n,\varepsilon}) \to \mathrm{Ent}_{\mathrm{m}}(\nu^n)$ as $\varepsilon \downarrow 0$.

We still need to modify a bit the plan $\overline{\gamma}^{n,\varepsilon}$ as it could be that the density of $\overline{\pi}^2_\#\gamma^{n,\varepsilon}$ is not bounded from below and that property (ii) of the definition of GP is not satisfied. Thus we first define

$$\overline{\gamma}^{n,\varepsilon,\varepsilon'} := (1-\varepsilon')\overline{\gamma}^{n,\varepsilon} + \varepsilon'(\mathrm{Id},\mathrm{Id})_\#\mathrm{m},$$

so that $\pi^i_\#\overline{\gamma}^{n,\varepsilon,\varepsilon'}$ has density bounded away from 0 and ∞ for $i = 1, 2$ and, obviously, $C(\overline{\gamma}^{n,\varepsilon,\varepsilon'}_\mu) \to C(\overline{\gamma}^{n,\varepsilon}_\mu)$ and $\mathrm{Ent}_\mathrm{m}\left(\pi^2_\#\overline{\gamma}^{n,\varepsilon,\varepsilon'}_\mu\right) \to \mathrm{Ent}_\mathrm{m}\left(\pi^2_\#\overline{\gamma}^{n,\varepsilon}_\mu\right)$ as $\varepsilon' \downarrow 0$.

We conclude with a truncation argument. Pick a continuous function $\chi : [0,\infty) \to \mathbb{R}$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $[0,1]$ and $\chi \equiv 0$ on $(2,\infty)$, fix $R > 0$ big enough and define

$$d\overline{\gamma}^{n,\varepsilon,\varepsilon',R}(x,y) := \frac{\chi\left(\dfrac{d(x,y)}{R}\right)}{\displaystyle\int \chi\left(\frac{d(x,y)}{R}\right) d\overline{\gamma}^{n,\varepsilon,\varepsilon'}(x,y)} d\overline{\gamma}^{n,\varepsilon,\varepsilon'}(x,y).$$

By construction, $\overline{\gamma}^{n,\varepsilon,\varepsilon',R} \in \mathrm{GP}$ and

$$\mathrm{Ent}_\mathrm{m}\left(\pi^2_\#\overline{\gamma}^{n,\varepsilon,\varepsilon',R}_\mu\right) \to \mathrm{Ent}_\mathrm{m}\left(\pi^2_\#\overline{\gamma}^{n,\varepsilon,\varepsilon'}_\mu\right), \qquad R \to \infty,$$
$$C(\overline{\gamma}^{n,\varepsilon,\varepsilon',R}_\mu) \to C(\overline{\gamma}^{n,\varepsilon,\varepsilon'}_\mu), \qquad R \to \infty,$$

therefore the conclusion follows from a diagonalization argument.

□

Now, given $\gamma \in \mathrm{GP}$ consider the following functional, which can be thought as a γ-dependent variation of the Entropy:

$$\mathrm{DE}_\gamma(\mu) := \begin{cases} \mathrm{Ent}_\mathrm{m}(\mu) - \mathrm{Ent}_\mathrm{m}(\nu_{\gamma,\mu}), & \text{if } \mathrm{Ent}_\mathrm{m}(\mu) < \infty, \\ +\infty & \text{otherwise} \end{cases}$$

The importance of the functional DE_γ is due to the following result:

Proposition 11 *The functional $\mu \mapsto \mathrm{DE}_\gamma(\mu)$ is convex w.r.t. linear interpolation and lower semicontinuous w.r.t. the weak topology on the subleves of the Entropy.*

Proof **Step 1: Convexity.** Fix $\mu_0, \mu_1 \in \mathscr{P}_2(X)$. If either $\mathrm{Ent}_\mathrm{m}(\mu_0) = +\infty$ or $\mathrm{Ent}_\mathrm{m}(\mu_1) = +\infty$ the claim is trivial. So we can assume $\mu_0 = f_0\mathrm{m}$, $\mu_1 = f_1\mathrm{m}$ both with finite Entropy. By Lemma 8 we know that $\nu_{\gamma,\mu^i} = g_i\mathrm{m}$, has finite Entropy as well, $i = 0, 1$. Let $\mu_t := (1-t)\mu_0 + t\mu_1 = f_t\mathrm{m}$, $\nu_t := \nu_{\gamma,\mu_t} = g_t\mathrm{m} = ((1-t)g_0 + tg_1)\mathrm{m}$ and observe that both the functions $t \mapsto \mathrm{Ent}_\mathrm{m}(\mu_t)$ and $t \mapsto \mathrm{Ent}_\mathrm{m}(\nu_t)$ are convex and continuous on [0,1]. In particular $t \mapsto \mathrm{DE}_\gamma(\mu_t)$ is continuous and real valued. Also, notice that the ratio f_0/f_t is uniformly bounded from above by $\frac{1}{1-t}$ as soon as $t < 1$, similarly, $f_1/f_t \le \frac{1}{t}$ for $t > 0$. This means that

$$\int \frac{(f_1 - f_0)^2}{f_t} d\mathrm{m} < +\infty, \qquad \forall t \in (0,1).$$

It holds:

$$\frac{d^2}{dt^2}\mathrm{Ent}_\mathrm{m}(\mu_t) = \frac{d^2}{dt^2}\int e(f_t)d\mathrm{m} = \int e''(f_t)(f_1 - f_0)^2 d\mathrm{m} = \int \frac{(f_1 - f_0)^2}{f_t} d\mathrm{m}, \tag{5}$$

for every $t \in (0,1)$.

Now recall that $\overline{f}, \overline{g}$ are the densities of $\pi^1_\# \boldsymbol{\gamma}$, $\pi^2_\# \boldsymbol{\gamma}$ w.r.t. m and observe that it holds:

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{Ent}_{\mathrm{m}}(\nu_t) &= \frac{d^2}{dt^2}\int e(g_t(y))d\mathrm{m}(y) = \int \frac{(g_1(y)-g_0(y))^2}{g_t(y)}d\mathrm{m}(y) \\ &= \int \frac{\left(\int \overline{g}(y)\frac{f_1(x)-f_0(x)}{\overline{f}(x)}d\boldsymbol{\gamma}_y(x)\right)^2}{\int \overline{g}(y)\frac{f_t(x)}{\overline{f}(x)}d\boldsymbol{\gamma}_y(x)}d\mathrm{m}(y).\end{aligned} \tag{6}$$

Now apply Jensen inequality to the convex and lower semicontinuous function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Psi(a,b) := \begin{cases} \dfrac{a^2}{b} & \text{if } b > 0, \\ 0 & \text{if } a = b = 0, \\ +\infty & \text{if } a \neq 0,\ b = 0 \text{ or } b < 0, \end{cases} \tag{7}$$

to obtain

$$\begin{aligned}&\int \frac{\left(\int \overline{g}(y)\frac{f_1(x)-f_0(x)}{\overline{f}(x)}d\boldsymbol{\gamma}_y(x)\right)^2}{\int \overline{g}(y)\frac{f_t(x)}{\overline{f}(x)}d\boldsymbol{\gamma}_y(x)}d\mathrm{m}(y) \\ &= \int \Psi\left(\int \overline{g}(y)\frac{f_1(x)-f_0(x)}{\overline{f}(x)}d\boldsymbol{\gamma}_y(x), \int \overline{g}(y)\frac{f_t(x)}{\overline{f}(x)}d\boldsymbol{\gamma}_y(x)\right)d\mathrm{m}(y) \\ &\leq \int\int \Psi\left(\overline{g}(y)\frac{f_1(x)-f_0(x)}{\overline{f}(x)}, \overline{g}(y)\frac{f_t(x)}{\overline{f}(x)}\right)d\boldsymbol{\gamma}_y(x)d\mathrm{m}(y) \\ &= \int\int \frac{(f_1(x)-f_0(x))^2}{f_t(x)}\frac{\overline{g}(y)}{\overline{f}(x)}d\boldsymbol{\gamma}_y(x)d\mathrm{m}(x) \\ &= \int \frac{(f_1(x)-f_0(x))^2}{f_t(x)}d\mathrm{m}(x),\end{aligned}$$

where in the last step we used equality (4). This inequality, valid for any $t \in (0,1)$, together with (5) and (6) gives

$$\frac{d^2}{dt^2}\Big(\mathrm{Ent}_{\mathrm{m}}(\mu_t) - \mathrm{Ent}_{\mathrm{m}}(\nu_t)\Big) \geq 0, \qquad \forall t \in (0,1),$$

which gives the convexity of $\mathrm{DE}_{\boldsymbol{\gamma}}$ in $(0,1)$. By continuity, we have convexity on the whole [0,1].

Step 2: Semicontinuity. We claim that for every f such that $\mathrm{Ent}_{\mathrm{m}}(f\mathrm{m}) < \infty$ it holds

$$\mathrm{DE}_{\boldsymbol{\gamma}}(f\mathrm{m}) = \sup_{\mu} \mathrm{DE}_{\boldsymbol{\gamma}}(\mu) + \int (f-\tilde{f})e'(\tilde{f}) - \int (g-\tilde{g})e'(\tilde{g}),$$

where the supremum is taken among all $\mu = \tilde{f}\mathrm{m} \in \mathscr{P}_2^{ac}(X)$ with density bounded away from 0 and ∞, $g, \tilde{g}$ are the densities of $\nu_{\boldsymbol{\gamma},f\mathrm{m}}, \nu_{\boldsymbol{\gamma},\mu}$ respectively.

The inequality $\geq$ follows immediately from the convexity, so we have to prove the converse inequality. If f is bounded, then we can find a sequence (f_n) of densities bounded away from 0 and infinity converging to f in the sup norm. In this case it is immediate to verify that $\mathrm{DE}_{\boldsymbol{\gamma}}(f_n\mathrm{m}) \to \mathrm{DE}_{\boldsymbol{\gamma}}(f\mathrm{m})$, $\int (f - f_n)e'(f_n) \to 0$ and $\int (g - g_n)e'(g_n) \to 0$ as $n \to \infty$, where g_n are the densities of $\nu_{\boldsymbol{\gamma},f_n\mathrm{m}}$. Thus we have only to deal with the case of f unbounded. In this situation define f_n as:

$$f_n := \min\{\max\{f, a_n\}, n\},$$

where $a_n > 0$ is chosen such that $\int f_n d\mathrm{m} = 1$. With this choice, it is obvious that $\mathrm{Ent}_{\mathrm{m}}(f_n\mathrm{m}) \to \mathrm{Ent}_{\mathrm{m}}(f\mathrm{m})$ and $\int (f - f_n)\log(f_n) \to 0$ as $n \to \infty$, so that

$$\lim_{n\to\infty} \mathrm{Ent}_{\mathrm{m}}(f_n\mathrm{m}) + \int (f - f_n)\log(f_n) = \mathrm{Ent}_{\mathrm{m}}(f\mathrm{m}). \tag{8}$$

Also, by the convexity of the Entropy we have that

$$\mathrm{Ent}_{\mathrm{m}}(g_n\mathrm{m}) + \int (g - g_n)\log(g_n) \leq \mathrm{Ent}_{\mathrm{m}}(g\mathrm{m}),$$

which gives

$$\varliminf_{n\to\infty} -\mathrm{Ent}_{\mathrm{m}}(g_n\mathrm{m}) - \int (g - g_n)\log(g_n) \geq -\mathrm{Ent}_{\mathrm{m}}(g\mathrm{m}). \tag{9}$$

Adding equations (8) and (9) we get the claim.

To conclude we need only to show that the map

$$f\mathrm{m} \qquad \mapsto \qquad \mathrm{DE}_{\boldsymbol{\gamma}}(\mu) + \int (f - \tilde{f})e'(\tilde{f}) - \int (g - \tilde{g})e'(\tilde{g}),$$

is continuous w.r.t. weak convergence on sublevels of the Entropy. But this is clear, as on sublevels of the Entropy we have equi-integrability at the level of the f's and, from Lemma 8 and its proof we have a uniform bound on the Entropy of $\nu_{\boldsymbol{\gamma},f\mathrm{m}}$ which gives equi-integrability at the level of the g's. Since $\tilde{f}, \tilde{g}$ are bounded away from 0 and ∞ the thesis follows. $\square$

3.2 Convexity and lower semicontinuity of the squared slope

Theorem 12 (Representation formula for the slope) *For every μ such that $\mathrm{Ent}_{\mathrm{m}}(\mu) < \infty$ it holds*

$$\begin{aligned}
&\sup_{\substack{\nu\in\mathscr{P}_2(X)\\ \nu\neq\mu}} \frac{\Big(\mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}(\nu) - \frac{\lambda^-}{2}W_2^2(\mu,\nu)\Big)^+}{W_2(\mu,\nu)} \\
&= \sup_{\boldsymbol{\gamma}\in\mathrm{GP}} \frac{\Big(\mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}\big(\nu_{\boldsymbol{\gamma},\mu}\big) - \frac{\lambda^-}{2}C(\boldsymbol{\gamma}_\mu)\Big)^+}{\sqrt{C(\boldsymbol{\gamma}_\mu)}},
\end{aligned} \tag{10}$$

where the value of the second expression is taken by definition as 0 if $C(\boldsymbol{\gamma}_\mu) = 0$.

Proof We start with $\geq$. Observe that the following inequality trivially holds:

$$a, b, c \in \mathbb{R}, \quad 0 < b \leq c \qquad \Rightarrow \qquad \frac{(a-b)^+}{\sqrt{b}} \geq \frac{(a-c)^+}{\sqrt{c}}.$$

Thus pick $\boldsymbol{\gamma} \in \mathrm{GP}$ and assume, without loss of generality, that $C(\boldsymbol{\gamma}_\mu) \neq 0$ and $\nu_{\boldsymbol{\gamma},\mu} \neq \mu$. From $C(\boldsymbol{\gamma}_\mu) \geq W_2^2(\mu, \nu_{\boldsymbol{\gamma},\mu}) > 0$ we get

$$\begin{aligned}&\frac{\Big(\mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma},\mu}) - \frac{\lambda^-}{2} W_2^2(\mu, \nu_{\boldsymbol{\gamma},\mu})\Big)^+}{W_2(\mu, \nu_{\boldsymbol{\gamma},\mu})} \\ &\geq \frac{\Big(\mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma},\mu}) - \frac{\lambda^-}{2} C(\boldsymbol{\gamma}_\mu)\Big)^+}{\sqrt{C(\boldsymbol{\gamma}_\mu)}},\end{aligned}$$

which gives the inequality $\geq$.

To prove the converse inequality we will use Lemma 10. Fix $\nu \in \mathscr{P}_2(X)$ with finite Entropy and different from μ. Use Lemma 10 to find a sequence $(\boldsymbol{\gamma}^n) \subset \mathrm{GP}$ such that $\mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma}^n,\mu}) \to \mathrm{Ent}_{\mathrm{m}}(\nu)$ and $C(\boldsymbol{\gamma}^n_\mu) \to W_2^2(\mu, \nu)$ as $n \to \infty$ to get

$$\begin{aligned}&\frac{\Big(\mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}(\nu) - \frac{\lambda^-}{2} W_2^2(\mu, \nu)\Big)^+}{W_2(\mu, \nu)} \\ &= \lim_{n\to\infty} \frac{\Big(\mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma}^n,\mu}) - \frac{\lambda^-}{2} C(\boldsymbol{\gamma}^n_\mu)\Big)^+}{\sqrt{C(\boldsymbol{\gamma}^n_\mu)}}\end{aligned}$$

□

Corollary 13 (Convexity and lower semicontinuity of the squared slope) *The squared slope is convex w.r.t. linear interpolation of measures and lower semicontinuous w.r.t. weak convergence on sublevels of the Entropy.*

Proof Thanks to formula (10) it is enough to show that for every $\boldsymbol{\gamma} \in \mathrm{GP}$ the map

$$\mu \qquad \mapsto \qquad \frac{\left(\Big(\mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma},\mu}) - \frac{\lambda^-}{2} C(\boldsymbol{\gamma}_\mu)\Big)^+\right)^2}{C(\boldsymbol{\gamma}_\mu)},$$

is convex w.r.t. linear interpolation of measures and lower semicontinuous w.r.t. weak convergence on sublevels of the Entropy.

Consider the map

$$\mu = f\mathrm{m} \qquad \mapsto \qquad C(\boldsymbol{\gamma}_\mu) = \int d^2(x,y) \frac{f(x)}{\overline{f}(x)} d\boldsymbol{\gamma}(x,y) = \int f(x) h(x) d\mathrm{m}(x),$$

where $\overline{f}$ is the density of $\pi^1_\# \boldsymbol{\gamma}$ and $h(x) := \int d^2(x,y) d\boldsymbol{\gamma}_x(y)$. It is clearly linear. Also, from property *(ii)* in the definition of GP we have that h is (essentially) bounded; finally, on sublevels of the Entropy we have equi-integrability. Therefore weak convergence implies convergence in duality with bounded functions and thus $\mu \mapsto C(\boldsymbol{\gamma}_\mu)$ is weakly continuous on sublevels of the Entropy.

Thus, from Proposition 11 we know that the map

$$\mu \qquad \mapsto \qquad \mathrm{Ent}_{\mathrm{m}}(\mu) - \mathrm{Ent}_{\mathrm{m}}(\nu_{\boldsymbol{\gamma},\mu}) - \frac{\lambda^-}{2} C(\boldsymbol{\gamma}_\mu),$$

is convex w.r.t. linear interpolation of measures and lower semicontinuous w.r.t. weak convergence on sublevels of the Entropy. Thus the same is true for its positive part. The conclusion follows from the fact that the function $\Psi : [0, +\infty)^2 \to \mathbb{R} \cup \{+\infty\}$ defined by (7) is convex, continuous on $[0,\infty)^2 \setminus \{(0,0)\}$ and increasing in a. □

3.3 Proof of existence and uniqueness

We start with uniqueness. The proof is based on the convexity of the squared slope and of the squared metric derivative:

Lemma 14 (Convexity of squared metric derivative) *Let $(\mu_t^0), (\mu_t^1)$ be two absolutely continuous curves on some interval $I \subset \mathbb{R}$ and define $\mu_t^{1/2} := (\mu_t^0 + \mu_t^1)/2$, $t \in I$. Then $(\mu_t^{1/2})$ is absolutely continuous and the following bound on its metric derivative holds:*

$$|\dot{\mu}_t^{1/2}|^2 \leq \frac{|\dot{\mu}_t^0|^2 + |\dot{\mu}_t^1|^2}{2}.$$

Proof Fix, $t, s \in I$ and pick $\boldsymbol{\gamma}^0 \in \mathrm{OPT}(\mu_t^0, \mu_s^0)$, $\boldsymbol{\gamma}^1 \in \mathrm{OPT}(\mu_t^1, \mu_s^1)$. The plan $(\boldsymbol{\gamma}^0 + \boldsymbol{\gamma}^1)/2$ belongs to $\mathrm{ADM}(\mu_t^{1/2}, \mu_s^{1/2})$ and therefore it holds

$$\begin{aligned} W_2^2(\mu_t^{1/2}, \mu_s^{1/2}) \leq& \int d^2(x,y) d\frac{\boldsymbol{\gamma}^0 + \boldsymbol{\gamma}^1}{2}(x,y) \\ =& \frac{\int d^2(x,y) d\boldsymbol{\gamma}^0(x,y) + \int d^2(x,y) d\boldsymbol{\gamma}^1(x,y)}{2} \\ =& \frac{W_2^2(\mu_t^0, \mu_s^0) + W_2^2(\mu_t^1, \mu_s^1)}{2}, \end{aligned}$$

which gives the absolute continuity. Dividing by $(s-t)^2$ and letting s go to t we get the conclusion. $\square$

Theorem 15 (Uniqueness) *Let (X, d, m) be a normalized metric measure space with Ricci curvature bounded from below, and let $\overline{\mu} \in \mathscr{P}_2(X)$ be such that $\mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) < \infty$. Then there exists at most one curve of maximal slope starting from $\overline{\mu}$.*

Proof Argue by contradiction and assume that there are two curves of maximal slope (μ_t^0) and (μ_t^1), starting from $\overline{\mu}$. Since $\mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) < \infty$ this means that

$$\begin{aligned} \mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) &= \mathrm{Ent}_{\mathrm{m}}\left(\mu_t^0\right) + \frac{1}{2}\int_0^t |\dot{\mu}_s^0|^2 ds + \frac{1}{2}\int_0^t |\nabla \mathrm{Ent}|^2(\mu_s^0) ds, \\ \mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) &= \mathrm{Ent}_{\mathrm{m}}\left(\mu_t^1\right) + \frac{1}{2}\int_0^t |\dot{\mu}_s^1|^2 ds + \frac{1}{2}\int_0^t |\nabla \mathrm{Ent}|^2(\mu_s^1) ds. \end{aligned}$$

Assume that these two curves are different, i.e. for some T_0 it holds $\mu_{T_0}^0 \neq \mu_{T_0}^1$. Now define

$$\mu_t^{1/2} := \frac{\mu_t^0 + \mu_t^1}{2}, \qquad \forall t \geq 0.$$

From the *strict* convexity of the Entropy, the convexity of the squared metric derivative and the convexity of the squared slope we have that

$$\mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) > \mathrm{Ent}_{\mathrm{m}}\left(\mu_{T_0}^{1/2}\right) + \frac{1}{2}\int_0^{T_0} |\dot{\mu}_s^{1/2}|^2 ds + \frac{1}{2}\int_0^{T_0} |\nabla \mathrm{Ent}|^2(\mu_s^{1/2}) ds,$$

which contradicts the inequality (1). $\square$

Now we turn to the existence. Here we will make an additional assumption: we assume that X is boundedly compact, i.e. closed balls are compact (recall that since X is geodesic, this is equivalent to assume that X is locally compact - see e.g. Theorem 2.5.28 of [2]).

This assumption is needed to obtain the following well known result:

Lemma 16 *Assume that X is boundedly compact. Then for every $x_0 \in X$ and $C > 0$ the subset of $\mathscr{P}_2(X)$ of those measures μ such that*

$$\int d^2(x, x_0) d\mu(x) \leq C,$$

is tight.

Proof Just observe that

$$C \geq \int d^2(x, x_0) d\mu(x) \geq \int_{X \setminus B_R(x_0)} d^2(x, x_0) d\mu(x) \geq R^2 \mu(X \setminus B_R(x_0)).$$

□

Theorem 17 (Existence) *Let (X, d, m) be a locally compact, normalized metric measure space with Ricci curvature bounded from below, and let $\overline{\mu} \in \mathscr{P}_2(X)$ be such that $\mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) < \infty$. Then there exists a curve of maximal slope starting from $\overline{\mu}$.*

Proof The proof immediately follows from the arguments presented in [1, Chapter 3] and the weak lower semicontinuity of the slope on sublevels of the Entropy proven here. We briefly recall the main steps of the proof.

Notation and discrete estimate. For any $\tau > 0$ and any $\nu \in \mathscr{P}_2(X)$ let $J_\tau(\nu)$ be the minimum of

$$\sigma \mapsto \mathrm{Ent}_{\mathrm{m}}(\sigma) + \frac{W_2^2(\sigma, \nu)}{2\tau},$$

(which exists and is unique). Define recursively the *discrete solution* as: $\mu_0^\tau := \mu$, $\mu_{n+1}^\tau := J_\tau(\mu_n^\tau)$. Then define the curve $t \mapsto \mu^\tau(t)$ by reparametrization and *variational interpolation*:

$$\begin{aligned} \mu^\tau(n\tau) &:= \mu_n^\tau, \\ \mu^\tau(t) &:= J_{t-n\tau}(\mu_n^\tau), \qquad \forall t \in \big(n\tau, (n+1)\tau\big). \end{aligned}$$

Finally, let $|\dot{\mu}^\tau|(t)$ be the discrete speed defined by

$$|\dot{\mu}^\tau|(t) := \frac{W_2(\mu_n^\tau, \mu_{n+1}^\tau)}{\tau}, \qquad \forall t \in \big[n\tau, (n+1)\tau\big)$$

Recall inequalities 3.2.3 and 3.2.4 of [1]:

$$\frac{1}{2}\int_0^T |\dot{\mu}^\tau|^2(t)dt + \frac{1}{2}\int_0^T |\nabla \mathrm{Ent}|^2(\mu^\tau(t))dt \leq \mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) - \mathrm{Ent}_{\mathrm{m}}(\mu^\tau(T)) \tag{11}$$

Compactness. From (11) we have

$$\begin{aligned} W_2^2(\mu^\tau(T), \overline{\mu}) &\leq \left(\int_0^T |\dot{\mu}^\tau|(t)dt\right)^2 \leq T\int_0^T |\dot{\mu}^\tau|^2(t)dt \\ &\leq 2T(\mathrm{Ent}_{\mathrm{m}}(\overline{\mu}) - \mathrm{Ent}_{\mathrm{m}}(\mu^\tau(T))) \leq 2T\mathrm{Ent}_{\mathrm{m}}(\overline{\mu}), \end{aligned}$$

which shows that for every T the distance between $\mu^\tau(T)$ and $\overline{\mu}$ is bounded uniformly on τ. Thus also the second moments are uniformly bounded, and therefore the set

$\{\mu^\tau(T)\}_\tau$ is tight for any T.
Passage to the limit. Using the tightness, the lower semicontinuity of W_2 w.r.t. weak convergence and the discrete estimate it is not hard to see that there exists a sequence $\tau_n \downarrow 0$ such that $\mu^{\tau_n}(t)$ weakly converge to some $\mu(t)$ as $n \to \infty$ for every $t \geq 0$. By the lower semicontinuity of W_2 w.r.t. weak convergence we have that $t \mapsto \mu(t)$ is absolutely continuous. By the Fatou Lemma and the lower semicontinuity of the Entropy w.r.t. weak convergence we know that the limit curve satisfies

$$\mathrm{Ent}_{\mathrm{m}}(\mu(t)) + \frac{1}{2}\int_0^T |\dot{\mu}|^2(t)dt + \frac{1}{2}\int_0^T \varliminf_{n\to\infty} |\nabla \mathrm{Ent}|^2(\mu^{\tau_n}(t))dt \leq \mathrm{Ent}_{\mathrm{m}}(\mu).$$

Here we make the crucial use of Corollary 13: since the squared slope is lower semicontinuous w.r.t. weak convergence on the sublevels of the Entropy (observe that $\mathrm{Ent}_{\mathrm{m}}(\mu^\tau(t)) \leq \mathrm{Ent}_{\mathrm{m}}(\mu)$ for every t, τ) we have:

$$\varliminf_{n\to\infty} |\nabla \mathrm{Ent}|^2(\mu^{\tau_n}(t)) \geq |\nabla \mathrm{Ent}|^2(\mu(t)),$$

and thus

$$\mathrm{Ent}_{\mathrm{m}}(\mu(t)) + \frac{1}{2}\int_0^T |\dot{\mu}|^2(t)dt + \frac{1}{2}\int_0^T |\nabla \mathrm{Ent}|^2(\mu(t))dt \leq \mathrm{Ent}_{\mathrm{m}}(\mu),$$

which is the thesis. □

4 Stability

The main result of this section is Theorem 21, where it is proven stability of curves of maximal slope of $\lambda-$geodesically convex functionals. We will see thereafter how this Theorem applies to the flow of the Entropy.

We start recalling the following two propositions:

Proposition 18 *Let (X_n, d_n) be compact metric spaces converging to (X, d) via some maps f_n and let $t \mapsto x_n(t) \in X_n$ be curves on $[0,1]$. Assume that these curves are equi-absolutely continuous, that is*

$$\sup_{n\in\mathbb{N}} d_n\big(x_n(t), x_n(s)\big) \leq \int_t^s h(r)dr, \qquad \forall t \leq s, \tag{12}$$

for some $h \in L^1(0,1)$. Then, possibly extracting a subsequence, the sequence (x_n) converges to (x) on $t \in [0,1]$, via the f_n's. The curve $t \mapsto x(t)$ is absolutely continuous and satisfies

$$\int_t^s |\dot{x}(r)|dr \leq \varliminf_{n\to\infty} \int_t^s |\dot{x}_n(r)|dr, \qquad \forall 0 \leq t < s \leq 1. \tag{13}$$

Also, if the function h in (12) belongs to $L^2(0,1)$ (so that the curves $t \mapsto x_n(t)$ are equi-2-absolutely continuous) it holds

$$\int_t^s |\dot{x}(r)|^2dr \leq \varliminf_{n\to\infty} \int_t^s |\dot{x}_n(r)|^2dr, \qquad \forall 0 \leq t < s \leq 1. \tag{14}$$

Proof Possibly extracting a subsequence, not relabeled, we may assume that $n \mapsto x_n(t)$ converges to some $x(t)$ via the f_n's for any $t \in [0,1] \cap \mathbb{Q}$. Fix $t < s \in [0,1] \cap \mathbb{Q}$ and observe that

$$d\big(x(t), x(s)\big) = \lim_{n\to\infty} d\big(x_n(t), x_n(s)\big) \le \int_t^s h(r)dr.$$

This shows that the map $x(t)$ can be extended to the whole $[0,1]$ to an absolutely continuous curve. The fact that this curve is actually the limit of $n \mapsto x_n(t)$ for any t is obvious. To prove (13) notice that

$$\begin{aligned}\int_t^s |\dot{x}(r)|dr &= \sup \sum_i d\big(x(t_i), x(t_{i+1})\big) = \sup \lim_{n\to\infty} \sum_i d_n\big(x_n(t_i), x_n(t_{i+1})\big)\\ &\le \varliminf_{n\to\infty} \sup \sum_i d_n\big(x_n(t_i), x_n(t_{i+1})\big) = \varliminf_{n\to\infty} \int_t^s |\dot{x}_n(r)|dr,\end{aligned}$$

where the supremum is taken among all the partitions of $[t,s]$. Similarly for (14):

$$\begin{aligned}\int_t^s |\dot{x}(r)|^2 dr &= \sup \sum_i \frac{d^2\big(x(t_i), x(t_{i+1})\big)}{t_{i+1}-t_i} = \sup \lim_{n\to\infty} \sum_i \frac{d_n^2\big(x_n(t_i), x_n(t_{i+1})\big)}{t_{i+1}-t_i}\\ &\le \varliminf_{n\to\infty} \sup \sum_i \frac{d_n^2\big(x_n(t_i), x_n(t_{i+1})\big)}{t_{i+1}-t_i} = \varliminf_{n\to\infty} \int_t^s |\dot{x}_n(r)|^2 dr,\end{aligned}$$

□

Proposition 19 *Let (X_n, d_n) be compact metric spaces converging to (X,d) in the Gromov-Hausdorff convergence via the maps f_n. Let $\mathscr{E}_n : X_n \to \mathbb{R} \cup \{+\infty\}$ be λ-geodesically convex functionals, with λ independent on n, equibounded from below and Γ-converging to some $\mathscr{E} : X \to \mathbb{R} \cup \{+\infty\}$. Then $\mathscr{E}$ is bounded from below and λ-geodesically convex.*

Proof The fact that $\mathscr{E}$ is bounded from below is trivial. Pick $x, y \in X$ and choose $x_n, y_n \in X_n$ such that $\mathscr{E}_n(x_n) \to \mathscr{E}(x)$ and $\mathscr{E}_n(y_n) \to \mathscr{E}(y)$ (this is possible by definition of Γ-limit). Choose a geodesic $t \mapsto \gamma_n(t) \in X_n$ from x_n to y_n parametrized on $[0,1]$ and with constant speed. Up to passing to a subsequence, not relabeled, we may assume that these geodesics converge to a limit geodesic γ from x to y. Now observe that from the λ-geodesic convexity of $\mathscr{E}_n$ we have

$$\mathscr{E}_n\big(\gamma_n(t)\big) \le (1-t)\mathscr{E}_n(x_n) + t\mathscr{E}_n(y_n) - \frac{\lambda}{2} t(1-t) d_n^2(x_n, y_n)$$

while from the $\Gamma-$convergence of the $\mathscr{E}_n$'s and the fact that $\gamma_n(t) \to \gamma(t)$ for any t we have

$$\mathscr{E}\big(\gamma(t)\big) \le \varliminf_{n\to\infty} \mathscr{E}_n\big(\gamma_n(t)\big).$$

Plugging together these two inequalities and recalling the choice made of x_n and y_n we get

$$\mathscr{E}\big(\gamma(t)\big) \le (1-t)\mathscr{E}(x) + t\mathscr{E}(y) - \frac{\lambda}{2} t(1-t) d^2(x,y).$$

This shows the λ-geodesic convexity of $\mathscr{E}$. □

The proof of stability is based on the following bound on the slope of the limit functional $\mathscr{E}$:

Proposition 20 *Let (X_n, d_n), (X, d), $\mathscr{E}_n$ and $\mathscr{E}$ as in the statement of the previous proposition. Then on the domain of $\mathscr{E}$ it holds*

$$|\nabla \mathscr{E}| \leq \Gamma - \varliminf_{n\to\infty} |\nabla \mathscr{E}_n|.$$

Proof Fix $x \in X$ such that $\mathscr{E}(x) < \infty$ and recall that

$$|\nabla \mathscr{E}|(x) = \sup_{y\in X} \left(\frac{\mathscr{E}(x) - \mathscr{E}(y)}{d(x,y)} - \frac{\lambda^-}{2} d(x,y) \right)^+$$

and similarly for the $\mathscr{E}_n$'s. Fix $y \in X$ and find a sequence $n \mapsto y_n \in X_n$ converging to y via the f_n's such that $\mathscr{E}_n(y_n) \to \mathscr{E}(y)$. Also, let $n \mapsto x_n \in X_n$ be any sequence converging to x. We have

$$\lim_{n\to\infty} d_n(x_n, y_n) = d(x,y), \qquad \varliminf_{n\to\infty} \mathscr{E}_n(x_n) \geq \mathscr{E}(x)$$

and therefore

$$\begin{aligned}\left(\frac{\mathscr{E}(x) - \mathscr{E}(y)}{d(x,y)} - \frac{\lambda^-}{2} d(x,y) \right)^+ &\leq \varliminf_{n\to\infty} \left(\frac{\mathscr{E}_n(x_n) - \mathscr{E}_n(y_n)}{d_n(x_n,y_n)} - \frac{\lambda^-}{2} d_n(x_n,y_n) \right)^+ \\ &\leq \varliminf_{n\to\infty} |\nabla \mathscr{E}_n|(x_n).\end{aligned}$$

□

Theorem 21 (Stability of curves of maximal slope) *With the same notation and assumptions of Proposition 19, assume that $n \mapsto x_n \in X_n$ is a sequence converging to some $x \in X$ via the f_n's such that $\mathscr{E}_n(x_n) \to \mathscr{E}(x)$ as $n \to \infty$.*

Then for any choice of $t \mapsto x_n(t)$ of curves of maximal slope for $\mathscr{E}_n$ in X_n starting from x_n it holds:

i) every subsequence of $n \mapsto (x_n(t))$ admits a further extraction which converges to a limit curve $(x(t))$ in X.

ii) any limit curve is a curve of maximal slope for $\mathscr{E}$ starting from x.

Proof **(i): compactness of the set of curves.** By equation 2.4.26 and inequality 2.4.24 of [1] we have

$$|\dot{x}_n|(t) \leq \sqrt{\frac{(1 + 2\lambda^+ t)e^{-2\lambda t}(\mathscr{E}_n(x_n) - \inf \mathscr{E}_n)}{t}},$$

so that from the assumptions we get that the curves $(x_n(t))$ are equi-absolutely continuous. To get (i) just apply Proposition 18.

(ii): any limit curve is of maximal slope. We know from Proposition 19 that $\mathscr{E}$ is bounded from below and λ-geodesically convex. Now assume for notational simplicity that the full sequence of curves $t \mapsto x_n(t)$ converges to some $t \mapsto x(t)$. We know that

$$\mathscr{E}_n(x_n) \geq \mathscr{E}_n(x_n(t)) + \frac{1}{2}\int_0^t |\dot{x}_n(s)|^2 ds + \frac{1}{2}\int_0^t |\nabla \mathscr{E}_n|^2(x_n(s)) ds,$$

for every $n \in \mathbb{N}$, $t \geq 0$ and we want to prove that

$$\mathscr{E}(x) \geq \mathscr{E}(x(t)) + \frac{1}{2}\int_0^t |\dot{x}(s)|^2 ds + \frac{1}{2}\int_0^t |\nabla \mathscr{E}|^2(x(s))ds.$$

Since we know that

$$\begin{array}{lll}
\lim\limits_{n\to\infty} \mathscr{E}_n(x_n) & = \mathscr{E}(x) & \text{by assumption,} \\
\varliminf\limits_{n\to\infty} \mathscr{E}_n(x_n(t)) & \geq \mathscr{E}(x(t)) & \text{by } \Gamma - \text{convergence,} \\
\varliminf\limits_{n\to\infty} \displaystyle\int_0^t |\dot{x}_n(s)|^2 ds & \geq \displaystyle\int_0^t |\dot{x}(s)|^2 ds & \text{by Proposition 18,} \\
\varliminf\limits_{n\to\infty} |\nabla \mathscr{E}_n(x_n(t))| & \geq |\nabla \mathscr{E}(x(t))| & \text{by Proposition 20,}
\end{array}$$

the conclusion follows. □

It is immediate to apply this result to the spaces $(\mathscr{P}(X_n), W_2)$ and the Entropy functionals $\mathrm{Ent}_{\mathrm{m_n}}(\cdot)$, where (X_n, d_n, m_n) are normalized compact metric measure spaces converging to some (X, d, m) in the MGH sense.

Indeed, recall that if (X_n, d_n) converges to (X, d) in the GH sense and $f_n : X \to X_n$ are approximate isometries, then the spaces $(\mathscr{P}(X_n), W_2)$ (which are compact) converge to $(\mathscr{P}(X), W_2)$ in the GH sense as well. Also, a natural choice of approximate isometries is given by $(f_n)_\# : \mathscr{P}(X) \to \mathscr{P}(X_n)$.

Thus we need only to show that $\mathrm{Ent}_{\mathrm{m_n}}(\cdot)$ Γ-converges to $\mathrm{Ent}_{\mathrm{m}}(\cdot)$ as $n \to \infty$. This is just a restatement of well known results in this setting. Observe that the $\Gamma - \varliminf$ inequality follows by:

$$\varliminf_{n\to\infty} \mathrm{Ent}_{\mathrm{m_n}}(\mu_n) \geq \varliminf_{n\to\infty} \mathrm{Ent}_{(\mathrm{g_n})_\#\mathrm{m_n}}((g_n)_\#\mu_n) \geq \mathrm{Ent}_{\mathrm{m}}(\mu), \tag{15}$$

where (μ_n) is any sequence converging to μ via the maps $(f_n)_\#$'s (here the g_n's are approximate inverse of the f_n's, and it is immediate to verify that in this case the $(g_n)_\#$'s are approximate inverses of the $(f_n)_\#$'s). For the validity of the first inequality in (15) see Lemma 9.4.5 of [1], while the second one is a consequence of the joint lower semicontinuity of the relative Entropy, see e.g. Lemma 9.4.3 of [1] for a proof of this in Hilbert spaces and [12], [9] for general metric spaces.

The $\Gamma - \varlimsup$ inequality is more technical. Given $\mu = \rho\mathrm{m}$, it can be proved either by smoothing the density $\rho \circ g_n$, as in the proof of Theorem 4.15 of [9], or via a coupling of the metric measures spaces as in Section 4.5 of [12], we omit the details (observe that the construction of our map $\mu \mapsto \nu_{\gamma,\mu}$ is actually pretty similar to Sturm's construction - although our case is technically simpler, because we work on a fixed space, while Sturm was coupling measures in different spaces).

Let us repeat that these inequalities concerning limit of Entropies are not new at all. Actually, they are the heart of the proof of stability of Ricci curvature bounds, and here we are just restating these results in the terminology of Γ-convergence.

5 Final comments

- **Not necessarily finite variation**. We never assumed $\mathrm{m} \in \mathscr{P}_2(X)$.
- **Lack of linearity**. Let us recall that it is *not* true that the Heat flow, in this generality, is linear. This means that if (μ_t) and (ν_t) are two curves of maximal

slope of the Entropy, then not necessarily $(\frac{\mu_t+\nu_t}{2})$ is. Indeed, as Sturm and Ohta showed in [13], in the case of Finsler manifolds with Ricci curvature bounded from below, the Heat equation is *not* linear. On the other side, Savaré showed that if a normalized metric metric space satisfy a certain *Local Angle Condition*, then the flow of the Entropy satisfies: the Evolution Variational Inequality, the λ-exponential contractivity and the linearity (see [11]).

- **Porous media**. As we argued, the Heat flow can be naturally defined as curve of maximal slope of the Entropy in a space with Ricci curvature bounded from below. Now, we know from [12] and [9] that a suitable definition of bound on the Ricci curvature plus bound from above on the dimension concerns the study of the (distorted) geodesic convexity properties of the functional

$$\rho \qquad \mapsto \qquad U_m(\rho) := \int e_m(\rho), \qquad\qquad e_m(z) := \frac{1}{m-1} z^m,$$

for suitable $m < 1$ (see [12], [9] or Chapter 29 of [14] for the definition of distorted geodesic convexity).

We also know that the gradient flow of such functional on $\mathbb{R}^d$ w.r.t. W_2 produces solutions of the porous medium equation (as shown in the seminal paper of Otto [10]). It is then natural to ask whether the same techniques that we used here for the study of the Entropy functional can be applied to these other functionals.

For what concerns the stability, the answer is yes, at least for the case of non-negative Ricci curvature. Indeed in this case the functional U_m is geodesically convex and there is Γ-convergence as soon as the sequence of metric measure spaces converges in the MGH sense, as shown in [9] and [12].

For what regards existence and uniqueness, quite surprisingly, the approach proposed here gives no insights. The reason is the following: the key fact that we used to prove the convexity of the squared slope is the convexity of the map $\mu \mapsto \mathrm{DE}_{\boldsymbol{\gamma}}(\mu) = \mathrm{Ent_m}(\mu) - \mathrm{Ent_m}(\nu_{\boldsymbol{\gamma},\mu})$. Now, shortly said, the convexity of $\mu \mapsto \mathrm{DE}_{\boldsymbol{\gamma}}(\mu)$ follows by application of the Jensen inequality to the function:

$$(0,\infty)^2 \ni (a,b) \qquad \mapsto \qquad a^2 e''(b),$$

as shown in the proof of Proposition 11. Thus, if we want to replicate the approach for the functionals U_m, we need the convexity of the map

$$(0,\infty)^2 \ni (a,b) \qquad \mapsto \qquad a^2 e''_m(b).$$

However, this map is *never* convex for $m < 1$ (while it is for $1 < m \leq 2$). And it is not hard to cook up from this observation an explicit counterexample to the convexity of $\mu \mapsto U_m(\mu) - U_m(\nu_{\boldsymbol{\gamma},\mu})$. We actually found pretty curious to discover that our arguments work only because of some very special properties of the Entropy.

- **Contractivity**. There is no contractivity result behind our proof of uniqueness (and I personally don't believe that in this generality there is λ-exponential contraction of W_2 along two flows).

- **Other uses of the convexity of the slope**. The fact that convexity of the squared slope plus strict convexity of the functional implies uniqueness of the curve of maximal slope is true in general. Thus it could be applied, in principle, also to non-geodesically convex functionals. A first attempt in this direction could be the Entropy functional on the Heisenberg group, in order to obtain uniqueness of the flow without calling into play the hypoelliptic Heat equation (recall that the Heisenberg group has no bound from below on the Ricci curvature, or, which is the same, the Entropy is not λ-geodesically convex for any $\lambda \in \mathbb{R}$. See [8]).

Other situations of potential use are those coming from the study of gradient flows w.r.t. the distance Wb_2 introduced in [5].

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